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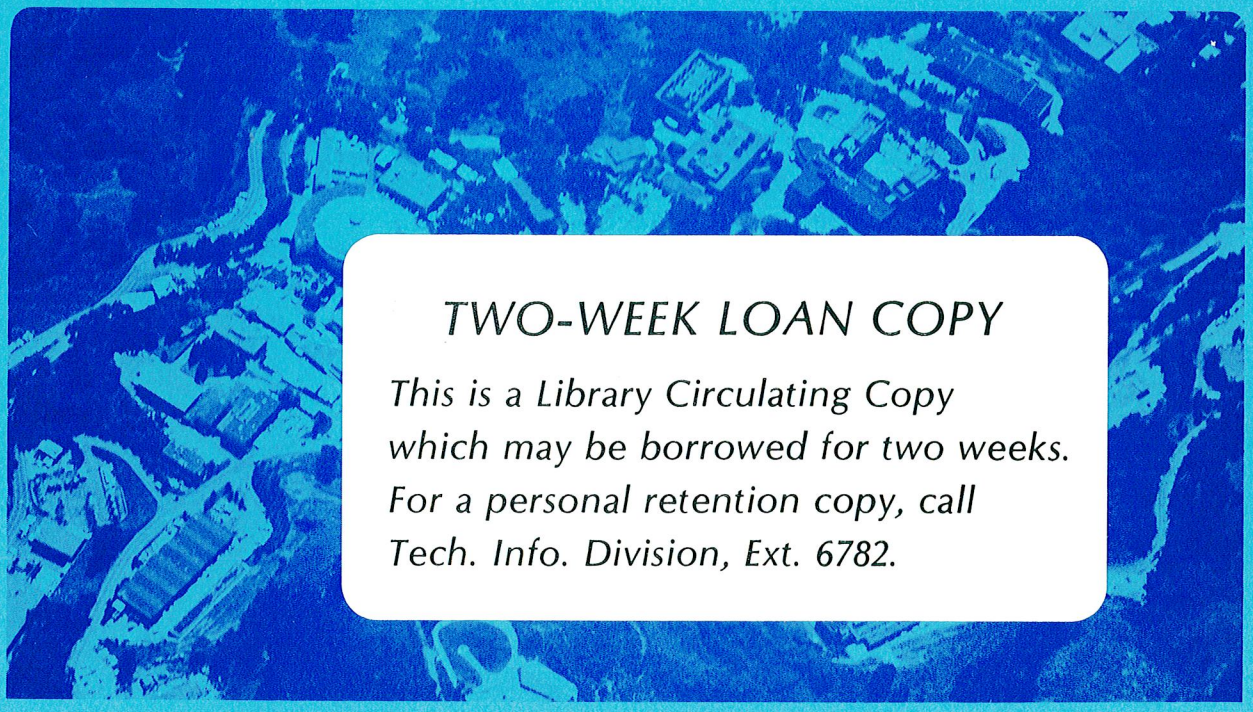

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Abstract

Multiphase fluid displacement in a porous medium gives rise naturally to the occurrence of steep fronts, for example between different fluids or between regions of differing chemical concentrations. Such fronts pose substantial difficulty for most numerical methods. However, the recently developed random choice numerical method has been found capable of following effectively even perfectly sharp fronts. We discuss an application to the calculation of immiscible displacement in a petroleum reservoir, including the effects of capillary pressure and gravity. Previous work with W. Proskurowski has considered the limiting hyperbolic case of zero capillary pressure with gravity neglected. Numerical results of our current work for solving a model problem of two-phase displacement in two dimensions indicate that the effects of the additional possible interactions of shock and expansion waves permitted by the inclusion of gravity can be handled efficiently within the framework of the random choice method.

Introduction

Multiphase fluid displacement in a porous medium gives rise naturally to the occurrence of propagating steep fronts, for example between different fluids or between regions of differing chemical concentrations. Even though these fronts may not be present initially, they can develop with time as a consequence of the inherent nonlinearities of a given problem. For problems in which the diffusive effects of capillary pressure are small the fronts can be very steep, becoming sharp discontinuities in the limiting case of zero capillary pressure.

The following of steep fronts poses substantial difficulty for conventional numerical methods, which are based on differencing or other discretization of the governing partial differential equations. Such methods generally rely on underlying assumptions on the smoothness of solutions that are violated when steep fronts are present. As a result, devices such as the introduction of artificial dissipation may be required for stability or to ensure that the correct solution is obtained.

In an attempt to circumvent these difficulties a study was initiated at the Lawrence Berkeley Laboratory several years ago to adapt the random choice numerical method, a novel method that is inherently

capable of following even perfectly sharp fronts, to solving problems of fluid displacement in porous media. The initial phase of that study was reported in [6], where a standard model problem of two-phase, immiscible, incompressible displacement with sharp fronts was solved. In this model problem the effects of gravity and of capillary pressure are neglected, and one obtains for one space dimension a single conservation law, the Buckley-Leverett equation,

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial x} f(s) = 0 \ . \tag{1}$$

The quantity $s(x,t)$ is the saturation of the wetting fluid (the fraction of the available pore volume occupied by the fluid), and x and t are the space and time variables. Of particular interest is the non convexity of $f(s)$, the fractional volumetric flow rate of the wetting fluid, which for this model problem has exactly one inflection (Fig. 1).

The framework for utilizing the random choice method for the model problem was developed in [6], where it was found that the numerical solution could be obtained accurately and efficiently. The tracking of discontinuities was accomplished without difficulty for the test problems considered, which were for one space dimension and also for two space dimensions for cases in which the solutions were essentially one dimensional in nature. The correct weak solution was obtained, a feature that need not be present with other numerical methods (see, for example, [11]).

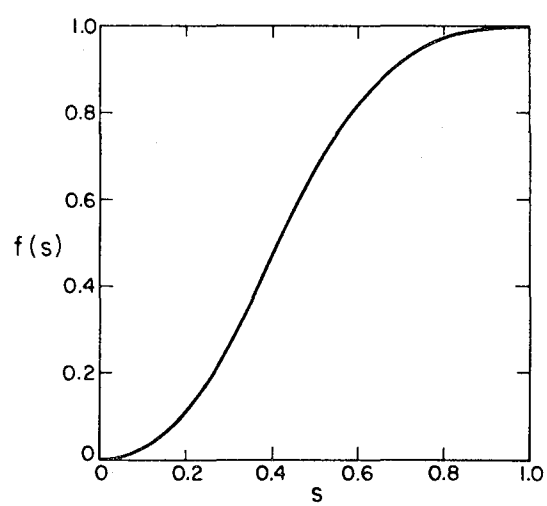

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Figure 1

In [2], the general two space dimensional case was considered, for which (1) becomes

$$\frac{\partial s}{\partial t} + \underset{\sim}{q} \cdot \nabla[f(s)] = 0 \ . \tag{2}$$

The quantity $\underset{\sim}{q}$ is the total velocity and satisfies equations simultaneous with (2), but, for the purposes of the study, $\underset{\sim}{q}$ was taken simply to be a fixed, physically representative function of x and y. (q does not appear explicitly in (1), since in one space dimension in the absence of sources it is a constant that can be absorbed into the other variables.) By means of fractional splitting, the one-dimensional results of [6] can be applied directly to the solution of (2). It was found in [2] that the random choice method can obtain a numerical solution of the two dimensional test problems efficiently and accurately.

In [1] the full two space dimensional case, including the simultaneous equations for $\underset{\sim}{q}$, was considered and solved numerically. For the test problems investigated, the computational demands of calculating $\underset{\sim}{q}$ were several times greater than those of advancing s in time from (2) by the random choice method, but the total computational efficiency was still found to be most satisfactory.

Subsequent work at Rockefeller University by Glimm, Marchesin, and McBryan [8],[9] considered in detail the problems arising from front instability (fingering) that can occur for the full two space dimensional case. Such instability does not arise when (2) is considered alone for fixed $\underset{\sim}{q}$, but results from the coupling between (2) and the simultaneous equations for $\underset{\sim}{q}$. A two-dimensional front-tracking procedure was developed for the random choice method that is not subject to the inaccuracies that may be introduced by one space dimensional fractional splitting; such inaccuracies can be particularly evident for a fluctuating, unstable front.

In the present paper we include a discussion of some of our current work in extending the random choice method to fluid displacement that includes the effects of gravity and of capillary pressure. Attention is on the case for which the capillary pressure is small so that fronts will be sufficiently steep for the random choice method to be of interest. For larger capillary pressure and more diffuse fronts, standard numerical methods generally are suitable.

Equations of Two-Phase Fluid Displacement

The equations, suitably normalized, for two-phase, immiscible, incompressible displacement in a porous medium are (see, for example, Ch. 1 of [10])

$$\frac{\partial s}{\partial t} + \underset{\sim}{q} \cdot \nabla f(s) - b \frac{\partial}{\partial z} g(s) - c \nabla \cdot [h(s) \nabla s] = 0 \tag{3}$$

$$\nabla \cdot \underset{\sim}{q} = Q \tag{4}$$

$$\underset{\sim}{q} = -\lambda(s)\nabla p + b\hat{g}(s)\underset{\sim}{e}_k + c\hat{h}(s)\nabla s \ . \tag{5}$$

Here $s(\underset{\sim}{x},t)$ and $\underset{\sim}{q}(\underset{\sim}{x},t)$ are the saturation and total velocity, as discussed in the previous section, and $p(\underset{\sim}{x},t)$ is pressure. (s is the saturation of the wetting fluid; $1 - s$ is the saturation of the non-wetting fluid.) The coordinates $\underset{\sim}{x} = (x,y,z)$ are chosen so that gravity acts downward in the negative z direction ($\underset{\sim}{e}_k$ is the unit vector in the positive z direction). The quantity Q represents the sources and sinks, and the porosity, taken to be constant, has been absorbed into the other variables. Eq. (3) is the saturation equation, (4) arises from the incompressibility condition, and (5) from Darcy's law.

The quantities $f(s)$, $g(s)$, $h(s)$, $\lambda(s)$, $\hat{g}(s)$, and $\hat{h}(s)$ are functions of the relative permeabilities and capillary pressure, which are empirically determined functions of saturation alone. Typically, for petroleum reservoir problems the fractional flow $f(s)$ has an S-shaped graph with one inflection, which can be conveniently modeled by

$$f(s) = \frac{s^2}{s^2 + a(1-s)^2} \ , \tag{6}$$

where a is the ratio of the viscosities of wetting to non-wetting liquids [11]. The graph in Fig. 1 is for $a = 1/2$. Eq. (6) is convenient for numerical computation while, at the same time, representing adequately the essential features of petroleum reservoir behavior. It is based on taking the (normalized) relative permeabilities being equal simply to the square of the saturations.

Corresponding models for the other functions are $\hat{g}(s) = a(1-s)^2$, $g(s) = \hat{g}(s)f(s)$, and $\lambda(s) = s^2 + a(1-s)^2$. If the (normalized) capillary pressure is denoted by $p_c(s)$, then $h(s) = (-dp_c/ds)g(s)$ and $\hat{h}(s) = \frac{1}{2}(-dp_c(s)/ds)[a(1-s)^2-s^2]$. The quantity $-dp_c(s)/ds$ is positive. Its precise form is not of importance here, as the effects of capillary pressure will not be discussed in detail. For purposes of numerical experimentation it can often be taken, as in [11], simply to be constant.

In (3) and (5) the terms with coefficient b arise from the effects of gravity and the terms with coefficient c from capillary pressure. These coefficients give more or less the relative magnitudes of the terms, as $g(s)$, $\hat{g}(s)$, $h(s)$, and $\hat{h}(s)$ have all been normalized. The coefficient c is non-negative (and presumably small); b may be positive, negative, or zero.

Eqs. (3), (4), and (5) are to be solved for $s(\underset{\sim}{x},t)$ and $p(\underset{\sim}{x},t)$ given the initial value of s and suitable conditions on s and p on the boundary of the domain of interest. The boundary conditions on p typically ensure that $\underset{\sim}{q}$ could be determined uniquely from (4) and (5) were s known. In [1], [2], and [6] the case $b = c = 0$ was considered. Here we permit b and c to be nonzero and consider solving (3), (4), and (5) numerically using the random choice method.

Random Choice Method

The random choice method (also called Glimm's method or the piecewise sampling method) is based on an elegant mathematical construction of Glimm [7] that was developed into a practical and efficient computational algorithm by Chorin [3],[4]. For a single nonlinear conservation law in one space dimension of the form (1), the method advances a solution one step in time by first approximating the solution $s(x,t_j)$ at the time t_j by a piecewise-constant function on a spatial grid. The function is equal to $s_i^j = s(x_i,t_j)$ in each interval $x_i - \frac{1}{2}\Delta x < x \leq x_i + \frac{1}{2}\Delta x$, where Δx is the mesh spacing. Then the exact (weak) solution of (1) is constructed analytically for the piecewise-constant initial data, and this analytic solution is sampled at the new time to obtain values for the new piecewise-constant approximation.

If the time increments Δt are chosen sufficiently small so that the Courant-Friedrichs-Lewy condition $(\Delta t/\Delta x) \max |f'(s)| < 1$ is satisfied, then the waves propagating from the discontinuities in the initial piecewise constant data will not interact. This permits the exact solution of (1) to be carried out for a given time step by joining together the separate solutions of the Riemann problems, (1) with initial data

$$s(x,t_j) = \begin{cases} s_i^j , & x \leq x_i + \frac{1}{2}\Delta x \\ s_{i+1}^j , & x > x_i + \frac{1}{2}\Delta x \end{cases} . \tag{7}$$

The practical success of the random choice method depends upon being able to solve these associated Riemann problems efficiently.

In [6] the Riemann problem solutions were obtained for the typical S-shaped porous-flow $f(s)$ depicted in Fig. 1. Because this function has one inflection point, the solutions may have combination shock and expansion wave solutions, a complexity that is not possible in the usual gas-dynamics case for which $f(s)$ is strictly convex. Nevertheless, the calculation of these solutions can be carried out quite efficiently, as described in [6]. In the next section we shall investigate the complications introduced in porous-flow problems by the inclusion of the function $g(s)$ arising from gravitational effects.

An important feature of the random choice method in one space dimension is that discontinuities are propagated sharply in a stable manner -- no numerical dissipation is introduced. Also of significance is that the correct weak solution for a purely hyperbolic problem is obtained, corresponding to the limiting solution of parabolic problems as the dissipation approaches zero.

Solution of Riemann Problems

In solving (3) by the random choice method with fractional splitting, one "splits off" the capillary pressure term and then solves successive one-dimensional Riemann problems such as (1),(7).

For the x- and y-components of (3) these problems are the same as those considered in [6] for the case without gravity, since only the function f(s) enters. For the z-component, however, one obtains the equation

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial z}\,[\phi(s)] = 0 \ , \tag{8}$$

where

$$\phi(s) = wf(s) - bg(s) \ .$$

Here w is a constant representing the z-component of the total velocity in the mesh interval under consideration.

For convenience, the Riemann-problem initial discontinuity may be translated to the (z,t) origin, so that the initial condition for (8), corresponding to (7), becomes

$$s(z,0) = \begin{cases} s_B \ , & z \leq 0 \\ s_A \ , & z > 0 \end{cases} \tag{9}$$

where s_B and s_A are the values respectively below and above the z mesh point under consideration.

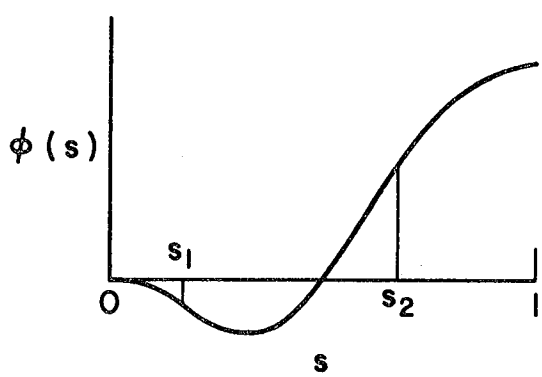

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Figure 2

The qualitative nature of the possible solutions of (8),(9) depends upon the relative values of w and b, since as w varies over the mesh so does the shape of $\phi(s)$ and its number of inflections. This complication does not arise for the case in which gravity is neglected ($b = 0$), for which the fractional flow function has the same shape at each mesh interval, always with one inflection. The three possible Riemann problem solution configurations for $b = 0$ are derived in [6].

The more complex configurations possible when b is not zero are considered here. If $|w/b|$ is not too large, then the number of inflections in $\phi(s)$ typically increases from one to two. For the model $f(s)$ and $g(s)$ this occurs if $-b < w < ab$ for $b > 0$, or if $ab < w < -b$ for $b < 0$.

Fig. 2 depicts a typical graph of $\phi(s)$ for the two-inflection case with $b > 0$ and $w > 0$. Inflections are at s_1 and s_2.

The unique weak solution of (8),(9) for this $\phi(s)$ can be determined from the Rankine-Hugoniot and generalized entropy conditions (see [6] and the references therein). These conditions lead to the following:

If the values of s_A and s_B are such that the interval between them contains either no inflection points or one inflection point of $\phi(s)$, then the situation is covered by the discussion in [6], which shows that an initial discontinuity can propagate only as a single shock, an expansion wave, or a combination single shock in contact with an expansion wave.

If the interval between s_A and s_B contains two inflection points, then, assuming as depicted in Fig. 2 $\phi'(0) \leq 0$, $\phi'(1) \geq 0$, the case for the model $f(s)$ and $g(s)$ considered here, one obtains for $b > 0$

(i) If $s_B < s_A$ holds:

The solution of (8),(9) consists of two shocks separated by an expansion wave. One shock propagates in the positive z direction and the other in the negative z direction. The advancing shock has speed $dz/dt = [\phi(s_A)-\phi(\bar{s}_A)]/(s_A-\bar{s}_A)$, where $\bar{s}_A < s_A$ is the value of s at which the line segment from $(s_A,\phi(s_A))$ lying below $\phi(s)$ is tangent. The receding shock has speed $dz/dt = -[\phi(s_B)-\phi(\bar{s}_B)]/(\bar{s}_B-s_B)$, where $\bar{s}_B > s_B$ is the value of s at which the line segment from $(s_B,\phi(s_B))$ lying below $\phi(s)$ is tangent.

(ii) If $s_B > s_A$ holds.

The solution of (8),(9) consists of a single shock and

(a) no expansion wave, if the chord joining $(s_A,\phi(s_A))$ with $(s_B,\phi(s_B))$ does not intersect $\phi(s)$ in (s_A,s_B). The shock speed in this case is $dz/dt = [\phi(s_B)-\phi(s_A)]/(s_B-s_A)$.

(b) an expansion wave in contact with the shock, if the chord joining $(s_A, \phi(s_A))$ with $(s_B, \phi(s_B))$ intersects $\phi(s)$ in (s_A, s_B). The shock speed and relative position of the expansion wave vary with differing configurations of the intersection point and the sign of w.

For $b < 0$ analogous results hold.

In practice, a computational algorithm will determine the number of inflection points lying between s_A and s_B for a given Riemann problem and branch accordingly to the appropriate solution. In many cases it is necessary to construct the convex or concave hull of $\phi(s)$ to determine shock propagation speeds, but this can be accomplished with a standard root-finding subroutine (see [6]).

Sampling procedure

Once the Riemann problem solution is obtained, it is evaluated at a sampling point in the new mesh interval in z at the new time to obtain the new value for s in the interval. Experience has substantiated that the van der Corput equidistributed quasi-random sampling procedure proposed in [5] is favorable for use with the random choice method. The nth number a_n in the basic van der Corput sampling sequence is

$$a_n = \sum i_k 2^{-(k+1)} ,$$

where the binary expansion for n is

$$n = \sum i_k 2^k .$$

For a discussion of the merits of this procedure and its generalization see [5].

Numerical Solution Procedure

A computer program written previously with W. Proskurowski has been extended to solve (3),(4),(5) with the random choice method on a two space dimensional domain for $c \neq 0$ and $b \neq 0$. For the unit square $0 \leq x \leq 1$, $0 \leq z \leq 1$, a uniform grid of mesh length $\Delta x = \Delta z = k$ is placed on the square, and p is approximated by its value at the mesh points $x_i = ik$, $z_j = jk$. The saturation s is approximated at points on a staggered grid that bisects the p mesh.

Let $s^{(n)}$ denote the approximate solution for the saturation at time $t = t_n$. Then $p^{(n)}$ and $q^{(n)} = (u^{(n)}, w^{(n)})$ are obtained by solving discrete aproximations to (4),(5) and accompanying boundary conditions, using $s^{(n)}$ for s. Standard discretization methods can be used with acceptable accuracy on the uniform mesh if saturation discontinuity fronts are reasonably smooth. Eq. (3) is advanced one step in time to obtain $s^{(n+1)}$ by solving successively

$$\frac{\partial s}{\partial t} + u^{(n)} \frac{\partial}{\partial x} f(s) = 0$$

$$\frac{\partial s}{\partial t} + w^{(n)} \frac{\partial}{\partial z} f(s) - b \frac{\partial}{\partial z} g(s) = 0$$

by the one-dimensional random choice method, and

$$\frac{\partial s}{\partial t} - c \nabla \cdot [h(s) \nabla s] = 0$$

by a standard explicit method. Special two-dimensional shock-tracking techniques are not included in the numerical experiments reported here.

As a particular example we solve a cross-sectional problem similar to that described in [11]. The unit square is initially occupied completely by the wetting fluid $(s = 1)$. For $t > 0$ a line source injects the nonwetting fluid $(s = 0)$ uniformly along $x = 0$, and fluid is removed at a line sink of the same total strength located at $x = 1$, $0.4 \leq y \leq 0.6$. The normal component of $\tilde{q}$ and normal derivative of s are taken to be zero at the boundaries. The capillary pressure coefficient c is taken to be zero, for which the solution has a sharp propagating discontinuity front.

The contour plots shown in Fig. 3 indicate the solution obtained on a mesh with spacing $k = 1/40$ for a test problem with $b = 5$ and $a = 5$. The line source and sink each have total strength of 20, which results in values of w ranging from about 20 near the injection line to about -2 near $x = 1$. The contours are curves of constant saturation in increments of 0.1, increasing from left to right. The coordinate tick marks indicate the mesh intervals.

The first plot is for a time $(t = 0.2)$ shortly after injection indicating the rapid upward buoyant motion of fluid from the source to the top of domain, along which it spreads out toward the right boundary. The successive plots $(t = 0.11, 0.18, 0.25)$ show positions of the interface as the injected fluid moves to the right boundary and down toward the sink. Because the contour plotting program performs interpolation on the data, contours that should lie on top of one another are smeared out over a mesh interval. Thus the advancing front, which is calculated by the random choice method as essentially a sharp discontinuity, is made to appear as a one mesh-width band of parallel contour lines in the figure. The occasional undulations in the advancing front, of the order of one mesh spacing, are typical of the random choice method. These random fluctuations do not cause numerical difficulty, however, even for the present case of $c = 0$, as the front is seen to move in a stable manner.

Initial experiments have indicated that the computer time required for the present program, although greater than that reported in [1] for the $c = b = 0$ case, is quite satisfactory. The greater part of the time appears to be devoted to solving (4),(5) by the fast-Poisson preconditioned conjugate gradient method, as was the case in [1]. Thus it is indicated that the inclusion of gravitational and capillary pressure effects in (3) fit comfortably within the framework of the random choice method. It is planned to discuss these matters more fully in a subsequent study.

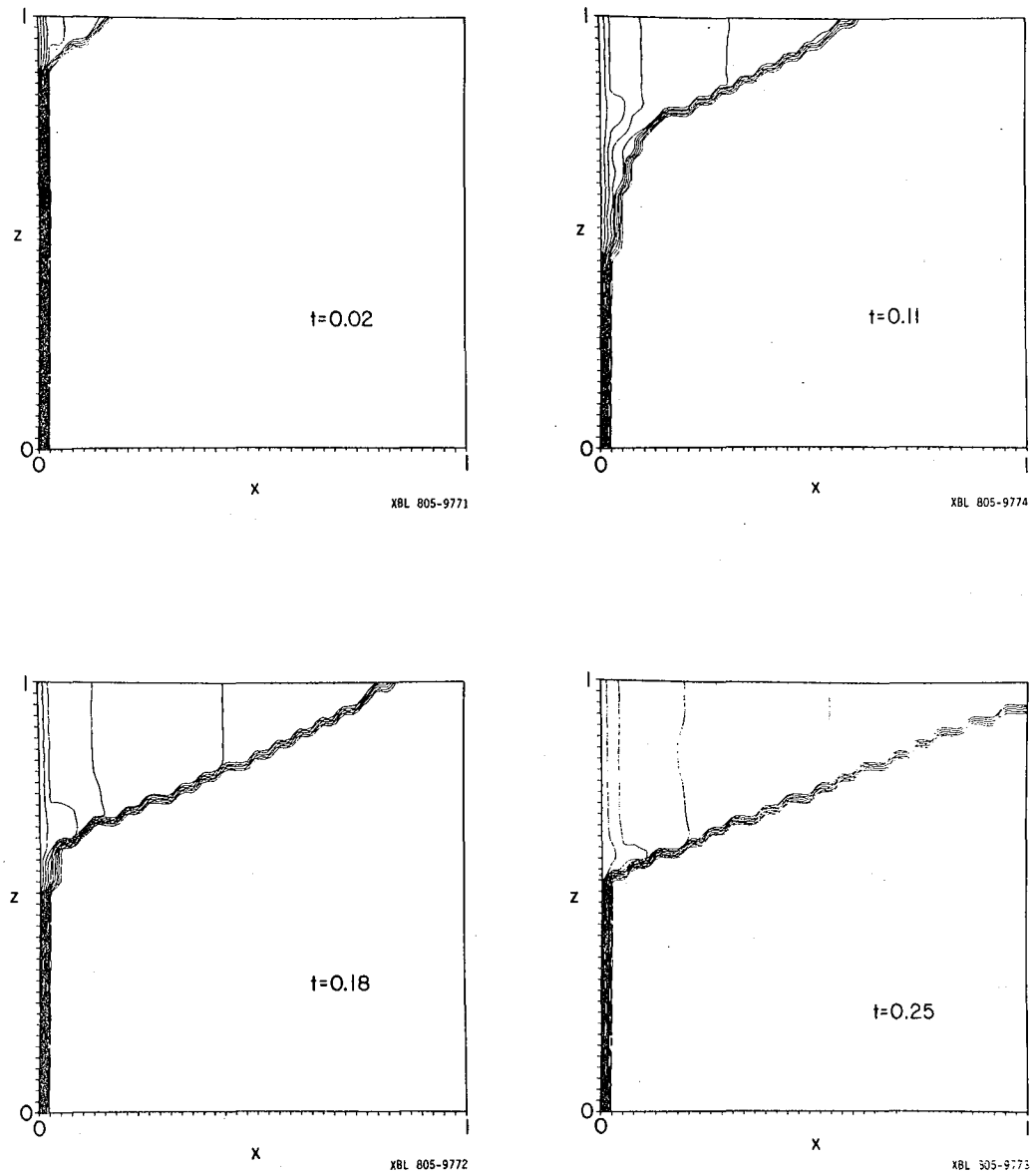
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Figure 3

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