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HIGGS PHASE IN NON-ABELIAN GAUGE THEORIES*

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ABSTRACT

In this work, we consider a non-Abelian gauge theory involving scalar fields with non-tachyonic mass terms in the Lagrangian and we wish to construct a finite energy density trial vacuum for this theory. The usual scalar potential arguments suggest that the vacuum of such a theory would be in the perturbative phase. However, the obvious choices for a vacuum in this phase, the Axial gauge and the Coulomb gauge bare vacua, do not have finite energy densities even with an ultraviolet cutoff. Indeed it is a non-trivial problem to construct finite energy density vacua for non-Abelian gauge theories and this is intimately connected with the gauge fixing degeneracies of these theories. Since the gauge fixing is achieved in the Unitary gauge, this suggests that the Unitary gauge bare vacuum might be a finite energy trial vacuum and, despite the form of the scalar potential, the vacuum of this theory might be in a Higgs phase rather than the perturbative phase.

In the first Chapter, we give a general discussion of the phases of a gauge theory and its gauge fixing degeneracies.

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In the second Chapter, starting from an Axial gauge canonical formulation, we construct the Unitary gauge bare vacuum expressed in terms of Axial gauge variables, within the context of Abelian Scalar Q.E.D.. This state which has the form of a coherent plasma of charge, is shown to have infrared finite energy density and invariant under the residual group of the Axial gauge. Due to ultraviolet reasons, we then consider a slight modification of this state.

In the third Chapter, these considerations are generalized to an SU(2) gauge theory with fundamental representation scalars. Within the same Axial gauge formulation, we thus have two trial states for the vacuum ; the Axial gauge bare vacuum for the perturbative phase and a coherent plasma of color for the Higgs phase. Since the latter has a finite energy density, this indicates that the vacuum of this non-Abelian gauge theory is more likely to be in the Higgs phase, rather than the perturbative phase with real massless gluons.

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CHAPTER 1

INTRODUCTION

A) Phases of a Gauge Theory

At the present time, it is generally believed that the fundamental interactions in Nature are describable in terms of non-Abelian Gauge Field Theories. We now have the Weinberg-Salam Model, incorporating successfully the phenomenology of the electroweak interactions, and we have the Quantum Chromodynamics which is an asymptotically free gauge field theory for the strong interactions. The principle of local gauge invariance under a non-Abelian gauge group is therefore an important symmetry principle in high energy physics.

The realization of this type of a symmetry in a gauge theory is however much more subtle than the usual way of realizing the Wigner-type symmetries of a quantum theory. In a gauge theory, the local gauge invariance is not in general implemented in terms of multiplets in the space of physical states. In the Weinberg-Salam type theories, one has the spontaneous breakdown of the gauge symmetry via a Higgs mechanism and the symmetry is hidden from the physical states, whereas in Q.C.D., only the singlet states are considered physical, in accordance with the color confinement hypothesis. We thus have various possibilities for the realization of a gauge symmetry and in general, every distinct way of realizing such a symmetry corresponds to a

distinct phase of a gauge theory.

In this work, we shall primarily be concerned with the Higgs phase of non-Abelian Gauge Theories. In order to put our considerations into a more general context, we would like to begin by giving a brief description of the possible phases of a gauge theory.[1]

In a confinement phase, the fundamental representation particles -which will be called quarks, regardless of their spin- are confined. The standart picture of confinement is the binding of a quark-antiquark pair by a thin tube of electric flux which cannot be spread out in the transverse directions.[2] This picture is qualitatively consistent with the dual string picture of hadrons. It can also be obtained in the strong coupling limit of lattice gauge theories. Whatever the mechanism for its justification might be, it is believed that the ability to support thin electric flux tube is an essential feature of the confinement phase.

This picture is meant to apply to a pure gauge theory with static external quarks, since in the presence of dynamical quarks, the flux tubes could be broken by the creation of quark-antiquark pairs from the vacuum. On the other hand, hadron phenomenology suggests that the quark-gluon couplings are small inside the hadrons; the latter do not contain too many quark-antiquark pairs. As a leading approximation to Q.C.D., one can therefore ignore the dynamical quarks altogether in which case the electric flux tu-

bes will be stable objects. We therefore conclude that, a large ring of electric flux cannot diffuse into a confining vacuum and , so long as it does not shrink to zero size, such an object will be stable. The overlap of such a loop state with the confining vacuum should then vanish as the loop size,and hence the loop stability, increases and considering a linear potential between a quark-antiquark pair, it is possible to show that, such an overlap should vanish as the exponential of the area defined by the loop. This is Wilson's criterion for confinement.[3] Since a loop of electric flux is a gauge invariant object, an operator which creates such a loop will be a gauge invariant order parameter, its surface clustering indicating that we are in a confinement phase.

It is remarkable that in a Higgs phase, one obtains the electric-magnetic dual of the above picture. Nielsen and Olesen showed that, in a Higgs phase completely broken by the adjoint scalars, there was a stringlike solution to the Yang-Mills equations which was stable by topological reasons.[4] The string was in the form of a thin magnetic flux tube, and in the Abelian case of the Nielsen-Olesen Model, Nambu[5] showed that these quantized vortices led to the confinement of monopole-antimonopole pairs. Mandelstam extended these considerations to non-Abelian models[6] and he suggested that a coherent plasma of monopoles could confine color in the same way as an ordinary Higgs vacuum, which is a coherent superconducting

plasma of color, confines the magnetic charges.[7] 't Hooft had also advanced a similar suggestion.[8] 't Hooft then showed that an operator which creates a loop of magnetic flux is a gauge invariant order parameter for the Higgs phase and that in a phase without massless excitations, either this operator or the Wilson loop operator should exhibit a surface clustering.[9] These developments have shed considerable light into the meaning of Higgs and Confinement phases in non-Abelian Gauge Theories.

One may ask whether, in addition to the Higgs and Confinement phases, there exists a phase where the gauge symmetry is realized with a multiplet structure in the space of physical states. Such a phase would contain physical massless gluons and it would correspond to the usual perturbative phase of an Abelian gauge theory. The fundamental distinction of a non-Abelian gauge theory however, is the fact that the gauge field quanta carry color, and any object with a definite color spin may end up being an arbitrary superposition of different color spin states, by emitting any number of soft gluons. It is then not clear what the physical interpretation of such states should be and it would be difficult to determine the color spin of any colored object.[10] The existence of such a phase is therefore doubtful, and in fact, the main purpose of this work is to provide an argument to the effect that the vacuum of a non-Abelian gauge theory is more likely to be in a Higgs phase rather than a perturbative phase with physical mass-

less gluons.

We have thus covered some of the phenomenological and kinematical characteristics of the various possible phases of a gauge theory. Which of these phases is actually realized in a given gauge theory is a fundamental dynamical question, requiring to identify the phase described by the lowest energy eigenstate of the theory. We have seen that we do have gauge invariant order parameters for the various phases so that, once we have a trial state for the vacuum of the theory, it should not be difficult to identify the phase in question.

It is however a nontrivial problem to construct physically acceptable states in non-Abelian gauge theories. In order to have finite energy densities, the physical states have to be chosen with special care, due to the gauge fixing degeneracies of the quantum theory of gauge fields, and it is to these issues that we now address ourselves in the following Section.We should like to point out that we consider all the time, a gauge theory defined with some kind of an ultraviolet regulation so that, any infinity that we may encounter here, is not related to the usual short distance problems of quantum field theories.

B) Residual Gauge Freedom in a Gauge Theory

It is well-known that, a quantal description of a gauge theory in terms of local fields necessitates the imposition of a gauge condition on the theory. Such a condition is needed to eliminate the redundant degrees of freedom from the theory so that the dynamical degrees of freedom could be uniquely defined. If however, the gauge condition itself is invariant under a residual subgroup of gauge transformations, then the gauge fixing is not complete and it is in general not clear how the residual degrees of freedom should be eliminated from the theory.

A case in point is the Coulomb gauge condition in a non-Abelian gauge theory. Gribov[11] has shown that, such a gauge condition does not fix the gauge completely. Although this gauge condition could be used for quantizing small oscillations around a classical configuration, it is not clear how the theory should be defined from a nonperturbative point of view. Moreover, Mandelstam[7] has pointed out that the trial states with finite energy density would have to satisfy certain nontrivial requirements, and in view of the complicated nature of this gauge in a non-Abelian theory, it is not known how to obtain a finite energy density, translational invariant trial state for the vacuum of such a theory. In particular, the bare vacuum does not have a finite energy density. This is in sharp contrast with the case of an Abelian gauge theory where the Coulomb gauge condition is an especially convenient way of

identifying the physical degrees of freedom.

In view of these difficulties with the Coulomb gauge, we shall now examine another commonly used gauge condition, namely the Axial gauge. This gauge has the advantage of being a ghost free gauge so that the Hilbert space structure of a theory is simpler in this gauge than in any other one. The gauge condition is :

$$A_a^3 = 0 \tag{1-1}$$

where the subscript denotes a group index. The canonical variables in this gauge are :

$$A_a^i \ , \quad F_a^{i0} \quad ; \qquad i=1,2$$

We note that F_a^{30} is here a dependent variable which can be expressed in terms of the canonical variables via the constraint equation of Gauss' law. We now consider the quantity:

$$G_a(x_1,x_2) = \int_{-\infty}^{\infty} dz \ \partial_z F_a^{30}(x_1,x_2,z) \tag{1-2}$$

It can now be shown that, as a result of canonical commutation relations, $G_a(x_1,x_2)$ has nontrivial commutators with the canonical variables. As an example, we have :

$$\left[G_a(x_1,x_2) \ , \ A_c^i(y_1,y_2,y_3) \right] = -i \ \delta_{ac} \ \partial_i \left\{ \delta(x_1-y_1)\,\delta(x_2-y_2) \right\} \tag{1-3}$$

This equation can easily be verified by expressing (1-2) in terms of the canonical variables, as we will do in the following Chapters. We now observe that, according to this equation, the expression ,

$$G_a(x_1,x_2) = F_a^{30}(x_1,x_2,+\infty) - F_a^{30}(x_1,x_2,-\infty)$$

cannot be taken as zero, and consequently, associated with any color fluctuation, there will be a non-vanishing field strength, infinitely far away from the fluctuation. The appearance of any net color on the system will result in an infinite energy. These remarks could be made more quantitative, by observing that the Hamiltonian of this general gauge theory will be of the form :

$$H = \frac{1}{2} \int d^3\underset{\sim}{x}\; F_a^{30}(\underset{\sim}{x})\; F_a^{30}(\underset{\sim}{x}) \quad + \text{positive definite terms}$$

It can now be shown that, because of the first term of this expression, the energy density of a trial state will be infinite, unless the quantity $G_a(x_1,x_2)$ vanishes on that state.[7] In other words, for finite energy density, the physical states must satisfy :

$$G_a(x_1,x_2) \mid \text{state} > \; = \; 0 \qquad (1\text{-}4)$$

This condition is related to the residual gauge freedom of the Axial gauge. The gauge condition (1-1) is

invariant under the subgroup of gauge transformations local in (x_1,x_2) and for each such point in space, we have a residual group which is not fixed by the gauge condition. It can easily be seen from (1-3) that the quantities $G_a(x_1,x_2)$ are the generators of this residual gauge group and, in view of this, the condition (1-4) implies the invariance of the physical states under the residual gauge group, the gauge fixing of the residual degrees of freedom being thus achieved on the space of physical states.

Another way to arrive at the same condition is to analyze the Poincare invariance of a gauge theory quantized in the Axial gauge. It can then be shown that[12], the Poincare algebra contains anomalies proportional to the generator of the residual group, and the Poincare invariance can only be achieved in a subspace satisfying (1-4).

The bare vacua of Abelian and non-Abelian gauge theories in the Axial gauge,[13] do not satisfy this condition. This means that the usual perturbative vacua will not have finite energy densities, and as such, they will not be good trial states for the ground states of these theories. Our aim in this work will be to modify the Axial gauge bare vacuum to obtain a lowering of the energy density in the infrared, thereby presenting a better trial state for the vacuum of a non-Abelian gauge theory. It is clear that, the key to such a procedure will be to construct a state which does satisfy the condition (1-4).

In the case of an Abelian theory, we can always

go to the Coulomb gauge where there is no gauge fixing degeneracies. In fact, it can explicitly be shown that[7], the Coulomb gauge bare vacuum expressed in the Axial gauge satisfies the condition (1-4) so that, it is a better trial state than the Axial gauge bare vacuum. A similar procedure does not work in non-Abelian gauge theories, because of the above-mentioned difficulties with the Coulomb gauge.

The foregoing arguments suggest that a formulation of a gauge theory where there is no gauge fixing degeneracies, should be preferable to other formulations.It is well-known that the choice of the Unitary gauge does provide such a formulation and we therefore expect thatthe bare vacuum in the Unitary gauge should be a finite energy density trial state. That such a gauge is especially convenient for displaying the physical degrees of freedom in a gauge theory, has also been suggested recently by 't Hooft. (Ref. 14)

In this work, we consider gauge theories involving scalar fields for which, the meaning of the unitary gauge is clear. These theories will be quantized in the Axial gauge. The scalar fields are not tachyonic in the Lagrangian and the usual scalar potential arguments suggest that we take a symmetric trial state for the vacuum. The obvious candidate for such a state, the Axial gauge bare vacuum, is not a satisfactory trial state as we have seen above. In view of this, we will suppose that the scalar fields develop an expectation value, which will enable us to go to a

Unitary gauge formulation of the theory. We will consider the theory at a fixed time and we shall express the Unitary gauge bare vacuum in terms of the Axial gauge variables, which expression will turn out to be in the form of a coherent plasma of color. We wish to show that, such a plasma satisfies the finite energy condition (1-4) and provides a lowering of the energy density compared to the Axial gauge bare vacuum. This will then indicate that, a state in the Higgs phase is a better trial state for the vacuum of the theory than the bare vacuum in the perturbative phase.

Before dealing with a non-Abelian gauge theory, it will be instructive to display our argument, firstly, within the context of an Abelian gauge theory. We therefore take the theory of Scalar Q.E.D. in the Second Chapter and we give a detailed presentation of the Unitary gauge bare vacuum in terms of the Axial gauge variables. Several technical details will be discussed along the way, and in particular, the ultraviolet difficulties of the Unitary gauge will require a slight modification in our approach.Nevertheless, a relatively simple expression will be obtained for a trial state in the Higgs phase and we shall then complete our argument to the effect that such a state is a better trial vacuum than the Axial gauge bare vacuum. The Second Chapter is a preparation for the discussion of the non-Abelian theory, and it should not be concluded that we are advocating a Higgs phase for this non-tacyonic Abelian theory. We have already remarked that the Coulomb gauge

formulation was satisfactory for such a theory, and therefore this Chapter should be considered just an introduction to our considerations of the non-Abelian theory.

The issues related to the non-Abelian nature of the gauge group will be discussed in the Third Chapter, where we take the scalar field in the fundamental representation of the gauge group SU(2). The generalization of the argument of the previous Chapter will then indicate that, a non-Abelian gauge theory is more likely to have its vacuum in the Higgs phase, rather than a perturbative phase with real massless gluons.

CHAPTER 2

UNITARY GAUGE IN AN ABELIAN THEORY

A) The Scalar Q.E.D. in the Axial Gauge

In order to separate out the issues related to the Unitary gauge formulation of a gauge theory, from the problems having to do specifically with the non-Abelian nature of the gauge group, we would like to expose our considerations firstly within the context of an Abelian gauge theory in this Chapter. The theory that we are interested in is the Scalar Q.E.D. quantized in the Axial gauge. Being Abelian, this theory can be quantized in the Coulomb gauge without any associated Gribov difficulties. We are however interested in the Axial gauge, since this is the convenient gauge choice for the non-Abelian theory where a consistent Coulomb gauge formulation is not available. This Chapter is then a preparation for the discussion of the non-Abelian theory of the next Chapter.

We begin by establishing our notation for the Scalar Q.E.D. . In an SO(2) notation, the scalar field will be represented by :

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{2-1}$$

where ϕ_a are Hemitian. The covariant derivative is :

$$D_\mu = \partial_\mu - g\,\epsilon\, A_\mu \;;\quad \epsilon_{12}=-\epsilon_{21} = 1 \;;\; \epsilon_{11}= \epsilon_{22}= 0$$

The Lagrangian density is given by ,

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi)_a(D^\mu\phi)_a - V(\phi)$$

where,

$$V(\phi) = \frac{1}{2}\mu^2(\phi_a\phi_a) + \frac{1}{4}\lambda^2(\phi_a\phi_a)^2 \quad ; \quad \mu^2 > 0 \qquad (2\text{-}2)$$

A note about our <u>Convention</u> : In this Chapter, all the Latin indices will run from 1 to 2, unless otherwise noted. In such a convention, the canonical variables are as follows :

The canonical coordinates are : A^i , ϕ_a

The conjugate momenta are : $F^{oi} \equiv -E^i$, $(D^0\phi)_a \equiv \Pi_a$

We note that, $E^3 = F^{30}$ is not an independent variable but depends on the canonical variables via the Gauss' law :

$$\underset{\sim}{\partial}\cdot\underset{\sim}{E} = g\,\epsilon_{ab}\Pi_a\phi_b \quad ,$$

or,

$$\mathscr{G}(\underset{\sim}{x}) \equiv \partial_3 E^3(\underset{\sim}{x}) = -\partial_i E^i(\underset{\sim}{x}) + g\,\epsilon_{ab}\Pi_a\phi_b \qquad (2\text{-}3)$$

The Hamiltonian density of the theory, expressed in terms of the canonical variables of the Axial gauge, is :

$$\left.\begin{aligned}\mathcal{H} = \frac{1}{2} E^i E^i + \frac{1}{2} [\partial_3^{-1}\mathcal{G}]^2 + \frac{1}{2} F^{12}F^{12} + \frac{1}{2}(\partial_3 A^i)(\partial_3 A^i) \\ + \frac{1}{2} \Pi_a \Pi_a + \frac{1}{2} (\partial_3 \phi_a)(\partial_3 \phi_a) + \frac{1}{2} (D_i\Phi)_a (D_i\Phi)_a + V(\Phi)\end{aligned}\right\} \quad (2\text{-}4)$$

where we define :

$$\partial_3^{-1} f(x_i, x_3) \equiv \int_{-\infty}^{\infty} dz\ \varepsilon(x_3 - z)\ f(x_i, z) \quad (2\text{-}5)$$

with,

$$\varepsilon(x) \equiv \frac{1}{2} [\ \theta(x) - \theta(-x)\ ] \quad (2\text{-}6)$$

It turns out that such a definition of the inverse derivative will present an advantage over other definitions, in that , certain manipulations of the non-Abelian theory in the Third Chapter will simplify due to the odd function character of $\varepsilon(x)$.

The quantization is achieved by the equal-time commutation relations :

$$\left.\begin{aligned}\left[E^i(\underset{\sim}{x},t)\ ,\ A^j(\underset{\sim}{y},t)\right] &= i\ \delta^{ij}\ \delta(\underset{\sim}{x}-\underset{\sim}{y}) \\ \left[\Pi_a(\underset{\sim}{x},t)\ ,\ \phi_b(\underset{\sim}{y},t)\right] &= -i\delta_{ab}\ \delta(\underset{\sim}{x}-\underset{\sim}{y})\end{aligned}\right\} \quad (2\text{-}7)$$

As a result of these relations and (2-3), the quantity,

$$G(x_1, x_2) \equiv \int_{-\infty}^{\infty} dz\ \mathcal{G}(x_1, x_2, z) \quad (2\text{-}8)$$

has nontrivial commutation relations with the canonical variables. We have already considered this quantity in (1-2) of the First Chapter ; the commutation relation of that Chapter , (1-3), can now be verified by using the canonical commutation relations (2-7). We also recall that, in order that the physical states have a finite energy density, we have as a necessary condition :

$$G(x_1,x_2) \quad | \text{ physical state } \rangle \;=\; 0 \qquad (2\text{-}9)$$

It can now be shown that the bare vacuum of the Hamiltonian (2-4) does not satisfy this condition. The operator $[\partial_3^{-1}]^2$ is potentially singular around the low frequency modes ($k_3 \approx 0$), and special care is needed to construct states for which this operator has a finite expectation value.The diagonalization of the quadratic part of the above Hamiltonian does not encompass such a care, since the above mentioned operator has a quartic piece, as can easily be seen from (2-3).Consequently, the Axial gauge bare vacuum does not have a finite energy density.

Since μ^2 is positive in (2-2), the above Hamiltonian does not contain any tachyonic scalar field at the quadratic level, and one may expect that the usual perturbative analysis might apply, leading to a vacuum in the perturbative phase. The failure of the Axial gauge bare vacuum to be a satisfactory state constitutes a serious impasse for such an expectation. For the present Abelian theory,

this is actually of no consequence, since one can always go to a Coulomb gauge formulation and the bare vacuum of that gauge will be a satisfactory trial state for the perturbative phase of the theory. Since however we cannot repeat the same procedure for the non-Abelian theory, we expect that the vacuum of such a theory will be in a phase other than the perturbative phase.

Our aim in this Chapter will be to construct the Unitary gauge bare vacuum in terms of the Axial gauge canonical variables and to show that , inter alia, it does satisfy the finite energy condition (2-9). To this end, we now introduce the Unitary Gauge Transformation which is the mapping necessary to transform the present theory, from the Axial gauge to the Unitary gauge.

B) The Unitary Gauge Transformation

We begin by parametrizing the scalar field Φ in terms of the polar variables viz ,

$$\Phi = \Phi(\rho,\theta) \equiv \exp(-\theta\varepsilon)\begin{pmatrix}\rho\\0\end{pmatrix} = \exp(-\theta\varepsilon)\ \Phi(\rho,\theta=0) \qquad (2\text{-}10)$$

The covariant derivative can also be parametrized in the same fashion :

$$D_\mu\Phi \equiv (\partial_\mu - g\varepsilon A_\mu)\Phi(\rho,\theta) = \exp(-\theta\varepsilon)[\partial_\mu - g\varepsilon B_\mu]\Phi(\rho,\theta=0) \qquad (2\text{-}11)$$

where we have defined :

$$B^{\mu} \equiv A^{\mu} + \frac{1}{g}\,\partial^{\mu}\theta$$

i.e.,

$$B^{j} \equiv A^{j} - \frac{1}{g}\,\partial_{j}\theta \tag{2-12}$$

$$B^{3} \equiv \quad - \frac{1}{g}\,\partial_{3}\theta \tag{2-13}$$

We thus have two equivalent sets (ϕ_a, A^j) and (ρ, B^j, B^3) for the canonical coordinates at a fixed time and the standart formulation of the Unitary gauge amounts to expressing the Lagrangian of the previous Section in terms of the second set, and to showing that after the symmetry breaking, one obtains the Lagrangian of massive vector bosons in the usual way. We will not follow such a Lagrangian procedure here. We will rather consider the above theory as a Hamiltonian field theory and seek a formulation of the Unitary Gauge Transformation as a canonical transformation on the canonical variables of the theory.

To this end, we introduce a parameter v -which will later be interpreted as a variational parameter- and we map the polar variables (ρ, B^3, B^j) into the canonical variables (ϕ_a, A^j) following the actions of the mappings M_1 & M_2 that we now describe. Firstly, we define M_1 :

$$M_1 : \begin{cases} \rho \longrightarrow v + \phi_1 \\ B^3 \longrightarrow -\dfrac{1}{gv}\,\partial_3\phi_2 \\ B^j \longrightarrow A^j - \dfrac{1}{gv}\,\partial_j\phi_2 \end{cases} \tag{2-14}$$

The second equation of this set, together with the definition (2-13) implies that :

$$M_1 : \quad \theta \longrightarrow \frac{1}{v}\,\phi_2 \tag{2-15}$$

The defining equations (2-10) and (2-12) can now be used to express the action of M_1 on the canonical variables :

$$M_1 : \begin{cases} \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \longrightarrow \exp(-\frac{1}{v}\phi_2\varepsilon) \begin{pmatrix} v + \phi_1 \\ 0 \end{pmatrix} \\ \\ A^j \longrightarrow A^j \end{cases} \tag{2-16}$$

We next define M_2 :

$$M_2 : \begin{cases} \phi_a \longrightarrow \phi_a \\ \\ A^j \longrightarrow A^j + \frac{1}{gv}\partial_j\phi_2 \end{cases} \tag{2-17}$$

The product map $M = M_2.M_1$ then satisfies ,

$$M : \begin{cases} \phi \longrightarrow v + \phi_1 \\ \\ B^i \longrightarrow A^i \\ \\ B^3 \longrightarrow -\frac{1}{gv}\,\partial_3\phi_2 \end{cases} \tag{2-18}$$

In view of the second line of this set, M is a gauge transformation from the Unitary gauge to the Axial gauge ; it is thus the required Unitary Gauge Transformation.

So far, we have displayed the actions of M_1 & M_2 on the canonical coordinates ; since we are working in a Hamiltonian formalism, we need to determine the actions of these mappings on the conjugate momenta as well.This, we shall now do, by demanding that M be a canonical transformation. In view of the simple action of M_2 in (2-17), it is easy to see that the following rules,

$$M_2 : \begin{cases} \Pi_2 \longrightarrow \Pi_2 - \frac{1}{gv}\partial_i E^i \\ \Pi_1 \longrightarrow \Pi_1 \\ E^i \longrightarrow E^i \end{cases} \tag{2-19}$$

will render M_2 a canonical transformation. In the case of M_1, the situation is considerably more complicated, since the action of this mapping on the canonical coordinates ϕ_a is highly nonlinear. Indeed this is transparent when we reexpress the equation (2-16) as ,

$$M_1 : \begin{cases} \phi_1 \longrightarrow (v+\phi_1)\cos(\frac{1}{v}\phi_2) \\ \phi_2 \longrightarrow (v+\phi_1)\sin(\frac{1}{v}\phi_2) \\ A^j \longrightarrow A^j \end{cases} \tag{2-20}$$

We are now faced with the determination of the action of M_1 on the conjugate momenta so as to make M_1 a canonical transformation. Within the context of a classical field theory, this issue is resolved in the Appendix and a simple generalization of that analysis suggests tha we consider :

$$M_1 : \begin{cases} \Pi_1 \longrightarrow \Pi_1 \cos(\frac{1}{v}\phi_2) - \frac{1}{2}(1+\frac{1}{v}\phi_1)^{-1}\{\Pi_2, \sin(\frac{1}{v}\phi_2)\}_+ \\ E^i \longrightarrow E^i \\ \Pi_2 \longrightarrow \Pi_1 \sin(\frac{1}{v}\phi_2) + \frac{1}{2}(1+\frac{1}{v}\phi_1)^{-1}\{\Pi_2, \cos(\frac{1}{v}\phi_2)\}_+ \end{cases} \tag{2-21}$$

Since the right-hand sides of these equations are linear in the momenta Π_a, it is not difficult to verify that the canonical commutation relations are preserved under the action of M_1. Hence, M_1 is canonical. Since we now have the actions of M_1 & M_2 on both the coordinates and momenta, we are now in a position to compute explicitly the action of M on any quantity of interest.We find, for example, that :

$$M_1 : \Pi_1\Pi_1 + \Pi_2\Pi_2 \rightarrow \Pi_1\Pi_1 + (1+\frac{1}{v}\phi_1)^{-2}\left[\Pi_2\Pi_2 - \frac{1}{4}v^{-2}\,\delta^{\epsilon}(0)\right] \tag{2-22}$$

$$M_1 : \quad \epsilon_{ab}\Pi_a\phi_b \rightarrow -v\,\Pi_2 \tag{2-23}$$

These equations illustrate two important points about M_1. The first one is that, from an ultraviolet point of view, M_1 is a singular operation. This is apparent in

(2-22) where there is an explicit $\delta^6(0)$ singularity as a esult of the action of M_1. The second aspect is illustrated in (2-23) where we observe that, an operator which is a quadratic functional of the canonical fields, namely the charge density operator, reduces to a linear functional under the action of M_1. This shows that the generator of the residual group, $G(x_1, x_2)$, as given by (2-8), is mapped under M_1 to a linear functional of the fields and consequently, it should now be easy to construct states annihilated by such an operator, thus satisfying the finite energy condition (2-9). Later on in this Chapter we shall have more to say about these points.

It is now easy to evaluate the action of M on the Hamiltonian density (2-4). We find,

$$M : \mathcal{H} \longrightarrow \mathcal{H}^{(1)} + \mathcal{H}^{(2)} \qquad \text{(2-24-a)}$$

where we have,

$$\mathcal{H}^{(1)} = \frac{1}{2}\Pi_1\Pi_1 + \frac{1}{2}\underset{\sim}{\partial}\phi_1\underset{\sim}{\partial}\phi_1 + \frac{1}{2}\mu^2(v+\phi_1)^2 + \frac{1}{4}\lambda^2(v+\phi_1)^4 ; \qquad \text{(2-24-b)}$$

$$\left.\begin{aligned}\mathcal{H}^{(2)} = {} & \frac{1}{2}\sum_{i=1}^{3}\xi^i\xi^i + \frac{1}{4}\sum_{i,j=1}^{3}(\partial^i W^j - \partial^j W^i)(\partial^i W^j - \partial^j W^i) \\ & + \frac{1}{2}\left[\, gv(1+\frac{1}{v}\phi_1)\,\right]^2 \left(\sum_{i=1}^{3} W^i W^i\right) \\ & + \frac{1}{2}\left[\, gv(1+\frac{1}{v}\phi_1)\,\right]^{-2}\left[(\underset{\sim}{\partial}\cdot\underset{\sim}{\xi})^2 - \frac{1}{4}g^2\,\delta^6(0)\right]\end{aligned}\right\} \qquad \text{(2-24-c)}$$

and where we have defined,

$$\left.\begin{array}{lll} \xi^i \equiv E^i \quad i=1,2 & ; & \xi^3 \equiv -\, gv\, \partial_3^{-1}\Pi_2 \\ W^i \equiv A^i \quad i=1,2 & ; & W^3 \equiv -\, \frac{1}{gv}\, \partial_3\phi_2 \end{array}\right\} \qquad (2\text{-}25)$$

Except perhaps for the term proportional to $\delta^6(0)$, the expressions in (2-24) will be recognized as the usual Hamiltonian of massive vector bosons in the Unitary gauge. The equations in (2-25) express the field operators of such a theory in terms of the canonical variables of the previous Section, and these expressions, together with the canonical commutation relations (2-7), now imply that :

$$\left[\xi^i(\underset{\sim}{x},t)\ ,\ W^j(\underset{\sim}{y},t)\right] = i\ \delta^{ij}\delta(\underset{\sim}{x}-\underset{\sim}{y})\ ;\ i,j=1,2,3 \qquad (2\text{-}26)$$

These are the usual commutation relations of a theory of massive vector bosons.

The expression (2-24) for the Hamiltonian in the Unitary gauge can also be obtained starting from a temporal gauge formulation of the theory. Such a result was reached in a paper by Creutz and Tudron[15], where an identical expression for $\mathcal{H}^{(2)}$ was found, including the singular $\delta^6(0)$ term. This is reassuring, since $\mathcal{H}^{(2)}$ expresses the gauge invariant degrees of freedom of the theory, and any formulation should lead to the same expression for such a quantity. Reference 15 also explains the reason for the term

proportional to $\delta^6(0)$, as well as its perturbative cancellation by tadpole diagrams.[16] From a nonperturbative point of view, such a term points out to the singular ultraviolet behaviour of $\mathscr{K}^{(2)}$ and in fact, all the terms proportional to the inverse powers of the dimensionful parameter v may be expected to show a similar ultraviolet behaviour. Later on we will assume that v^{-1} is a small quantity in which limit these terms will be unimportant and we shall be careful not to base our arguments on such terms. It is however clear that, what is responsible for such a behaviour, is the above-mentioned singular nature of the mapping M_1.

We have so far displayed the Unitary Gauge Transformation as a canonical mapping on the algebra of canonical operators of the theory. In a quantum theory, such a mapping should be implemented in the Hilbert space of states. Our theory is defined in terms of the canonical operators in the Axial gauge together with a Hilbert space which carries a representation of the canonical commutation relations (2-7), and by a mapping on the theory, we really mean a representation of such a mapping on the Hilbert space of states. It is clear that such an implementation for M will be difficult ; M_1 is a nonlinear mapping on the algebra of canonical operators, and as such, there is no natural way of implementing this transformation on the space of states. To simplify this task of implementation, we will now present an approximation to the Unitary Gauge Transformation.

In principle, to implement M_1 as given by (2-20), we would need a unitary operator U_1 such that ,

$$\left.\begin{aligned} U_1^\dagger\ \phi_1\ U_1 &= (v + \phi_1)\ \cos(\tfrac{1}{v}\phi_2) \\ U_1^\dagger\ \phi_2\ U_1 &= (v + \phi_1)\ \sin(\tfrac{1}{v}\phi_2) \end{aligned}\right\} \qquad (2\text{-}27)$$

The right-hand sides of these equations are rather complicated functionals and in general, the existence of such a unitary operator U_1 is doubtful. We observe however that these expressions simplify considerably as v tends to infinity ; in that limit ϕ_2 is invariant under U_1 , whereas ϕ_1 is just shifted by the quantity v :

$$(v + \phi_1)\ \cos(\tfrac{1}{v}\phi_2) = v + \phi_1 + \text{order}(\tfrac{1}{v})$$

$$(v + \phi_1)\ \sin(\tfrac{1}{v}\phi_2) = \phi_2 + \text{order}(\tfrac{1}{v})$$

It is easily seen that, in this limit, U_1 is given by the operator $\exp\{-iv\int d^3\underset{\sim}{x}\ \Pi_1(\underset{\sim}{x})\}$. In the general case, we shall represent U_1 by :

$$U_1 = \exp\{-iv\int d^3\underset{\sim}{x}\ \Pi_1(\underset{\sim}{x})\}\ \exp\{iF\} \qquad (2\text{-}28)$$

where F is a formal power series in v^{-1} with operator coefficients :

$$F = \sum_{n=1}^{\infty} v^{-n} F_{(n)} \tag{2-29}$$

The generators $F_{(n)}$ can now be determined by expanding the right-hand side of the equation (2-27) in inverse powers of v and generating these terms by using the Baker-Hausdorff formulae. The first few terms in this expansion are :

$$F_{(1)} = \frac{1}{2} \int d^3\underset{\sim}{x} \left[\Pi_1 \phi_2 \phi_2 - \phi_1 \{\Pi_2, \phi_2\} \right]$$

$$F_{(2)} = -\frac{1}{24} \int d^3\underset{\sim}{x} \left\{ \Pi_2, \phi_2 (\phi_2 \phi_2 - 6\phi_1 \phi_1) \right\} \tag{2-30}$$

$$F_{(3)} = -\frac{1}{24} \int d^3\underset{\sim}{x} \left[\{\Pi_1, \phi_1 \phi_1\} \phi_2 \phi_2 - 2\phi_1 \{\Pi_2, \phi_2 (\phi_2 \phi_2 - 2\phi_1 \phi_1)\} \right]$$

The expression (2-29) is formal ; indeed it is not clear in what sense the series for F converges, if at all, and the operator U_1 which is obtained by exponentiating such a formal series will probably be too formal to exist. We therefore approximate F by a first few terms of (2-29) by considering the limit in which v tends to infinity. We have already seen that, such a limit was also necessary from an ultraviolet point of view, the canonical mapping M_1 being singular otherwise.

We will now display the details of this approximation by considering the computation of the quantity , $U_1^+ \mathcal{H} U_1$, where $\mathcal{H}$ is the Hamiltonian density in (2-4). $\mathcal{H}$ is quartic in the canonical fields and consequently,

it isn't difficult to check that,

$$\left.\begin{aligned} &\exp\{iv\int d^3\underset{\sim}{x}\ \Pi_1\}\ \mathscr{H}\exp\{-iv\int d^3\underset{\sim}{x}\ \Pi_1\} = \\ &\qquad = av^4 + bv^3\phi_1 + v^2O_2 + vO_3 + \mathscr{H} \end{aligned}\right\} \quad (2\text{-}31)$$

where O_2 and O_3 are quadratic and cubic functionals of the canonical fields and a & b are constants independent of v. This is the first step in the computation of $U_1^{\pm}\mathscr{H}U_1$ and we now need to conjugate this expression by the operator exp{iF} which is a coherent sum of small operators in the large v limit. These small operators are to be multiplied by the large operators proportional to powers of v in the above expression (2-31) and the final result for $U_1^{\pm}\mathscr{H}U_1$ will then be a sum of terms with positive, zero or negative net powers of v . The large v approximation consists of neglecting terms with net negative powers of v.

Apart from a constant term, the leading term in (2-31) is third order in v . In the computation of $U_1^{\pm}\mathscr{H}U_1$ we therefore need to retain only the first three terms in the expansion of F in (2-29) ; the remaining terms are proportional to v^{-4} or even smaller in the large v limit and they will not contribute to the conjugation of (2-31) by the operator exp{iF}. These three terms were displayed in (2-30). We therefore take :

$$U_1 = \exp\{-iv\int d^3\underset{\sim}{x}\ \Pi_1\}\ \exp\{i\sum_{n=1}^{3} v^{-n}F_{(n)}\} \qquad (2\text{-}32)$$

as the approximate implementation of M_1 in the Hilbert space, the approximation being sensible for large values of v.

A comment is now in order. We have arrived at this explicit form for U_1, starting from the assumed action of M_1 on the canonical fields ϕ_a in (2-27), but without ever having to assume the form of the action of M_1 on the conjugate momenta Π_a. We can now apply to Π_a the explicit form of U_1 in (2-32), and since U_1 is a representation of M_1 in the large v limit, we can in this way determine how the conjugate momenta transform under the Unitary Gauge Transformation, without going through an analysis that led to the expressions in (2-21). In fact, we obtain,

$$\left.\begin{aligned} U_1^{+}\,\Pi_1\,U_1 &= \Pi_1 - \frac{1}{2v}\{\Pi_2,\phi_2\} + \text{order}(v^{-2}) \\ U_1^{+}\,\Pi_2\,U_1 &= \Pi_2 + \frac{1}{v}\,\varepsilon_{ab}\Pi_a\phi_b + \text{order}(v^{-2}) \end{aligned}\right\} \qquad (2\text{-}33)$$

It can be checked that the right-hand sides of these equations agree with the large v expansion of the right-hand sides of (2-21). So long as we are interested only in a few leading terms in the large v expansion, the knowledge of the exact action of M_1 is not necessary.

Now that the meaning of U_1 has been clarified by the formula in (2-32), we can evaluate the conjugation of any quantity with U_1. We have, for example :

$$U_1^{+}\,\varepsilon_{ab}\Pi_a\phi_b\,U_1 = \exp\{-i\sum_{n=1}^{3} v^{-n}F_{(n)}\}(-v\Pi_2+\varepsilon_{ab}\Pi_a\phi_b)\exp\{i\sum_{n=1}^{3} v^{-n}F_n\}$$

$$= -v\Pi_2 + \left[-\frac{i}{v} F_{(1)}, -v\Pi_2\right] + \varepsilon_{ab}\Pi_a\phi_b + \text{order}(\frac{1}{v})$$

From (2-30), we have :

$$i \left[F_{(1)}, \Pi_2\right] = - \varepsilon_{ab}\Pi_a\phi_b$$

and we finally obtain,

$$U_1^\dagger \varepsilon_{ab}\Pi_a\phi_b U_1 = -v\Pi_2 + \text{order}(\frac{1}{v}) \qquad (2\text{-}34\text{-a})$$

In fact, with the form of U_1 given by (2-32), this result is valid to order(v^{-3}) and this then constitutes an explicit way of obtaining the formal result of (2-23), in the large v limit. For future use, we also note that this result immediately implies :

$$U_1^\dagger \mathscr{G} U_1 = -\partial_i E^i - gv\Pi_2 + \text{order}(v^{-3}) \qquad (2\text{-}34\text{-b})$$

where we have also used the definition (2-3). We have gone through the details of this derivation to illustrate various manipulations that were necessary in the evaluation of the action of U_1 on the quantities of interest and from now on such details will usually be suppressed. By similar manipulations, one can now obtain the action of U_1 on the Hamiltonian density $\mathscr{H}$. With $\mathscr{H}^{(1)}$ given by (2-24-b), we find :

$$\left.\begin{aligned} U_1^{\dagger}\,\mathcal{H}\,U_1 = {} & \frac{1}{2}\,E^iE^i + \frac{1}{2}\,[\partial_3^{-1}(gv\Pi_2+\partial_iE^i)]^2 + \frac{1}{2}\,F^{12}F^{12} \\ & + \frac{1}{2}(\partial_3A^i)(\partial_3A^i) + \frac{1}{2}\,\Pi_2\Pi_2 + \frac{1}{2}(\partial_3\phi_2)(\partial_3\phi_2) \\ & + \frac{1}{2}(gvA^j-\partial_j\phi_2)(gvA^j-\partial_j\phi_2) + \mathcal{H}^{(1)} \\ & + g^2v\,A^jA^j\phi_1 - 2g\,A^j\phi_1\partial_j\phi_2 + \frac{1}{2}\,g^2A^jA^j\phi_1\phi_1 \\ & + \text{order}\ (\tfrac{1}{v}) \end{aligned}\right\} \quad (2\text{-}35)$$

The unitary implementation of M_2 poses no problem, since this mapping is linear in the canonical variables. In fact, the equations,

$$\left.\begin{aligned} U_2^{\dagger}\,\Pi_2\,U_2 &= \Pi_2 - \frac{1}{gv}\,\partial_iE^i \\ U_2^{\dagger}\,A^i\,U_2 &= A^i + \frac{1}{gv}\,\partial_i\phi_2 \end{aligned}\right\} \quad (2\text{-}36)$$

are satisfied by :

$$U_2 = \exp\{\,\frac{i}{gv}\int d^3\underset{\sim}{x}\;E^i(\underset{\sim}{x})\;\partial_i\phi_2(\underset{\sim}{x})\,\} \quad (3\text{-}37)$$

This then concludes the implementation of the Unitary Gauge Transformation M by the unitary operator $U = U_1.U_2$ in a Hilbert space carrying a representation of the canonical commutation relations (2-7).We now turn to an explicit construction of such a representation.

C) The Unitary Gauge Bare Vacuum

In this Section, we wish to construct a Hilbert space on which to represent the canonical commutation relations (2-7) and we shall take this to be the Fock space associated with the Unitary gauge bare vacuum.

To begin with, it is important to be clear about the approximation scheme which obtains the Unitary gauge bare vacuum as the leading approximation to the ground state of the theory. Such an approximation scheme is the loop expansion in the perturbation theory of the Higgs phenomenon.[17] In the notation of the previous Section, the standart assumption of such a scheme is to consider the couplings g and λ as small quantities, the parameter v as a large quantity, in such a way that the products gv and λv are kept finite. Under this assumption and for the case where the scalar fields have tachyonic mass terms in the Lagrangian, the Unitary gauge bare vacuum becomes the leading approximation to the vacuum of the theory. Our construction will make this point quite clear.

In this Section, we therefore adopt this standart assumption for the loop expansion of the Higgs phenomenon and we consider the quantity $U^{\pm}\mathcal{H}U$ in such an approximation. We recall that the expression (2-35) for $U_1^{\pm}\mathcal{H}U_1$ was evaluated in a large v approximation so that it is appropriate for our present purpose as well. We now need to conjugate this expression by the operator U_2 in (2-37) and we observe that the generator of this operator depends on v

through the combination gv, which we are now considering as a finite quantity. We cannot therefore consider the generator in question as a small quantity and we need to evaluate the action of U_2 without any approximation. As a matter of fact, in view of the particularly simple form of this operator, its action given by (2-36) does not involve any approximation and we can easily evaluate the conjugation of the quantity $U_1^+ \mathscr{H} U_1$ by this operator. We thus obtain ;

$$\left.\begin{aligned} U^+ \mathscr{H}\, U &= U_2^+ (U_1^+ \mathscr{H}\, U_1) U_2 = \\ &= \frac{1}{4}\lambda^2 v^4 + \frac{1}{2}\mu^2 v^2 + v(\mu^2+\lambda^2 v^2)\phi_1 + \mathscr{H}_Q \\ &+ g^2 v (A^j A^j)\phi_1 + \frac{1}{2} g^2 (A^j A^j)\phi_1\phi_1 + \lambda^2 v (\phi_1)^3 \\ &+ \frac{1}{4}\lambda^2 (\phi_1)^4 + \text{order}\left(\frac{1}{v}\right) \end{aligned}\right\} \quad (2\text{-}38\text{-}a)$$

where,

$$\left.\begin{aligned} \mathscr{H}_Q = {} & \frac{1}{2} E^i E^i + \frac{1}{2} g^2 v^2 (\partial_3^{-1}\Pi_2)^2 + \frac{1}{2} F^{12} F^{12} \\ & + \frac{1}{2}\left(\partial_3 A^i - \partial_i \frac{1}{gv}\partial_3\phi_2\right)^2 + \frac{1}{2} g^2 v^2 A^j A^j \\ & + \frac{1}{2}\left(\Pi_2 - \frac{1}{gv}\partial_i E^i\right)^2 + \frac{1}{2}\partial_3\phi_2\partial_3\phi_2 \\ & + \frac{1}{2}\Pi_1\Pi_1 + \frac{1}{2}\underset{\sim}{\partial}\phi_1\cdot\underset{\sim}{\partial}\phi_1 + \frac{1}{2}(\mu^2+3\lambda^2 v^2)\phi_1\phi_1 \end{aligned}\right\} \quad (2\text{-}39)$$

According to our perturbative assumption, the terms in $U^{\dagger}\mathscr{H}\,U$ can now be classified as follows : There is the constant term

$$\frac{1}{4}\,\lambda^2 v^4 \;+\; \frac{1}{2}\,\mu^2 v^2 \tag{2-40}$$

which is the leading term, going like v^2 in the above limit. The next-to-leading term is the tadpole term

$$v(\,\mu^2 + \lambda^2 v^2\,)\;\varphi_1 \tag{2-41}$$

Then, there are terms which form the expression (2-39) for $\mathscr{H}_Q$; these terms are all quadratic in the fields and their dependence on the parameter v is always through the combination gv so that these terms are to be considered finite. Finally, there are cubic and quartic interaction terms in (2-38), and they should be considered as small quantities since they are proportional to the coupling constants. Note that these terms are of the same order as some of the terms that we have neglected ; our approximation scheme considers a term proportional to g as of the same order as a term of order $(\frac{1}{v})$. In a leading order approximation, all these terms should be neglected and we obtain :

$$\left.\begin{aligned} U^{\dagger}\mathscr{H}\,U = {} & \frac{1}{4}\,\lambda^2 v^4 + \frac{1}{2}\,\mu^2 v^2 + v(\mu^2 + \lambda^2 v^2)\,\varphi_1 + \mathscr{H}_Q \\ & + \text{ non-leading terms} \end{aligned}\right\} \tag{2-38-b}$$

In a theory with negative μ^2, one proceeds by minimizing the leading term (2-40) and this results in the tadpole condition ($\mu^2+\lambda^2 v^2=0$), thus eliminating the linear term (2-41). In that case, the leading order approximation to $U^+ \mathcal{H} U$ is the quadratic Hamiltonian $\mathcal{H}_Q$. The Unitary gauge bare vacuum is, by definition, the ground state of $\mathcal{H}_Q$ and we shall denote this state by $|\Omega^\circ\rangle$. We therefore see that, in the above perturbative limit and for the case where μ^2 is negative, the state $U|\Omega^\circ\rangle$ becomes the leading approximation to the vacuum of $\mathcal{H}$.

Our case however is different. Since we do assume that μ^2 is positive, the above analysis does not apply and the tadpole term will be present in the expression for $U^+ \mathcal{H} U$. The state $U|\Omega^\circ\rangle$ will not be eigenstate of our Hamiltonian in any kind of approximation. Nevertheless, we shall still choose our Fock space following the diagonalization of $\mathcal{H}_Q$ and the vacuum of this Fock space will be the Unitary gauge bare vacuum $|\Omega^\circ\rangle$.

By expressing $\mathcal{H}_Q$ in terms of the quantities (ξ ,W^j; i=1,2,3) as given by (2-25), we obtain ;

$$\left.\begin{aligned}\mathcal{H}_Q = {} & \frac{1}{2}\sum_{i=1}^{3}\xi^i\xi^i + \frac{1}{4}\sum_{i,j=1}^{3}(\partial^i W^j - \partial^j W^i)(\partial^i W^j - \partial^j W^i) \\ & + \frac{1}{2}(gv)^{-2}\left(\sum_{i=1}^{3}\partial_i\xi^i\right)^2 + \frac{1}{2}(gv)^2\sum_{i=1}^{3}W^i W^i \\ & + \frac{1}{2}\xi_1\xi_1 + \frac{1}{2}\partial\xi_1\cdot\partial\xi_1 + \frac{1}{2}(\mu^2+3\lambda^2 v^2)\xi_1\xi_1\end{aligned}\right\} \quad (2\text{-}42)$$

The first part of this expression will be recognized as the standart free field Hamiltonian density for vector bosons of mass gv, whereas the last line describes a free scalar field of mass$=(\mu^2+3\lambda^2 v^2)^{\frac{1}{2}}$. We also note that the quantities $(\xi^i, W^j;\ i,j=1,2,3)$ satisfy the standart commutation relations of massive vector bosons as explained in (2-26). By representing these relations in the usual Fock space associated with $\mathcal{H}_Q$, we may therefore obtain a representation of the original commutation relations (2-7), and we shall now display the specific form of such a representation. To this end, the Fourier decomposition of the various fields are as follows :

$$E^i(\underset{\sim}{x}) = \frac{i}{(2\pi)^{3/2}} \int d^3\underset{\sim}{k}\ \exp(-i\underset{\sim}{k}.\underset{\sim}{x}) \sum_{r=1}^{2} e_r^i(\underset{\sim}{k})\, p_r(\underset{\sim}{k}) \qquad (2\text{-}43\text{-a})$$

$$A^i(\underset{\sim}{x}) = \frac{i}{(2\pi)^{3/2}} \int d^3\underset{\sim}{k}\ \exp(\ i\underset{\sim}{k}.\underset{\sim}{x}) \sum_{r=1}^{2} e_r^i(\underset{\sim}{k})\, q_r(\underset{\sim}{k}) \qquad (2\text{-}43\text{-b})$$

$$\pi_a(\underset{\sim}{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\underset{\sim}{k}\ \exp(-i\underset{\sim}{k}.\underset{\sim}{x})\ \hat{\pi}_a(\underset{\sim}{k}) \qquad (2\text{-}43\text{-c})$$

$$\phi_a(\underset{\sim}{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\underset{\sim}{k}\ \exp(\ i\underset{\sim}{k}.\underset{\sim}{x})\ \hat{\phi}_a(\underset{\sim}{k}) \qquad (2\text{-}43\text{-d})$$

where we have :

$$e_1^i = \epsilon^{ij}\frac{k^j}{|k_T} \qquad e_2^i \quad \frac{k^i}{|k_T|} \quad ; \quad i=1,2$$

$$|k_T|^2 \quad (k_1)^2 + (k_2)^2 \qquad (2\text{-}44)$$

The Hermiticity condition is :

$$O^{\dagger}(\underset{\sim}{k}) = O(-\underset{\sim}{k}) \quad ; \quad O = p_r \, , \, q_r \, , \, \pi_a \, , \, \phi_a \tag{2-45}$$

The canonical commutation relations (2-7) become, in this notation :

$$\left.\begin{aligned} \left[p_r(\underset{\sim}{k}) \, , \, q_s(\underset{\sim}{l}) \right] &= -i \, \delta_{rs} \, \delta(\underset{\sim}{k}-\underset{\sim}{l}) \\ \left[\pi_a(\underset{\sim}{k}) \, , \, \phi_b(\underset{\sim}{l}) \right] &= -i \, \delta_{ab} \, \delta(\underset{\sim}{k}-\underset{\sim}{l}) \end{aligned}\right\} \tag{2-46}$$

The diagonalization of $\mathcal{H}_Q$ amounts to a Bogoliubov transformation into a new set of variables $P_i(\underset{\sim}{k}), Q_i(\underset{\sim}{k})$:

$$p_1(\underset{\sim}{k}) = P_1(\underset{\sim}{k}) \tag{2-47-a}$$

$$q_1(\underset{\sim}{k}) = Q_1(\underset{\sim}{k}) \tag{2-47-b}$$

$$p_2(\underset{\sim}{k}) = \frac{|k^3|}{|\underset{\sim}{k}|} P_2(\underset{\sim}{k}) + \frac{m}{\omega} \frac{|k_T|}{|\underset{\sim}{k}|} P_3(\underset{\sim}{k}) \tag{2-47-c}$$

$$q_2(\underset{\sim}{k}) = \frac{|k^3|}{|\underset{\sim}{k}|} Q_2(\underset{\sim}{k}) + \frac{\omega}{m} \frac{|k_T|}{|\underset{\sim}{k}|} Q_3(\underset{\sim}{k}) \tag{2-47-d}$$

$$\pi_2(\underset{\sim}{k}) = \frac{|k^3|}{m} \frac{|k_T|}{|\underset{\sim}{k}|} P_2(\underset{\sim}{k}) - \frac{(k_3)^2}{|\underset{\sim}{k}|\omega} P_3(\underset{\sim}{k}) \tag{2-47-e}$$

$$\phi_2(\underset{\sim}{k}) = \frac{|k_T|}{|\underset{\sim}{k}|} \frac{m}{|k^3|} Q_2(\underset{\sim}{k}) - \frac{\omega}{|\underset{\sim}{k}|} Q_3(\underset{\sim}{k}) \tag{2-47-f}$$

with $\omega = (\underset{\sim}{k}^2+m^2)^{1/2}$, $m = gv$ (2-47-g)

and the relations (2-46) imply :

$$\left[P_i(\underset{\sim}{k}) \ , \ Q_j(\underset{\sim}{l}) \right] = -i \, \delta_{ij} \, \delta(\underset{\sim}{k}-\underset{\sim}{l}) \quad ; \ i,j=1,2,3 \qquad (2\text{-}48)$$

Using these quantities, the expression (2-39) becomes :

$$H_Q = \int d^3x \ \mathcal{H}_Q = \frac{1}{2} \int d^3\underset{\sim}{k} \ \sum_{i=1}^{3} \left[|P_i(\underset{\sim}{k})|^2 + \omega^2 \, Q_i(\underset{\sim}{k})^2 \right]$$

$$+ \frac{1}{2} \int d^3\underset{\sim}{k} \left[|\dot{Q}_1(\underset{\sim}{k})|^2 + (\omega_1)^2 \, \dot{Q}_1(\underset{\sim}{k})^2 \right]$$

where $(\omega_1)^2 = \underset{\sim}{k}^2 + \omega^2 + 3\omega^2 v^2$, and the representation,

$$P_i(\underset{\sim}{k}) = -i\left(\frac{\omega}{2}\right)^{\frac{1}{2}} \left[a_i(-\underset{\sim}{k}) - a_i(\underset{\sim}{k}) \right] , \ i=1,2,3 \qquad (2\text{-}49\text{-}a)$$

$$Q_i(\underset{\sim}{k}) = \left(\frac{1}{2\omega}\right)^{\frac{1}{2}} \left[a_i(\underset{\sim}{k}) + a_i(-\underset{\sim}{k}) \right] , \ i=1,2,3 \qquad (2\text{-}49\text{-}b)$$

$$\dot{Q}_1(\underset{\sim}{k}) = -i\left(\frac{\omega_1}{2}\right)^{\frac{1}{2}} \left[a_0(-\underset{\sim}{k}) - a_0(\underset{\sim}{k}) \right] \qquad (2\text{-}49\text{-}c)$$

$$\dot{Q}_1(\underset{\sim}{k}) = \left(\frac{1}{2\omega_1}\right)^{\frac{1}{2}} \left[a_0(\underset{\sim}{k}) + a_0(-\underset{\sim}{k}) \right] \qquad (2\text{-}49\text{-}d)$$

diagonalizes $\mathcal{H}_Q$. The usual algebra of creation&annihilation operators now implies (2-48) and the Hermiticity requirement (2-45) is also satisfied. The Fock space vacuum annihilated by the destruction operators of (2-49) is the the Unitary gauge bare vacuum $\overset{c}{\ }$.

We have thus obtained that, the expression

$$U|\Omega^{\circ}\rangle = \exp(-iv\int d^3\underset{\sim}{x}\ \Pi_1)\ \exp(iF)\ \exp(\frac{i}{gv}\int d^3\underset{\sim}{x}\ E^i\partial_i\Phi_2)\,|\Omega^{\circ}\rangle \quad (2\text{-}50)$$

in the above perturbative limit, is the Unitary gauge bare vacuum expressed in terms of the Axial gauge variables. It is in the form of a coherent plasma of color and we shall now examine some important properties of such a state.

We begin by gauge invariance considerations and we wish to show explicitly that the Unitary gauge bare vacuum, expressed in terms of the Axial gauge variables, is invariant under the residual group of the Axial gauge. To this end, we need to conjugate the generator $G(x_1,x_2)$ by the operator U. The first step of this procedure was already evaluated in (2-34-b) which we now conjugate by U_2. Using the action of U_2 given by (2-36), we obtain,

$$U^{+}\ \mathcal{G}(\underset{\sim}{x})\ U = -gv\ \Pi_2(\underset{\sim}{x}) + \text{order}(v^{-3})$$

i.e.,

$$U^{+}\ G(x_1,x_2)\ U = -gv\int_{-\infty}^{\infty} dz\ \Pi_2(x_1,x_2,z) + \text{order}(v\text{-}3). \quad (2\text{-}51)$$

We now observe from our Fock space representation (2-47-e) that :

$$\Pi_2(k_1,k_2,0) = 0 \quad (2\text{-}52\text{-}a)$$

which immediately implies,

$$\int_{-\infty}^{\infty} dz\ \Pi_2(x_1,x_2,z) = 0 \quad (2\text{-}52\text{-}b)$$

We therefore conclude that the state $U|\Omega^{\circ}>$ satisfies:

$$||G(x_1,x_2)U|\Omega^{\circ}>||^2=<\Omega^{\circ}|(U^{+}G(x_1,x_2)U)^2|\Omega^{\circ}>=$$
$$= \text{order}(v^{-6}) \to 0 \qquad (2\text{-}53)$$

Thus the finite energy condition is satisfied and this state is invariant under the residual group, in the large v limit. We note that this result depended on two important facts about our trial state. The first point is that, $G(x_1,x_2)$ which is a quadratic functional of the canonical fields, is mapped under U_1 to a linear functional. This was previously noted in (2-23) & (2-34) as an important property of U_1. Our second point is that, once we have transformed the generator into a linear functional, any representation which satisfies the condition (2-52-a) will be satisfactory since it will lead to the vanishing of $||G(x_1,x_2)U|\Omega^{\circ}>||^2$ in the large v limit, where $|\Omega^{\circ}>$ is the Fock space vacuum of the representation in question. This remark will be useful in the next Section.

The energy density $\mathscr{E}^{\circ}$ of the state $U|\Omega^{\circ}>$ can now be evaluated to leading order, by taking the expectation value of (2-38-b). We first note that, by construction,

$$<\Omega^{\circ}|\mathscr{H}_Q|\Omega^{\circ}> = \frac{1}{2(2\pi)^3}\int_0^{\Lambda} d^3k\left[3(\underset{\sim}{k}^2+g^2v^2)^{\frac{1}{2}}+(\underset{\sim}{k}^2+m^2+3\lambda v^2)^{\frac{1}{2}}\right] \qquad (2\text{-}54)$$

(In this and the following expressions, we shall use the ultraviolet cutoff Λ to confine the range of integrations

into a region $|k|<\Lambda$). We therefore obtain,

$$\langle\Omega^{\circ}|U^{+}\mathcal{H}\,U|\Omega^{\circ}\rangle = \mathcal{E}^{\circ}(v^2) + \text{nonleading terms} \tag{2-55}$$

$$\mathcal{E}^{\circ}(v^2) \equiv \frac{1}{4}\lambda^2 v^4 + \frac{1}{2}\mu^2 v^2 + \langle\Omega^{\circ}|\mathcal{H}_Q|\Omega^{\circ}\rangle$$

This expression is obviously infrared finite.

The next step in the variational argument should be to minimize the energy density $\mathcal{E}^{\circ}(v^2)$ with respect to the variational parameter v^2 and to find the optimum value for this parameter. It is easy to see that the derivative of $\mathcal{E}^{\circ}(v^2)$ with respect to v^2 is positive definite for positive v^2, so that $\mathcal{E}^{\circ}(v^2)$ is an increasing function of v^2. Hence the minimum of such a function is at $v=0$.

This result present a serious problem to the consistency of our approximation. We have been assuming all the time that v was a large quantity, but the leading order expression of such an approximation predicts that v is in fact zero and that the terms proportional to the inverse powers of v that we have been neglecting, are in fact much more important than what we have been calling the leading terms. The only way out of this difficulty is to consider not so large a value for the parameter v , i.e., to start considering the case where the terms of order $(\frac{1}{v})$ make an appreciable contribution to the energy density of $U|\Omega^{\circ}\rangle$.

Before considering these higher order terms, we would like to stress once more that our approximation scheme

considers the products gv and λv as finite quantities. This is also apparent in the Fock space representation of our canonical operators given by Eqns. (2-47) & (2-49). These quantities have a quite complicated dependence on the products gv and λv, so that, if we were to worry about such dependences, the evaluation of the expression (2-50) for the Unitary gauge bare vacuum would be uncontrollably difficult. If the canonical operators cannot be considered as finite quantities, the generators $F_{(n)}$ in (2-30) would now contain both small and large expressions in the large v limit, so that, it would practically be impossible to represent the Unitary Gauge Transformation of Section B in a sensible way. Therefore, if we wish to use the above Fock space representation and consider the Unitary gauge bare vacuum, we should organize our considerations according to the assumption of the loop expansion and treat gv and λv as finite quantities.

We now examine the non-leading terms of the expression (2-38-a) for $U^{\pm}\mathcal{H}U$. The terms which give a non-vanishing expectation value are :

$$\frac{1}{2}\, g^2\, A^j A^j :_1 :_1 \tag{2-56}$$

as well as the $\frac{1}{4}\, \lambda^2 (:_1)^4$ term. These are second order quantities in our approximation and we should therefore consider order(v^{-2}) terms that we have neglected, in equal footing with these terms. Now, it can easily be shown that

order(v^{-2}) terms give a positive definite expectation value in the state $|\Omega^{\circ}>$ and, this is obviously sufficient to establish that the minimizing value of the parameter v^2 is no longer zero. Therefore, the inclusion of the first non-vanishing contribution to the energy density from the nonleading terms, resolved our problem of consistency. We should consider values for the variational parameter v, large enough to justify the neglect of higher order terms than order(v^{-2}), but not terribly large, so that order(v^{-2}) terms still make an appreciable contribution.

We now recall that the terms which were proportional to the inverse powers of v, were ultravioletwise more divergent than the energy densities of the usual vacua of four dimensionful field theories. This means that, in order to resolve the problem of consistency, we have had to consider ultravioletwise too singular terms as important ones in the evaluation of the energy density of the Unitary gauge bare vacuum. Our conclusion is that, although the Unitary gauge bare vacuum has an infrared finite energy density, this density is more ultraviolet-divergent than the energy density of the Axial gauge bare vacuum, and as such, it is not clear whether this is a better trial state than the Axial gauge bare vacuum.

We therefore have to modify our considerations. We would like to arrive at an expression for the energy density of a trial state which finite in the infrared and not too singular in the ultraviolet. We now claim that,

the four point coupling given by (2-56) is just such a term. Needless to say, we cannot consider this term as an important one without at the same time considering order(v^{-2}) terms as equally important, within our approximation scheme, so that, any lesson that we might derive from the four - point coupling $\frac{1}{2}\, g^2 A^j A^j \phi_1 \phi_1$ will not alter our conclusion of the previous paragraph about the Unitary gauge bare vacuum. Nevertheless, this consideration will be relevant in the next Section, where the approximation scheme of this Section will be modified so as to enable us to consider terms like the Higgs-gluon coupling , without having to consider at the same time, ultravioletwise too singular an expression.

We now consider the vacuum expectation value of the above term. We have,

$$\langle 0 | A^j(x) A^j(x) | 0 \rangle = 2\,(g^2 v^2) + \frac{1}{2(2\pi)^3} \frac{1}{(gv)^2} I(gv), \quad (2\text{-}57)$$

where, $$I(gv) = \int_0^{\Lambda} d^3\underset{\sim}{k} \frac{(k_T)^2}{(\underset{\sim}{k}^2 + g^2 v^2)^{\frac{1}{2}}}$$

and where we define , for s nonnegative :

$$(s) = \frac{1}{2(2\pi)^3} \int_0^{\Lambda} \frac{d^3\underset{\sim}{k}}{(\underset{\sim}{k}^2 + s)^{\frac{1}{2}}} \quad (2\text{-}58)$$

We also note that,

$$\langle 0 | \phi_1(x) \phi_1(x) | 0 \rangle = (\mu^2 + 3\lambda^2 v^2)$$

The second term in (2-57) is an order(v^{-2}) term which we will not consider here. We then have :

$$<\Omega^{\circ}|\ \frac{1}{2}g^2A^jA^j\phi_1\phi_1|\Omega^{\circ}> = g^2\ \Delta(g^2v^2)\ \Delta(\mu^2+3\lambda^2v^2)\ +\ \text{order}(v^{-2}) \tag{2-59}$$

Now,the function $\Delta(s)$ is a positive definite quantity which has a finite limit as s tends to zero from the positive s axis.We observe however that its derivative becomes singular in the same limit. This can be seen by differentiating the integral representation (2-58) with respect to s, and observing that the resulting expression becomes an infrared divergent integral as s tends to zero.We may also arrive at the same result by an explicit evaluation of $\Delta(s)$:

$$\Delta(s) = \frac{1}{8\pi^2}\left[\Lambda(\Lambda^2+s)^{\frac{1}{2}} + \frac{1}{2}\ s\ \log\left(\frac{s}{[\Lambda+(\Lambda^2+s)^{\frac{1}{2}}]^2}\right)\right] \tag{2-60}$$

Due to the presence of the s.log(s) term , the derivative of $\Delta(s)$ contains a log(s) term and therefore,

$$\lim_{s\to 0}\frac{d\Delta(s)}{ds} = -\infty \tag{2-61}$$

Thus $\Delta(s)$ is a sharply decreasing function of s around s=0.

Let us now come back to the expression (2-59).If we now include (2-59) to the leading order expression $\mathscr{E}^{\circ}(v)$ for the energy density in (2-55-b), we will get an expression (with s v^2) :

$$\mathscr{E}(s) \equiv g^2\Delta(g^2 s)\ \Delta(\mu^2+3\lambda^2 s) + \mathscr{E}^{\circ}(s) \tag{2-62}$$

We now observe that, due to the first term of this expression and the result (2-61), $\mathscr{E}(s)$ is a sharply decreasing function of s near s=0, and its minimum cannot be at s=0. Since the remaining terms coming from $\mathscr{E}^{\circ}(s)$ are increasing functions of s, we conclude that $\mathscr{E}(s)$ does have a minimum corresponding to a finite value of s. This situation is depicted in Figure 1. On the other hand, this expression is infrared finite and it goes like the fourth power of of the ultraviolet cutoff Λ, in the high Λ limit, and this is the typical ultraviolet divergence of the energy densities of the usual bare vacua in four dimensional field theories. We therefore conclude that, if we can concoct a scheme where we can consider the Higgs-gluon coupling (2-56) without having to consider at the same time, ultraviolet-wise too singular expressions, we would obtain a desirable trial state. An example of such a procedure is now provided in the following Section.

D) A Modified Bare Vacuum

In order to include the Higgs-gluon coupling of (2-56) among the leading terms of an approximation scheme, it is clear that we must consider nonperturbative values of the coupling constant g. On the other hand, the analysis of the Section B on the unitary implementation of the mapping M_1 by the operator U_1, assumed in an essential way that the

parameter v was a large quantity; otherwise the expression (2-32) would not be a sensible approximation to the operator U_1 as defined in (2-27). Furthermore, even if one were able to construct an operator U_1 which would satisfy (2-27) without any approximation, the discussion of the Section B showed that, due to the ultraviolet singularities, such an expression could be satisfactory only for large values of the parameter v. In this Section, we will therefore continue to assume that v is a large quantity, but the coupling constants g and λ will be considered as finite, nonperturbative quantities. Since our considerations in Section B nowhere made any assumption on the coupling constants, we shall continue our analysis from where we left at the end of Section B, but incorporating in it the various lessons that we have learned from the construction of the Unitary gauge bare vacuum in Section C.

Now that we are not keeping the products gv and λv as fixed quantities, the Fock space representation given in the previous Section will have to be modified. We shall assume that the scalar fields $\tilde{\phi}_a(\underset{\sim}{k})$ and their momenta $\tilde{\pi}_a(\underset{\sim}{k})$ do not have any dependence on v. As a convenient example, we take :

$$\tilde{\pi}_1(\underset{\sim}{k}) = -i\left(\frac{\omega_o}{2}\right)^{\frac{1}{2}}\left[a_0(-\underset{\sim}{k}) - a_0^+(\underset{\sim}{k})\right] \qquad (2\text{-}63\text{-a})$$

$$\tilde{\phi}_1(\underset{\sim}{k}) = \left(\frac{1}{2\omega_o}\right)^{\frac{1}{2}}\left[a_0(\underset{\sim}{k}) + a_0^+(-\underset{\sim}{k})\right] \qquad (2\text{-}63\text{-b})$$

$$\tilde{\pi}_2(\underset{\sim}{k}) = -i \frac{|k_3|}{(2\omega_o)^{\frac{1}{2}}} \left[a_3(-\underset{\sim}{k}) - a_3^{\dagger}(\underset{\sim}{k})\right] \tag{2-63-c}$$

$$\tilde{\phi}_2(\underset{\sim}{k}) = \frac{1}{|k_3|} \frac{\omega_o}{2}^{\frac{1}{2}} \left[a_3(\underset{\sim}{k}) + a_3^{\dagger}(-\underset{\sim}{k})\right] \tag{2-63-d}$$

where $\omega_o = (\underset{\sim}{k}^2 + \mu^2)^{\frac{1}{2}}$. We can now use the large v approximation of Section B to evaluate for example the quantity $U_1^{+} \mathcal{H} U_1$ and we arrive at the formula (2-35) of that Section. This should now be conjugated by the operator U_2 and the generator of this operator can now be written as :

$$\frac{-i}{gv} \int d^3\underset{\sim}{x}\ \partial_i E^i(\underset{\sim}{x})\ \phi_2(\underset{\sim}{x}) \tag{2-64}$$

Although we can evaluate the action of this operator without any approximation, for the sake of further simplicity, we shall assume that the quantity $\partial_i E^i(\underset{\sim}{x})$ is also independent of v in its Fock space representation. We note that, according to (2-43),

$$\partial_i E^i(\underset{\sim}{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\underset{\sim}{k}\ \exp(-i\underset{\sim}{k}.\underset{\sim}{x})\ k_T\ p_2(\underset{\sim}{k})$$

so that, it involves only one helicity degree of freedom. Consequently, we choose,

$$p_2(\underset{\sim}{k}) = -i\left(\frac{|\underset{\sim}{k}|}{2}^{\frac{1}{2}}\right)\left[a_2(-\underset{\sim}{k}) - a_2^{\dagger}(\underset{\sim}{k})\right] \tag{2-65-a}$$

$$q_2(\underset{\sim}{k}) = (2|\underset{\sim}{k}|)^{-\frac{1}{2}}\left[a_2(\underset{\sim}{k}) + a_2^{\dagger}(-\underset{\sim}{k})\right] \tag{2-65-b}$$

In this connection, we note that the quantity,

$$A_T^i \equiv \left[\delta^{ij} - \frac{\partial^i \partial^j}{\partial_T^2}\right] A^j$$

(where $\partial_T^2 = \sum_{i=1}^{2} \partial^i \partial^i$), is easily seen to be invariant under the residual group of the Axial gauge. From the Fourier representation (2-43-b) we see that,

$$A_T^i = \frac{i}{(2\pi)^{3/2}} \int d^3\underset{\sim}{k} \exp(i\underset{\sim}{k}.\underset{\sim}{x})\ e_1^i(\underset{\sim}{k})\ q_1(\underset{\sim}{k})$$

so that q_1 corresponds to a degree of freedom for which the gauge fixing is complete. The generator (2-64) thus involves only those degrees of freedom which are not fixed by the Axial gauge condition. For (p_1, q_1) we shall choose a representation corresponding to a mass $m=gv$, viz,

$$p_1(\underset{\sim}{k}) = -i\left(\frac{\omega}{2}\right)^{\frac{1}{2}} \left[a_1(-\underset{\sim}{k}) - a_1^{+}(\underset{\sim}{k})\right] \qquad (2\text{-}65\text{-}c)$$

$$q_1(\underset{\sim}{k}) = \left(\frac{1}{2\omega}\right)^{\frac{1}{2}} \left[a_1(\underset{\sim}{k}) + a_1^{+}(-\underset{\sim}{k})\right] \qquad (2\text{-}65\text{-}d)$$

where $\omega^2 = \underset{\sim}{k}^2 + g^2v^2$. Although in the non-Abelian theory, the analogous operator to A_T^i will not be invariant under the residual group - which will be non-Abelian -, a representation like (2-65) will still be a simple and useful choice.

Denoting the Fock vacuum of the present representation by $|0\rangle$, we are therefore interested in a trial state which is the same as (2-50), except for the different

choice of Fock space, namely,

$$U|\Omega\rangle = \exp(-iv\int d^3\underset{\sim}{x}\,\Pi_1)\ \exp(iF)\ \exp(\frac{i}{gv}\int d^3\underset{\sim}{x}\ E^i\partial_i\phi_2)|\Omega\rangle \qquad (2\text{-}66)$$

It is now obvious that the residual gauge invariance considerations will still lead to the expression (2-51) and, since we have chosen $\tilde{\Pi}_2(\underset{\sim}{k})$ in (2-63-c) with the condition,

$$\tilde{\Pi}_2(k_1,k_2,0) = 0 \qquad (2\text{-}52\text{-}a)$$

in mind, we can still conclude that this trial state is annihilated by the generator $G(x_1,x_2)$ of the residual group by following the identical steps that led to (2-53).

The energy density of $U|\Omega\rangle$ can now be explicitly evaluated. We now have the generator (2-64) as a small quantity in the large v limit and in the conjugation of $U_1^{\dagger}\mathcal{H}U_1$ by U_2, we neglect all the inverse powers of v. From (2-35), we now obtain :

$$\left.\begin{aligned} U^{\dagger}\mathcal{H}U = U_2^{\dagger}(U_1^{\dagger}\mathcal{H}U_1)U_2 = {} & \frac{1}{2}E^iE^i + \frac{1}{2}(gv)^2(\partial_3^{-1}\Pi_2)^2 \\ & + \frac{1}{2}F^{12}F^{12} + \frac{1}{2}(\partial_3A^i)(\partial_3A^i) + \frac{1}{2}\Pi_2\Pi_2 \\ & + \frac{1}{2}(\partial_3\phi_2)(\partial_3\phi_2) + \frac{1}{2}(gv)^2A^jA^j + g^2v\,\phi_1\phi_1A^jA^j \\ & + \frac{1}{2}g^2\phi_1\phi_1A^jA^j + \mathcal{H}^{(1)} + \text{order}(\frac{1}{v}) \end{aligned}\right\} \qquad (2\text{-}67)$$

Note that the quadratic part of this expression is different from the free field Hamiltonian of massive vector bosons $\mathcal{H}_Q$ as given in (2-39) and it can be obtained from $\mathcal{H}_Q$ by neglecting the terms proportional to the inverse powers of gv.

It is now trivial to evaluate the expectation value of (2-67). With $s \equiv v^2$, $\omega^2=k^2+g^2s$, $\omega_o^2=k^2+\mu^2$, we obtain explicitly that

$$\langle\Omega|U^{+}\mathcal{H}\,U|\Omega\rangle \;=\; \mathcal{E}(s) \;+\; \text{order}\left(\frac{1}{v}\right)$$

$$\begin{aligned}\mathcal{E}(s) \;=\; & \frac{1}{4}\lambda^2 s^2 + \frac{1}{2}\mu^2 s + \frac{1}{2} g^2 s \left[\Delta(\mu^2)+\Delta(0)+\Delta(g^2 s)\right] \\ & \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left[\omega_o + |k| + \frac{\omega}{2} + \frac{(k_3)^2}{2\omega} + \frac{(k_3)^2}{\omega_o}\right] \\ + & \frac{3}{2}\lambda^2 s\,\Delta(\mu^2) + \frac{3}{4}\lambda^2\,[\Delta(\mu^2)]^2 \\ + & \frac{1}{2} g^2\,\Delta(\mu^2)\,[\,\Delta(0) + \Delta(g^2 s)\,]\end{aligned}$$

We recall that the function $\Delta(s)$ was defined in (2-58). Similar to our considerations at the end of the last Section, we have thus arrived at an expression for $\mathcal{E}(s)$ which is an infrared finite quantity and which is not too singular in the ultraviolet. Because of the last term

$$\frac{1}{2} g^2 \quad (\mu^2)\;\Delta(g^2 s)$$

which comes from the Higgs-gluon interaction term (2-56), as well as the property (2-61) of the function $\Delta(s)$, we observe that $\mathcal{E}(s)$ is a sharply decreasing function of s near s=0. It is furthermore a positive definite expression which increases indefinitely as s gets very large,and as we have already concluded in (2-62) and in Figure 1, such an expression must necessarily have a minimum at a point where s is non-zero.

The situation can now be summarized as follows : In this Section, we have seen that a simple modification of the expression (2-50) for the Unitary gauge bare vacuum led to a trial state with a finite energy density and with an ultraviolet divergence which was typical for the bare vacua of four dimensional field theories. This was achieved in the large v limit and for the nonperturbative values of the coupling constants. We have assumed that some helicity component of the gauge field was represented with a mass gv, as in (2-65-d), and consequently the four point Higgs-gluon coupling term led to the conclusion that the parameter was nonzero. We conclude that such a state is a better trial state than the Axial gauge bare vacuum, for the ground state of the theory. In the absence of a Coulomb gauge bare vacuum, this result indicates that the vacuum of a gauge theory with scalar fields is more likely to be in a broken symmetry-Higgs phase, rather than the perturbative phase with an unbroken symmetry.

CHAPTER 3

UNITARY GAUGE IN A NON-ABELIAN THEORY

A) Description of the Theory

In this Chapter, we would like to generalize the argument of the previous Chapter into a non-Abelian gauge theory. Specifically, we are interested in an SU(2) gauge theory with scalar fields in the fundamental representation. Our conventions are as follows : In this Chapter, the indices i,j,k will be ordinary space indices running from 1 to 2 whereas the letters a,b,c,d,e,f will denote the group indices running from 1 to 3. The vector notation $\underset{\sim}{A}$ will only be employed for the group space and the dot and cross products of vectors will be interpreted accordingly.

The isospinor scalar field ϕ is represented in terms of the canonical fields ϕ_0 and ϕ_a as follows :

$$\phi = \begin{pmatrix} \phi_2 + i\phi_1 \\ \phi_0 - i\phi_3 \end{pmatrix} = (\phi_0 + i\underset{\sim}{\phi}\cdot\underset{\sim}{\tau}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3-1}$$

The notation $\underset{\sim}{\phi}\cdot\underset{\sim}{\tau}$ will be very handy throughout but we of course do not want to suggest that the quantities ϕ_a transform according to the adjoint representation.

The covariant derivative is defined as :

$$D^\mu \phi = \partial^\mu \phi - igA_a^\mu \frac{\tau^a}{2} \phi \tag{3-2}$$

and the field strength tensor as :

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g\,\epsilon_{abc} A_b^\mu A_c^\nu \tag{3-3}$$

The Lagrangian of the theory is :

$$\mathscr{L} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu,a} + \frac{1}{2}(D^\mu \phi)_a^\dagger (D_\mu \phi)_a - V(\phi^\dagger \phi)$$

where,

$$V(\phi^\dagger \phi) = \frac{1}{2} \mu^2 \phi^\dagger \phi + \frac{1}{4} \lambda^2 (\phi^\dagger \phi)^2 \tag{3-4}$$

We now describe the canonical formalism of such a theory in the Axial gauge.

The canonical coordinates are : ϕ_0 , ϕ_a , A_a^i

The conjugate momenta are : π_0 , π_a , E_a^i

with $D^0 \phi = (\pi_0 + i\underset{\sim}{\pi} \cdot \underset{\sim}{\tau}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $E_a^i = F_a^{i0}$

The dependent variable F_a^{30} can be expressed in terms of the canonical variables via the constraint equation of Gauss' law, leading to the definition of the density $\mathscr{G}_a(x)$:

$$\mathscr{G}_a(x) = \partial_3 F_a^{30} = -\partial_i E_a^i + g\,\rho_a \tag{3-5}$$

where the charge density ρ_a is given by :

$$\underset{\sim}{\rho} = - \underset{\sim}{E}^i \times \underset{\sim}{A}^i + \frac{1}{2} [\underset{\sim}{\Pi} \phi_0 - \Pi_0 \underset{\sim}{\phi} + \underset{\sim}{\Pi} \times \underset{\sim}{\phi}] \tag{3-6}$$

The Hamiltonian density of the theory is :

$$\left.\begin{aligned} \mathcal{H} = & \frac{1}{2} \underset{\sim}{E}^i . \underset{\sim}{E}^i + \frac{1}{2} [\partial_3^{-1} \underset{\sim}{\rho}]^2 + \frac{1}{4} \sum_{i,j=1}^{3} F^{ij} . F^{ij} \\ & + \frac{1}{2} (\Pi_0 \Pi_0 + \underset{\sim}{\Pi} . \underset{\sim}{\Pi}) + \frac{1}{2} [(\partial_3 \phi_0)(\partial_3 \phi_0) + (\partial_3 \underset{\sim}{\phi})(\partial_3 \underset{\sim}{\phi})] \\ & + \frac{1}{2} (D_j \psi)^{\dagger} (D_j \psi) + V(\psi^{\dagger} \psi) \end{aligned}\right\} \tag{3-7}$$

where,

$$\left.\begin{aligned} (D_j \psi)^{\dagger} (D_j \psi) = & \partial_j \psi_0 \partial_j \psi_0 + \partial_j \underset{\sim}{\psi} . \partial_j \underset{\sim}{\psi} \\ & - q \underset{\sim}{A}^j . [\underset{\sim}{\psi} \partial_j \psi_0 - \psi_0 \partial_j \underset{\sim}{\psi} + \underset{\sim}{\psi} \times \partial_j \underset{\sim}{\psi}] \\ & + (\frac{q}{2})^2 \underset{\sim}{A}^j . \underset{\sim}{A}^j [\psi_0 \psi_0 + \underset{\sim}{\psi} . \underset{\sim}{\psi}] \end{aligned}\right\} \tag{3-8}$$

The equal time canonical commutation relations are :

$$\left.\begin{aligned} \left[\Pi_m (\underset{\sim}{x},t) , \psi_n (\underset{\sim}{y},t) \right] &= -i \delta_{mn} \delta (\underset{\sim}{x}-\underset{\sim}{y}) \\ \left[E_a^i (\underset{\sim}{x},t) , A_b^j (\underset{\sim}{y},t) \right] &= -i \delta_{ab} \delta^{ij} \delta (\underset{\sim}{x}-\underset{\sim}{y}) \end{aligned}\right\} \tag{3-9}$$

where, as a further convention, the isospinor indices m & n run from 0 to 3 .

As usual, the generators of the residual gauge transformations within the Axial gauge are :

$$G_a(x_1,x_2) \equiv \int_{-\infty}^{\infty} dz \; \mathcal{G}_a(x_1,x_2,z) \qquad (3\text{-}10)$$

and the finite energy condition that the physical states must satisfy is given by ,

$$G_a(x_1,x_2) \mid \text{state} > \; = \; 0 \qquad (1\text{-}4)$$

In order to obtain such a state by the methods of the previous Chapter, we shall now describe the definition and the implementation of the Unitary Gauge Transformation, which will map the present non-Abelian theory from the Axial-gauge canonical variables into the gauge invariant polar variables.

B) The Unitary Gauge Transformation

We first express the isospinor field ψ in terms of the polar variables ρ, θ_a :

$$\psi = \psi(\rho,\theta_a) = \exp(i\underset{\sim}{\theta}.\frac{\underset{\sim}{\tau}}{2}) \begin{pmatrix} 0 \\ \rho \end{pmatrix} = \exp(i\underset{\sim}{\theta}.\frac{\underset{\sim}{\tau}}{2}) \; \psi(\rho,\underset{\sim}{\theta}=0) \qquad (3\text{-}11)$$

We then introduce the variables B_a^μ by ,

$$D^\mu\psi = D^\mu\psi(\rho,\underset{\sim}{\theta}) = \exp(i\underset{\sim}{\theta}.\frac{\underset{\sim}{\tau}}{2}) \; [\partial^\mu - igB_a^\mu \frac{\tau_a}{2}] \; \psi(\rho,\underset{\sim}{\theta}=0) \qquad (3\text{-}12)$$

The expression of the polar quantities B_a^μ in terms of the canonical variables A_a^μ is considerably more difficult in the present non-Abelian theory than the formulae (2-12) and (2-13) of the previous Chapter. Indeed, the main purpose of this Chapter is to show that such complications can be overcome and the analysis of the previous Chapter can be generalized to a non-Abelian gauge theory. Now, it can easily be shown that[18], the equation

$$[\partial^\mu - ig\underset{\sim}{B}^\mu \cdot \frac{\underset{\sim}{\tau}}{2}] = \exp(-i\underset{\sim}{\theta}\cdot\frac{\underset{\sim}{\tau}})\ [\partial^\mu - ig\underset{\sim}{A}^\mu \cdot \frac{\underset{\sim}{\tau}}{2}]\ \exp(i\underset{\sim}{\theta}\cdot\frac{\underset{\sim}{\tau}})$$

leads to the expression

$$B_a^\mu = [\exp(\underset{\sim}{\theta}\cdot\underset{\sim}{T})]_{ab}\ A_b^\mu - \frac{1}{g}\ [S(\theta)]_{ba}\ \partial^\mu\theta_b \qquad (3\text{-}13\text{-}a)$$

where we define,

$$(T^b)_{ac} \equiv \varepsilon_{abc} \qquad (3\text{-}13\text{-}b)$$

$$S(\theta) = \int_0^1 du\ \exp(-u\underset{\sim}{\theta}\cdot\underset{\sim}{T}) = \frac{1-\exp(-\underset{\sim}{\theta}\cdot\underset{\sim}{T})}{\underset{\sim}{\theta}\cdot\underset{\sim}{T}} \qquad (3\text{-}13\text{-}c)$$

In other words we have ,

$$B_a^j = [\exp(\underset{\sim}{\theta}\cdot\underset{\sim}{T})]_{ab}\ A_b^j + \frac{1}{g}\ [S(\theta)]_{ba}\ \partial_j\theta_b \qquad (3\text{-}14)$$

$$B_a^3 = \frac{1}{g}\ [S(\theta)]_{ba}\ \partial_3\theta_b \qquad (3\text{-}15)$$

The last equation shows that $\underset{\sim}{B}^j$ and $\underset{\sim}{\theta}$ are related to each other in a highly nonlinear way, due to the non-Abelian nature of the gauge group. For the same reason, the gauge field $\underset{\sim}{A}^j$ is now rotated in isospace by the action of the adjoint representation as shown in (3-14).

By analogy with (2-18), we wish to define a mapping M such that,

$$M : \begin{cases} \rho \rightarrow 2v + \phi_0 \\ B_a^j \rightarrow A_a^j \\ B_a^3 \rightarrow \frac{1}{gv} \partial_3 \phi_a \end{cases} \tag{3-16}$$

This will then be the Unitary Gauge Transformation of the present Chapter. Following the analysis of the previous Chapter, we are going to describe this mapping, step by step. We first define a mapping M_1 such that,

$$M_1 : \begin{cases} \rho \rightarrow 2v + \phi_0 \\ \underset{\sim}{\theta} \rightarrow \frac{1}{v} \underset{\sim}{\phi} \equiv \underset{\sim}{\beta} \end{cases} \tag{3-17}$$

The last equation is clearly analogous to (2-15) and for the purpose of making some future expressions simpler looking, we have introduced the parameter $\underset{\sim}{\beta}$. The action of M_1 on the canonical variables is as follows :

$$M_1 : \begin{cases} \phi = (\phi_0 + i\underset{\sim}{\phi}\cdot\underset{\sim}{\tau})\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \exp(i\underset{\sim}{\beta}\,\frac{\underset{\sim}{\tau}}{2})\begin{pmatrix}0\\ \phi_0+2v\end{pmatrix} \\ A_a^j \rightarrow A_a^j \end{cases} \tag{3-18}$$

The definitions (3-14) and (3-15) imply that,

$$M_1 : \begin{cases} B_a^j \rightarrow [\exp(\underset{\sim}{\beta}\cdot\underset{\sim}{T})]_{ab}\left[A_b^j + \frac{1}{gv}[S(\underset{\sim}{\beta})]_{bc}\,\partial^j\phi_c\right] \\ B_a^3 \rightarrow \frac{1}{gv}[S(\underset{\sim}{\beta})]_{ab}\,\partial^j\phi_b \end{cases} \tag{3-19}$$

where in the first line, we have made use of the fact that,

$$\exp(\underset{\sim}{\beta}\cdot\underset{\sim}{T})\,S(\underset{\sim}{\beta}) = S^{T}(\underset{\sim}{\beta})$$

which can easily be derived by using the definition (3-13) and the antisymmetry of T^a.

From now on, the Higgs field ϕ will be invariant under all the mappings to come. We now define the second mapping M_2 as,

$$M_2 : \begin{cases} A_a^j \rightarrow A_a^j - \frac{1}{gv}[S(\underset{\sim}{\beta})]_{ab}\,\partial^j\phi_b \\ \phi_a \rightarrow \phi_a \end{cases} \tag{3-20}$$

Using (3-14), we see that we have so far :

$$M_2 . M_1 : \begin{cases} B_a^j & \rightarrow & [\exp(\underset{\sim}{\beta} . \underset{\sim}{T})]_{ab} \, A_b^j \\ B_a^3 & \rightarrow & \frac{1}{gv} \, [S(\underset{\sim}{\beta})]_{ab} \, \partial_3 \Phi_b \end{cases} \tag{3-21}$$

It is then clear that, by defining

$$M_3 : \begin{cases} A_a^j & \rightarrow & [\exp(-\underset{\sim}{\beta} . \underset{\sim}{T})]_{ab} \, A_b^j \\ \phi_a & \rightarrow & \phi_a \end{cases} \tag{3-22}$$

and,

$$M_4 : \begin{cases} [S(\underset{\sim}{\beta})]_{ab} \, \partial_3 \phi_b & \rightarrow & \partial_3 \phi_b \\ A_a^j & \rightarrow & A_a^j \end{cases} \tag{3-23}$$

we shall have the required action $M = M_4 . M_3 . M_2 . M_1$ as described by the equation (3-16) which was in fact the starting point of this discussion. M is thus the Unitary Gauge Transformation.

Our plan is now as follows : We shall first introduce a Fock space representation for the canonical variables, in a completely analogous way to the analysis of the Section D of the last Chapter. We shall consider large values of the parameter v and nonperturbative values of the coupling constants. In such a setting, we shall examine the unitary implementations of the above mappings. These

considerations will lead us to a trial state which will be a better one compared to the Axial gauge bare vacuum.

We begin by introducing the Fourier decomposition of the canonical fields :

$$E_a^i(\underset{\sim}{x}) = \frac{i}{(2\pi)^{3/2}} \int d^3\underset{\sim}{k} \exp(-i\underset{\sim}{k}.\underset{\sim}{x}) \sum_{r=1}^{2} e_r^i(\underset{\sim}{k}) p_a(\underset{\sim}{k},r)$$

$$A_a^i(\underset{\sim}{x}) = \frac{i}{(2\pi)^{3/2}} \int d^3\underset{\sim}{k} \exp(i\underset{\sim}{k}.\underset{\sim}{x}) \sum_{r=1}^{2} e_r^i(\underset{\sim}{k}) q_a(\underset{\sim}{k},r)$$

$$\pi_m(\underset{\sim}{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\underset{\sim}{k} \exp(-i\underset{\sim}{k}.\underset{\sim}{x}) \tilde{\pi}_m(\underset{\sim}{k})$$

$$\phi_m(\underset{\sim}{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\underset{\sim}{k} \exp(i\underset{\sim}{k}.\underset{\sim}{x}) \tilde{\phi}_m(\underset{\sim}{k})$$

The canonical commutation relations become in this notation,

$$\left[p_a(\underset{\sim}{k},r) \ , \ q_b(\underset{\sim}{l},s) \right] = -i\ \delta_{ab}\delta_{rs}\ \delta(\underset{\sim}{k}-\underset{\sim}{l})$$

$$\left[\tilde{\pi}_m(\underset{\sim}{k}) \ , \ \tilde{\phi}_n(\underset{\sim}{l}) \right] = -i\ \delta_{mn}\ \delta(\underset{\sim}{k}-\underset{\sim}{l})$$

and they will be represented by the usual algebra of creation-annihilation operators as ,

$$\tilde{\pi}_0(\underset{\sim}{k}) = -i \left(\frac{\omega_0}{2}\right)^{\frac{1}{2}} [\ a_0(-\underset{\sim}{k}) - a_0^{+}(\underset{\sim}{k})\ ] \qquad (3\text{-}24\text{-a})$$

$$\tilde{\phi}_0(\underset{\sim}{k}) = \left(\frac{1}{2\omega_0}\right)^{\frac{1}{2}} [\ a_0(\underset{\sim}{k}) + a_0^{+}(-\underset{\sim}{k})\ ] \qquad (3\text{-}24\text{-b})$$

$$\tilde{\pi}_c(\underset{\sim}{k}) = -i \frac{k_3}{(2\omega_0)^{\frac{1}{2}}} [\ a_c(-\underset{\sim}{k}) - a_c^{+}(\underset{\sim}{k})\ ] \qquad (3\text{-}24\text{-c})$$

$$\tilde{\phi}_c(\underset{\sim}{k}) = \frac{1}{|k_3|}\left(\frac{\omega_0}{2}\right)^{\frac{1}{2}} [\, a_c(\underset{\sim}{k}) + a_c^{\dagger}(-\underset{\sim}{k}) \,] \qquad (3\text{-}24\text{-}d)$$

$$p_c(\underset{\sim}{k},1) = -i\left(\frac{\omega}{2}\right)^{\frac{1}{2}} [a_c(-\underset{\sim}{k},1) - a_c^{\dagger}(\underset{\sim}{k},1)] \qquad (3\text{-}24\text{-}e)$$

$$q_c(\underset{\sim}{k},1) = \left(\frac{1}{2\omega}\right)^{\frac{1}{2}} [a_c(\underset{\sim}{k},1) + a_c^{\dagger}(-\underset{\sim}{k},1)] \qquad (3\text{-}24\text{-}f)$$

$$p_c(\underset{\sim}{k},2) = -i\,\frac{|\underset{\sim}{k}|^{\frac{1}{2}}}{2} [a_c(-\underset{\sim}{k},2) - a_c^{\dagger}(\underset{\sim}{k},2)] \qquad (3\text{-}24\text{-}g)$$

$$q_c(\underset{\sim}{k},2) = \frac{1}{2|\underset{\sim}{k}|^{\frac{1}{2}}} [a_c(\underset{\sim}{k},2) + a_c^{\dagger}(-\underset{\sim}{k},2)] \qquad (3\text{-}24\text{-}h)$$

where $\omega_0 = (\underset{\sim}{k}^2+\mu^2)^{\frac{1}{2}}$ and $\omega = (\underset{\sim}{k}^2+g^2v^2)^{\frac{1}{2}}$, in complete analogy with (2-63) & (2-65). The Fock space vacuum of this representation will be denoted by $|0\rangle$.

We now describe the unitary implementation of the various factor mappings of the Unitary Gauge Transformation M. We begin by M_1. From (3-18), what we require is an approximation to the operator formally defined by :

$$U_1^{\dagger}\,(\Gamma_0 + i\,\underset{\sim}{\Gamma}\cdot\underset{\sim}{\sigma})\begin{pmatrix}0\\1\end{pmatrix}\Gamma_1 = \exp\left(i\,\frac{\underset{\sim}{\Gamma}}{v}\cdot\frac{\underset{\sim}{\tau}}{2}\right)\begin{pmatrix}0\\ \Gamma_0+2v\end{pmatrix}$$

and a sensible approximation in the large v limit is provided by :

$$U_1 = \exp(-i2v\int d^3\underset{\sim}{x}\,\Gamma_0)\ \exp\left(i\sum_{n=1}^{3}(2v)^{-n}\,F_{(n)}\right) \qquad (3\text{-}25)$$

where,

$$F_{(1)} = \frac{1}{2}\int d^3\underset{\sim}{x}\left[\Pi_0\underset{\sim}{\phi}\cdot\underset{\sim}{\phi} - \phi_0\{\Pi_a,\phi_a\}\right]$$

$$F_{(2)} = \frac{1}{24}\int d^3\underset{\sim}{x}\left[\Pi_a\ ,\ \phi_a(6\phi_0\phi_0 - \underset{\sim}{\phi}\cdot\underset{\sim}{\phi})\right]$$

$$F_{(3)} = \frac{-1}{24}\int d^3\underset{\sim}{x}\left[\{\Pi_0,\phi_0\phi_0\}\underset{\sim}{\phi}\cdot\underset{\sim}{\phi} + 2\phi_0\{\Pi_a,\phi_a(2\phi_0\phi_0 - \underset{\sim}{\phi}\cdot\underset{\sim}{\phi})\}\right]$$

These formulae are the obvious generalizations of (2-30). We also observe that these generators do not have any dependence on the parameter v through the Fock space representation (3-24) so that the large v approximation is straightforward. The action of U_1 on the various quantities of interest will now be computed. We have :

$$U_1^{-1}\,\mathscr{B}_a\,U_1 = \mathscr{B}_a^{(1)} + \text{order}(v^{-2}) \qquad (3\text{-}26\text{-a})$$

$$\left.\begin{aligned}\mathscr{B}_a^{(1)} \quad & gv\,\Pi_a - \Pi_i E_a^i - g\,f_{abc}E_b^i A_c^i + \frac{g}{2}\,\epsilon_{abc}\Pi_b\phi_c \\ & + \frac{g}{24v}\left[f_{ad}f_{bc} - f_{ab}f_{cd}\right]\{\Pi_b,\phi_c\phi_d\}\end{aligned}\right\}(3\text{-}26\text{-b})$$

Evidently, U_1 does not reduce this operator into a linear functional of the canonical operators. The cancellation of the infrared divergencies will thus be less transparent in the present non-Abelian theory. We shall also retain order$(\frac{1}{v})$ terms for the transforms of $\mathscr{B}_a(x)$ since this operator enters quadratically into our Hamiltonian and order$(\frac{1}{v})$ terms may interfere with the $g\,\Pi_a$ term of (3-26-b), to

give a non-vanishing result in the large v limit. We can similarly transform the Hamiltonian density of (3-7):

$$U_1^{+} \mathcal{H} U_1 = \mathcal{H}^{(1)} + \mathcal{H}_{(0)} + \text{order}\left(\frac{1}{v}\right) \tag{3-27-a}$$

$$\left.\begin{aligned} \text{where, } \mathcal{H}^{(1)} = {} & \frac{1}{2}\, \underset{\sim}{E}^{i}.\underset{\sim}{E}^{i} + \frac{1}{2}[\partial_3^{-1}\underset{\sim}{\mathcal{B}}^{(1)}]^2 + \frac{1}{2}\, \underset{\sim}{\Pi}.\underset{\sim}{\Pi} \\ & + \frac{1}{4} \sum_{i,j=1}^{3} \underset{\sim}{F}^{ij}.\underset{\sim}{F}^{ij} + \frac{1}{2} \sum_{i=1}^{3} \partial_i \underset{\sim}{\phi}.\partial_i \underset{\sim}{\phi} \\ & - \frac{g}{2}\, \underset{\sim}{A}^{j}.(\underset{\sim}{\phi} \times \partial_j \underset{\sim}{\phi}) + g(v+\phi_0)\underset{\sim}{A}^{j}.\partial_j \underset{\sim}{\phi} \\ & + \frac{1}{2}\left(\frac{g}{2}\right)^2 \underset{\sim}{A}^{j}.\underset{\sim}{A}^{j}\, (2v+\phi_0)^2 \end{aligned}\right\} \tag{3-27-b}$$

and,

$$\mathcal{H}_{(0)} = \frac{1}{2}\, \Pi_0 \Pi_0 + \frac{1}{2} \sum_{i=1}^{3} \partial_i \phi_0 \partial_i \phi_0 + V[(2v+\phi_0)^2] \tag{3-28}$$

The transformations that we will describe in what follows, will all leave the (Π_0, ϕ_0) Higgs degrees of freedom in peace.

We now consider the implementation of the mapping M_2 defined in (3-20). Since this simply adds a quantity to the operator A_a^j, we clearly have,

$$U_2 = \exp\left(\frac{-i}{gv} \int d^3\underset{\sim}{x}\; E_a^i\; [S(\underset{\sim}{x})]_{ab}\; \partial_i \phi_b \right) \tag{3-29}$$

where $\underset{\sim}{\phi} = \frac{1}{v}\, \underset{\sim}{\phi}$. This operator leaves E_a^i invariant and its

action on the conjugate momen a Π_a can be obtained by some algebra,

$$U_2^{+} \Pi_a U_2 = \Pi_a + \frac{1}{gv}[S(\beta)]_{ab} \partial_i E_b^{i}$$

$$- \frac{1}{gv^2} [\delta_{ad}\delta_{ce} - \delta_{ae}\delta_{cd}] \partial_i \phi_c [\frac{\partial S(\beta)}{\partial \beta_d}]_{eb} E_b^{i} \qquad (3\text{-}30\text{-}a)$$

We can therefore evaluate the action of U_2, without approximation, on any quantity of interest; however, we are only interested in the large v limit of this action. We therefore approximate the above action by,

$$U_2^{+} \underset{\sim}{\Pi} U_2 = \underset{\sim}{\Pi} + \frac{1}{gv} \partial_i \underset{\sim}{E}^{i} - \frac{1}{2gv^2}[(\partial_i \underset{\sim}{E}^{i} \times \underset{\sim}{\phi}) + 2(\underset{\sim}{E}^{i} \times \partial_i \underset{\sim}{\phi})]$$

$$+ \text{order}(v^{-3}) \qquad (3\text{-}30\text{-}b)$$

and the original transformation (3-20) by,

$$U_2^{+} \underset{\sim}{A}^{j} U_2 = \underset{\sim}{A}^{j} - \frac{1}{gv} \partial_j \underset{\sim}{\phi} + \frac{1}{2gv^2} \underset{\sim}{\phi} \times \partial_j \underset{\sim}{\phi} + \text{order}(v^{-3}) \qquad (3\text{-}30\text{-}c)$$

At this point, one aspect of our approximation scheme has to be clarified ; (3-24-e) shows that the Fourier component $p_a(k,l)$ of the electric field E_a^i does have a v dependence, and since the generator of U_2 depends on E_a^i, one might wonder how such a dependence should be taken into account in evaluating the large v limit. We now observe that, E_a^i in the generator as well as in the expression (3-30-b) is

always accompanied by a factor of $(gv)^{-1}$. Now the Fourier transform $p_a(\underset{\sim}{k},1)$ depends on v through $(\omega)^{\frac{1}{2}} = (\underset{\sim}{k}^2+g^2v^2)^{\frac{1}{4}}$ and since the quantity $(gv)^{-1}(\omega)^{\frac{1}{2}}$ goes to zero as v tends to infinity, it is easy to see that any expression whose only dependence on v and E_a^i is through the combination $\frac{1}{v}E_a^i$ will have a negligible vacuum expectation value in the large v limit. Therefore, using the formula (3-30-b), we can effectively treat E_a^i as a quantity independent of v.

We are now interested in the action of U_2 on $\mathscr{G}_a^{(1)}$ of (3-26-b) and $\mathscr{H}^{(1)}$ of (3-27-b). We obtain,

$$U_2^{+}\mathscr{G}_a^{(1)}\,U_2 = \mathscr{G}_a^{(2)} + \text{order}(v^{-2}) \tag{3-31-a}$$

$$\left.\begin{aligned}\mathscr{G}_a^{(2)} \equiv\ & gv\,\Pi_a - g\epsilon_{abc}E_b^iA_c^i + \frac{g}{2}\,\epsilon_{abc}\Pi_b\phi_c \\ & + \frac{g}{24v}\,[\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd}]\{\Pi_b,\phi_c\phi_d\}\end{aligned}\right\} \tag{3-31-b}$$

and

$$U_2^{+}\mathscr{H}^{(1)}\,U_2 = \mathscr{H}^{(2)} + \text{order}(\tfrac{1}{v}) \tag{3-32-a}$$

$$\left.\begin{aligned}\mathscr{H}^{(2)} \equiv\ & \frac{1}{2}\,\underset{\sim}{E}^i\cdot\underset{\sim}{E}^i + \frac{1}{2}[\partial_3^{-1}\underset{\sim}{\mathscr{G}}^{(2)}]^2 + \frac{1}{4}\sum_{i,j=1}^{3}\underset{\sim}{F}^{ij}\cdot\underset{\sim}{F}^{ij} \\ & + \frac{1}{2}\,\underset{\sim}{\Pi}\cdot\underset{\sim}{\Pi} + \frac{1}{2}\,\partial_3\underset{\sim}{\Phi}\cdot\partial_3\underset{\sim}{\Phi} + \frac{1}{2}\,g^2v^2\,\underset{\sim}{A}^j\cdot\underset{\sim}{A}^j \\ & + \frac{1}{2}\,g^2v\,\phi_0\underset{\sim}{A}^j\cdot\underset{\sim}{A}^j + \frac{1}{8}\,g^2\phi_0\phi_0\underset{\sim}{A}^j\cdot\underset{\sim}{A}^j\end{aligned}\right\} \tag{3-32-b}$$

We shall now implement the mapping M_3. The equation,

$$U_3^{+} A_a^j U_3 = [\exp(-\underset{\sim}{\beta}.\underset{\sim}{T})]_{ab} A_b^j \tag{3-33}$$

is easily seen to be satisfied by the operator,

$$U_3 = \exp[\frac{i}{v}\int d^3\underset{\sim}{x}\ \underset{\sim}{\phi}.(\underset{\sim}{E}^i \times \underset{\sim}{A}^i)] \tag{3-34}$$

and we have,

$$U_3^{+} \Pi_a U_3 = \Pi_a + \frac{1}{v}[S(\underset{\sim}{\beta})]_{ab}(\underset{\sim}{E}^i \times \underset{\sim}{A}^i)_b$$

$$U_3^{+} E_a^i U_3 = [\exp(-\underset{\sim}{\beta}.\underset{\sim}{T})]_{ab} E_b^i$$

These equations do not involve any approximation and in a large v limit, they reduce to,

$$U_3^{+} \underset{\sim}{A}^i U_3 \quad \underset{\sim}{A}^i + \frac{1}{v}\underset{\sim}{A}^i \times \underset{\sim}{\phi} + \frac{1}{2v^2}\underset{\sim}{\phi} \times (\underset{\sim}{\phi} \times \underset{\sim}{A}^i)$$

$$U_3^{+} \underset{\sim}{E}^i U_3 \quad \underset{\sim}{E}^i + \frac{1}{v}\underset{\sim}{E}^i \times \underset{\sim}{\phi} + \frac{1}{2v^2}\underset{\sim}{\phi} \times (\underset{\sim}{\phi} \times \underset{\sim}{E}^i)$$

$$U_3^{+} \underset{\sim}{\Pi} U_3 \quad \underset{\sim}{\Pi} + \frac{1}{v}\underset{\sim}{E}^i \times \underset{\sim}{A}^i - \frac{1}{2v^2}\underset{\sim}{\phi} \times (\underset{\sim}{E}^i \times \underset{\sim}{A}^i)$$

It is now easy to evaluate that,

$$U_3^{+} \mathscr{B}_a^{(2)} U_3 = \mathscr{B}_a^{(3)} + \text{order}(v^{-2})$$

where,

$$\mathcal{G}_a^{(3)} \equiv gv\,\Pi_a + \frac{g}{2}\,\varepsilon_{abc}\Pi_b\phi_c$$
$$+ \frac{g}{24v}\,[\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd}]\{\Pi_b, \phi_c\phi_d\} \tag{3-35}$$

We finally consider the mapping M_4. In view of (3-23), what we need is a unitary operator U_4 such that,

$$U_4^{+}\,[S(\xi)]_{ab}\,\partial_3\phi_b\,U_4 = \partial_3\phi_a$$

This is of course a formal expression and a meaning can only be given by expanding the expression for $S(\xi)$ in inverse powers of v and generating these terms by the action of U_4. In a large v limit, we then require U_4 such that,

$$U_4\,\partial_3\underset{\sim}{\phi}\,U_4^{+} = \partial_3\underset{\sim}{\phi} + \frac{1}{2v}(\underset{\sim}{\phi} \times \partial_3\underset{\sim}{\phi}) + \frac{1}{6v^2}\,\underset{\sim}{\phi} \times (\underset{\sim}{\phi} \times \partial_3\underset{\sim}{\phi}) \tag{3-36}$$

We shall represent,

$$U_4 = \exp[i(v^{-1}K_{(1)} + v^{-2}K_{(2)})]$$

To construct the generators $K_{(n)}$ is in principle straightforward but in practice it requires an extremely messy algebra. In this connection, the particular definition of the inverse derivative ∂_3^{-1}, involving the odd function $\varepsilon(x)$ as in (2-5) & (2-6) leads to notable simplifications.

This is especially advantageous in evaluating the commutators of $K_{(n)}$ with the conjugate momenta. The generators $K_{(n)}$ are found to be :

$$K_{(1)} = \frac{-1}{2}\int d^3\underset{\sim}{x} \quad \partial_3^{-1}\underset{\sim}{\Pi}.(\underset{\sim}{\phi} \times \partial_3\underset{\sim}{\phi})$$

$$K_{(2)} = \frac{-1}{48}[\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}]\int d^3\underset{\sim}{x}\left\{\partial_3^{-1}\Pi_a, [\phi_b\phi_d\partial_3\phi_e + 3\partial_3\phi_b\partial_3^{-1}(\phi_d\phi_e)]\right\}$$

We have thus implemented all the factor mappings of the Unitary Gauge Transformation, and by defining

$$U \equiv U_1 \cdot U_2 \cdot U_3 \cdot U_4$$

and using (3-37), it is now possible to show that,

$$U^{+}\,\mathcal{G}_a(x)\;U \;=\; g\,\frac{\partial}{\partial x_3}\,\mathcal{F}_a(x) + \text{order}(v^{-2}) \qquad (3\text{-}37)$$

where,

$$\mathcal{F}_a(x) = v\,\partial_3^{-1}\Pi_a + \epsilon_{abc}(\partial_3^{-1}\Pi_b)\phi_c$$

$$- \frac{1}{4v}[\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}]\left[\{\partial_3^{-1}\Pi_c, \phi_b\phi_d\} + \{\partial_3^{-1}\Pi_b, \partial_3^{-1}(\phi_c\partial_3\phi_d)\}\right]$$

Now, we have chosen the Fock space representation (3-24-c) for $\hat{\Pi}_a(\underset{\sim}{k})$, with the condition

$$\hat{\Pi}_a(k_1, k_2, 0) = 0$$

in mind, In other words, we have :

$$\int_{-\infty}^{\infty} dz\ \Pi_a(x_1,x_2,z) = 0$$

According to the definition (2-5) of the inverse derivative this means that,

$$\lim_{x_3\to\pm\infty} (\partial_3^{-1}\Pi_a)(x_1,x_2,z) = \pm \int_{-\infty}^{\infty} dz\ \Pi_a(x_1,x_2,z) = 0. \quad (3\text{-}38)$$

We now observe from (3-37) that, all the terms of the quantity $\mathcal{F}_a(x)$ contain a factor $\partial_3^{-1}\Pi_a$. Therefore when the latter vanishes, so must $\underset{\sim}{\mathcal{F}}(x)$. (3=38) therefore establishes,

$$\lim_{x_3\to\pm\infty} \mathcal{F}_a(x_1,x_2,x_3) = 0$$

which in turn implies that,

$$G_a(x_1,x_2) \equiv \int_{-\infty}^{\infty} dz\ \mathcal{G}_a(x_1,x_2,z) = g \int_{-\infty}^{\infty} dz \frac{\partial}{\partial z}\ \mathcal{F}_a(x_1,x_2,z)$$

$$= 0 + \text{order}(v^{-2}) \qquad (3\text{-}39)$$

A second noteworthy feature of the expression (3-37) for $\mathcal{F}_a(x)$ is that, the term proportional to v is orthogonal in group space to the term independent of v, which in turn is ortogonal to the terms proportional to $(\frac{1}{v})$. A similar feature is much more transparent in the expression (3-36). As a result of this fact, we have :

$$\underset{\sim}{\mathcal{F}} \cdot \underset{\sim}{\mathcal{F}} = v^2 \; (\partial_3^{-1} \underset{\sim}{\Pi}) \cdot (\partial_3^{-1} \underset{\sim}{\Pi}) \; + \; \text{order}(\tfrac{1}{v}) \tag{3-40}$$

Using now (3-23-b) and (3-40), we can arrive at the result,

$$\left.\begin{aligned} U^{+} \cdot \mathcal{H} \; U \; = \; & \frac{1}{2} \underset{\sim}{E}^{i} \cdot \underset{\sim}{E}^{i} + \frac{1}{2}(gv)^2 (\partial_3^{-1} \underset{\sim}{\Pi}) \cdot (\partial_3^{-1} \underset{\sim}{\Pi}) + \frac{1}{2} \underset{\sim}{\Pi} \cdot \underset{\sim}{\Pi} \\ & + \frac{1}{4} \sum_{i,j=1}^{3} \underset{\sim}{F}^{ij} \cdot \underset{\sim}{F}^{ij} + \frac{1}{2} \partial_3 \underset{\sim}{\phi} \cdot \partial_3 \underset{\sim}{\phi} + \frac{1}{2}(gv)^2 \underset{\sim}{A}^{j} \cdot \underset{\sim}{A}^{j} \\ & + \frac{1}{2} g^2 v \; \phi_0 \underset{\sim}{A}^{j} \cdot \underset{\sim}{A}^{j} + \frac{1}{8} g^2 \; \phi_0 \phi_0 \underset{\sim}{A}^{j} \cdot \underset{\sim}{A}^{j} \\ & + \mathcal{H}_{(0)} \end{aligned}\right\} \tag{3-41}$$

This expression is now to be compared with expression (2-67) that we derived at the end of the last Chapter. In these expressions, all the terms are in one to one correspondence with each other, with the exception of the gluon-self coupling terms

$$g^2 \sum_{i,j=1}^{2} (\varepsilon_{abc} A_b^i A_c^j)(\varepsilon_{ade} A_d^i A_e^j) \tag{3-42}$$

as well as the triple gluon vertices which come from the magnetic field term of (3-41). Our choice of the Fock space was also an obvious generalization from the Section D of the last Chapter. Therefore, except possibly for the four-gluon term (3-42), the vacuum expectation value of the expression (3-41) must have the same characteristics as the

energy density $\mathscr{E}(s)$ of that Section as depicted in Figure 1. Let us now analyze the vacuum expectation value of the four gluon term. We have

$$\langle\Omega|A_a^i(x)A_b^j(x)|\Omega\rangle = \frac{1}{2}\,\delta^{ij}\,\delta_{ab}[\,\Delta(0) + \Delta(g^2 s)\,]^2$$

where the function $\Delta(s)$ was defined in (2-58) and $s=v^2$. We then find,

$$\langle\Omega|\sum_{i,j=1}^{2}(\underset{\sim}{A}^i \times \underset{\sim}{A}^j).(\underset{\sim}{A}^i \times \underset{\sim}{A}^j)\,|\Omega\rangle = 6g^2[\Delta(0) + \Delta(g^2 s)]^2$$

We now observe that the behaviour of this expression as a function of s, is extremely similar to that of the expectation value of the Higgs-gluon coupling term :

$$\langle\Omega|\,\frac{1}{8}\,g^2\,\phi_0\phi_0 A^j.A^j\,|\Omega\rangle = \frac{3}{8}\,g^2\,\Delta(v^2)\,[\Delta(g^2 s) + \Delta(0)]$$

Both these terms are ultravioletwise not too singular and infrared finite;they both are sharply decreasing around s=0. We can now adapt the conclusions of the previous Chapter to the present non-Abelian theory and we then conclude that the expectation value of $U^{+}\mathscr{H}U$ on $|\Omega\rangle$ is an infrared finite expression which is not too singular in the ultraviolet and for which the variational parameter v is nonzero. The presence of the four gluon term does not change any of these conclusions and in fact, we have seen that this term, by itself can drive v to a nonzero value.

C) Conclusions

In this Chapter, we have shown that the trial state $U|\Omega\rangle = U_1.U_2.U_3.U_4|\Omega\rangle$ in the large v limit, is a finite energy trial state and due to the result (3-39), it does satisfy the residual gauge invariance condition (1-4). As such, it is a better trial state for the vacuum of our non-Abelian gauge theory than the Axial gauge bare vacuum. The latter is in the perturbative phase. In our trial state, the scalar field develops a vacuum expectation value and the gauge symmetry is broken. Based on these results, we conclude that the vacuum of our non-Abelian gauge theory is more likely to be in a Higgs phase than the perurbative phase.

In this analysis, we have assumed that the scalar field was in the fundamental representation. This led to a case where the symmetry breakdown was complete.We think that the construction of a finite energy trial state would be more difficult for the case of a partial symmetry breaking, due to the gauge fixing degeneracies associated with the surviving subgroup. On the other hand, considerations similar to ours, might be applied to the case of the Nielsen-Olesen Model in its non-Abelian form[4] and it would be interesting to see whether one can explicitly verify the area law associated with the magnetic 't Hooft loop, in such a coherent plasma.

Our main purpose in this work was to construct a trial state for the Higgs phase, starting from an Axial

gauge formulation of the non-Abelian gauge theory and to show that such a state was a better trial state for the vacuum of the theory, compared to the Axial gauge bare vacuum. We have explicitly shown how this could be done. On the other hand, our trial state is not as such a realistic one. One problem is of course the rotational invariance, but in this connection we note that the Axial gauge bare vacuum is also not rotationally invariant. On the other hand, for the present type of a theory with scalar fields in the fundamental representation, a realistic Higgs phase vacuum has been argued to be a color singlet.[19] It is far from obvious whether our trial state can lead to such a result. Nevertheless, the arguments that have been presented in this work do indicate that the vacuum of a non-Abelian gauge theory is likely to be in a phase other than the perturbative phase with physical massless gluons.

APPENDIX

Our purpose in this Appendix is to determine within the context of a classical field theory, the action of the mapping M_1 of Equation (2-20) on the conjugate momenta Π_1 & Π_2, so as to render this mapping a canonical transformation. We have :

$$Q_1 = (v + \phi_1) \cos(\frac{1}{v} \phi_2)$$

$$Q_2 = (v + \phi_1) \sin(\frac{1}{v} \phi_2)$$

We wish to find two functionals P_1 & P_2 of the canonical fields such that their Poisson brackets with Q_a satisfy :

$$\{ P_a(\underset{\sim}{x}) , Q_b(\underset{\sim}{y}) \} = \delta_{ab} \delta(\underset{\sim}{x}-\underset{\sim}{y}) \qquad \text{(A-1)}$$

We note that the Poisson bracket of two functionals F and G is defined as :

$$\{F,G\} = \int d^3\underset{\sim}{x} \sum_{a=1}^{2} \left[\frac{\delta F}{\delta \Pi_a(\underset{\sim}{x})} \frac{\delta G}{\delta \phi_a(\underset{\sim}{x})} - \frac{\delta F}{\delta \phi_a(\underset{\sim}{x})} \frac{\delta G}{\delta \Pi_a(\underset{\sim}{x})} \right]$$

and that we have :

$$\{ \Pi_a(\underset{\sim}{x}) , \phi_b(\underset{\sim}{y}) \} = \delta_{ab} \delta(\underset{\sim}{x}-\underset{\sim}{y}) \qquad \text{(A-2)}$$

In view of (A-1) and (A-2), the mapping M ,

$$M \; : \; \begin{cases} \phi_a \rightarrow Q_a \\ \Pi_a \rightarrow P_a \end{cases}$$

will be a canonical transformation.

Since the new coordinates $Q_a(\underset{\sim}{x})$ do not involve the old momenta $\Pi_a(\underset{\sim}{x})$, the new momenta $P_a(\underset{\sim}{x})$ can easily be determined following the standart textbooks on Classical Mechanics[20]. We then have,

$$P_a(x) = \int d^3\underset{\sim}{y} \; \Pi_b(\underset{\sim}{y}) \; \frac{\delta}{\delta Q_a(\underset{\sim}{x})} \; b(\underset{\sim}{y})$$

and a simple calculation then gives

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \exp\left(\frac{-1}{v} \Pi_2 \frac{\delta}{\delta \Pi_1}\right) \begin{pmatrix} \Pi_1 \\ \Pi_2 \left(1 + \frac{1}{v}\right)^{-1} \end{pmatrix} \qquad \text{(A-3)}$$

Writing this expression in terms of components and replacing the factors which do not commute in the quantum theory, by their anticommutators, we can arrive at the formulae (2-21).

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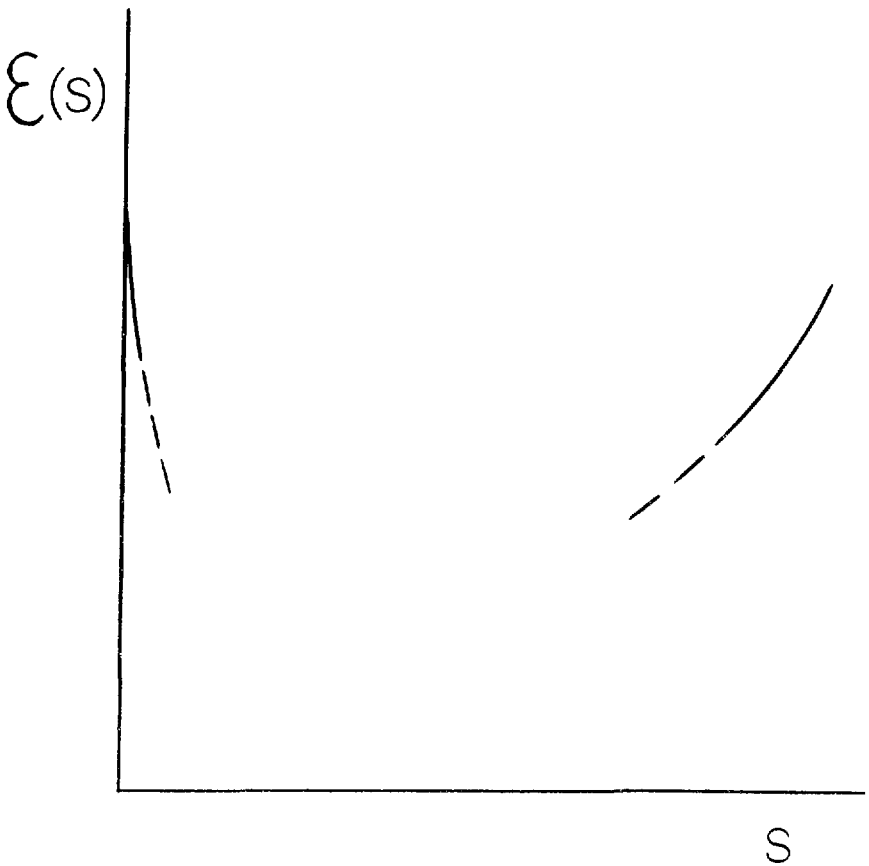
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