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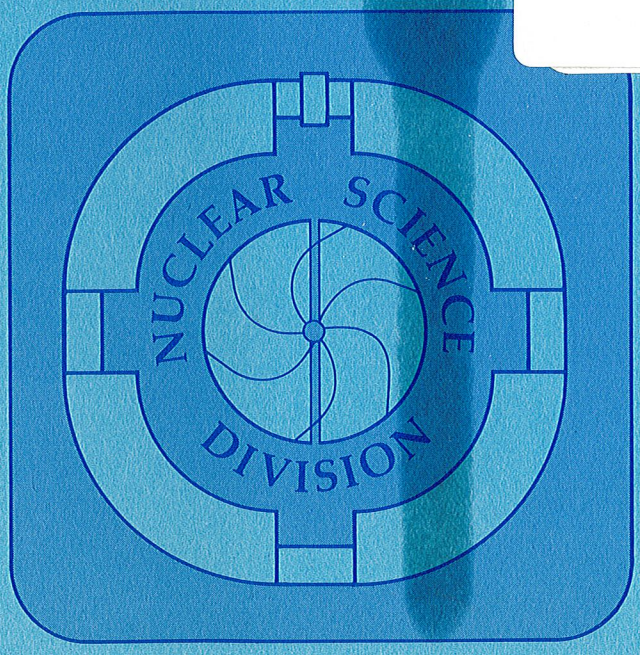

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ABSTRACT

We present solutions and some of their properties of a classical vector field model in two-dimensional Minkowski space with internal symmetry Z(3)--the cyclic group of order three.

1. INTRODUCTION

Cyclic symmetric field theories are of interest in solid state physics where the symmetry follows from the lattice structure of the crystals[1,2] and in elementary particle physics where the color theory of quarks and gluons imposes cyclic transformations of the quantum numbers[3].

In this short communication we are dealing with the mathematical structure of a system of nonlinear differential equations that is interesting in itself. Some physical applications are given elsewhere[4].

2. THE Z(3) MODEL

A. The Lagrangian density of the model[5] is a functional of a two-component vector field $\vec{\phi} = [\phi_1(t,x), \phi_2(t,x)] \in R_2$ in one time, one space dimensions $x_\mu = (t,x) \in R_2$, $\mu,\nu = 0,1$:

$$L = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - V(\phi_1,\phi_2) \quad . \tag{1}$$

$V(\phi_1,\phi_2)$ is the potential part of the Lagrangian that contains the polynomial self-interaction of the scalar fields:

$$V(\phi_1,\phi_2) = \lambda(\phi_1^2 + \phi_2^2)^2 - \nu(\phi_1^3 - 3\,\phi_1\phi_2^2) - \mu(\phi_1^2 + \phi_2^2) - \gamma \quad . \tag{2}$$

The metric is $g_{\mu\nu} = \mathrm{diag}(+1,-1)$, $\mu, \nu = 0,1$; $\partial\mu = \partial/\partial x\mu$, $\partial_\mu^2 = \partial_0^2 - \partial_1^2$. The parameters λ, μ are positive $\lambda > 0$, $\mu > 0$. The constant γ is determined such that the potential (2) is positive definite $V \geqslant 0$. The parameter ν is chosen to be positive $\nu > 0$.

The group transformations of the fields that leave the Lagrangian density (1) invariant are (i) a reflection of ϕ_2:

$$\phi_1 \to \phi_1 \quad , \quad \phi_2 \to -\phi_2 \quad , \tag{3}$$

and (ii) a Z(3) discrete rotation by 120° of the fields:

$$\phi_1 \to \phi_1 \cos(2\pi n/3) - \phi_2 \sin(2\pi n/3) \quad , \tag{4a}$$

$$\phi_2 \to \phi_1 \sin(2\pi n/3) + \phi_2 \cos(2\pi n/3) \quad , \tag{4b}$$

with $n = 0,1,2$.

The discrete symmetry Z(3) Eq. (4) is imposed by the cubic term in the potential (2). When ν is set zero the Lagrangian obviously has full SO(2) rotational symmetry.

There exist three more representations of the Lagrangian (1): in terms of complex fields, in polar coordinates, and in terms of a matrix field.

B. With the complex fields $\phi = \phi_1 + i\phi_2$, $\phi^* = \phi_1 - i\phi_2$ the Lagrangian becomes:

$$L = \frac{1}{2}\partial_\mu \phi \partial_\mu \phi^* - [\lambda |\phi|^4 - \frac{1}{2}\nu(\phi^3 + \phi^{*3}) - \mu |\phi|^2 - \gamma] \quad . \tag{5}$$

The reflection symmetry (3) becomes a complex conjugation:

$$\phi \to \phi^* \quad \text{and} \quad \phi^* \to \phi \quad , \tag{6}$$

and the discrete rotation (4) becomes a phase transformation:

$$\phi \to \phi \exp(i2\pi n/3) \quad , \ \phi^* \to \phi^* \exp(-i2\pi n/3) \quad . \tag{7}$$

C. In polar coordinates the Z(3) invariance is easily seen. The fields are $\phi_1 = \rho\cos\theta$, and $\phi_2 = \rho\sin\theta$. The potential part (2) of the Lagrangian becomes:

$$V(\rho,\theta) = \lambda\rho^4 - \nu\rho^3 \cos(3\theta) - \mu\rho^2 - \gamma \quad . \tag{8}$$

The reflection transformation (3) becomes a reflection of the angle θ:

$$\theta \to -\theta \quad , \tag{9}$$

and the rotational transformation (4) a translational operation:

$$\theta \to \theta + 2\pi n/3 \quad , \ n = 0,1,2 \quad . \tag{10}$$

The only θ-dependent factor in (8) $\cos(3\theta)$ is evidently invariant under the mappings (9) and (10).

D. The last form is in terms of a matrix field that links the model to a SU(2) representation. Define the matrix field:

$$\hat{\phi} = \begin{pmatrix} \phi_1 & -\phi_2 \\ -\phi_2 & -\phi_1 \end{pmatrix} = \sigma_3\phi_1 - \sigma_1\phi_2 \tag{11}$$

where σ_1, σ_2, σ_3 are the Pauli matrices. Then we can write the Lagrangian density (1) as a trace (Tr) expression:

$$L = \frac{1}{4}\,\mathrm{Tr}(\partial_\mu \hat{\phi})^2 - \frac{1}{2}\,\mathrm{Tr}\,[\lambda\hat{\phi}^4 - \nu(\sigma_3\hat{\phi})^3 - \mu\hat{\phi}^2 - \gamma I] \quad , \tag{12}$$

I is the 2 x 2 dimensional unit matrix.

The reflection symmetry (3) becomes the mapping:

$$\hat{\phi} \rightarrow \sigma_3 \hat{\phi} \sigma_3 \quad , \tag{13}$$

and the Z(3) transformation the mapping:

$$\hat{\phi} \rightarrow O^+ \hat{\phi} O \quad , \tag{14}$$

with the operator:

$$O = \exp(-i\pi n\sigma_2/3) \quad , \quad O^3 = I \quad , \quad n = 0,1,2 \quad . \tag{15}$$

This matrix representation is a scalar version of the SU(2) Yang-Mills field[6]:

$$A_\mu = \sum_{a=1}^{3} A_\mu^a \sigma_a \quad ,$$

with the imaginary component set to zero $A_\mu^2 = 0$. The representation (11) is suitable for generalization to more than two field components by an expansion in terms of the generators of the SU(3) group.

3. SOLITON SOLUTIONS

The Euler-Lagrange field equations in two dimensions are:

$$(\partial_0^2 - \partial_1^2)\phi_1 = -4\lambda(\phi_1^2 + \phi_2^2)\phi_1 + 3\nu(\phi_1^2 - \phi_2^2) + 2\mu\phi_1 \quad , \tag{16a}$$

$$(\partial_0^2 - \partial_1^2)\phi_2 = -4\lambda(\phi_1^2 + \phi_2^2)\phi_2 - 6\nu\phi_1\phi_2 + 2\mu\phi_2 \quad . \tag{16b}$$

First we are looking for time independent solutions of the field equations[7]. One dimensional solutions of the φ^4 theory have found applications in polyacetylene[8]. For further discussion of solitons in $(CH)_x$ see Ref. 9. In addition, references to this problem can be found in this volume of the proceedings.

Plane wave solutions in higher dimensional space can be obtained from the one dimensional solution by the mapping[10] $x \rightarrow k_\mu x_\mu / k_\mu k_\mu$.

In order to solve the field equations, it is useful to consider the topology of the potential $V(\phi_1,\phi_2)$ as a surface defined on the ϕ_1,ϕ_2 space. There are three minima—also called vacua—denoted by $\Omega_1,\Omega_2,\Omega_3$, where the potential is zero, located on a circle of radius ϕ_V at an angle of 120° from each other. The coordinates of the minima are:

$$\Omega_1 = (\phi_V,0) \quad ; \; \Omega_2 = (-\tfrac{1}{2}\phi_V, \tfrac{\sqrt{3}}{2}\phi_V) \quad ; \; \Omega_3 = (-\tfrac{1}{2}\phi_V, -\tfrac{\sqrt{3}}{2}\phi_V) \quad . \tag{17}$$

ϕ_V is the value of the vacuum field determined as the positive root of the equation $\partial V(\rho,0)/\partial\rho = 0$. In the center of the ϕ_1,ϕ_2 plane the potential is elevated $V(0,0) = -\gamma > 0$. For large values of the fields $\phi_1^2 + \phi_2^2 >> \phi_V^2$ the potential has steep walls due to the positivity of λ. There are three saddle points located at 60°, 180° and 240°. The distance from the center of the saddle points is given by the positive root of the equation $\partial V(\rho,\pi/3)/\partial\rho = 0$.

Solutions tunneling from one vacuum to the other are obtained by the method of trajectories in field space[11].

When the parameters λ, ν, μ, γ are related to each other in the following way:

$$\mu = \lambda\phi_V^2 \quad , \quad \nu = \frac{2}{3}\lambda\phi_V \quad , \quad \gamma = -\frac{2}{3}\lambda\phi_V^4 = -\nu\phi_V^3 \quad , \tag{18}$$

the geometry of the potential arranges itself such that the saddle point between two vacua lies on the straight line connecting these two vacua. The tunneling from one vacuum to the other will now occur along this straight line through the saddle point.

There are six pairs of solutions to the field equations connecting the minima pairwise. Three pairs of solutions are generated by the mapping (4); the other three obtained by the reflection (3). We use the notation $\alpha = (3\lambda/2)^{1/2}\phi_V$. The basic set of solutions is:

1. The solution going from Ω_1 to Ω_2, i.e. $\vec{\phi}(+\infty) \in \Omega_1$, $\vec{\phi}(-\infty) \in \Omega_2$ is

$$\phi_1(x) = \frac{1}{4}\phi_V[1 + 3\tanh(\alpha x)]; \quad \phi_2(x) = \frac{\sqrt{3}}{4}\phi_V[1 - \tanh(\alpha x)] \quad . \tag{19}$$

The trajectory is the algebraic relation independent of x:

$$\phi_1 + \sqrt{3}\,\phi_2 = \phi_V \quad . \tag{20}$$

The next two pairs are obtained from (19) by rotation of 120° and 240°.

2. The solution tunneling from Ω_2 to Ω_3 with the asymptotic values $\vec{\phi}(+\infty) \in \Omega_2$, $\vec{\phi}(-\infty) \in \Omega_3$ is

$$\phi_1(x) = -\frac{1}{2}\phi_V \quad ; \quad \phi_2(x) = \frac{\sqrt{3}}{2}\phi_V \tanh(\alpha x) \quad . \tag{21}$$

The trajectory is

$$\phi_1 = -\frac{1}{2}\phi_V \quad . \tag{22}$$

3. The soliton tunneling from Ω_3 to Ω_1 $\vec{\phi}(+\infty) \in \Omega_3$, $\vec{\phi}(-\infty) \in \Omega_1$ is

$$\phi_1(x) = \frac{1}{4}\phi_V[1 - 3\tanh(\alpha x)] \quad ; \phi_2(x) = -\frac{\sqrt{3}}{4}\phi_V[1 + \tanh(\alpha x)] \quad , \tag{23}$$

with the trajectory

$$\phi_1 - \sqrt{3}\,\phi_2 = \phi_V \quad . \tag{24}$$

All trajectories together [Eqs. (20),(22), and (24)] form an equilateral triangle with the corners in the minima of the potential.

4. PROPERTIES OF THE SOLITON SOLUTIONS

The energy-momentum tensor is a functional of the fields and their derivatives defined as:

$$T_{\mu\nu} = \sum_{k=1}^{2} \frac{\partial L}{\partial \phi_{k,\mu}} \phi_{k,\nu} - g_{\mu\nu} L; \quad \mu,\nu = 0,1 \quad . \tag{25}$$

The kinetic and potential energy of the field configuration given by any of the Eqs. (19),(21), or (23) are equal. Their sum is the energy density:

$$T_{00}(x) = \frac{9}{8}\lambda\phi_V^4[\cosh(\alpha x)]^{-4} \quad . \tag{26}$$

The integrated field energy gives the mass of the soliton:

$$M = \int_{-\infty}^{\infty} dx\ T_{00}(x) = \alpha\phi_V^3 \quad . \tag{27}$$

The other components of the field tensor--the field momentum T_{01} and the field pressure T_{11}—vanish.

The linear extension (analog of the radius in one dimension) of the energy distribution in configuration space is defined by:

$$R = [3\int_0^{\infty} dx\ x\ \ T_{00}(x)/\int_0^{\infty} dx\ T_{00}(x)]^{1/2} \quad , \tag{28}$$

$[T_{00}(x) = T_{00}(-x)]$, such that a box-like energy distribution of arbitrary height and length 2s gives the correct answer R = s. The dependence on λ and ϕ_V is explicit:

$$R = (0.80305...)\lambda^{-1/2}\phi_V^{-1} \quad . \tag{29}$$

With $B = -\gamma > 0$ the mass of the soliton can be written in terms of the linear radius:

$$M = (1.1438...)B(2R) \quad . \tag{30}$$

5. RESONANCES

Excited states of the scalar fields are given by fluctuations in time about the classical fields ϕ_1 and ϕ_2. Again we consider the solution Eq. (19). The perturbative expansion of the fields[12] (Q^2 and Q^3 are neglected in the field equations):

$$\phi_1^*(t,x) = \phi_1(x) + Q(t,x) \quad ; \ \phi_2^*(t,x) = \phi_2(x) - \frac{1}{\sqrt{3}}Q(t,x) \quad , \tag{31}$$

is chosen such that the new fields ϕ_1^* and ϕ_2^* satisfy the same algebraic relation (20) as the classical fields. With this expansion the energy spectrum of the resonance has two levels:

$$M^{*(k)} = M + E^{(k)} \quad , k = 1,2 \quad ; \tag{32}$$

where the energies and the perturbations are:

$$E^{(1)} = \sqrt{3}\ \alpha \quad ; Q^{(1)}(t,x) = A\cos(Et)\ \mathrm{sech}(\alpha x)\ \tanh(\alpha x) \quad , \tag{33}$$

and

$$E^{(2)} = 2\alpha \quad ; \ Q^{(2)}(t,x) = A\cos(Et)\left[1 - \frac{3}{2}\,\mathrm{sech}^2(\alpha x)\right] \quad . \tag{34}$$

6. TWO SOLITON INTERACTIONS

In physical applications the behavior of multicenter solutions is important. I have not found any exact multicenter solution in this model.

I will shortly analyze an ansatz for a field configuration with two centers at x_1 and x_2. There is an ordering of the centers implied $x_2 \leq x_1$. The trial function contains linear[13,14] and quadratic forms of the single soliton solutions Eq. (19) located at x_1 and the solution Eq. (23) with the sign of ϕ_2 changed [symmetry Eq. (3)] located at x_2:

$$\phi_1^{(2)}(x,x_1,x_2) = \cos^2(\varepsilon)\,\frac{1}{4}\,\phi_V\Big\{1 + 3[1 + \tanh(\alpha(x - x_1)) - \tanh(\alpha(x - x_2))]\Big\}$$

$$+ \sin^2(\varepsilon)\,\frac{1}{8}\,\phi_V[1 + 3\tanh(\alpha(x - x_1))][-1 + 3\tanh(\alpha(x - x_2))] \quad , \tag{35}$$

$$\phi_2^{(2)}(x,x_1,x_2) = \cos^2(\varepsilon)\frac{\sqrt{3}}{4}\,\phi_V\Big\{1 - [1 + \tanh(\alpha(x - x_1)) - \tanh(\alpha(x - x_2))]\Big\}$$

$$+ \sin^2(\varepsilon)\frac{\sqrt{3}}{8}\,\phi_V[1 - \tanh(\alpha(x - x_1))][1 + \tanh(\alpha(x - x_2))] \quad . \tag{36}$$

ε is a variational parameter with values $\varepsilon \in [0,\pi/2]$.

If one of the centers is removed to infinity, the fields (35,36) reduce to exact single soliton solutions. This limit process is independent of the value of ε. If $x_2 \to -\infty$, the fields (35,36) reduce to Eq. (19) and for $x_1 \to +\infty$ the fields reduce to Eq. (23) with the sign of ϕ_2 inverted.

The static potential energy between two solitons at a distance $d = x_1 - x_2$ is the difference between the total field energy and twice the mass of an isolated soliton:

$$P(d,\varepsilon) = \int_{-\infty}^{\infty} dx\; T_{00}^{(2)}(x,d,\varepsilon) - 2M \quad . \tag{37}$$

A numerical calculation shows that the potential is negative, i.e. the two solitons attract each other when they come close together. For $\varepsilon = 0$ (pure linear combination) we obtain a lower bound and for $\varepsilon = \pi/2$ (pure quadratic form) an upper bound for the potential energy.

$$P(d,0) \leq P(d,\varepsilon) \leq P(d,\pi/2) \leq 0 \quad ; \quad \varepsilon \in (0,\pi/2) \quad . \tag{38}$$

The potential converges fast to zero when the distance d becomes slightly larger than the linear extension of theparticles $d > 2R$.

The quality of the ansatz Eqs. (35,36) can be checked by computing the action for this field configuration. The action per unit time is:

$$S(d,\varepsilon) = -\int_{-\infty}^{\infty} dx\; T_{00}^{(2)}(x,d,\varepsilon) \quad . \tag{39}$$

For large distances $d \gg R$ the action is $-2M$ independent of ε. For fixed distances in the interaction region, the action has a minimum for $\varepsilon = \pi/2$ and a maximum for $\varepsilon = 0$.

$$-2M \leq S(d,\pi/2) \leq S(d,\varepsilon) \leq S(d,0) \leq 0 \quad ; \; \varepsilon \in (0,\pi/2) \quad . \tag{40}$$

In the static trial functions(35,36) the quadratic form of two single soliton solutions minimizes the action at any distance and seems therefore to be preferred to the linear form.

The net effect is that the static solitons attract each other. This is consistent with the findings in the φ^4 model at low velocities. High velocity solitons colliding with each other may scatter backward showing a repulsive potential at high speed[14,15].

CONCLUSIONS

We analyzed a field theory model with Z(3) internal symmetry in 1 + 1 dimensions. Exact soliton solutions are found in one space dimension. An ansatz for the two-center field shows that the static potential between two solitons is attractive.

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REFERENCES

1. R.B. Potts, Some generalized order-disorder transformations, Proc. Cambr. Phil. Soc. 48 (1952) 106-109.
2. A.N. Berker, S. Ostlund and F.A. Putnam, Renormalization-group treatment of a Potts lattice gas for krypton adsorbed onto graphite, Phys. Rev. B17 (1978) 3650-3665.
3. A. de Rujula, H. Georgi and S.L. Glashow, A theory of flavor mixing, Ann. of Phys. 109 (1977) 258-313.
4. H.M. Ruck, Polynomial chromodynamics in 1+1 dimensions I. Confinement of quarks and the structure of composite particles, Lawrence Berkeley Laboratory preprint LBL-12316 (February 1981).
5. F. Constantinescu and H.M. Ruck, Quantum field theory Potts model, J. Math. Phys. 19 (1978) 2359-2361; Phase transitions in a continuous three states model with discrete gauge symmetry, Ann. of Phys. 115 (1978) 474-495.
6. C.N. Yang and R. Mills, Conservation of isotopic spin and isotopic gauge invariance, Phys. Rev. 96 (1954) 191-195.
7. H.M. Ruck, Solitons in cyclic symmetry field theories, Nucl. Phys. B167 (1980) 320-326.
8. M.J. Rice, Charged π-phase kinks in lightly doped polyacetylene, Phys. Lett. 71A (1979) 152-154.
9. W.P. Su, J.R. Schrieffer and A.J. Heeger, Solitons in polyacetylene, Phys. Rev. Lett. 42 (1979) 1698-1701; Soliton excitations in polyacetylene, Phys. Rev. B22 (1980) 2099-2111.
10. P.B. Burt, Solitary waves in nonlinear field theories, Phys. Rev. Lett. 32 (1974) 1080-1081.
11. R. Rajaraman, Solitons of coupled scalar field theories in two dimensions, Phys. Rev. Lett. 42 (1979) 200-204.
12. H.M. Ruck, Quark confinement in field theories with discrete gauge symmetry Z(3), in: K.B. Wolf (ed.), Group theoretical methods in physics, Lecture Notes in Physics, 135 (Springer Verlag, Berlin, 1980).
13. A.E. Kudryavtsev, Solitonlike solutions for a Higgs scalar field, JETP Lett. 22 (1975) 82-83.
14. V.G. Makhankov, Dynamics of classical solitons (In non-integrable systems), Phys. Rep. C35 (1978) 1-128.
15. D.K. Campbell, private communication.