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GAUGE INVARIANT DESCRIPTION OF HEAVY QUARK BOUND STATES IN QUANTUM CHROMODYNAMICS

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Gauge Invariant Description of Heavy Quark Bound States in Quantum Chromodynamics

by

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Abstract

A model for a heavy quark meson is proposed in the framework of a gauge-invariant version of quantum chromodynamics. The field operators in this formulation are taken to be Wilson loops and strings with quark-antiquark ends. The fundamental differential equations of point-like Q.C.D. are expressed as variational equations of the extended loops and strings. The 1/N expansion is described and it is assumed that non-leading effects such as intermediate quark pairs and nonplanar gluonic terms can be neglected.

The action of the Hamiltonian in the $A_o = 0$ gauge on a string operator is derived. A trial meson wavefunctional is constructed consisting of a path-averaged string operator applied to the full vacuum. A Gaussian in the derivative of the path location is assumed for the minimal form of the measure over paths. A variational parameter is incorporated in the measure as the exponentiated coefficient of the squared path location.

The expectation value of the Hamiltonian in the trial state is evaluated assuming the negative logarithm of the expectation value of a Wilson loop is proportional to the loop area. The energy is then minimized by deriving the equivalent quantum mechanical Schroedinger's

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equation and using the quantum mechanical 1/n expansion to estimate the effective eigenvalues. It is found that the area law behavior of the Wilson loop implies a nonzero best value of the variational parameter corresponding to a quantum broadening of the flux tube.

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I. Introduction

All current evidence on the spectrum and internal structure of hadrons is consistent with the notion that they are color singlet bound states of the more point-like fermions and vector bosons of quantum chromodynamics. Not all properties of these composite states are yet derivable from the field theory. Most notably, models which give an accurate description of at least some part of the spectrum introduce assumptions about the structure of the supposed nonperturbative confining force: that it is flavor-independent, has the color structure of single gluon exchange, and has little overlap with the deep perturbative forces, for example. While this suffices for some heavy quark phenomenology, it gives rise to double counting problems in the calculation of perturbative corrections and special treatment is required for the lowest-lying pseudoscalar mesons and the lightest singlet pseudoscalar.

In addition to lacking a direct proof of confinement, there remains the parallel problem of developing an appropriate field theoretic description of the bound states. Evidence from spectral fits and from analogue models, such as two dimensional field theories and lattice gauge theories, suggest different limiting forms for the field theoretic bound states corresponding to distinct, not clearly compatible, perturbative treatments. Weak coupling perturbative effects are familiar from

scaling behavior in deep inelastic scattering and, when combined with a static (heavy) quark approximation or a $1/M_q$ expansion, show up in the splitting of charm and bottom meson states. On the other hand, the effective linear potential which seems to provide the main binding of heavy quark states arises naturally in a strong coupling expansion on the lattice. The suppression of intermediate quark-antiquark states, seen in the minimal $3x3^*$ and 3x3x3 structure of the lowest hadron multiplets and in the narrowness of the new quark resonances, has received an explanation only from topological arguments, corresponding to an expansion in 1/N, where N is the number of colors. While any given spectral detail can be handled, at least descriptively, within a known perturbative scheme, no single method has been extended to all phenomena.

We begin with a brief survey of some approaches to the study and characterization of bound states in field theory. No presumption of completeness or chronological accuracy is made. The methods here were chosen for the light they shed on the physical structure of the hadrons, to the extent that they faithfully represent hadron properties, and for their demonstration of the efficacy of various perturbation schemes in different regimes. The assorted schemes reflect the use of small oscillations about distinct dominant modes.

1. Weak coupling approaches

Weak coupling approaches are the most familiar, having well-established precedents in atomic physics and quantum electrodynamics. The first major class of weak coupling treatments is founded on the notion that certain interactions among constituents, or between constituents and external sources, can be treated perturbatively even though the bound state itself represents an infinite superposition of interactions. As a phenomenological description, this is an old idea, having appeared first in the context of quantum electrodynamics[1]. Asymptotic hadron states were introduced as (fermionic or bosonic) fields with unknown internal structure, expressed as a charge distribution to be determined experimentally. The interaction of photons with elementary (pointlike) charged particles was believed to be completely described by quantum electrodynamics and quantum scalar electrodynamics. However, hadrons were manifestly not pointlike, having charge, mass, and other properties distributed throughout a finite region. Hadrons were believed to be multi- and possibly infinitely- composite; on electromagnetic time scales, a proton, for instance, could pass through many virtual states containing pions, rhos, neutrons, etc. Despite an understanding of the local properties of photon scattering, hadrons remained black boxes. Distribution functions then served as effective photon-hadron vertices. Consistency conditions reducing the set of unknown functions could be derived from dispersion relations[2] but this was always limited by the small number of usable hadron targets and by an

ignorance of the full spectrum.

Distribution functions were brought closer to field theory when they evolved to the phenomenological wavefunction envelopes of partons[3-5]. To account for scaling behavior observed in deep inelastic processes, it was proposed that at large momentum transfer only pointlike fermions interact with photons. Distribution functions then describe the probability of finding fermions of a particular charge and mass or momentum within a hadron. As there was originally no field theoretic justification for the presence of free quarks in hadrons, partons were not necessarily identified as such. The parton model was not itself a field theory so distribution functions could not be derived. There were, however, constraints imposed by sum rules[3] and consistency conditions[4] and it was expected that many theories would be eliminated by the data. If the theory is not eventually predictive, this approach is begging the question. In the absence of an underlying field theory, it was necessary to rely on assumptions such as the saturation of sum rules by low-mass or -angular momentum states and restriction of singularities to narrow width poles. Even if an effective theory could be extrapolated, the rules for working with effective field theories are ambiguous.

Following the work of Politzer[6] and Gross and Wilczek[7] demonstrating asymptotic freedom in Yang-Mills theories, it became feasible for the first time to view hadrons as conventional bound states in a field theoretic framework. The vanishing of the effective coupling at

short distances gave a qualitative resolution of the former paradox of the quark model - permanently bound but quasi-free constituents. Restored to the status of elementary particles, quarks could be treated in standard ways in bound state models. Potential models exploited the experience with atomic composites by proposing the existence of a confining force between quarks representable as a quantum mechanical potential.

Potential models had long been used in attempts to reproduce the hadron spectrum by making the simplifying assumptions that only the valence quarks are present to any great degree, that these quarks have well-defined constituent masses, and that the dominant binding force is flavor-independent, two-body, and increasing with separation. For the lighter hadron states, and before the advent of Q.C.D., potential models repeatedly failed to produce the correct mass splittings and often predicted multiplets of a size far in excess of anything observed. Except that it made the minimal quark structure more mysterious, this result was not surprising since the pion mass was known - and the light quark masses believed - to be on the order of nuclear binding energies. Relativistic and inelastic effects rendered suspect any quasi-quantum mechanical models of the lowest-lying hadrons. It was not clear whether such models could ever be justified in hadron physics.

The narrowness of the psi signaled its weak coupling to hadrons composed of the three lighter quark varieties. Its mass being over three times the proton mass, it seemed likely that the charmed quark

could be treated quantum mechanically as it is very heavy relative to pions. These speculations received confirmation when the spectrum of radial excitations for states containing only charmed quarks were fit accurately assuming only that the potential is smooth and confining[8-10]. Two or three parameters - typically the quark mass and the coefficient of a linear or quadratic potential - were all that was required.

Inverse scattering calculations served to verify the general assumed form of the effective heavy quark potential[11]. When data was available on a second heavy quark system, the bottom quark states, it was possible to extract information on the quark mass dependence of energy splittings and, to some extent, the wavefunction at the origin. General quantum mechanical arguments relate these quantities to the power law behavior of the potential. For a monomial potential, best fits were found for logarithmic dependence[12], but this is equivalent for these purposes to linear plus Coulomb, the latter arising from the perturbative part of the interaction.

Additional fine structure had long been observed in the mass splittings between states which, in the standard quark model, differ only in the alignment of the constituent quark spins. A hyperfine or spin-spin interaction occurs in perturbative Q.C.D. for single gluon exchange[FtN]. The leading graph is of order $1/M_q$ because it represents nondiagonal spinor coupling, so there were general reasons to expect this effect to be present but relatively suppressed in the heavy quark states. Inclusion of a hyperfine term in the effective potential with a flavor-

independent coefficient and color spin structure given by single gluon exchange seems to account for spin-dependent mass splittings among the heavy mesons[13]. Application of these techniques to baryons has met similar success[14-15]. Even for the low-lying octet, careful treatment of the three-body problem with a universal two-body confining force plus hyperfine corrections structured like single gluon exchange gives the correct multiplicities and mass splittings, succeeding where pre-Q.C.D. SU(N flavors), SU(6), and SU_L(N flavors)xSU_R(N flavors) had all failed.

The quantum mechanical potential of a two-body bound state is, in the context of the underlying field theory, the static limit of the two particle irreducible kernel. In an attempt to tie potential models more closely to perturbative quantum chromodynamics, several groups investigated the 2PI kernel perturbatively. A Foldy-Wouthuysen transformation is first performed to isolate leading $1/M_q$ contributions. Quark recoil effects and intermediate heavy quark states are then suppressed by factors of the ratio of typical momentum exchange in the bound state to the quark mass and the kernel is determined by the pure gauge theory.

In quantum electrodynamics, the coupling approaches a small infrared fixed point, $\alpha = \frac{1}{137}$ and it remains weak at atomic lengths. The lowest order kernel in the Coulomb gauge is a single, timelike photon exchange and $V(r) = \frac{\alpha}{r}$, the Coulomb potential. Analysis for the nonabelian gauge theory is complicated by gluon self-interaction. The leading contribution to the static 2PI kernel includes single Coulombic gluon exchange as well as multiple Coulombic exchanges connected by

intermediate transverse gluons. It was found to two loops[16-18] that the renormalized, renormalization group improved potential in the nonabelian version was of the same form as the abelian kernel with the replacement of α by $\alpha(r^2\Lambda^2)$, the running, strong coupling, and multiplication by a function $F(\alpha(r)) = 1 + O(\alpha(r))$.

These calculations assumed weak fields and a naive vacuum. The function $F(\alpha(r))$ only modifies the Coulombic potential by powers of logarithms of r and does not appear likely to produce positive powers of r at any finite order α. The strong coupling behavior of α may be radically different from logarithmic, or there may arise nonpolynomial α dependence in the strong coupling series for F, but these effects do not seem accessible perturbing about weak coupling eigenstates.

A systematic combination of the small coupling (α) and static or heavy quark ($1/M_q$) expansions has been performed in a functional integral framework[19,20]. The heavy quark-antiquark four-point function is expressed as a functional integral over the canonical positions, $X_i(\tau)$ and momenta, $P_i(\tau)$, of the quarks, and over the gauge field. The latter cannot be performed except perturbatively in α and it must be assumed that confinement arises nonperturbatively. The original quark part of the action in the exponent of the functional integral is replaced by:

$$X_i(\tau)\cdot P_i(\tau) - H_q(\frac{1}{M})$$

where the second term is the usual $1/M_q$ expanded Hamiltonian. The double expansion in the interaction and the quark kinetic energy produces all the qualitative features noted above, i.e. a leading Coulombic exchange with running coupling and a spin-spin interaction suppressed by $1/M_q$. In addition, if the gluon path integration is converted to Hamiltonian form, one reproduces the semi-classical operator equations of motion in the Heisenberg picture derived from variational arguments by another group[21].

Several interesting results can be gleaned from potential model fits to quarkonia spectra. First they confirm the traditional expectation that for large enough constituent masses in the presence of a non-pathological, mass-independent interaction, any few-body bound state will display a familiar nonrelativistic level scheme. Secondly, they demonstrate the insensitivity of the spectra to the precise form of the confining potential. In a sense, this is a necessary concomitant of the accuracy of the perturbative hyperfine corrections. If the potential serves chiefly to restrict the quarks to a region of space where the potential is not felt, as a square well, for example, would do, then in low angular momentum bound states, subtle distinctions arising from spin-orbit interactions will not be wiped out by interactions of similar structure hidden in the long range part of the force. It also reflects a rapid, smooth transition from weak to strong coupling: heavy quarks are heavy enough that the coupling is indeed weak throughout much of the bound state wavefunction and surface effects are not evident.

Potential models continue to be suspect for the lightest hadrons. It seems unlikely that the observation of new quarkonium states will significantly increase our understanding of the dynamical origin of the confining force. There are intrinsic limitations on the degree to which perturbative and nonperturbative effects can be distinguished in spectral structure. There is mixing, for example, between the confining potential and pertubative contributions of the same structure, as seen in the near-equivalence of logarithmic and linear plus Coulombic trial potentials. Velocity-dependent effects such as spin-orbit coupling will also be affected by both nonperturbative and perturbative components of the force. Attempts to determine the wavefunction at the origin for S-wave states, related quantum mechanically to the expectation value of the gradient of the potential, are frustrated by the presence of annihilation channels, relativistic corrections, intermediace quark-antiquark states, and the limited number of such states available below threshold for strong decays, effects which may not decrease in importance as the valence quark mass increases. It appears from the perturbative standpoint that lowest order Q.C.D. Feynman diagrams involving individual gluon exchanges, quark loops, etc. must still be superimposed on a nonperturbative hadron wavefunction, and that as a result it will continue to be nesessary to determine the relative weights of the various terms in the potential phenomenologically.

In the second class of weak coupling approaches, the context is again perturbative Q.C.D. but a restricted class of nonperturbative

effects are included. The presumed source of confinement is isolated in some function - Green's function, propagator, n-point function, etc, - and a guess is made as to the explicit form of this function. Substitution into the perturbative field equations permits improvement on this guess and/or investigation of its effects.

Many of these techniques originated long before quantum chromodynamics, including the familiar use of an ansatz kernel in the Bethe-Salpeter equation. The Bethe-Salpeter equation is, within the context of perturbative quantum field theory, an exact integral equation for bound state wavefunctions. It expresses the wavefunction, which is the residue of the four point function at the bound state pole, as a convolution of itself with the two particle irreducible kernel and the full constituent propagators. The Bethe-Salpeter equation includes potential models as a limiting case. Practical calculations are usually restricted to the iteration of simple kernels. Lorentz invariance is usually maintained but gauge invariance is sacrificed.

Iteration of the Coulomb kernel, for example, has been investigated in quantum electrodynamics[22,23]. A bound state will be produced having the approximate spectrum of hydrogen (unequal masses) or positronium (equal masses). To match experimental accuracy, it is necessary to include crossed graphs or, roughly equivalently, time-reversed fermion lines and, for positronium, annihilation channels in the kernel.

An R^4-symmetric kernel corresponding to a potential $V(r) = cr^2$ was used in early attempts to clarify the structure of the confining

force[24]. Since spin-dependent effects were not included, this produces an SU(6) symmetric spectrum with the excess multiplicities noted above. It also does not produce the appropriate linear Regge trajectories or the correct power law fall-off of form factors[25,26]. Later systematic investigations of confining kernels with better high angular momentum behavior have failed to reveal a candidate lowest order kernel for the four dimensional bound states.

The Bethe-Salpeter equation for mesons in two-dimensional Q.C.D. was solved numerically by 'tHooft[27] in the planar limit. The gauge field has no nontrivial degrees of freedom in two dimensions: in the lightcone gauge, it is simply an instantaneous Coulomb potential between the quarks and there are no gluon self-interactions. Instead of the usual perturbative treatment, a topological expansion was performed by replacing the original coupling g^2 by an effective coupling $\lambda^2 = g^2N$, which is then held fixed as N becomes large. Diagrams are then distinguished by their topology, and the class of diagrams of a given topology includes all orders in the effective coupling. In leading order in 1/N, fermion loops or crossed gluons are suppressed. With such simplifications, it was possible to derive an exact expression for the quark propagator which exhibited a pole (physical mass) tending to infinity as the infrared cutoff was removed. Substitution of this propagator into the homogeneous part of the Bethe-Salpeter equation for a two-body bound state produced a discrete spectrum of excitations identified as a family of fixed valence quark mesons.

This result differs in substance from the electrodynamic bound state in four dimensions because of the permanence of the binding and because the kernel, instead of being merely a single (or finite) iteration of the Coulomb interaction, is an infinite sum of planar graphs. Thus even in two dimensions, the achievement of confinement in Q.C.D. is a highly collective effort. Questions about the legitimacy of the two-dimensional planar solution[28-31], and about the convergence of the 1/N expansion in general[32,33] have since been raised, without clear sign that there are serious pathologies.

The perturbative structure of the gluon propagator in four dimensions is similar enough to the photon propagator that no sign of permanent binding appears using finite sets of gluons for the kernel even with the inclusion of the running, logarithmically growing coupling[34]. The planar limit in four dimensions does not appear simple enough to permit exact solution for the quark or gluon propagators. In the ultraviolet, these are just free propagators due to asymptotic freedom. Several workers have tackled the problem of the leading infrared behavior by attempting to approximately solve the Schwinger-Dyson equations, truncated to enhance long wavelength modes[35,36]. The assumptions made about the dominance of certain vertices or n-point functions have been difficult to justify, and no clear evidence emerged that confinement indeed occurs. Less ambitious attempts proposed the replacement of the usual running coupling with an ansatz form $g(k^2/\Lambda^2)$ which interpolates smoothly between the asymptotically free result and a small

momentum form which produces a gluon propagator having an instantaneous piece growing linearly with distance[37]. This is, of course, practically equivalent to phenomenological potential models. The ad hoc adjustment of the β function in the cross-over region between weak and strong coupling corresponds to the choice of coefficients of terms in the potential. Also imitative of the two-dimensional results is the substitution of a fermion propagator with a pole at infinite mass into the four-dimensional Bethe-Salpeter equation with a Coulombic kernel, i.e. a standard perturbative gluon propagator[38].

Other semi-nonperturbative approaches derive from the discovery of classical, finite action Euclidean solutions to pure gauge theories. Such instantons[39,40] or monopoles[41] may radically modify the structure of the vacuum. To take proper account of such classical solutions, which are not in evidence perturbatively, it is considered necessary to add effective terms to the pure gauge part of the action. In the case of instantons, this contribution amounts to a total divergence and since the Lagrangian is only defined up to total divergences, one must in general include this term. Instantons may radically alter the quark-containing particle spectrum through their coupling to the chiral current. These effects may be related to the observed instability of the classical Coulomb interaction between nonabelian charges. It was shown in a three dimensional gauge theory that a dilute gas of instantons induces an effective confining potential between charged particles[42]. Despite ferocious speculation,[43] no results of similar

persuasiveness have been forthcoming in four dimensions.

Recent reformulations of the bag models[44-49] appeal to the formal possibility of vacuum structures to produce a so-called dielectric constant of the clothed vacuum. The bare vacuum is supposed to exist inside hadrons. The transition between these different dielectric media, it is argued, occurs at the surface of hadrons producing, for example, a change in energy density which accounts for the bag constant. It has not been proven, however, that such a vacuum dielectric materializes in the quantum theory (although classical[50-52] and semi-classical[53] arguments suggest the effect). Moreover, pions are not incorporated in the model.

2. Analogue models and strong coupling expansions

The relative simplicity of the color singlet sector may indicate that both the incorporation of the confinement force naturally and systematically in the perturbative framework and the handling of light mass collective modes may come from an understanding of collective phenomena in gauge theories. The term collective phenomena in the context of field theory refers to canonical state bases, usually nonlocal or nonlinear in the original fields, which represent the particle content of the theory. The field equations expressed in the collective basis have the form of a free field theory with a truly perturbative interaction.

Examples of such collective coordinates occur most clearly in two-dimensional field theories because these are often exactly soluble. The

demonstration of the equivalence of the sine-Gordon and massive Thirring models[54,55] led to a dual interpretation of the solitons of the former theory as the fermions of the latter. For certain values of the coupling, these fermions would repel or attract their antifermion partners, forbidding or permitting the formation of bosonic bound states which would then correspond to the fields in the sine-Gordon theory. Depending on the strength of the fundamental coupling, then, the sine-Gordon or the Thirring Lagrangian is the proper choice as the stable excitations are then, respectively, bosonic or fermionic.

Perhaps more appropriate as analogues to Q.C.D. are the asymptotically free theories in two dimensions. In these, as in the four dimensional gauge theories, the coupling is dimensionless and cannot therefore be put in by hand. Masses in the theory arise either as solutions to the gap equation, or remain trivial, determining only the renormalization point. In the O(N) nonlinear sigma model[56], the first term of the nonlocal 1/N-expanded Lagrangian expressed in terms of the bilocal composites $\sigma^i(x)\sigma^i(y)$ describes a free field theory of massive O(N) degenerate particles[57], while perturbation in the original $\sigma(x)$ fields is an interacting field theory with broken symmetry. Similarly, the O(2N) Dashen-Hasslacher-Neveu[58] action can be partially linearized by substitution of Lagrange multiplier, bilocal fields. Functional Gaussian integration over the original Fermionic modes produces a nonlocal effective action whose leading order-1/N term is soluble[59,60].

Both these models are also solvable via a 1/N expansion of the

Hamiltonian[61], analogous to the pseudospin formulation of the O(N) BCS model. The large N limit is determined in this case by the real-time solutions of the original field equations with nonlinear constraints. The effective coupling, which is the dimensionless coefficient of the interaction Hamiltonian, is in these cases proportional to 1/N. Neither method, however, has been shown applicable to theories with matrix-e.g. SU(N)- rather than vector-e.g. O(N)- type symmetry. The problem seems to spring from the fact that the volume of phase space for a matrix symmetry increases like N^2 rather than N; the gauge-fixing term is then of the same order as the leading term in the action[33].

A strong coupling in the original theory is, in these two dimensional systems, exchanged for a weak effective coupling by choosing as a state basis nonlocal bound states of the original fields. A major attempt to construct an extended field theory for hadrons came as an off-shoot of dual resonance models[62,63]. The focus in the dual string theories was, initially, to reproduce the dual, crossing-symmetric amplitudes that were the generalizations of the four-point Veneziano amplitude. The spectrum of states is de facto precisely that of dual theories: infinitely narrow resonances on linearly rising Regge trajectories. Transverse modes of the string are identified with these hadronic resonances. The original string endpoints carried no mass and thus traveled at the speed of light. Massive quarks were added later for the purpose of producing a more realistic spectrum but they were not incorporated in a dynamical way. The lightest scalar in the theory was

tachyonic. Low-lying states and the possible internal structure of the string were inherently inaccessible in these models.

There also remained formal problems involved in the transition from the classical to the quantum field theory of extended objects with the chosen action:

$$S_{string} = \iint d\sigma \, d\tau \, \sqrt{-g}$$

$$g = \det \left[\frac{\partial (x^{\mu}, x^{\nu})}{\partial (\sigma, \tau)} \right]$$

Such an action is invariant under a large class of redefinitions of the proper "time" and "length" parameters, σ and τ. It thus represents a gauge theory, but one with a nonpolynomial action except in certain linear gauges.

Wilson's strong coupling expansion of gauge theories on a lattice provided the first explicit connection between particles which exist as stringlike excitations and the fundamental pointlike fields of a gauge theory[64]. Local gauge invariance on a lattice requires that non-singlet operators, such as quarks, at separated lattice sites be connected by links carrying units of electric flux. States of pure flux consist of traces around closed link loops. Lattice models provide intuition about the likely properties of the strong coupling regime. In the strong coupling limit, the electric part of the Hamiltonian dominates so eigenstates are distinguished by their link (and quark)

content[65-68]. The perturbing, magnetic part of the Hamiltonian adds bumps to flux paths, introducing some stochasticity in the eigenstates and allowing for the propagation of strings.

In order to make the transition from the lattice to the continuum, it is necessary to find the phase structure and the critical points of the theory[65,69]. A fruitful area of research in this regard has been the use of computers to simulate finite-size lattices. Two important results have come from computer "experiments" with lattice gauge theories. First there is no sign of a first or second order phase transition in the system as the effective coupling moves from weak to strong coupling (low to high temperature) regimes. This was observed in Monte Carlo simulations of in which the lattice is fixed initially in an ordered (disordered) state corresponding to low (high) temperature. A weighted heat bath of varying temperature is applied to each link successively and the evolution of the average action per link is observed[70]. This order parameter passes smoothly through the same values whether the system was initially hot or cold. In contrast, it was demonstrated that abelian gauge systems in four dimensions or a nonabelian gauge systems in five dimensions has discontinuities in the evolution of the order parameter, as had been expected from renormalization group arguments. The same general conclusions were reached by other groups using Padé approximants[71] to interpolate from the lower end of a strong coupling calculation to the perturbative region. Both these groups also demonstrated that the coupling appears to become

strong rapidly and at fairly short distances, on the scale of lattice spacings[49,50].

The double expansion in 1/N and the inverse effective coupling $1/g^2N = 1/\lambda^2$ has also been investigated on the lattice[72,73]. The behavior of the weak to strong transition of $\beta(g)$ as a function of N showed an increasingly narrow cross-over region. It was speculated that a kink develops in β at infinite N. This would be in line with the type of nonanalyticity observed in exact solutions of toy lattice models. Gross and Witten[74] considered the large N limit of two-dimensional lattice Q.C.D. Plaquettes decouple at infinite N so the problem reduces to solving the Schwinger-Dyson equation for a single plaquette. A discontinuity in the β-function occurs at finite coupling representing a third order phase transition. Similar behavior was found in a three plaquette model[75] and in a 2 + 1 -dimensional model in which a single plaquette propagates in continuous time[76]. These results display the expected extreme case of a discontinuously sharp, weak to strong transition region at infinite N; they do not imply, however, that the two regimes become inaccessible to each other.

All of these approaches can be expanded as well in the inverse quark mass, for bound heavy quark systems. Light or zero mass quarks pose special problems for lattice gauge theories. Treatment of fermions on the lattice involve either explicit breaking of chiral symmetry, in which case the possible spontaneous breakdown will be hidden, or the inclusion of nonlocal interactions which must then be carefully

truncated as successive renormalizations are performed[77,78].

3. Loop space

The continuum formulation of nonabelian gauge theories in terms of loops and strings appears to be formally consistent, as first suggested by Mandelstam[79,80]. The most general fundamental excitations in a gauge theory in which physical states are expected to be gauge invariant are the parallel transports of ordinary pointlike fields in nonsinglet representations. These are the straightforward continuum analogues of Wilson's lattice configurations. Extended fields are thus introduced naturally while preserving the underlying asymptotically free theory. It is not known how to perform a strong coupling expansion in the continuum. Nevertheless, it is possible that coherent hadron states may ultimately be more easily described as loop and string composites than as quark and gluon composites. The construction of an extended field theory based on Q.C.D. was largely ignored as a practical formulation until recently[81-83]. Exact results in the large N limit demonstrate decoupling of continuum Wilson loops which leads to the hope that the lowest excitation of the strong coupling expansion is accessible to leading order-1/N solution.

If the leading order-1/N solution is found and if the 1/N expansion can be shown free of fatal pathologies in the continuum as N goes to infinity, then this may be sufficient to produce higher states by providing a dynamical origin of string-like gluonic modes or equivalently

an effective linear potential. To the extent that they can be observed in isolation, what are believed phenomenologically to be the collective modes behave in a predictable, fairly linear fashion. We refer in particular to the regularity of radial excitations in heavy quark systems and the linearity of Regge trajectories for high angular momentum states of the lighter quark mesons. If these modes reflect a 1/N pattern, then it is possible that the nonleading corrections are also free of pathologies.

There are major differences between these QCD flux tube strings and older dual strings. Their dynamics are not put in by hand but are inherent in the underlying field theory. The action would be an effective action and may bear little resemblance, except in leading order or for high angular momentum states, to that of the original relativistic string. Open strings terminate on quarks. Free, quarkless strings must be closed and correspond to the Wilson loops or their collective manifestation.

All of this requires better understanding of regularization, both in the context of a dual strong coupling and 1/N expansion, and for manifestly extended fundamental fields. It is here that the usefulness of lattice analogues is most evident. The explicit space of link states provides a concrete realization of functional operations, such a keyboard displacements and the breaking and joining of flux lines, and may also give clues to regularization by isolating nonanalytic N-dependence and identifying the phase structure and relevant order parameters.

As in ordinary perturbation theory, one would like higher order corrections to merely refine the mapping between bare and clothed fields, not to introduce a nonlinear change of state basis as apparently occurs in the transition from Q.C.D. to hadron physics. Such a result would partially compensate for the added difficulty of working with extended instead of pointlike fields. These ideas will be discussed in more detail in the body of the paper.

The purpose of the present work is to perform a rough practice calculation in the framework of a loop space description of a heavy quark bound state. The larger issue of establishing a complete field theory of loops and strings, although briefly discussed, will essentially be avoided. Neither will any attempt be made to reproduce the fine structure of the hadron spectrum as this has been done successfully elsewhere. Rather, a few technical problems arising en route between the raw definition of loop space and the interpretation of the wavefunctionals that emerge will be considered.

The heavy quark system has been chosen to minimize complications due to familiar kinematic effects and couplings to intermediate states. A trial wavefunctional for a mesonic state is acted on by the loop space Hamiltonian. The resulting variational problem minimizing the ground state energy highlights certain problems associated with an extended field formulation - in particular, those relating to the use of the path measure and regularization of path space - while circumventing others involved with special systems such as glueballs and pions.

The partial machinery of a tentative loop space field theory is displayed in Section II. The connection to perturbative gauge theory is presented to clarify properties of the extended operators. Questions pertaining to the regularization of functional differential operators are discussed. Simplifications accruing from an expansion in 1/N are described.

The Hamiltonian formulation is developed in Section III. The principles involved in the integration over path degrees of freedom are considered. The variational technique in field theory is reviewed in Section IV.

The trial wavefunctional for a meson is presented in Section V. The action of the Hamiltonian operator on this state is derived under a set of assumptions about the properties of the full vacuum. The assumption of confinement is also introduced in order to exhibit a possible truncated form of the variational equation for the ground state. The resulting path-averaged expression is interpreted as a quantum mechanical expectation value in the limit of heavy quark masses and suppressed communication with vacuum channels.

A quantum mechanical $\frac{1}{(d+2L)}$ expansion, where d is the number of spatial dimensions and L is the orbital angular momentum, is used in Section VI to complete the solution of the approximate variational problem. Conclusions and possible extensions of this work are discussed in Section VII.

II. Loop space formulation

The loop space formulation of Q.C.D. is motivated by the desire to work with gauge invariant fundamental fields since the presumed physical states of the theory are strictly color singlets. The seeds of this technique can be found in the quantum mechanical treatment of electromagnetism by Dirac[84] who introduced the path ordered integral of the gauge field to invariantly describe an electron trajectory. Mandelstam[79,80] extended this notion to the quantum gauge field theories for both abelian and nonabelian groups by displaying the formal representation of gauge-invariant Green's functions in terms of path dependent operators. Residual gauge dependence remained only in the point at infinity where all paths ended. It was shown that all physical, hence gauge-invariant, expectation values calculable perturbatively with the standard fields could be recovered, also perturbatively, in the extended field formulation.

Recent loop space work[81-83] has been directed toward moving beyond a reliance on perturbation in the coupling. The growth of the QCD coupling in the infrared and the existence of nonperturbative, finite energy field configurations make a perturbative treatment untrustworthy for large loops. The 1/N expansion, which distinguishes topological rather than extensive properties, shows the greatest promise as a tool for simplifying the dynamics of extended configurations

independent of their size.

1. General gauge-invariant operators

To establish some notation, we first define the usual pointlike fields that occur in a gauge theory with fermions in four spacetime dimensions. The gauge field is denoted:

$$A_\mu = igA^a_\mu T^a$$

The fermions will be written as:

$$q(x), \quad \bar{q}(x) = q^+(x)\gamma_0$$

Color and Dirac indices are implicit. Flavor degrees of freedom are suppressed. The matrices T^a are in the adjoint representation of the gauge group, which will be taken to be U(N) or SU(N) for various N's as convenience dictates. They satisfy commutation relations:

$$[T^a, T^b] = C^{abc}T^c$$

We will be concerned exclusively with fermions in the fundamental representation so the q(x)'s are N vectors in color and the T^a's are N by N matrices. Under a gauge rotation:

$$\Lambda(x) = \exp\,[ig\theta^a(x)T^a]$$

$$\Lambda^{-1}(x) = \Lambda^{\dagger}(x)$$

the $A_\mu(x)$'s transform as:

$$A_\mu(x) \rightarrow \Lambda^{-1}(x)\;[A_\mu + \partial_\mu]\;\Lambda(x)$$

and the fermions transform as:

$$q(x) \rightarrow \Lambda^{-1}(x)q(x)$$

$$\bar{q}(x) \rightarrow \bar{q}(x)\Lambda(x)$$

We can easily construct invariants in these fields and their derivatives. First consider bilinear combinations of the fermions. If $O(x)$ is any local operator transforming covariantly under $\Lambda(x)$:

$$O(x) \rightarrow \Lambda^{+}(x)O(x)\Lambda(X)$$

then the combination $\bar{q}(x)O(x)q(x)$ is invariant. The covariant derivative, for example, is given by:

$$D_\mu = \partial_\mu + A_\mu$$

since under a gauge rotation:

$$D_\mu(x) \rightarrow \delta_m + [\,\Lambda^+(x)\, D_\mu\, \Lambda(x)\,]$$

$$= \Lambda^+(x)\, D_\mu(x)\, \Lambda(x)$$

The derivative here acts on everything to its right unless enclosed in parentheses. The general invariant form constructed from covariant operators at a point is:

$$\mathrm{Tr}\; O_1(x) O_2(x)$$

In particular, the connection $F_{\mu,\sigma} = [D_\mu, D_\sigma]$ is covariant and the bilinear trace is invariant.

The full, gauge invariant Lagrangian is given by:

$$L_{QCD} = L_{YM} + L_Q + L_{FP} + L_{GF}$$

$$L_{YM} = -\frac{1}{4g^2}\, \mathrm{Tr} F_{\mu,\sigma}(x) F_{\mu,\sigma}(x)$$

$$L_Q = \bar{q}(x)\; [\gamma_\mu D^\mu + M]\; q(x)$$

The gauge-fixing and Fadeev-Popov ghost terms are necessary in any noninvariant formulation using nonlinear gauges. Though we will

occasionally work out examples perturbatively, we will always use linear gauges sosyntax error in equation 27 file two, between lines 202 and 202 these term:

potentials are used. To display the canonical commutation relations and equations of motion, choose the $A_o = 0$ gauge (see also Section III). The momenta conjugate to the gauge fields are then:

$$\Pi_i^a(x) = \frac{\partial L}{\partial A_i^a(x)}$$

$$= E_i^a(x) = \frac{1}{g^2} F_{oi}^a(x)$$

giving commutation relations:

$$[\ E_i^a(x),\ A_j^b(y)\ ]$$

The fermions obey the anticommutation relations:

$$\{q_\beta^a(x),\ q_\lambda^{+\ b}(y)\}_{(x_o=y_o)} = \delta_{\beta,\lambda}^{a,b}\ \delta^3(x-y)$$

Writing the fermionic current $\bar{q}(x)\ \gamma_\mu\ q(x)$ as $J_\mu(x)$, the equations of motion can be simply expressed as:

$$D_\sigma(x) F^{\sigma,\mu}(x) = J^\mu(x)$$

$$\gamma_\mu D^\mu(x)\ q(x) = 0$$

The gauge condition and the Bianchi identity

$$D^*_\sigma F^{\sigma,\mu} = \epsilon_{\mu\sigma\lambda\rho} D_\sigma [D_\lambda, D_\rho] = 0$$

complete the formal set of information necessary to specify the theory.

Although we began with an invariant Lagrangian, variation with respect to the local fields to get the equations of motion destroys manifest gauge invariance. Even if it were contrived to reformulate the equations of motion in terms of invariant products of operators, most regularization procedures require point splitting at intermediate stages of the renormalization program. The forms $\mathrm{Tr}\, O_1(x+\delta)O_2(x)$, for example, are no longer invariant but tranform like:

$$\mathrm{Tr}\ O_1(x+\delta)O_2(x) \rightarrow \mathrm{Tr}\ \Lambda_\delta^{-1}\ O_1(x+\delta)\ \Lambda_\delta\ O_2(x)$$

$$\Lambda_\delta = \Lambda(x+\delta)\ \Lambda^{-1}(x)$$

There are other, physical reasons for expecting a local gauge invariant formulation to be impossible for a physically interesting gauge theory. If the fermions can indeed be treated in all respects, i.e. at all phases of the calculation, like bilinear singlets, then the relevant fundamental fields would appear to be noninteracting bosons.

We therefore extend the class of general gauge-invariant operators to nonlocal combinations of the form:

$$\bar{q}(x)\ U_{(x,y)}[P]\ q(y)$$

$$\mathrm{Tr}\ U_{(x,y)}[P]\ O_1(y)\ U_{(y,x)}\ O_2(x)$$

These will be invariant if $U_{(x,y)}[P]$ transforms as:

$$U_{(x,y)}[P] \rightarrow \Lambda^{-1}(x)\ U_{(x,y)}[P]\ \Lambda(y)$$

and to regain the proper local forms:

$$U_{(x,x)}[P] = I$$

These equations are solved by the path-ordered exponential:

$$:\exp \int_P dx_\mu A^\mu(x):$$

The path ordering convention is that operators are arranged right to left as their arguments range from the lower to the upper limits of integration. Gauge invariance requires only that the path of integration P be continuous. The nontrivial singlet operators of a pure gauge theory include as well the trace of U's defined on closed continuous paths or loops, i.e. operators of the form:

$$W\ [L] = \mathrm{Tr}\ :\exp \int_L dx_\mu\ A^\mu(x):$$

These will be referred to as Wilson loops. The matrix (untraced) version of the same operator:

$$U_{x_o}[L] = :\exp\oint_L dx_\mu A^\mu(x):$$

will be called simply loop operators. They are implicit functions of the point x_o chosen as the starting place of the path ordering operation and tranform as local covariant operators at x_o. The loop operators can thus be used to represent the local covariants of the original formulation. In particular, taking the limit of zero area for a loop surrounding the infinitesimal region $\sigma_{\mu,\lambda}$ at x_o, we recover the gauge field $F_{\mu\lambda}(x_o)\sigma_{\mu\lambda}(x_o)$.

2. Differential operations

The path dependence appended to make all operators gauge invariant introduces ambiguity in the canonical formalism. To treat the W[L]'s or some variant thereof, literally as canonical fields would require expressing the original Lagrangian in terms of the generalized invariants. The path dependence of the new operators does not appear explicitly, however, since the theory is independent of paths. As in the case of gauge dependence, path dependence appears only when we progress to the equations of motion. Although it is possible to derive these equations from a Lagrangian functional of path-dependent operators, we find it simpler to translate the covariant equations derived in the

usual way directly into differential equations for the strings and loops.

In order to do so, it is first necessary to formally define differential operations on strings, which is what we shall call the restriction of the ordered exponential to a particular path in spacetime. Since the differentiation acts only on path points, it has the character of a functional differentiation. For a functional $x_\mu(s)$ describing the spatial location of a point on a path parameterized by s, the derivative $\frac{\delta}{\delta x_\mu(s)}$ is defined to have the effect of moving the string at $x_\mu(s)$ by an infinitesimal amount in the μ-direction. Consider a string which has an endpoint at x_μ, neglecting for the moment other fields to which it may be attached. The effect of adding a segment dx_μ is[85]:

$$\delta_\mu U(x,*) = [e^{\int_{x(s)}^{x(s)+\delta x_\mu(s)} ds' \frac{dy^\lambda}{ds'} A_\lambda(s')} - 1]\, U(x,*)$$

$$= \delta x_\mu(s) \int ds'\, \delta(s-s') \left[\frac{dy_\lambda(s')}{ds'} \frac{\delta A_\lambda(s')}{\delta y_\mu(s')} - \delta_{\mu\lambda} \frac{d}{ds'} A_\lambda(s') \right] U[x(s'),*]$$

$$= \delta x_\mu(s) \frac{dy_\lambda(s)}{ds} \left[\frac{\delta A_\lambda}{\delta y_\mu(s)} - \frac{\delta A_\mu(s)}{\delta y_\lambda} + A_\mu(s) A_\lambda(s) \right]$$

Now embedding x(s) in the middle of a string segment gives the complete form:

$$\frac{\partial}{\delta x_\mu(s)} U[P] = : F_{\lambda\mu}[x(s)] \frac{dx_\lambda(s)}{ds} U[P]:$$

The symbols [:,:] are meant to order all operators between them, outer symbols taking precedence over inner ones (e.g. those in the righthand U[P] above).

This definition gives a manifest representation of parameterization invariance. If $s \rightarrow f(s)$, then:

$$\frac{dx_\mu(s)}{ds} \rightarrow \frac{df}{ds} \frac{dx_\mu(f)}{df}$$

and

$$\frac{dx_\mu(s)}{ds} \frac{\partial}{\delta x_\mu(s)} U[P] = \frac{df}{ds} \frac{dx_\mu(f)}{df} \frac{\partial}{\delta x_\mu(f)} U[P]$$

but by symmetry, both sides vanish and are thus equal to:

$$= \frac{dx_\mu(f)}{df} \frac{\partial}{\delta x_\mu(f)} U[P]$$

which is the statement of reparameterization invariance. We will also require the differential operation representing the addition of an infinitesimal bump in the string. This has already appeared in the representation of $F_{\mu\lambda}$, which was seen to be equivalent to a small loop of area $\sigma_{\mu\lambda}$. Pinching a string at a point x(s) is like adding a loop

bit there so we can identify:

$$\frac{\partial}{\partial\sigma_{\mu\lambda}[x(s)]} U[P] = : F_{\mu\lambda}[x(s)]\ U[P]:$$

The Bianchi identity, which is an expression of the fact that $F_{\mu\lambda}$ is a connection, becomes in the context of strings the statement that the change in a string due to a change in paths is independent of the surface traversed in moving between paths. Considering two small surfaces between loop segments near x separated by dx_k, then the change in the string due to the change in choice of surface is:

$$\delta_k \int_S \sigma_{\mu\lambda} F_{\mu\lambda} = \int dv\ \epsilon_{krst}\ \partial_r F_{st}$$

$$= 0$$

The equations of motion similarly translate to differential properties of strings. Applying the functional $\frac{\partial}{\partial x_\mu(s)}$ twice gives[86]:

$$\frac{\partial^2}{\partial x_\mu(s)\ \partial x^\mu(s')} U[P] = \delta(s-s')\ :D^\mu F_{\mu\lambda}[x(s)]\ \frac{dx_\lambda}{ds} U[P]:$$

$$+ :F_{\sigma\mu}[x(s)]\ \frac{dx^\sigma}{ds}\ F_{\lambda\mu}[x(s')]\ \frac{dx^\lambda}{ds'} U[P]:$$

The first term vanishes in the pure gauge theory and is proportional to

the quark current $J_\lambda(x(s)) = \bar{q}(x(s))\gamma_\lambda q(x(s))$ in Q.C.D.

3. Regularization

The new questions to be asked about regularization in the loop space formulation concern the nature of loop operators as infinitely composite - albeit countable - functionals of the gauge fields or as extended operators which may be fundamental excitations in a 1/N expansion and which are defined on paths rather than at points. Regularization problems may be circumvented by working on a lattice. The form of the expectation value of products of Wilson loops in this case is:

$$\langle 0| \prod_{i=1}^{n} W[\phi] \,|0\rangle = \int \prod_{l_i} dU_{l_i} \, e^{-\frac{1}{g^2} H[U]} \, \mathrm{Tr} \prod_{l_i \subset \phi} U[l_i]$$

where the U's are defined on the links of the lattice and are integrated over the group. Ultraviolet divergences are controlled by the finite lattice spacing. Infrared divergences do not appear in the strong coupling expansion because the state space contains only color singlet operators whose minimum energy configurations are minimal structures on the lattice. Calculations are performed in an inverse coupling expansion, which has a finite radius of convergence. The crucial problem remaining is to show that a second order phase transition occurs at critical coupling $g_c=0$, which would give asymptotic freedom, and that in the limit $g \to g_c$ and $a \to 0$, the physical parameters approach finite values

integrations around the loops and must be dealt with in the averages over paths discussed below.

A second class of regularizations is needed to handle the singularities peculiar to operators defined only on a curve. This is related to the possibility mentioned above that infinitely many field operators are relevant to the short distance behavior. Kinks and self-intersections may occur even for single loops. We need to consider the possibility of rough paths such that, for example, a path functional $F[\phi]$ may be finite and well-defined for any ϕ smooth, but develops infinities for sufficiently irregular paths. Lacking a complete regularization program based on the calculated short distance properties of infinite sets of gluonic excitations along the paths, one can nevertheless introduce the operation of smearing over the location of the paths, which must be done in any case in order to define physical expectation values. In general, wherever a functional of a fixed path appears, a smooth distribution around the path can be substituted. For example, choosing an infinitely smoothing operator Σ of compact support in R^3, we may express all functionals of paths $F[\phi]$ as:

$$[\Sigma\ F](\phi) = \oint_{\phi} ds \int d^3x\ \Sigma(x)\ F[\phi(x(s))]$$

Each point on the string is smeared around the original path. The smearing need not necessarily be removed later, but it must be chosen to preserve all physical symmetries and should not introduce unphysical

parameter dependence. The physical parameter which defines the regularization point for path averaging will be related to the string tension at some momentum value (slope of the Regge trajectories in the linear region) or equivalently to the mass gap, the energy relative to the vacuum of the lowest excited state. (This assumes the ratio of these quantities is fixed by the theory and is well-behaved in the planar limit.)

The operation of path averaging will generally have the effect of suppressing contributions of rough, self-intersecting paths relative to smooth, non-self-intersecting ones. This is because two adjacent paths in R^3 almost always miss each other, and under the action of the smoothing operation, nonsmooth paths look like smooth ones. It is not expected that purely geometric singularities will introduce significant physical complications.

In the context of a strong coupling expansion, one considers the full expression:

$$\langle \Omega \, | W[\phi_1] \cdots W[\phi_n] | \, \Omega \rangle$$

Regularization is needed to make sense of the case where loops overlap. For continuum as opposed to lattice formulations, the functional operators of differentiation with respect to the path that occur in the equations of motion and Bianchi identity:

with suitable redefinition of the bare parameters, which would restore Lorentz invariance and establish the renormalizability of the strong coupling expansion.

The composite nature of the Wilson loop adds no complication in the framework of perturbative renormalization if terms are truncated at finite order in the coupling. This will destroy manifest gauge invariance and will require imposition of the Ward-Takahashi identies. The full set of terms in every order 1/N, however, involves an infinite set of graphs. Renormalizability is usually proven in the context of a weak coupling expansion having finite sets of graphs in each order. Lowenstein and Speer[87] have developed renormalization procedures for the 1/N-expanded ϕ^4 theory in three dimensions. There is probably no insurmountable difficulty involved with the 1/N expansion, though it will be necessary to develop techniques for systematic application of 1/N for the entire scheme to be feasible.

The choice of regularization in the continuum is contingent on the class of objects being investigated. In lieu of a complete understanding of the planar limit, for deep inelastic processes it is often convenient to perform calculations perturbatively in the coupling. Proofs of relationships within the 1/N expansion are also often more easily accomplished using the g^2N expansion. In these cases, one must regularize the expectation values of multiple operator products of the gauge field at nearby spacetime points on fixed paths arising from fixed order- g^2N terms in the expansion of products of Wilson loops:

$$\int_{\phi_1} \cdots \int_{\phi_n} \prod_{i=1}^{n} d\phi_i \, \langle \Omega \, | A_1[\phi_i] \cdots A_n[\phi_n] | \, \Omega \rangle$$

The ϕ_i's in the integrand may represent the same or distinct paths and the $A_i[\phi_i]$'s are ordered along the several paths. The expectation value of path-ordered n-products of the gauge fields can be regularized in the same way as any time-ordered product. To each order, a well-defined finite, cutoff-dependent expression is obtained. It is possible that interesting short distance physics derives from the limit of infinite numbers of gluons on finite string segments. Additional regulariztion or proof of regularizability may be needed in this case.

The fact that points approach each other along a fixed path rather than from arbitrary directions does not change the structure of the singularities in perturbation theory. However, points approaching each other along a string retain the same degree of singularity regardless of the ambient dimension so dimensional regularization is not an appropriate choice of technique.

From the standpoint of renormalization, it has been pointed out that the Wilson loop is equivalent to the propagator of a quark-antiquark pair after integration of fermionic modes[19]. The fermion-antifermion pair are here restricted to follow a specific trajector corresponding to the path. It can thus be renormalized perturbative. as any ordinary propagator i.e. cutoff dependence can be removed from physical expectation values after a finite number of parameter redinitions. Kinkiness of the paths themselves will show up in the

$$\frac{\partial^2}{\partial x(s)^2}$$

$$\frac{\partial}{\partial \sigma_{ij}(s)}$$

will require separate regularization. This is a new feature not seen in the fully perturbative version of Q.C.D. where the equations of motion involve standard partial derivatives. Regulator functions can be defined on the space of loops such that they depend only on one-dimensional parameters, equivalent to angular variables. Examples of regulator functions are:

Mass-dependent version: $R(x,y) = (\frac{\pi}{t})^{\frac{1}{2}} e^{-\frac{D}{4t}(x-y)}$

Mass-independent version: $R(x,y) = D^{-s}(x-y)$

$$D = -\frac{\partial^2}{\partial x^2} : S^1 \rightarrow S^1$$

We will also be interested in the regularization of the Hamiltonian operator which is bilinear in the gauge field. When acting on functionals of Wilson loops, the Hamiltonian acts like a second order functional derivative The electric term is effectively a Laplacian on the space of loops since its action on a functional of a loop ϕ is:

$$[\int d^3x\ (E_i^a(x))^2,\ F](\phi) = \int d^3x\ [\int D\phi\ [E_i^2(x),W[\phi]]\ \frac{\delta}{\delta W[\phi]}$$

$$+ \int D\phi_1\ D\phi_2\ [E_i^a(x),W[\phi_1]]\ [E_i^a(x),W[\phi_2]]\ \frac{\delta^2}{\delta W[\phi_1]\ \delta W[\phi_2]}\ ]F[\phi]$$

In order to soften the $\delta^2(0)$ singularity that occurs due to the double functional differentiation, it is necessary to point-split the product of $E_i^a(x)$'s. To maintain gauge invariance, it is then necessary to insert strings between the separated operators, i.e. to use the generalized invariant form of of the inner product:

$$\int d^3x\ E_i^a(x)\ E_i^a(x) \rightarrow \int d^3x\ d^3y\ R^{1,2}(x,y)\ \mathrm{Tr}[\ U_1(y,x)E_i(x)U_2(x,y)E_i(y)\ ]$$

where we choose the U_j's to be non-self-intersecting. The function $R^{1,2}(x,y)$ is the three dimensional version of the regulator on the loop. The pointsplit form of the commutator of the electric term with a non-self-intersecting Wilson loop, if we choose the arbitrary path segments to lie along the loop, then produces a factor of:

$$\int d\sigma\ d\sigma'\ [\frac{d\phi_i(\sigma)}{d\sigma}\ \frac{d\phi_i(\sigma')}{d\sigma'}]\ R_\phi[\phi(\sigma),\phi(\sigma')]$$

instead of the two dimensional delta function. Upon removal of the regularization, usually at some value of a parameter on which R depends, $R(x,y)$ should tend toward $\delta^3(x-y)$. For the choice of non-self-intersecting paths (1,2), the U_j's tend to unity and the original

expression is recovered. The same regulation should be performed on the $B_1^2(x)$ term.

4. Results from 1/N expansion

Although it is not expected that the bound state spectrum will be obtainable in ordinary perturbation theory, there are indications that the Schwinger-Dyson equations may become workable in the limit of an infinite number of colors. There are qualitative results from the 1/N expansion which persist for all N and which appear to account for certain phenomenological features of the hadrons. The suppression of fermion loops regardless of their mass provides a possible explanation of both the minimal valence quark structure of the known hadrons and of the O.Z.I. rule, or the suppression of decays forced to proceed via quark pair creation.

The strong coupling expansion on the lattice is effectively an expansion in inverse powers of g^2N. In the continuum, it is expected that the coupling will grow large naturally at large distances, from extrapolation of renormalization group arguments. If we permit the color group to grow also, the effective coupling can be made as large as needed to simplify the dynamics.

It is desirable to have a natural way to scale the ordinary coupling as the number of colors varies. One criterion of naturalness is to maintain fixed the relation between the effective potential in a quark-antiquark bound state and the gluon propagator. The effective

potential can be expressed as the expectation value in a meson state of single gluon exchange to lowest order. We obtain a factor of $\frac{1}{\sqrt{N}}$ squared for the meson normalization, two powers of the ordinary coupling from the vertices, and a factor of N^2 (for large N, $N^2 \approx N^2-1$) for the number of types of gluons that must be summed over. Insensitivity to N can thus be achieved by replacing the usual coupling g by $\frac{g}{\sqrt{N}}$ at every vertex[88].

In addition, it is necessary to include appropriate factors of 1/N to give normalized states, as in a meson:

$$|M> \approx \frac{1}{\sqrt{N}} \bar{q}(x) U(x,y) q \mid \Omega >$$

This case can be covered by the rule that the trace operation occurring in measurable operator combinations should always be replaced by 1/N times the trace. In calculating the meson state normalization, for instance, with bare fermions, we obtain after commuting:

$$< M \mid M > = < \Omega \mid \frac{1}{N} \mathrm{Tr}\ U^{+}(x,y)\ U(x,y) \mid \Omega >$$

$$= < \Omega \mid \frac{1}{N} W(x,y) \mid \Omega >$$

The required normalizability of meson states can thus be translated to the condition that 1/N times the Wilson loop, not the plain Wilson loop, is the relevant measurable operator in the large N limit. Application

of these rules to all diagrams in either weak or strong coupling perturbation theory proves sufficient to give a consistent 1/N expansion in the sense that no physical expectation values have terms proportional to positive powers of N.

When N is taken to be large at a fixed value of g^2N, there arise fairly simple criteria for determining the relative 1/N weighting of classes of Feynman graphs. Graphs with F fermion loops are suppressed by a factor N^{-F} relative to the same diagram containing only gluons. Within the gluon sector, purely topological distinctions arise. Planar graphs are defined to be those which can be mapped on a plane, without any crossed lines. Nonplanar graphs require handles appended to the plane to carry crossing lines. Graphs with H handles are suppressed by N^{-2H} relative to the corresponding planar graph.

In any order 1/N, there still remain infinitely many graphs to be summed so for finite N, the relative weighting of various orders topological complexity cannot be calculated exactly. They can be estimated in many cases by using leading g^2N or $(g^2N)^{-1}$ contributions.

A useful result is that, in the large N limit, the vacuum expectation value of Wilson loops factorizes so that [82,83]:

$$\lim\ (N\to\infty)\ \langle \frac{1}{N} W[\phi_1]\ \frac{1}{N} W[\phi_2] \rangle = \frac{1}{N} \langle W[\phi_1]\rangle\ \ \frac{1}{N} \langle W[\phi_2]\rangle$$

This encourages the hope that the large N spectrum of pure QCD may be exactly calculable since factorization is characteristic of soluble

field theories. In any case, the factorization property can be combined with ordinary perturbative expansions for the purposes of analyzing contributions to the expectation value of the Wilson loop. Maheenko and Migdal have shown, for example, that the Wilson loop is the generator of planar vacuum to vacuum diagrams[83].

There may exist infinite N properties, such as massless glueballs or discontinuities in the β function, which are not continuable to finite N. Although it has not been shown that Q.C.D. is soluble in leading order 1/N, or which large N results may be smoothly extrapolated and persist as leading effects at N=3, we will use 1/N suppression in circumstances where classes of effects are chiefly distinguished by their color combinatorics. This will apply to the elimination of intermediate fermion loops and will be used to single out planar gluonic structures where there is no obvious difference in other singularities.

III. Hamiltonian Formulation

In order to produce measurable functions in the loop space framework, it is necessary to dispose of the unphysical path dependence. The problem of whether stable states are stringlike is in some ways analogous to the familiar question in ordinary field theory as to whether the physical states are localized. This is a dynamical issue not a matter of choice of notation. It is necessary to demonstrate the persistence of stringlike configurations in the presence of interactions. Choosing an incorrect bare state basis will lead to radical alterations in the form of trial eigensystems in each order. Q.E.D. could be described in term of loop operators, for example, but the phase structure of a U(1) gauge theory indicates charged particles may be free finite energy states. Given the known large distance smallness of the coupling, perturbation theory about the unconfined phase is permissible in the infrared. This gives a Coulombic distribution of flux lines, i.e. stringlike states become diffuse in lowest order in the interaction.

It may be possible that the Hamiltonian of Q.C.D., to leading order in some form, possesses a known spectrum. The eigenstates, although simple in the transformed basis, would be coherent superpositions of ordinary field operators applies to the vacuum. With the expectation that the ultimate states of the system will be, even in lowest order, coherent states, it is of interest to investigate the question of

choosing bare states before the solution of the full theory which may be amenable to elucidating information.

There has been little work done with true coherent states in four-dimensional field theories. The most familiar models, ϕ^4 and Q.E.D., share the features that they are infrared stable so weak coupling at intermediate distances permits extrapoation to large distances and hence asymptotic states are directly related to the bare, pointlike fields.

In two dimensions, a greater variety of workable theories occur though none so far show precisely the structure expected in QCD. The sine-Gordon model above a critical coupling is more succinctly described by the Thirring model[55,56]. Here there is a distinct, discrete coupling above which a new stable particle appears as the fundamental excitation. The two-dimensional nonlinear O(N) sigma model[56,57] displays a perturbative spectrum of (N-1) degenerate massless particles and spontaneously broken O(N) symmetry. Transformation to a new set of generalized, nonlocal group invariants $\sigma(x)\sigma(y)$ permits the action to be rewritten in a form having a consistent 1/N expansion, soluble in lowest order. The natural bare spectrum consists of an N-degenerate massive multiplet.

Whatever the change in the state basis, the spectrum of the Hamiltonian remains formally the same for a wide class of tranformations, although its perturbative form may bear little resemblance to the original expansion. Numerical results from lattice, string, and bag models point to the possibility that, at least for heavy quark and high angular

momentum light quark systems, the coherent states are strikingly close to the naive strong coupling lattice states.

1. The Q.C.D. Hamiltonian

The general form of the Hamiltonian and the canonical commutation relations will be derived using the method of Dirac[89]. If the canonical coordinates for the gauge sector are chosen to be $A^a_\mu(x)$, the canonical momenta obtained from the Lagrangian are:

$$\Pi^a_i(x) = \frac{\partial}{\partial(\delta_o A^a_i(x))} L$$

$$= \frac{1}{g^2} F^a_{oi}(x)$$

$$= E^a_i(x)$$

$$\Pi^a_o(x) \approx 0$$

The second equation is a primary constraint, using Dirac's terminology. The $\approx$ sign indicates the operator can only be set to zero after all commutations have been performed. Primary constraints with arbitrary coefficients c_i must be included in the full Hamiltonian to obtain the correct equations of motion.

The Hamiltonian density at this stage is given by:

$$H_i(x) = \sum_{i=1,2,3} \Pi_i^a(x)\delta_o A_i^a(x) - L(x) + c_i^a \Pi_o^a(x)$$

$$= \mathrm{Tr}\ [\ \frac{g^2}{2} \Pi_i(x)\ \Pi_i(x) + \frac{1}{4g^2} F_{ij}(x)F_{ij}(x)$$

$$+ \Pi_i(x)D_i(x)A_o(x) \quad + c_i(x)\Pi_o(x)\ ]$$

The time development of operators should be given by their commutator with the Hamiltonian. Before imposing constraints, the naive canonical commutator between any (Bose) coordinate $Q_a(x)$ and its conjugate momentum $P_b(y)$ is:

$$[\ Q_a(x),\ P_b(y)\ ] = i\ \delta_{ab}\ \delta(x - y)$$

In particular, if the primary contraint is to be true for all times, we require the weak ($\approx$) vanishing of:

$$[\ \Pi_o^a(x), H\ ] = \Pi_i^b(x)D_i^{ba}(x) \approx 0$$

This gives Gauss' law, which is required to be true on the space of physical states. It is equivalent to a gauge rotation and is thus automatically satisfied by all states in the loop space formulation, to the extent that we do not destroy gauge invariance at intermediate stages by our regularization procedure.

The commutator of Gauss' law with the Hamiltonian H_1 vanishes identically, so it induces no new contraints. The form of the Hamiltonian with all constraints is thus:

$$H_2(x) = \mathrm{Tr}\ [\frac{g^2}{2}\Pi_i(x)\ \Pi_i(x) + \frac{1}{4g^2}F_{ij}(x)F_{ij}(x)$$

$$+ [A_o(x) + c_2(x)]D_i(x)\Pi_i(x) + c_1(x)\ \Pi_o(x)\ ]$$

Note that the time derivative of the $A_o(x)$ field is equal to $c_1(x)$. Since furthermore A_o only occurs added to the arbitrary coefficient of Gauss' Law, we are free to set $A_o = 0$ for all times.

The final form of the nonvanishing equal time commutation relations expressed in the $A_o = 0$ gauge are:

$$[\ E_i^a(x)\ ,\ A_j^b(y)\ ] = -i\ \delta^3(x - y)\ \delta_{ij}^{ab}$$

$$i,j = 1,2,3$$
$$a,b = 1,2,\ldots N$$

The quark anticommutators are straightforward:

$$\{\ q_m^+(x)\ ,\ q_n(y)\ \} = \delta^3(x - y)\ \delta_{mn}$$

The subscripts refer to Dirac, color. and flavor indices. Quark operators commute with all gauge field operators. The full Hamiltonian

density including fermions is thus:

$$H(x) = \frac{g^2}{2} \Pi^a(x)\, \Pi_i^a(x) + \frac{1}{4g^2} F_{ij}^a(x) F_{ij}^a(x) + q^+(x)\, [\, \alpha_i\, D_i + \gamma_0 m\,]\, q(x)$$

2. Application to Wilson loop operators

States will be constructed by applying products of Wilson loop operators to the vacuum $|\Omega\rangle$ defined as the state annihilated by the normal ordered Hamiltonian and by all quark annihilation operators. All operators are defined at a fixed time. Paths are required to be continuous on this fixed time slice. The action of the normal ordered Hamiltonian on loop states:

$$\prod_n W[\phi_i]\ |\Omega\rangle$$

can then be obtained directly from the canonical commutators.

The commutator of the electric field with multiple loop states effects a functional differentiation with respect to the gauge field:

$$E_i^a(x) \rightarrow -i\, \frac{\partial}{\partial A_i^a(x)}$$

From the representation of $W[\phi]$ in terms of A_μ, commutation of E_i^a with a single loop produces:

$$[\ E_i^a(x), W[\phi]\ ] = \mathrm{Tr} \int_\phi ds\, \frac{d\phi^i(s)}{ds} : T^a(s)\ U[\phi]\ : \delta(\ \phi(s) - x\ )$$

The differential element of path dx_μ has been represented by the gradient of a three-vector function $\phi(s)$ with respect to an arbitrary path parameter, s. The notation $T^a(s)$ means the matrix T^a is to be ordered in the operator product as if it were at the point $\phi(s)$ on the loop. Commutation of the full electric term gives:

$$[\ \mathrm{Tr}\ (E_i^a(x))^2,\ W[\phi]\ ] =$$

$$\int_\phi\int ds\, ds'\ (\frac{d\phi_i}{ds}\frac{d\phi_i}{ds'}\ )\ :\ T^a(s)\ T^a(s')\ U[\phi]\ :\ \delta^3(x - \phi(s))\ \delta^3(x - \phi(s'))$$

$$+\ 2 \int ds\, \frac{d\phi_i}{ds}\ :\ T^a(s)\ U[\phi]\ :\ \delta^3(x - \phi(s))\ E_i^a(x)$$

Written in this form, it is evident that the electric energy operator distinguishes non-self-intersecting from self-intersecting paths. In the former case, the first term factorizes, after integration over x, to:

$$(T^a)^2 \int_\phi\int ds\, ds'\ (\frac{d\phi_i}{ds})^2\ \delta^3(\ \phi(s) - \phi(s')\ )\ \ W[\phi]$$

The coefficient of $W[\phi]$ is roughly the loop length times a two

dimensional delta function, reflecting the zero path width.

In the latter case, we use the identities:

$$U(N): \qquad \sum_a (T^a)^{\alpha}_{\beta} (T^a)^{\alpha'}_{\beta'} = \delta^{\alpha}_{\beta'} \delta^{\alpha'}_{\beta}$$

The analogous identity for SU(N) has the same form plus a piece of order 1/N, which is assumed suppressed. For a self-intersection at X dividing the loop into closed sections (ϕ_1, ϕ_2), a contribution to [E^2,W[ϕ]] is:

$$[\oint ds\, ds' \frac{d\phi_1}{ds} \frac{d\phi_2}{ds'} \delta(\phi_1(s) - \phi_2(s'))]_{s=s_1, s'=s_2} W[\phi_1]\, W[\phi_2]$$

Thus the electric term causes a splitting of Wilson loops at points of self-intersection. If E^2 acts on a product of Wilson loops for paths which overlap, there will also be contributions corresponding to the fusion of loops at common points. However, using the $\frac{Tr}{N}$ convention:

$$\frac{Tr}{N} :T^a(s)U[\phi_1]: \frac{Tr}{N} :T^a(s')U[\phi_2]: = \frac{1}{N} \frac{Tr}{N} :U[\phi_1]U[\phi_2]:$$

it can be seen that fusion of loops will be suppressed by a factor of 1/N relative to the other two types of terms.

The magnetic energy term commutes with states composed of Wilson loops. It has the form of a uniform density of infinitesimal spacelike loops since:

$$[F_{ij}(x)]^2 = \lim y \to 0 \quad [\frac{1}{y^2} < W[y] + W^+[y] >$$

where y is a small spacelike loop and the average is over all spatial orientations. In the expectation value of the magnetic term in a state $F[W]$:

$$< \Omega \mid F'[W] \; B_i^a(x)^2 \; F[W] \mid \Omega >$$

there will be contributions to the background, which will generally be absorbed in the normalization, and to correlations between states on the left and right differing only by factors of infinitesimal Wilson loops, suppressed by $1/N$. The explicit contribution of the magnetic terms will not be needed for the leading order contribution to the expectation value of the normal ordered Hamiltonian.

3. Physical state space

In order to define physical expectation values, it is necessary to dispose of the path dependence. This will be done by defining path-averaged wavefunctionals for the trial states:

$$|\Psi > = \int D\mu[\phi] \; \Psi[\phi] \mid \Omega >$$

with the inner product of state vectors given by:

$$< \Psi_1 \mid \Psi_2 > = \int D\mu[\phi]' \, D\mu[\phi] \, < \Omega \mid \Psi_1^{+}[\phi'] \, \Psi_2[\phi] \mid \Omega >$$

This definition retains the distinction between ordinary expectation values of Q.C.D. operators - such as Wilson loops - which may be uninteresting because of their loop dependence, and the path averaging used to form coherent superpositions which are both stable and measurable.

We assume the existence of a state $\mid \Omega >$, the full vacuum, which is the minimum eigenvalue eigenstate of the Q.C.D. Hamiltonian. The normal ordered Hamiltonian H_{no} is then defined with respect to this state so that:

$$H_{no} \mid \Omega > = [H_{QCD} - E_o] \mid \Omega > = 0$$

The gauge field operator structure of the full vacuum is left unspecified. The full vacuum is naive Fock vacuum for fermions since fermion bubbles (strings) are always suppressed by 1/N relative to leading gluonic terms. Except where quarks appear as valence components in external states, they will be deleted from what follows.

Excited trial states of the system are contructed by application to the full vacuum of valence loop space operators, averaged over paths. For example, a glueball centered at x may have the form:

$$\mid G(x) > = \int D\mu_x[\phi] \; W[\phi] \mid \Omega >$$

$$< \Omega \mid G(x) > = 0$$

where the subscript on $D\mu$ centers the measure at x. Information about the internal structure of states is presumably contained in the measure over paths, with a different measure, in general, applying to each state.

4. Properties of the path measure

If this method is to be workable, a measure should be used which enhances what are expected to be the dominant configurations in some limit - most likely large N. Although there is no rigorous theoretical reason to suppose that these configurations functionally resemble ordinary Wilson loops and fermionic strings, the geometric and group theoretic similarities between many physical hadron states and the bare string states of the strong coupling lattice lead us to attempt to absorb in a well-chosen path measure nonlinearities of the mappings from our original modes to the eigenmodes of the Hamiltonian.

The bounded measure $D\mu[\phi]$ will be required to have the properties that, first, if n ϕ_i's appear in the integrand, then:

$$D\mu[\phi] \rightarrow \prod_{i=1}^{n} D\mu[\phi_i]$$

Secondly, the measure should be invariant under rotations and translations of the paths appropriate to the symmetry of the state under

consideration. It can thus depend only on certain invariant combinations of $\vec{\phi}(s)$ and $\frac{d\vec{\phi}(s)}{ds}$, which are the three vectors of the path position and gradient at the point s. Finally, we require that the set of measurable functionals include, in both the small coupling and large N perturbative form, $\langle\Omega|W[\phi]|\Omega\rangle$ and $H_{op}\ W[\phi]$ for some types of loops. These conditions make all n-products of Wilson loop expectation values measurable, and from the factorization result for large N, expectation values of n-products of loops are also measurable.

The class of allowed paths is naturally intimately associated with the allowed measures. Path configurations can affect singularities in the expectation values of Wilson loops and of $HW[\phi]$. On the other hand, $D\mu[\phi]$ may be chosen to effectively exclude certain configurations, such as self-intersecting paths. We can immediately dispose of the complication of discontinuities in $\frac{d\phi(s)}{ds}$ by using the fact that smooth paths:

$$C_\infty : [0,1] \rightarrow R^3$$

are dense in the set of continuous paths $C_o(R^3)$.

There is some information available on the functional path dependence of $\langle\ \Omega|W[\phi]|\Omega\ \rangle$. From the small coupling perturbative form, we will get arbitrary order polynomials in the path gradient. Polynomials in the path separations might be expected in multiloop expectation values. A measure space over smooth paths which contains as measurable function all polynomials in ϕ and the gradient of ϕ has a measure which

is the Gaussian form[90]:

$$D\mu[\phi] = D\phi\, e^{-\frac{1}{2}[\phi, B\phi]}$$

B is a positive, 3 x 3 matrix containing no more than two derivatives. The symmetry of B determines the symmetry of the measure. The expression on the right is the limiting form of the product of Gaussians at each point s_i on the loop.

Using the perturbative information that loop expectation values may contain arbitrary order polynomials in only the path gradient, we choose a minimal form of the measure independent of the path location. By the translation invariance of the measure, we can choose B so the Gaussian is diagonal:

$$D\mu_{min}[\phi(s)] = D\phi(s)\, e^{-\frac{1}{2}\left(\frac{d\phi_i}{ds}\frac{d\phi_i}{ds}\right)}$$

If no $|\phi|$-dependence arises in the integrands, as is the case in lowest order electromagnetism, this measure is sufficient to produce Coulombic distributions. The perimeter dependence which arises perturbatively and which dominates configurations in the abelian theory contribute an effective measure of the form:

$$e^{-\int ds\, \left|\frac{d\phi^i(s)}{ds}\right|^{\frac{1}{2}}}$$

This is subsumed by the free measure above since it is well-known[91] that both effective actions are minimized along geodesics. The geodesics in Euclidean space without sources present are straight lines, with sources and with the requirement of flux conservation we obtain the usual Coulombic distributions[92].

IV. Variational Principle

The possibility of making a reasonable guess of the approximate form of the path weight suggests using a variational technique.

1. General characteristics

The principles of variational calculations in quantum field theory are much the same as those in quantum mechanics[93]. The full Hamiltonian is presumed given in terms of the canonical variables. We take as the space of possible physical states the set of normalizable functionals of the canonical fields (coordinate representation) or momenta (momentum representation) such that the Hamiltonian acting on this space is self-adjoint. A trial state of a particular structure is chosen from this space. Its form is partially constrained by known properties of the exact wavefunctional, such as the fermion content, or expected properties, such as similarity to the solution when some coupling is set to zero. An arbitrary parameter or parameter function is introduced in the structure of the wavefunctional so that variation of this parameter provides access to a subspace of the state space. The expectation value of the Hamiltonian gives an upper bound on the exact energy. Minimization of the trial energy determines the best wavefunction of the predicated form. Trial excited states are chosen to satisfy the additional criterion of orthogonality to the ground state.

2. Example in quantum field theory

In quantum mechanical variational calculations, the trial state is usually parameterized in such a way that the minimization condition gives an algebraic equation for the trial parameter. Such a parameter in field theoretic problems may be required for each spacetime point. As an illustration, we first consider a field theoretic calculation which can be viewed as an extension of a quantum mechanical problem. Gauge theories are not good examples in this regard because of their inherently extended character in the strong coupling regime. It is simpler to work in a theory having weakly bound composite states so the coordinates of the coordinate representation retain their pointlike character. Therefore, following Barnes and Ghandour[94], consider ϕ^4-theory in four spacetime dimensions. The corresponding quantum mechanical problem is that of an anharmonic oscillator, which in one spatial dimension has the Hamiltonian:

$$H(p,q) = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 + \lambda_o\, q^4$$

The space of states is $L^2(R)$ with an inner product:

$$< \Psi_1 | \Psi_2 > = \int dq\, \Psi_1^+(q)\, \Psi_2(q)$$

When λ is zero, the exact, normalized solution is the Gaussian:

$$\Psi_o(q) = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\omega}{2}q^2}$$

A possible trial solution is a Gaussian with an unknown, $\alpha(\omega,\lambda)$, substituted for ω. The trial energy is then:

$$E_\alpha = \langle \Psi_\alpha | H | \Psi_\alpha \rangle$$

$$= \frac{\alpha}{4} + \frac{\omega^2}{4\alpha} + \frac{3\lambda}{4\alpha^2}$$

and the equation for alpha coming from $\delta E_\alpha = 0$ is:

$$\alpha = \frac{6\lambda}{(\alpha^2 - \omega^2)}$$

The field theoretic calculation is done in the Heisenberg representation with all states and operators defined at a fixed time. The coordinate (ϕ) space representation of the ϕ^4 Hamiltonian:

$$H = \int d^3x \, \frac{-1}{2} \frac{\partial^2}{\partial\phi(x)^2} + \frac{1}{2} (\nabla\phi(x))^2 + \frac{m^2}{2} \phi(x)^2 + \frac{\lambda}{4!} \phi(x)^4$$

has the same structure as the anharmonic oscillator Hamiltonian with the substitution of $\phi(x)$ for q, $-i\frac{\partial}{\partial\phi(x)}$ for p, and the addition of the gradient term. State functionals in the coordinate representation are denoted by:

$$\Psi[\phi] \equiv \langle \Psi \mid \phi \rangle$$

$$\langle \phi(x) \mid \phi(y) \rangle = \delta^3(x - y)$$

with an inner product defined:

$$\langle \Psi_1 | \Psi_2 \rangle = \int D\phi(x)\ \Psi_1^+[\phi(x)]\ \Psi_2[\phi(x)]$$

Allowable $\Psi[\phi]$'s are those which are square integrable in the Wiener sense; ϕ's inside the functional integral are treated as classical fields.

The variational problem is set up as before by first noting that the λ=0 vacuum solution can be written:

$$\Omega_o[\phi] = N_f\ e^{-\frac{1}{2}\int d^3x\ d^3y\ \phi(x)\ f_o(x-y)\ \phi(y)}$$

with $f_o(x-y) = \delta^3(x-y)(-\nabla^2+m^2)^{\frac{1}{2}}$ the analogue of ω. A trial vacuum state is now chosen to have the same form:

$$\Omega_g[\phi] = N_g\ e^{-\frac{1}{2}\int d^3x\ d^3y\ \phi(x)g(x-y)\phi(y)}$$

with a trial parameter function $g(x-y) = \delta^3(x-y)(-\nabla^2+a)^{\frac{1}{2}}$. The same

results would follow from the more general choice $g(x-y) = \delta^3(x-y)g[\nabla_x]$. The expectation value of the Hamiltonian in this state is:

$$E_g = \frac{\delta^3(0)}{8} \left[\int \frac{d^3p}{g(p)} \left[1 + \frac{(p^2 + m^2)}{g^2(p)}\right]\right.$$

$$\left. + \frac{\lambda\, \delta^3(0)}{8(2\pi)^3} \left(\int \frac{d^3p}{g(p)}\right)^2 \right]$$

$$g(p) = (p^2 + a)^{\frac{1}{2}}$$

Minimization of the energy with respect to the parameter a produces the equation:

$$a = m^2 + \frac{\lambda}{4(2\pi)^3} \int \frac{d^3p}{g(p)}$$

The interacting vacuum wavefunctional has the form of the free vacuum with a new mass $\mu = \sqrt{a}$. For comparison, consider the renormalized mass found from ordinary perturbation theory by summing all nonoverlapping bubble diagrams. The contribution from each bubble is, up to a factor:

$$I_B = \int \frac{dp_o d^3p}{p_o^2 + p^2 + m^2}$$

$$= \pi \int \frac{d^3p}{(p^2 + m^2)^{\frac{1}{2}}}$$

and the sum of all bubbles produces:

$$\mu_B^{\ 2} = m^2 + \frac{\lambda}{2(2\pi)^4} I_B$$

which is the same as the variationally determined corrected mass. The coupling is unaffected.

Both the nature of the correction - improvement of the propagator - and the form of the renormalization - a sum of disconnected bubbles - follow directly from the assumed form of the vacuum wavefunctional. This has the advantage of permitting the choice of the wavefunctional to be dictated by the desired sensitivity to a particular measurable function, but it is disadvantageous in that the results are more or less predetermined. The correspondence between variational renormalization and standard perturbative renormalization does not follow a fixed pattern. The assumed form of the vacuum wavefunctional is not sufficient, for example, to produce the lowest terms in the renormalized coupling. Using a bound state wavefunctional or an effective potential[94], coupling renormalization occurs but with peculiar consequences: stability of the bound state seems to imply a repulsive renormalized coupling, in the former calculation, and in the latter, the coupling looks like (but is not quite equal to) the leading order M coupling in an O(M) ϕ^4

theory. These problems may be circumvented if four-point correlations are included in the trial vacuum.

Nevertheless, there seems to be nothing intrinsically wrong with the use of the variational method as long as its limitations are understood. It will not be relied upon here to produce a complete set of equations for renormalization, analogous to the boundary conditions in the covariant formulation. The key assumption in the ϕ^4 example was the form of the vacuum wavefunctional. It may be equally effective to assume a functional form for expectation values of operators especially in cases where these factorize to leading order. The vacuum variational calculation may then be avoided and one may pass directly to excited state calculations. Since renormalization must presumably be carried out on the level of the loops themselves, it is feasible that we need only consider finite, physical parameters as coefficients in the path space measure.

3. Path space version

The path degree of freedom in the gauge-invariant formulation introduces additional possibilities for the use of variational constructs. The Wilson loops as coordinate functionals can be used as the arguments of the wavefunctional, analogous to the use of the ϕ fields in the example. In addition, wavefunctionals of the paths, of unspecified operator origin, may be introduced.

In the former method, it is not supposed that only a finite number

of Wilson loops are present. The most general wavefunctional which should be considered has the form:

$$\Phi\,[W] = \int \prod_{i=1}^{n} D\mu[\phi_i] \sum_{n=0}^{\infty} \Phi[\phi_1 \cdots \phi_n]\; W[\phi_1] \cdots W[\phi_n]$$

The inner product in state space is defined:

$$(\,\Phi_1[W],\, \Phi_2[W]\,) = \int DW'[\phi]\; DW[\phi]\Phi_1^{+}[W']\; \Phi_2[W]$$

The functional integration over the loop operators is necessary to compensate for an ignorance of the actual expectation values. To measure the energy, we must require that the Hamiltonian acting on a state of class Φ, however that is defined, gives another element of the class so that the expectation value of H makes sense. The general conditions which the class of wavefunctionals must fulfill can be gotten directly from the W-operator form of the Hamiltonian. The procedure for defining the Hamiltonian operator on this space of functionals of the Wilson loop was described by Sakita[95]. For any particular choice of trial wavefuntion, it is a matter of taste whether one chooses to use this formal definition or not. It is entirely equivalent to use the explicit commutators.

An alternative way to construct the trial wavefunctional is to incorporate variational functions in the measure over path space. This certainly seems the most ecomomical for the heavy meson trial states.

There will be dynamical path dependence coming from the actual calculation of expectation values of Wilson loops, but one may use the variational path dependence to simulate higher order corrections. The Hamiltonian could be formally expressed as an operator on the space of path functionals[96], which has been done in the preceeding section for the electric term acting on a single Wilson loop. For a general wavefunctional, and in particular for the vacuum, it is necessary to consider also the magnetic term. The expression for this term as the limit of a functional in loop space was displayed in Section II. It is not necessary to express the Hamiltonian as an operator on the measure since its action on Wilson loops is known. The key assumption is that the wavefunctionals depend on paths only through the loop operators. One should choose a trial measure such that, for the class of Wilson loop functionals desired for state space, gives again a measurable function. Since the action of the Hamiltonian, to leading order $1/N$, is effectively multiplication by either Wilson loops or simple functions such as loop length, the criteria for the effective loop measure are straightforward. They will be discussed in more detail in the next section.

It will also be seen that in the leading order strong coupling and $1/N$ expansion, the Wilson loops decouple. In this case, it is entirely equivalent to assume a form for the vacuum expectation value of the loops as to assume a form for the trial vacuum, as it should be required of the trial vaccum that factorization occurs to leading order. It is then straighforward to perform a variational calculation using the

measure over paths as the trial wavefunctional as the Wilson loops themselves have been eliminated in favor of their expectation values, which are assumed explicit functionals of the paths.

V. Trial Meson Wavefunctional

The ideas and approximations of the preceeding two sections will now be applied to a rough variational calculation of a heavy, color singlet, quark-antiquark bound state wavefunctional.

1. Heavy quark meson

The valence composition of a meson is the minimal set of quarks whose additive quantum numbers give the quantum numbers of the meson. The valence wavefunctional of a model meson in the loop space formulation is defined analogously as the minimal gauge-invariant state with the appropriate quantum numbers, including net fermion and antifermion number at separate points. The valence meson wavefunctional is given by the fixed-time state:

$$| M_{(x,y)} > = \frac{1}{\sqrt{N}} \int D\mu[\phi] \; q^{+}(y) \; U_{(x,y)}[\phi] \; q(x) \; | \Omega >$$

The form $q^{+}(y)q(x)$ is abbreviated notation for a general quark-antiquark bilinear with Lorentz and additive quantum number structure also expressed minimally. We assume that the vacuum and the bound state are such that heavy bound quarks may be treated as free fields, so that we may define spinorial raising and lowering operators such that:

$$b(x) \mid \Omega > = d(x) \mid \Omega > = 0$$

and if $\mid M >$ is a physical meson state, then:

$$< M \mid b_i^+ d_j^+ \mid \Omega > = \mid 0$$

for some i,j. The b's and d's satisfy the canonical anticommutation relations.

The string measure $d\mu_S[\phi]$ has the general properties discussed in the previous section with the modification of boundary conditions that the strings end on the color charges:

$$\phi[0] = \text{quark position}$$

$$\phi[1] = \text{antiquark position}$$

There are more substantive changes due to the dynamics of the massive, charged endpoints such as the enhancement of longitudinal modes due to radial oscillations. A meson string may be more likely to break into a $\bar{q}q$ pair because the longitudinal modes, which are coupled to the quark current, are enhanced by the axial geometry of the meson.

With the restriction to heavy quarks in the ground state of the meson, the endpoints will be fixed to lowest order approximation. The notion of heaviness may be made more precise by defining it as a large

ratio between the free quark mass and the energy scale at which confinement sets in, roughly determined by the measured parameter Λ setting the scale for the falloff of the coupling:

$$g^2(q^2) = \log^{-2}\left(\frac{q^2}{\Lambda^2}\right)$$

A minor problem with this choice is the ambiguity in the value of Λ because of possible higher twist contributions.

Alternatively, the large ratio could be required between the quark mass and the coefficient of the linear potential λ in spectral fits. Since Q.C.D. has no intrinsic scale, these two dimensionful quantities, Λ and λ, are related in a way fixed by the theory, although not yet calculable. The linear potential measures the quark - antiquark binding in the radial direction and the meson radial oscillation frequency is on the order of this binding energy. The relevant inertial frequency comes from the quark mass. Heavy quarks, by these definitions, can be considered static in the ground state to order $\frac{\lambda}{(M_q)^2}$ or $\frac{\delta E_r}{M_q}$ where δE_r is a typical radial excitation energy splitting.

The normalization condition for the meson wavefunctional will, under these circumstances, provide a relation between string and loop measures. At each order of approximation, we require that:

$$1 = \int dx'\, dy'\, \langle M_{(x',y')} | M_{(x,y)} \rangle$$

$$= \int D\mu[\phi_1]\ D\mu[\phi_2]\ \langle \Omega |\ \frac{1}{N}\ \mathrm{Tr}U_1^{+}(\phi_1;x,y)U_2(\phi_2;x,y)\ |\Omega\rangle$$

$$= \int D\mu_{(x,y)}[\phi]\ \langle \Omega |\ \frac{1}{N}\ W[\phi]\ |\Omega\rangle$$

where the subscript on the loop measure constrains it to paths passing through the points (x,y). The trace over the left and right strings producing the Wilson loop came from the fermion anticommutators. Henceforth, we will implicitly use the 1/N-consistent definition of the trace so that vacuum expectation values of Wilson loops as so redefined will be written simply as:

$$\langle \Omega |\ \frac{1}{N}\ \mathrm{Tr}U[\phi]\ |\Omega\rangle = \langle \Omega |\ W[\phi]\ |\Omega\rangle$$

In the more general case in which arbitrary numbers of fermion - anti-fermion pairs (sea quarks) appear in U_T, the righthand side is a sum over products of Wilson loops.

2. Action of H_{QCD} on minimal meson

The operational Schroedinger equation for the ground state meson has the form:

$$[\ \int dx\ [\ \frac{g^2}{2N}(E_i^a(x))^2 + \frac{N}{2g^2}(B_i^a(x))^2 + q^{+}(x)(\ \alpha_i\nabla_i + M\ )q(x)\ ]\ -\ E_T]|\ M\rangle = 0$$

Note that the covariant derivative in the quark part has been replaced

by the ordinary derivative which gives the proper action on the quark fields within an extended gauge invariant wavefunctional. We are interested in the energy of the meson above the vacuum so we replace E_T by $E_o + E_M$. Using the valence form for the meson, which commutes with $B_i^a(x)^2$, and the strong vacuum condition:

$$[\ H_{QCD} - E_o\ ]|\Omega> = 0$$

we get the simplified equation:

$$0 = [\ H_{no} - E_M\ ]|M>$$

$$= \int d^3x\ \{\ \frac{g^2}{2N}\ [\ E^2(x), M_{(x,y)}]|\Omega>$$

$$+ [\ q^+(x)(\ \alpha_i \nabla_i + m\ )\ \frac{\partial}{\partial q^+(x)} - E_M\ ]M_{(x,y)}|\Omega>\ \}$$

The kinetic quark terms are, as mentioned above, suppressed by factors of $1/M_q$ relative to the mass term. They may be comparable to the gauge field part of the energy even in the ground state. However, as we are interested in determining the flux distribution rather than radial oscillatory modes, they will be dropped. Subtracting the quark masses from the meson energy now gives an expression for the contribution of the string to the total energy of the form:

$$E = \frac{g^2}{2N}\int D\mu_S[\phi_1]\ d\mu_S[\phi_2]\ \langle\Omega|\ \frac{Tr}{N}\ U^+[\phi_1][\ (E_i^a(x))^2, U[\phi_2]]\ |\Omega\rangle$$

We would like to further decompose the effect of the Hamiltonian in order to isolate leading order contributions which are sensitive to $\phi_1 - \phi_2$ correlations. To do this, rewrite the expectation value of the electric commutator in symmetrized form using the identity:

$$\langle\Omega|\ Tr\ U_1^+[E^2, U_2]\ |\Omega\rangle = \langle\Omega|\ Tr\ [U_1^+, E^2]U_2\ |\Omega\rangle$$

$$= \langle\Omega|\ Tr[U_i^+, E_i^a][E_i^a, U_2]\ |\Omega\rangle$$

$$= \frac{1}{2}\ \{\langle\Omega|\ Tr\ E_i^a U_1^+[E_i^a, U_2] + Tr[U_1^+, E_i^a]U_2 E_i^a + Tr[U_1^+, E_i^a]E_i^a U_2 + Tr U_1^+ E_i^a[E_i^a, U_2]\ |\Omega\rangle\}$$

Now us that fact that under the combined charge conjugation and time reversal operations, the gauge fields are unaffected and electric field changes sign. A typical $(n,m)^{th}$-order term in the perturbative expansion of the commutator has the form:

$$A_1 \cdots A_n[E^2(x), A'_m \cdots A'_1)] =$$

$$[\sum_{k=1}^{n}\sum_{k'=1}^{m}(A_1 \cdot\cdot [A_k] \cdot\cdot A_n A'_m \cdot\cdot [A'_k] \cdot\cdot A'_1)\delta(x-k)\delta(x-k')]$$

$$+ \frac{i}{2}\sum_{k=1}^{m}[A_1 \cdot\cdot A_n A'_m \cdot\cdot [A'_k] \cdot\cdot A'_1 E(x) + E(x) A_1 \cdot\cdot A_n A'_m \cdot\cdot [A'_k] \cdot\cdot A'_m]\delta(x-k)$$

where spatial arguments and group matrices have been suppressed and square brackets around an A_i indicates absence of the term in the product. The A's are ordered so that their indices increase as their arguments range on the path from $\phi[0]$ to $\phi[1]$. The CT invariance of the vacuum implies that the vacuum expectation value of any operator O satisfies:

$$\langle\Omega|\ O\ |\Omega\rangle = \langle\Omega|\ [(CT)^{-1}O(CT)]^{+}\ |\Omega\rangle$$

Now using the symmetrized expression for the expectation value of the Hamiltonian, it can be seen that, order by order, terms involving a single commutator of E_i^a with the string functional can be paired so that the combinations, such as:

$$\langle\Omega|\ A_1 \cdot\cdot A_n A'_m \cdot\cdot A'_1 E(x)\ |\Omega\rangle + \langle\Omega|\ E(x) A'_m \cdot\cdot A'_1 A_n \cdot\cdot A_1\ |\Omega\rangle$$

undergo a change of sign under $O \rightarrow [(CT)^{-1}O(CT]^{+}$. We have also made use of the symmetry in the integration over paths under interchange of the dummy path labels 1 and 2. The single commutator terms are therefore zero which can also be seen by using the transformation property of the U_i's under CT:

$$[(CT)^{-1}U(CT)]^{+} = [U^{+}]^{*}$$

and:

$$[(U)^*, E_i^a(x)] = -[U, E_i^a(x)]^*$$

The star indicates complex conjugation. The symmetrized single commutator terms such as:

$$< \Omega | \ \mathrm{Tr} U_1^+ [E(x), U_2] E(x) \ | \Omega > + < \Omega | \ \mathrm{Tr} E [U_1^+, E(x)] U_2 \ | \Omega >$$

thus transform to:

$$< \Omega | \ \mathrm{Tr} E(x) [(U_2^+)^*, E(x)] U_1^* \ | \Omega > + < \Omega | \mathrm{Tr} (U_2^+)^* [E(x), U_2^*] E(x) \ | \Omega >$$

$$= - [< \Omega | \ \mathrm{Tr} E(x) [U_2^+, E(x)] U_1 \ | \Omega > + < \Omega | \ \mathrm{Tr} [U_2^+, E(x)] U_2 E(x) \ | \Omega >]^*$$

but the symmetrized sum is also real so it vanishes. The surviving piece of the energy is:

$$< \Omega | \ \mathrm{Tr} U_1^+ [E^2(x), , U_2] \ | \Omega > = < \Omega | \ \mathrm{Tr} [U_1^+, E_i^a(x)] [E_i^a(x), U_2] | \Omega >$$

3. Estimate of $\langle \Omega \mid W[\phi] \mid \Omega \rangle$

This expression, although formally exact, will only be useful if it is possible to obtain at least an approximation to the explicit path dependence of the full-vacuum expectation value of the Wilson loop. The Wilson loop in sourceless quantum electrodynamics can be evaluated exactly. Photons have no self-interactions and satisfy the free, zero mass Klein-Gordon equation $\partial_\mu \partial^\mu A_i = 0$ in the radiation gauge, $A_o = 0$, $\vec{\nabla}\,\vec{A} = 0$. The full set of solutions can be represented in a plane wave basis by:

$$A_i(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega}} \sum_\lambda \epsilon_i(k,\lambda)\ [a(k,\lambda)e^{ikx} + a^+(k,\lambda)e^{-ikx}]$$

$$[a(k,\lambda), a^+(q,\rho)] = \delta_{\lambda\rho}\delta^3(k-q)$$

$$\sum_\lambda \epsilon_i(k,\lambda)\ \epsilon_j(k,\lambda) = \delta_{ij} - \frac{k_i k_j}{k^2}$$

The Nth order term in the expansion of the vacuum expectation value of the Wilson loop:

$$\langle 0 \mid \int dx_1 \int^{x_1} dx_2 \cdots \int^{x_{N-1}} dx_N\ A_1(x_1) \cdots A_N(x_N) \mid 0 \rangle$$

is given by the Wick expansion for a product of N gauge fields:

$$\langle 0| \prod_{m=1}^{N} A_m(x_m) |0\rangle = \sum_{\pi(N)} \prod_{(m,n=1)}^{\frac{N}{2}} D_{mn}(x_m - x_n) \quad N \text{ even}$$

$$= 0 \quad N \text{ odd}$$

where $D_{ij}(x-y)$ in Euclidean space is:

$$\langle 0| A_i(x)\, A_j(y) |0\rangle = \int \frac{d^4k}{(2\pi)^4\,(k_4^{\,2} + k^2)} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) e^{-ik(x-y)}$$

or the $t \to 0$ limit of the free propagator. The abbreviation $\pi(N)$ is the instruction to sum over all distinct permutations of the N indices among the $\frac{N}{2}$ D_{ij}'s. The integrand does not distinguish among the arguments since $D_{ij}(x-y) = D_{ij}(y-x)$

The ordered integration over the N x_m's can be converted to N independent integrations by averaging over the N! permutations of the variables. The sum over permutations $\Pi(N)$ is now redundant, giving a factor of $\frac{(N!)}{2^{\frac{N}{2}}\,(\frac{N}{2})!}$ for the number of ways to group N indices into $\frac{N}{2}$ indistinguishable pairs. The net result is that the vacuum expectation value of the abelian Wilson loop in the absence of interactions is:

$$\langle 0| W[\phi] |0\rangle = \sum_{(N \text{ even})} \frac{(-e)^{2N}}{2^{\frac{N}{2}}\,(\frac{N}{2})!} \left[\iint_{\phi} dx_i dy_j\, D_{ij}(x-y) \right]^{\frac{N}{2}}$$

$$= e^{-\left(\frac{e^2}{2}\right) \oint\oint dx_i dy_j \, D_{ij}(x-y)}$$

Using:

$$\int \frac{dk_4 d^3k}{k_4^2 + k^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) e^{ik(x-y)} = \frac{3}{4} \frac{\delta_{ij}}{(x-y)^2}$$

gives for the double integral around the path:

$$\oint\oint dx_i dy_i \frac{1}{(x-y)^2} \approx P[\phi] \int_{(\phi)} dr \frac{1}{r^2}$$

$$r \equiv (x-y)$$

$$P[\phi] = \text{loop perimeter}$$

Now let Λ be the momentum space cutoff so that $r_{min} = \Lambda^{-1}$:

$$\langle 0| \, W[\phi] \, |0\rangle \approx e^{-\frac{3e^2}{8} P[\phi] \, \Lambda}$$

which shows the origin of the perimeter dependence in the Wilson loop in QED.

When interactions are included, we expect the bare coupling to be replaced by the renormalized coupling but the perimeter dependence

should persist. The arguments for this are, first, that the renormalized coupling at large distances is small so large Wilson loops should be calculable perturbatively. Secondly, area dependence could only arise through intermediate interaction with charged particles. If these propagate over large distances, e.g. between distant legs of the loop, they should behave like free particles and their effect will be damped exponentially in their mass. At shorter distances, they only contribute to the renormalization of the photon propagator from which they arose.

The assumption of confinement will be expressed as an ansatz functional path dependence of the vacuum expectation value of large scale Wilson loops. In particular, we assume the functional $W[\phi]$ is dominated by:

$$< \Omega | W[\phi] | \Omega > \rightarrow W_o \, e^{-\lambda \int_0^{\frac{1}{2}} dt \, |\phi(t) - \phi(1-t)|}$$

for loops with diameter:

$$\max_t \, | \phi(s) - \phi(s+t) |$$

large relative to typical inverse bound state masses. This is the usual area law with a specific representation of the area chosen to conform to the notation used in the path measure. The parameter λ reflects the coefficient of the linear term in the nonrelativistic potential models

or the Regge slope in dual string theories. It is presumed fixed by another calculation or by phenomenological fits.

4. Reduction to quantum mechanical form

The ansatz functional form for the dominant infrared behavior of the Wilson loop in the full vacuum will now be inserted in the path space integral with a trial measure. We choose the minimal Gaussian form described in section III with an additional term in the exponent, quadratic in the path parameter $\phi(s)$, weighted by a variational parameter β:

$$D\mu_S[\phi] = D\phi(s)\, e^{-\int ds\, \frac{1}{2}\left(\frac{d\phi}{ds}\right)^2 + \frac{\beta}{2}[\phi(s)]^2}$$

The choice seems natural as it is the simplest, axially symmetric function of the only other vector in the problem. If the minimal term has the effect of damping paths which are too convoluted, the variational term in this context tends to suppress even fairly smooth paths which stray too far from the origin. The origin with respect to which $\phi(s)$ is measured will be fixed by the geometry of operators within the functional integrand, in this case the meson string.

We choose a particular parameterization of the string, identifying s with the coordinate z lying along the meson axis. The quark shall be located at the the origin and the quark-antiquark separation is a fixed constant R. The functional, $\frac{d\phi_i}{ds}$, is replaced by $\frac{d\phi_i}{dz}$ for i=(1,2), and by

a constant for i=3. This constant need not be retained in the measure. The $|\phi|^2$ term in the measure is defined relative to the z-axis so z will only appear as a parameter of the path and not as a functional. This should be a good parameterization for strings long enough so that endpoint effects do not significantly influence the quasi-free gluonic modes.

The z-parameterized form of the measure is thus:

$$\prod_{i=1}^{2} \int D\mu_S[\phi_i] = \prod_{i=1}^{2} \int D\phi_i(z_i)\, e^{-\frac{1}{2}\int dz_i \left[\left(\frac{d\phi_i(z_i)}{dz_i}\right)^2 + \beta(\phi_i(z_i))^2\right]}$$

We now have a distinct parallel between this method of path averaging and a Euclidean, quantum mechanical Feynman path integral. Used on a meson wavefunctional in normalized coordinates, the integration is subject to the endpoint constraints:

$$\phi^i(0) = 0 \quad \text{and} \quad \phi^i(1) = R\,\delta^{i3}$$

The path vectors ϕ must be continuous in z and can be taken to be infinitely differentiable. Subject to the endpoint constraints, the path position and its gradient can be independently varied, and the path measure will be peaked about path configurations $\phi(z)$ which minimize the exponent.

We can thus make a formal identification between the average over

paths of general path-dependent QCD expectation values and the Euclidean functional integral over representations of these expectation values as operators in a two plus two (spatial) dimensional quantum mechanical theory. The fact that the functionals $\phi(s), \frac{d\phi(s)}{ds}$ have a one-dimensional argument, s, makes this a quantum mechanical rather than quantum field theoretic functional integral. The exponent of the measure is the pseudo-action of the theory:

$$S_{\psi} = \int dt \; L_{\psi}[q(t), \dot{q}(t)]$$

$$= \frac{1}{2}\int dt \; \{ \; \dot{q}(t)^2 + \beta q(t)^2 \; \}$$

The canonical coordinates are:

$$q^i(t) \longleftrightarrow \phi^i(z)$$

and their conjugate momenta, defined as $\frac{\partial L}{\partial(\frac{dq}{dt})}$, are:

$$p^i(t) \longleftrightarrow \frac{d\phi^i(z)}{dz}$$

The assumption above of z-independance of the measure is equivalent to requiring that the pseudo-time axis is defined so that the pseudo-Lagrangian (the integrand of the z integration in the exponent of the measure) contains no explicit z-dependence. The expression for the

$\phi_1 - \phi_2$ correlation dependent form of the energy expectation value includes a three-dimensional delta between the ϕ_i's. This constrains the integration over the two paths to a single value of the parameter z. Both canonical coordinates, ϕ_1 and ϕ_2, are then defined with respect to a unique effective time variable. Hence we can construct an operator whose eigenstates, independent of z, minimize the exponent. The quantum mechanical Hamiltonian corresponding to the minimal measure is:

$$H^o_\psi = \frac{1}{2}[-\frac{d^2}{dq^2}]$$

and for the trial measure we have a harmonic oscillator Hamiltonian:

$$H^o_\psi(\beta) = H^o_\psi + \frac{\beta}{2}[\ q^2\ ]$$

The variational coefficient β is identified with the square of the oscillator frequency. If QCD opertor expectation values introduce only polynomials in ϕ and $\dot{\phi}$, as in perturbation theory, the corresponding path average is just the expectation value in a plane wave or harmonic oscillator state of polynomials in q and $\dot{q}$.

However, expectation values of Wilson loops generate exponential functions. Even for weak coupling, i.e. for small or abelian loops, we get a factor of $e^{-\int dt|\dot{q}(t)|}$. These sorts of terms modify the action since they affect the saddle point approximation. The perimeter law induces a trivial modification of the pseudo-Hamiltonian - a constant

shift in the canonical momenta. A similar conclusion applies to exponentials in q^1 or q^2. We will consider the perimeter dependence ignorable at the level we are calculating.

The area law, on the other hand, adds a definite interaction term in the pseudo-Lagrangian. In the calculation of operators involving large Wilson loops, there will be an additional piece in the exponent proportional to $|\phi_1(s) - \phi_2(s)|$. The functional form of the area law for a long Wilson loop $W[\phi_1 u \phi_2]$ having sides along string paths ϕ_1 and ϕ_2 will be chosen, in the z-parameterization, to be:

$$< \Omega \mid W[\phi_1 u \phi_2] \mid \Omega > \approx W_o \, e^{-\lambda \int dz \, \mid \phi_1(z) - \phi_2(z) \mid}$$

This is consistant with known properties of the Wilson loop. Since the Bianchi identity for the Wilson loop says that all surfaces spanning the loop are dynamically equivalent, the representation of the area as the fixed-z distance between opposite sides of the loop introduces no new assumptions. Because the meson axis selects a favored direction and the static quark-antiquark separation introduces a length scale, the original reparameterization invariance becomes irrelevant.

The resultant form for the expectation value of the Hamiltonian is, in functional notation:

$$\frac{\kappa^2}{2} \int_0^R dz \int D\phi_1(z) \, D\phi_2(z) \, \{[\frac{d\phi_1}{dz} \frac{d\phi_2}{dz} + 1] \quad \delta^2 \, (\phi_1(z) - \phi_2(z))\}$$

$$\times \quad \exp -\int dz \, \frac{1}{2} \left(\frac{d\phi_i}{dz}\right)^2 + \frac{\beta}{2} (\phi_i)^2 + \lambda \, |\phi_1(z) - \phi_2(z)|$$

At each value of z, the pseudo-Hamiltonian corresponding to the pseudoaction which has the simple form:

$$H_\psi = -\frac{1}{2} \left[\frac{d^2}{dq_1^2} + \frac{d^2}{dq_2^2}\right] + \frac{\beta}{2}[q_1^2 + q_2^2] + \lambda |q_1 - q_2|$$

We define quantum mechanical wavefunctions $\Psi_\beta(q_1,q_2)$ which are eigenstates of this Hamiltonian in the (q_1,q_2) coordinate space representation. They are required to satisfy the usual conditions of finiteness at the origin and normalizability. Normalizability in this context guarantees conservation of the color electric flux passing between the colored sources endpoints. The pseudotime (z) evolution of the wavefunctions is given by the exponential of their energy.

The functional expression for the expectation value of the Q.C.D. Hamiltonian can then be translated into the quantum mechanical expectation value of an equivalent operator, in a state which maximizes the flux weight at every point z:

$$E_\beta = \langle \Psi_\beta \mid \delta^2(q_1 - q_2) \left[\frac{\partial}{\partial q_1} \frac{\partial}{\partial q_2} + 1 \right] \mid \Psi_\beta \rangle$$

The variational problem which we wish to solve now requires the solution of the pseudo-Schroedinger's equation, which we proceed to do in the

next section.

VI. Reduced Variational Problem

1. Separation of modes

We seek the eigenstates and spectrum of the trial quantum mechanical Hamiltonian which describes the dominant mesonic string configurations in our approximation. The appellation pseudo-Hamiltonian is dropped in this section. When referring to QCD quantities it will be made explicit.

Define a new set of two-dimensional variables (vector indices suppressed):

$$q = \frac{1}{\sqrt{2}} (q_1 + q_2)$$

$$r = \frac{1}{\sqrt{2}} (q_1 - q_2)$$

Schroedinger's equation in these coordinates has the form:

$$[- \frac{1}{2}(\frac{d^2}{dq^2} + \frac{d^2}{dr^2}) + \frac{\beta}{2} (q^2 + r^2) + \lambda \ |r| - E_\beta] \ \Psi_\beta(q,r) = 0$$

which can be separated by substituting:

$$\Psi_\beta(q,r) = X(q) \ \psi(r)$$

$$E_\beta = E_q + E_r$$

The full spectrum for the q-equation is known since it describes a two-dimensional harmonic oscillator. We will only be making use of the lowest state. Redefining β as ω^2, the normalized ground state q-wavefunction is:

$$X_o(q) = \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\omega}{2} q^2}$$

with eigenvalue $E_q^o = \omega$.

The r-equation has the form:

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{\beta}{2} r^2 + \lambda|r| - E_r\right] \psi(r) = 0$$

The problem of a two-dimensional Schroedinger equation with a linear, or a linear plus quadratic, potential is not exactly solvable. In the context of the quark-antiquark bound state, the coefficient of the linear term is a fixed physical parameter whereas the quadratic term has been introduced variationally. There is no initial information on their relative size so ordinary perturbation theory, which would assume $\lambda \ll \omega$, may not reveal a true "best" action with $\omega \ll \lambda$. Experience with other problems indicates a variational calculation with a trial Gaussian wavefunction may not suffer a similar defect despite assuming a

form close to $\lambda = 0$ solutions. However, it would be desirable to have a cross-check on the variational result and, if possible, a technique capable of producing higher order corrections.

2. Quantum mechanical 1/n expansion

Such a technique, recently developed by Mlodinow and Papanicolaou[97], expresses the spectrum of Schroedinger's equation in arbitrary spatial dimensions d for potentials of the form $V = \frac{1}{2}\omega^2 r^2 + \mu\lambda r^\mu$, $-2 < \mu < \infty$, as an expansion in a parameter $x^2 = 2[d + 2L]^{-1}$ where L is the total angular momentum eigenvalue. Their method in $d = 3$ reproduces earlier results on the Coulomb (r^{-1}) and anharmonic oscillator (r^4) potentials. The accuracy of their calculations for $L = 0$ states ($x^2 = 2/3$) when compared to known exact solutions indicate their expansion may be generally good, even for fairly large x^2. For the ground state in the two-dimensional case, $x^2 = 1$. An explicit calculation of the spectrum, however, reveals that the $O(x^2)$ term is about an order of magnitude smaller than the leading term over the entire range of parameters. Furthermore, the ground state energy is manifestly lower than the energies computed using ordinary perturbation theory or a trial Gaussian wavefunction.

The Mlodinow-Papanicolaou (MP) x^2-expansion is based on the Holstein-Primakoff[98] representation of the general spherically symmetric Hamiltonian:

$$H = -\frac{1}{2}\nabla + \frac{1}{2}\omega^2 r^2 + \mu\lambda r^\mu$$

$$\nabla = \sum_{i=1}^{d} \frac{d}{dx_i}\cdot\frac{d}{dx_i} = -p_i\cdot p_i$$

$$r^2 = x_i\cdot x_i$$

Change variables to the 2d Heisenberg operators:

$$a_i = \sqrt{\frac{\hat{\omega}}{2}}\, x_i + \frac{i}{\sqrt{2\hat{\omega}}}\, p_i$$

$$a_i^+ = \sqrt{\frac{\hat{\omega}}{2}}\, x_i - \frac{i}{\sqrt{2\hat{\omega}}}\, p_i$$

with $\hat{\omega}$ undetermined at this point. The Hamiltonian:

$$H = \left[\frac{\omega^2}{4\hat{\omega}} - \frac{\hat{\omega}}{4}\right](a_i^+a_i^+ + a_ia_i) + \left[\frac{\omega^2}{4\hat{\omega}} + \frac{\hat{\omega}}{4}\right](a_i^+a_i + a_ia_i^+)$$

$$+ \mu\lambda(2\hat{\omega})^{-\frac{\mu}{2}}\left[a_ia_i + a_i^+a_i^+ + (a_i^+a_i + a_ia_i^+)\right]^{\frac{\mu}{2}}$$

is a function only of the combinations a_ia_i, $a_i^+a_i^+$, and $(a_i^+a_i + a_ia_i^+)$, reflecting the $O^+(3)$ symmetry. If we define a new set of operators:

$$A = \frac{1}{2} a_i a_i$$

$$A^+ = \frac{1}{2} a_i^+ a_i^+$$

$$2B = \frac{1}{2}[a_i^+ a_i + a_i a_i^+]$$

computation shows they satisfy the commutation relations:

$$[\ A,\ A^+\ ] = 2B$$

$$[\ A,\ B\ ] = A$$

$$[\ A^+,\ B\ ] = -A^+$$

We thus have a representation of the SO(2,1) algebra. The Casimir invariant S^2 of SO(2,1) is related to that of $O^+(3)$, J^2, by:

$$S^2 = B^2 - \frac{1}{4}[\ AA^+ + A^+A\ ] = \frac{J^2}{4} + \frac{d}{4}[\frac{d}{4} - 1]$$

where $J^2 = L(L + d - 2)$. Therefore:

$$S^2 = k(k - 1)$$

with:

$$k = \frac{1}{2}\left(L + \frac{d}{2}\right)$$

By restricting to a fixed L (hence fixed k) subspace, the operators A, A^+, and B can be represented by a single pair of Heisenberg operators (β, β^+) via the relations:

$$A = 2k\sqrt{1 + x^2\beta^+\beta}\ x\beta$$

$$A^+ = 2k\ x\beta^+\sqrt{1 + x^2\beta^+\beta}$$

$$2B = 2k\ (1 + 2x^2\beta^+\beta)$$

with $x^2 = (2k)^{-1}$. The usual commutator $[\beta, \beta^+] = 1$ reproduces the correct SO(2,1) algebra.

If we set $\lambda = 0$ and $\hat{\omega} = \omega$, $H_L(\beta, \beta^+)$ has the form:

$$x^2 H_L(\beta, \beta^+) = (1 + 2x^2\beta^+\beta)$$

which is the Holstein-Primakoff[98] representation for the unperturbed harmonic oscillator. The normalized β-Fock space states $|n\rangle_L$ are in one-to-one correspondence with harmonic oscillator states $|2n + L; L\rangle$; i.e., the solution space has been decomposed into a sum over fixed-L Fock spaces in which the ground state has priciple quantum number L.

The perturbed Hamiltonian ($\lambda \neq 0$) could now be expressed as a power series in x, expanding all powers and roots by the binomial formula. The lowest order nonconstant terms, however, are proportional to β and β^+, and the next order terms include β^2, β^{+2}, and $\beta^+\beta$. There is no simple interpretation of this truncated Hamiltonian. Instead, a canonical transformation of $H_L(\beta, \beta^+)$ is first performed:

$$H(p,p^+) = U(\sigma)\ H(\beta,\beta^+)\ U^+(\sigma)$$

where:

$$U(\sigma) = e^{\frac{1}{x}(\sigma^*\beta - \sigma\beta^+)}$$

corresponding to a shift in the operators:

$$\beta \rightarrow \beta + \frac{\sigma}{x} \equiv p + \frac{\sigma}{x}$$

$$\beta^+ \rightarrow \beta^+ + \frac{\sigma^*}{x} \equiv p^+ + \frac{\sigma^*}{x}$$

Now when we carry out the expansion in x, the parameters σ and $\hat{\Omega}$ can be chosen so as to eliminate the nonharmonic terms p, p^+, p^2, and p^{+2}, to order x^2.

The final form for the Hamiltonian is:

$$H_L(p,p^+) = \frac{M}{2x^2}\left[\frac{\mu+2}{\mu} + \frac{\mu-2\omega^2}{\mu M^2} + 2x^2(2c^2 - 1)\right]$$

$$+ \; 2Mc^2 p^+ p + O(x^2)$$

with the following defining equations:

$$c^2 = 1 + \sigma^2 = \cosh^2\phi$$

$$M = \hat{\omega}\, e^{-2\phi}$$

$$M^2 - \omega^2 = \mu^2 \lambda\, x^{2-\mu} M^{(1-\frac{\mu}{2})}$$

$$c^4 = 1 + \frac{(2-\mu)\,[\omega^2 - M^2]}{4M^2}$$

The problem of determining the spectrum for any d, L, and μ has thus been reduced to solving two algebraic equations. Higher order corrections are computed using standard nondegenerate perturbation theory.

3. Two-dimensional linear potential

For the Hamiltonian of interest, we set $\mu = 1$ to get:

$$x^2 H_L = \frac{M}{2}\left[3 - \frac{\omega^2}{M^2} + 2x^2(c^2 - 1)\right]$$

$$+ \quad 2Mc^2x^2p^+p \quad + \quad O(x^2)$$

where M is given by:

$$M = \frac{u}{2}\,[y + (2y - 1)^{\frac{1}{2}}]$$

$$y = [1 + \frac{4\,\omega^2}{u^2}]^{\frac{1}{2}}$$

$$u = [[\frac{\lambda^4}{4} + \frac{2\,\omega^6}{27}]^{\frac{1}{2}} + \frac{\lambda^2}{2}]^{\frac{1}{3}}$$

$$- [[\frac{\lambda^4}{4} + \frac{2\,\omega^6}{27}]^{\frac{1}{2}} - \frac{\lambda^2}{2}]^{\frac{1}{3}}$$

The ground state energy is given by the constant terms in H_L, denoted by E_0, and the coefficient of the quadratic term gives the level spacing. To compare the different order-x^2 terms, we look at two extreme values of the ratio of the quadratic to the linear terms in the original Hamiltonian. First, for the case where the quadratic term dominates, we define:

$$\delta = \frac{\lambda^2}{16}\,3^{3/2}\,\omega^{-3}$$

and expand E_0 for small δ:

$$E_0 \approx \omega\,[\,1 + 2\delta - \frac{1}{2}\delta^2 + \ldots] \;+\; x^2\omega\,[-\frac{\delta}{4} + \frac{3}{32}\delta^2 + \ldots]$$

For the complementary case of a dominant linear term, define a small parameter:

$$\epsilon = \frac{4}{3}\,\omega^2\lambda^{-4/3}$$

giving an ϵ - expanded E_0 of the form:

$$E_0 \approx 3\,\lambda^{2/3}\,[\,1 + \frac{1}{4}\epsilon + \ldots] + x^2\,\lambda^{2/3}[(\sqrt{3} - 2) + (\frac{7\sqrt{3}-16}{16})\,\epsilon + \ldots]$$

The order-x^2 pieces in both cases are inherently smaller than the leading terms: by a factor $\frac{1}{4}\delta$ in the first instance and by a factor 0.089 in the second. The latter is particularly surprising since it persists even for $\omega \to 0$, or the pure linear limit.

The small λ expansion can be compared to the energy calculated via standard perturbation theory about a harmonic oscillator with fundamental frequency ω. The lowest order ground state energy and wavefunction are:

$$E_p = \omega\,[1 + \frac{1}{2}\lambda\pi^{3/2}\omega^{-3/2}] + O(\lambda^2)$$

$$\Psi_p(r) = \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\omega r^2}$$

In terms of the parameter δ:

$$E_p = \omega[1 + \frac{2\sqrt{\pi}}{3^{3/4}} \delta^{\frac{1}{2}} + O(\delta)]$$

so that the lowest order MP correction is $1.13\ \delta^{\frac{1}{2}}$ times the lowest order perturbative correction, indicating more rapid convergence.

If, on the other hand, ω is the small parameter, $H_0 = -\frac{\nabla}{2} + \lambda r$ is not solvable so perturbation theory is not a good technique. A workable variational wavefunction os a Gaussian of undetermined frequency w for the ground state:

$$\Psi_w = \left(\frac{w}{\pi}\right)^{\frac{1}{2}} e^{-\frac{w}{2} r^2}$$

Minimizing $E_w = \langle\Psi_w]H[\Psi_w\rangle$ with respect to w gives a best value w_0 which is the solution of:

$$w^2 - \omega^2 = \frac{1}{2}\sqrt{\pi}\, \lambda w^{\frac{1}{2}}$$

and

$$E_{w_0} = \frac{1}{2} w_0 \left(3 - \frac{\omega^2}{w_0^2}\right)$$

Since w_0 satisfies the same equation as M with the replacements of λ by $\frac{1}{2}\sqrt{\pi}\,\lambda$, and since:

$$\frac{\partial w_0}{\partial \lambda} > 0$$

$$\frac{\partial E_{w_0}}{\partial \lambda} = \frac{1}{2}\frac{\partial w_0}{\partial \lambda}\left(3 + \frac{\omega^2}{w_0^2}\right) > 0$$

we have $E_{w_0} < E_{MP}$ to order x^0. However, the next order term in the MP expression is negative. When it is included and x^2 is set equal to 1, this inequality is reversed. In particular, we have the following expansions when $\delta << 1$:

$$E_{w_0} \simeq \omega \left[1 + \frac{\pi}{2}\delta - \frac{\pi^2}{32}\delta^2 + \cdots\right]$$

$$E_{MP} \simeq \omega \left[1 + \frac{3}{4}\delta - \frac{13}{32}\delta^2 + \cdots\right] \leq E_{w_0}$$

and when $\epsilon << 1$:

$$E_{w_0} \simeq \lambda^{2/3}\left[2.88 + 0.84\epsilon + \cdots\right]$$

$$E_{MP} \approx \lambda^{2/3} \, [2.73 + 0.51\epsilon + \cdots] \leq E_{w_0}$$

Corrections to E_{MP} to $O(x^4)$ can be shown to be down by about 10^{-2} so these inequalities persist.

4. Determination of variational parameter

Returning to the original problem, the calculation of the expectation value of the partial Q.C.D. Hamiltonian in the transverse flux mode ground state, we wish to evaluate the following quantity:

$$E_{QCD}(\omega) = \langle \Psi_o \, | \, \delta^2(r) \, [\frac{d^2}{dq^2} - \frac{d^2}{dr^2} + 1] \, | \, \Psi_o \rangle$$

$$= |\psi_r(0)|^2 \, (1 - \omega) - \int d^2r \, \delta^2(r) [\, \psi^*(r) \, \frac{d^2}{dr^2} \psi(r) \,]$$

Both $\psi_r(0)$ and $\langle\psi|\frac{d^2}{dr^2}|\psi\rangle$ are functions of the variational parameter ω. The apparent constant, 1, in the coefficient of the first term comes from $(\frac{dz}{ds})^2$ when the parameterization was fixed. It should be replaced, therefore, by a dimensionful quantity such as the square of the average length of strings passing between the quarks. It will be treated as a known quantity here.

To solve the variational equation for E_{QCD}, it is necessary to know the form of the wavefunction ψ_r, or at least its value at the origin and the expectation value of the squared momentum. These can be obtained in

the 1/n approximated form but we choose instead to work with the simpler trial Gaussian. Using the notation of Part 3, the expression for the area dependent part of the QCD energy now reads:

$$E_{QCD} = \left(\frac{w_o}{\pi}\right)^{\frac{1}{2}} [\ 1 - \omega + 2w_o\ ]$$

If λ were zero, $w_o = \omega$ and $E_{QCD}(\omega)$ is minimized when $\omega = 0$. The introduction of any area dependence in the effective action produces an asymmetry in the modes corresponding to the average position and separation of the flux lines. The eigenvalue associated with the average position is linear in ω; for any nonquadratic separation (r) dependence, the best value of the Gaussian frequency, w_o, will not be a positive power monomial in ω, as can be proven directly from the algebraic equation for w_o.

The variational equation to be solved for ω is:

$$0 = \frac{\partial E_{QCD}(\omega)}{\partial \omega} = \frac{\partial w_o}{\partial \omega} [\ \frac{1-\omega}{2\sqrt{w_o}} + 3\sqrt{w_o}\ ] - \sqrt{w_o}$$

We are not so much interested in the precise value of ω that minimizes the energy as in determining whether this value is finite and nonzero. An infinite ω would force the flux distribution along the z-axis. This would mean that despite the attempt to smear out the strings, we still have geometrically singular configurations. A zero result would indicate a highly diffuse flux configuration which can be shown to be

Coulombic.

It is straightforward to show that the minimization equation has a finite ω solution by using the asymptotic forms of the various terms as ω ranges from 0 to ∞. Referring to the solution for $M(\lambda)$, which is $w_o(\sqrt{\pi}\lambda)$, note first that the function $u(\omega)$ is continuous in ω, ranges from $\lambda^{\frac{2}{3}}$ to zero (decreasing like ω^{-2}) as ω ranges from 0 to ∞, and has no other zeros or singularities. Consequently, the function $y(\omega)$ is also continuous in ω, ranging monotonically from 1 to $\infty \approx \omega^3$ as ω ranges from 0 to ∞. Over this range, then, w_o is continuous, approaching a constant as ω goes to zero and becoming linear in ω for large ω. Thus $\frac{dw_o}{d\omega}$ ranges from 0 to 1 and $\frac{dE_{QCD}}{d\omega}$ ranges from $-\lambda^{\frac{1}{3}}$ to ∞. i.e. the latter must pass through zero at finite ω. This shows that there is a finite value of ω which minimizes E_{QCD}.

The energy is minimized at zero ω in this model only when λ, the coefficient of the area dependence, is exactly zero. None of these results will be qualitatively changed if area-independent or higher order area-dependent terms are included in the expression for the QCD energy. Using this approach, then, the quantum spreading of the flux tube is an automatic consequence of the assumed linear potential.

VII. Conclusions

We have shown how, in a variational framework, the area law behavior of the vacuum expectation value of the Wilson loop may induce a quantum mechanical broadening of the flux distribution. This effect can be viewed as the converse of results[99,100] showing that a path-dependent wavefunctional satisfying the massive loop wave equation:

$$\left[-\frac{\delta^2}{\delta x_\mu(s)\delta x_\lambda(s)} + M^4\left(\frac{dx_\mu}{ds}\right)^2 \right] \Psi[\phi(s)] = 0$$

and decaying smoothly when the loop $\phi(s)$ becomes large, falls off exponentially in the area of the loop. This roughly duplicates the behavior familiar from pointlike field theories that massive particle correlation functions decay exponentially in the particle mass. Here the massiveness of the string, reflected in the Gaussian damping of the flux spread, forces an area law suppression of large loops, and vice versa.

There are several routes one may take in attempting to generate higher order effects within this model. Given the overwhelming importance of the confinement ansatz, none of the obvious extrapolations will significantly add to an understading of meson wavefunctionals. Nevertheless, for the sake of completeness, we briefly describe them here.

The calculation of the energy in the quantum mechanical x^2-expansion may be carried out to arbitrary order and the wavefunction correspondingly amended using standard techniques. From experience with other systems, it is likely this will produce only minor refinements at the expense of vastly complicating the variational problem. Higher order states - excited modes of the transverse flux tube - are given directly from the ground state solutions. Of interest here, perhaps, is a quantitative comparison between L and N "transverse "angular" and "radial") energy levels, and between these and the scale of the true radial modes. The latter can be computed using standard two-body potential techniques. The radial mode calculation is most likely to be sensitive to order 1/M quark kinetic energy terms

Corrections which are higher order in 1/N would include contributions from intermediate quark-antiquark states, corrections to the gauge matrix sum rule which will introduce twisting of strings, as well as possible modifications of the functional area law behavior of the vacuum expectation value of the Wilson loop. Loop fusion terms might enter in the regularization of the length-dependent term in the expectation value of the energy. These more or less neglected length-dependent terms will affect calculations of radial modes, will contribute to the renormalization of the quark masses, and will require regularization consistent with that used in related calculations, but, as pointed out, will not significantly influence the large distance behavior of the wavefunctional.

Finally, one could attempt to drop the assumption of z-independence of the wavefunctional to allow for more spherical flux distributions appropriate to lighter quark mesons. This would certainly require a better understanding of the short distance properties of the strings as relativistic and inelastic effects become important. The introduction of a substantially more sophisticated variational wavefunction, given the plethora of higher order effects, would seem to hinge on a more explicit knowledge of the path dependence of the vacuum expectation value of the Wilson loop.

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