LBL-13310 Revised
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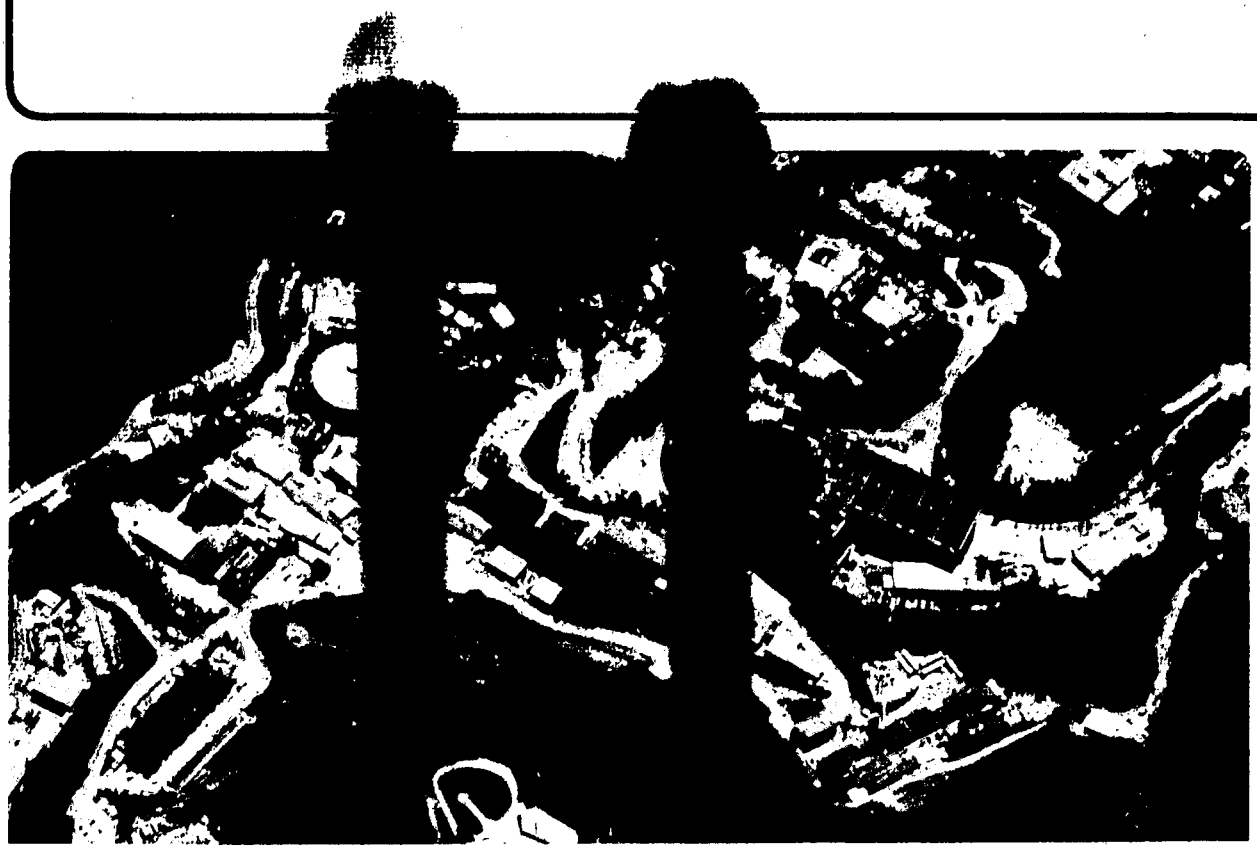
Henry P. Stapp

January 1982



Prepared for the U.S. Department of Energy under Contract W-7405-ENG-48

LBL-13310 Rev.
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September 1981

LBL-13310
Revised 1/82

# TOPOLOGICAL THEORY OF HADRONS I: MESONS*

Henry P. Stapp

Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720

ABSTRACT

Spin is incorporated into the hadronic topological expansion scheme. Spin analogs of Chan-Paton factors are introduced in a way that avoids the troubles encountered in earlier attempts. Those troubles, at the meson level, were, first, the occurrence of twice the wanted number of pseudo-scalar and vector mesons; second, the occurrence of parity-doublet partners of the pseudo-scalar and vector mesons; and third, the occurrence of these parity-doublet partners as particles of negative metric, called ghosts. These troubles are all avoided by introducing a new topological level, called zero- entropy, that lies below the ordered level. At the zero-entropy level quarks of opposite chirality are treated as distinct particles. The theory has been extended to all hadrons, and the basic particles are exactly those of the constituent quark model, which for baryons start with the $(56^+)$ and $(70^-)$. The theory is formulated in the M-function framework, where the "quarks" are represented by two-component spinors, and it entails $SU(6)_W$ symmetry of the hadronic vertices at a low level of the topological expansion.

* This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract No. W-7405-ENG-48.



## 1. INTRODUCTION

A scattering amplitude can be represented as a sum of contributions from all ways in which the process can occur. Each contribution has a phase factor, and the scattering amplitude between randomly chosen states tends to be small due to an averaging-out of these phase factors. The dominant transitions are between states in which the elements of order characterizing the initial state are carried into the final state in some "direct" way.

This tendency of the the dominant transitions to preserve order is particularly important in hadron physics, due to the inherent complexity of the hadrons and their interactions. Indeed, this order-preserving tendency has been made the basis of a successful approximation procedure for meson physics. This procedure is based not on the smallness of any coupling constant but rather on the smallness of contributions that do not preserve order. Order is defined so that it is preserved by contributions to the scattering amplitude that correspond to sequences of scattering events represented by graphs that can be drawn in a plane with no lines crossing. Contributions from non planar graphs generally have phase factors that tend to average to zero in high-energy regimes.

This topological approach to hadron dynamics, which originated in some works by Veneziano[1], and has been pursued by many workers, has been recently reviewed by Chew and Rosenzweig[2]. They show how the topological expansion procedure, combined with the requirements of unitarity, analyticity, duality, and Lorentz invariance, organizes and predicts many of the dominant features of meson physics.

The successes achieved in meson physics by this topological approach have motivated efforts to develop it into a comprehensive basis for particle physics. The most obvious deficiency of earlier work is its restriction to mesons. Accordingly, one major aim of the present two-part work is to extend the theory to baryons. Paper II[3] is devoted to that task.

But beyond this problem of baryons, there lie other problems of equal importance. To provide a satisfactory basis for particle physics, the theory must, first of all, provide a practical method of determining, through the nonlinear bootstrap conditions, the magnitudes of all coupling constants that occur in particle physics. Phenomenological analyses indicate that the ratios of the hadronic coupling constants satisfy $SU(6)_W$ symmetry to a degree unlikely to be purely accidental.[4] It is therefore probably essential to the practical viability of the topological approach, considered as a general basis for particle physics, that it be constructed so as to exhibit $SU(6)_W$ symmetry at a low level of topological complexity. Accordingly, a second major aim of this work is to construct a topological bootstrap framework that treats the spin degrees of freedom in a way that ensures $SU(6)_W$ symmetry at the lowest level of the topological



expansion.

Historically, this $SU(6)_W$ property emerged, within the present work, as an unexpected by-product of the effort to generalize the Chan-Paton factors[5] from isospin to ordinary spin. These original Chan-Paton factors enjoy the following important property: every product of amplitudes that contributes at the lowest topological level to any given amplitude has the same Chan-Paton factor. This product property ensures the existence, at the lowest topological level, of a solution to the isotopic spin part of the dual topological bootstrap dynamical conditions.

It is not obvious that this solution is unique. However, it is a simple solution that is quite possibly unique. For, a priori, the infinite number of dual bootstrap conditions need have no solution at all.

To ensure the solubility of the spin part of these dynamical conditions, the following requirement is here imposed: the spin-generalization of the Chan-Paton factors are required to satisfy the direct generalization of the product property enjoyed by the original Chan-Paton factors. This demand determines the basic character of the spin formalism described in this paper.

The problem of extending the Chan-Paton factors to ordinary spin was considered by Mandelstam in the late sixties. Mandelstam's work,[6] like the present work, was based on M functions.[7] These functions have the combined advantage of possessing simple crossing properties and a minimal number of spin components: crossing is represented by analytic continuation alone, and the redundant components that arise from describing spin-½ particles by four-component wave functions are avoided. Using these functions, one finds that the required product property (of the spin-generalizations of the Chan-Paton factor) cannot be reconciled with the demand of invariance under parity. Thus, Mandelstam, proceeding in the straight-forward way, summed two parity-reflected contributions, each of which individually satisfies the product property, in order to obtain a parity invariant form. However, this procedure of simply summing two separate terms, each of which enjoys also simple factorization properties, led first to a parity-doublet partner for each of the observed mesons $\pi, \rho, \eta, \omega$, and then to a second doubling of this set of mesons.

The procedure followed here differs from that of Mandelstam by its strict enforcement of the product-property requirement described above. The present procedure, originally introduced to ensure the



the solubility of the spin part of the dynamical conditions, generates a number of important further consequences. First, by keeping the treatment of spin closely parallel to the successful Chan-Paton treatment of iso-spin, it leads automatically to $SU(6)_W$ symmetry of the hadronic coupling constants, at the lowest topological level. Second, it automatically produces a basic set of mesons that accords exactly to the phenomenologically observed set ($\pi, \rho, \eta, \omega$): there is no parity doubling, or any other doubling, of the meson spectrum. Third, when combined with certain assumptions about the number of flavors, it leads to a super symmetry connection between the meson-meson-meson coupling constants and the meson-baryon-baryon coupling constants. This connection is in good agreement with experiment. Fourth, when supplemented by a plausible universality requirement, it leads to a value of the ratio of the strong-interaction coupling constants to the electromagnetic coupling constant. This ratio is also in satisfactory agreement with experiment.[8]

The technical basis of these achievements is the fact, recognized and exploited already in the work of Chew and Rosenzweig, that different levels of the topological expansion can act in different Hilbert spaces. In the paper of Chew and Rosenzweig,

the lowest level of the topological expansion was the "ordered" level, and each ordered amplitude was associated with a corresponding ordered Hilbert space. These ordered amplitudes were summed to form "planar" amplitudes, which were associated with new "planar" Hilbert spaces. Approximate correspondence to physical amplitudes was possible only at the planar and higher levels of the topological expansion. Yet, the ordered amplitudes were important, for the topological expansion concentrated all nonlinear aspects of the original unitarity equations in the ordered unitary equations satisfied by the ordered amplitudes. These ordered unitarity equations were much simpler than the original unitarity equations because they involved only planar Landau diagrams. Thus, the critical problem of determining the overall strength of the couplings was greatly simplified. Moreover, the represention of physical amplitudes by low-order terms in the topological expansion allowed many aspects of meson physics to be understood even without solving the nonlinear equations.[2]

The product-property requirement on the spin-generalization of the Chan-Paton factors places severe conditions on the theoretical structure. These conditions can be satisfied by introducing a new topological level, called zero entropy, that



lies below the ordered level. The individual zero-entropy amplitudes are not invariant under parity, and, like the ordered amplitudes, they cannot be regarded as approximations to the physical amplitudes. It is the planar amplitudes, which do conserve parity, that are again to be considered as the first approximation to the physical amplitudes.

In the work of Chew and Rosenzweig, the planar amplitudes were formed as finite sums of the ordered amplitudes. In the present framework, the planar amplitudes are formed as infinite sums of zero-entropy amplitudes. These infinite sums create new technical problems. But they also create the possibility, and, in fact, the necessity, of calculating, for example, the $\pi$-$\rho$ mass difference. For at zero entropy, these two masses are equal.

The present work is part of a long-term collaborative effort with Geoffrey Chew to construct a practical basis for particle physics based on dual topological bootstrap dynamics. Chew's ideas are woven into it in many ways. However, this paper deals only with certain spinor, topological, and group theoretic aspects of the whole theory; other important aspects are left untouched.

One problem not considered in this paper itself is the extension to all hadrons of the formalism developed herein for mesons. An extension is

described in paper II. It incorporates also the group-theoretic properties of the constituent-quark model. The whole work is formulated completely within the S matrix framework, and hence involves no microscopic description in terms of quark wave functions. Hence, it provides, in principle, the foundation of a Lorentz covariant approach to particle physics that has a basic set of particles that agrees, as far as spin, parity, and other group-theoretic properties are concerned, with those of the constituent-quark model, yet has no confinement problem. Moreover, it incorporates $SU(6)_W$ symmetry, at the lowest topological level.

The present paper is associated with a recent paper by Chew and Poenaru;[9] it describes technical results that have been used in the development of their ideas. However, the aims of Chew and Poenaru are broader than those of the present work, which simply accepts the group-theoretic structure of the constituent-quark model on the basis of its empirical success. Chew and Poenaru seek to derive the group-theoretic structures from topological considerations and consequently need a richer topological structure than the one used here. Their topological structure contains, in addition to the quark-particle graphs of the present theory, and surface upon which these graphs are imbedded, also a second surface, called the quantum



surface, in which the group-theoretic relations associated with flavor and other symmetries reside.

In the present work, flavor is an unconstrained variable. The flavor structure may, in fact, be determined by the nonlinear dynamical equations, but it is not determined within the present framework by topological considerations alone.

Proposals for extending the theory to electroweak interactions have been made recently by Chew, Finkelstein, McMurray and Poenaru[10,11,12]

A crucial problem not addressed in any of these works is the development of reliable methods for solving the nonlinear conditions. These conditions should determine the overall strength of the hadronic and electroweak interactions. However, several calculations have been performed,[13] and they all yield values that differ from the empirically observed overall strength of the hadronic-electroweak interaction by a factor of roughly two. This result seems significant, particularly because the spinor, topological, and group-theoretic considerations introduce as many as twenty different factors of two into this result. These theoretical factors were calculated <u>prior</u> to the calculation of the the approximate solutions to the nonlinear equations.

A second major problem not addressed in any of the published works is the development of reliable

methods for constructing the planar amplitude from zero-entropy amplitudes. This construction must yield, for example, a first approximation to the $\pi - \rho$ mass difference. Some calculations of this difference have been made, with encouraging results, but the work is still in a developmental stage.

Much of the work contained in this paper was completed several years ago,[14] but was not submitted for publication because of the above-mentioned elements of incompleteness of the whole theory. However, a number of recently-published papers[8-12] are based directly on the spin formalism developed in that earlier work. This fact, in conjunction with the encouraging character of works in progress, makes publication of this expanded version now appropriate.

The theory developed here is based on the M-function formalism. Since the original description of that formalism[7] was very brief, the key points are described here in Section 2, with particular emphasis on the results that are important in the context of the present work. The main body of the paper is contained in Section 3. The results are summarized in Section 4. Appendix A shows that discontinuity equations, though usually considered in S-matrix theory as being derived from unitarity, are actually more basic than unitarity. Appendix B



explains the failure of unitarity at the zero-entropy level. The planar discontinuity equations nevertheless continue to hold. Appendix C describes the connection of the two-component formalism used in the body of the paper to the four-component formalism based on Dirac matrices.

## 2. SPIN

### 2.1. Lorentz Transformations in Spin Space

Let $\sigma_\mu$ represent the Pauli spin-matrix four-vector

$$\sigma_\mu = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (1, \vec{\sigma}), \tag{2.1}$$

where $\sigma_0$ is the two-by-two unit matrix and $\sigma_1$, $\sigma_2$, and $\sigma_3$ are

the three Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

Let $A$ and $B$ be any two-by-two matrices with determinant one. Then the Lorentz transformation matrix $L^{\mu}{}_{\nu}(A,B)$ is defined by

$$A\sigma_{\mu} B = \sigma_{\nu} L^{\nu}{}_{\mu}(A,B) \equiv (\sigma\cdot L)_{\mu} \tag{2.3}$$

(Repeated vector and spinor indices are always to be summed.)

Let $\tilde{\sigma}_{\mu}$ represent the Pauli spin-matrix four vector

$$\tilde{\sigma}_{\mu} = (1, -\vec{\sigma}). \tag{2.4}$$

Then

$$\frac{1}{2} \mathrm{Tr}\, \sigma_{\mu}\tilde{\sigma}_{\nu} = g_{\mu\nu}, \tag{2.5}$$

where $g_{\mu\nu}$ is the Lorentz metric tensor with diagonal elements $(1, -1, -1, -1)$.

Let $C = -i\sigma_2 = -C^{Tr}$ be the (charge) conjugation matrix, and let $M$ be any two-by-two matrix. Then the Pauli identity

$$C^{-1} M^{Tr} C M = \det M. \tag{2.6}$$

entails that

$$C^{-1}\sigma_{\mu}{}^{Tr} C = \tilde{\sigma}_{\mu} \tag{2.7}$$

and that



$$B^{-1}\tilde{\sigma}_{\mu} A^{-1} = \tilde{\sigma}_{\nu} L^{\nu}{}_{\mu}(A,B) \tag{2.8}$$

$$\equiv (\tilde{\sigma}\cdot L)_{\mu} .$$

To specify four different ways of applying transforms to spin indices four different types of spinor indices are introduced. The spin transformation $\Lambda \equiv \Lambda(A,B)$ acts on the different types of spinor indices according to the rules:

$$\Lambda(\phi_{\alpha}) = A_{\alpha}{}^{\alpha'} \phi_{\alpha'} \equiv (A\phi)_{\alpha} \tag{2.9}$$

$$\Lambda(\phi_{\dot{\beta}}) = \phi_{\dot{\beta}'} B^{\dot{\beta}'}{}_{\dot{\beta}} = (\phi B)_{\dot{\beta}}$$

$$\Lambda(\phi^{\dot{\beta}}) = (B^{-1})^{\dot{\beta}}{}_{\dot{\beta}'} \phi^{\dot{\beta}'} = (B^{-1}\phi)^{\dot{\beta}}$$

$$\Lambda(\phi^{\alpha}) = \phi^{\alpha'} (A^{-1})_{\alpha'}{}^{\alpha} = (\phi A^{-1})^{\alpha} .$$

Thus the transformation to be applied is determined by the location of the index (upper or lower) and whether it is dotted or undotted.

The operator $\Lambda$ acts like the identity on any sum of the form $\phi^{\alpha}\psi_{\alpha}$ or $\phi_{\dot{\beta}}\psi^{\dot{\beta}}$. For example,

$$\Lambda(\phi^{\alpha}\psi_{\alpha}) = (\Lambda\phi^{\alpha})(\Lambda\psi_{\alpha})$$

$$= (\phi^{\alpha'} A_{\alpha'}^{-1\alpha})(A_{\alpha}{}^{\alpha''}\psi_{\alpha''})$$

$$= \phi^{\alpha}\psi_{\alpha} . \tag{2.10}$$

Let $a_1, a_2, \ldots, a_{2n}$ be any set of 2n four-vectors. Then

$$\frac{1}{2}\,\mathrm{Tr}\; a_1\cdot\sigma\; a_2\cdot\tilde{\sigma}\;\; a_3\cdot\sigma \;\ldots\; a_{2n}\cdot\tilde{\sigma} \tag{2.11}$$

is a Lorentz-invariant function of the four-vectors $a_1, \ldots a_{2n}$. To see this let the indices on $\sigma_\mu$ and $\tilde{\sigma}_\mu$ be specified always in the following way:

$$\sigma_\mu \to \sigma_{\mu\alpha\dot{\beta}} \qquad\qquad \tilde{\sigma}_\mu \to \sigma_\mu{}^{\dot{\beta}\alpha} \quad . \tag{2.12a}$$

Then (2.3) and (2.8) become

$$\Lambda\sigma_\mu \;=\; (\sigma\cdot L)_\mu \qquad\qquad \Lambda\tilde{\sigma}_\mu \;=\; (\tilde{\sigma}\cdot L)_\mu . \tag{2.12b}$$

Application of the operator $\Lambda$ leaves invariant the trace (2.11), due to (2.10). It gives, alternatively, by virtue of (2.12),

$$\frac{1}{2}\,\mathrm{Tr}\;(\sigma\cdot La_1)\;(\tilde{\sigma}_\nu\cdot La_2)\ldots\;(\tilde{\sigma}_{2n}\cdot La_{2n}). \tag{2.13}$$

Thus the trace is invariant under any Lorentz transformation of all the vectors $a_i$.

Two important special cases are

$$\frac{1}{2}\,\mathrm{Tr}\; a_1\cdot\sigma\; a_2\cdot\tilde{\sigma} \;=\; a_1\cdot a_2 \;, \tag{2.14a}$$

which follows from (2.5), and



$$\frac{1}{2}\,\mathrm{Tr}\; a_1\cdot\sigma\;\; a_2\cdot\tilde{\sigma}\;\; a_3\cdot\sigma\;\; a_4\cdot\tilde{\sigma}$$

$$= (a_1\cdot a_2)(a_3\cdot a_4) + (a_1\cdot a_4)(a_2\cdot a_3) - (a_1\cdot a_3)(a_2\cdot a_4)$$

$$+\; i\,[a_1, a_2, a_3, a_4] \;, \tag{2.14b}$$

where

$$[a_1, a_2, a_3, a_4] \;=\; a_1^\mu a_2^\nu a_3^\sigma a_4^\delta \;\; \varepsilon_{\mu\nu\sigma\delta} \quad . \tag{2.15}$$

Here $\varepsilon$ is the fully antisymmetric matrix with $\varepsilon_{0123} = 1$.

## 2.2 Covariant Spin-Projection Operators

Let $p = mv$ be the momentum-energy of a freely moving particle, as measured in some general Lorentz frame $\Sigma$. Let $s$ be a spin vector that satisfies $s\cdot p = 0$. Let $\Sigma^r(v)$ be the particle-rest-frame obtained by applying a "boost" to $\Sigma$. This boost is a Lorentz transformation that leaves unchanged any space component that is perpendicular to $\vec{v}$. The vectors $v$ and $s$ as measured in $\Sigma^r(v)$ are

$$v^r = (v^{r\mu}) \equiv (v^{ro}, \vec{v}^r) = (1,\, 0,0,0) \tag{2.16a}$$

and

$$s^r = (s^r_\mu) \equiv (s^{ro}, \vec{s}^r) = (0,\, s^r_1,\, s^r_2,\, s^r_3) \tag{2.16b}$$

The rest-frame projection operator is

$$\tilde{P}^r(s) = \frac{1}{2}(1 + \vec{s}^r \cdot \vec{\sigma})$$

$$= \frac{1}{2}(v^r + s^r)^\mu \tilde{\sigma}_\mu = \frac{1}{2}(v^r + s^r) \cdot \tilde{\sigma}. \qquad (2.17)$$

This operator projects onto the spin state in which the spin is directed along $s^r = (0, \vec{s}^r)$ as measured in $\Sigma^r(v)$, and hence along $s$ as measured in $\Sigma$.

The operator $\tilde{P}^r(s)$ refers to the rest frame $\Sigma^r(v)$. To eliminate this frame dependence one may apply the boost $\Lambda(A,B)$ that converts $\tilde{P}^r$ from its form in $\Sigma^r(v)$ to its form in the general coordinate frame $\Sigma$:

$$\tilde{P}^r(s) \to \Lambda(v)\, \tilde{P}^{(r)}(s)$$

$$= B^{-1}\tilde{P}^{(r)}(s)\, A^{-1}$$

$$= \frac{1}{2}\tilde{\sigma} \cdot (Lv^r + Ls^r)$$

$$= \frac{1}{2}\tilde{\sigma} \cdot (v + s)$$

$$= \frac{1}{2}(v + s) \cdot \tilde{\sigma}$$

$$\equiv \tilde{P}(s, v) \quad . \qquad (2.18)$$

Real Lorentz transformations are generated by matrices $A$ and $B$ that satisfy $A = B^\dagger$, where dagger denotes hermitian conjugation.



For rotations $A$ is unitary, but for boosts $A$ is hermitian. The boost $\Lambda(A,B)$ that converts the rest frame form $\tilde{P}^r$ into the general coordinate system form $\tilde{P}$ is

$$A(v) = B^\dagger(v) = A^\dagger(v) = \sqrt{v \cdot \sigma} \qquad (2.19a)$$

$$A^{-1}(v) = B^{\dagger -1}(v) = B^{-1}(v) = \sqrt{v \cdot \tilde{\sigma}} \qquad (2.19b)$$

where

$$\sqrt{v \cdot \sigma} = \exp \frac{\theta}{2}(\vec{\sigma} \cdot \vec{n}) = \cosh \frac{\theta}{2} + \vec{n} \cdot \vec{\sigma} \sinh \frac{\theta}{2} \qquad (2.20)$$

and

$$v \cdot \sigma = \exp \theta(\vec{\sigma} \cdot \vec{n}) = \cosh \theta + \vec{n} \cdot \vec{\sigma} \sinh \theta$$

$$= v^\mu \sigma_\mu$$

$$\equiv v^0 + \vec{v} \cdot \vec{\sigma}$$

$$\equiv v^0 + \vec{n} \cdot \vec{\sigma}\, |\vec{v}| \qquad (2.21)$$

Note that

$$v \cdot \sigma \quad v \cdot \tilde{\sigma} = 1 \qquad (2.22a)$$

and

$$\sqrt{v \cdot \sigma} \quad \sqrt{v \cdot \tilde{\sigma}} = 1 \quad . \qquad (2.22b)$$

Another useful form is

$$\sqrt{v \cdot \sigma} = \frac{v_0 + 1 + \vec{v} \cdot \vec{\sigma}}{(2v_0 + 2)^{\frac{1}{2}}} \quad . \tag{2.23}$$

The operator

$$\tilde{P}(s,v) = \sqrt{v \cdot \tilde{\sigma}} \; \frac{1}{2} (1 + \vec{s}^r \cdot \vec{\sigma}) \sqrt{v \cdot \tilde{\sigma}} \tag{2.24}$$
$$= \frac{1}{2} (v \cdot \tilde{\sigma} + s \cdot \tilde{\sigma})$$

is called a covariant spin operator. The vectors $v$ and $s$ occurring in $\tilde{P}(s \cdot v)$ have components $v^\mu$ and $s^\mu$ that refer to the general frame of reference $\Sigma$.

Because the boost operators $A^{-1}$ and $B^{-1}$ are hermitian, rather than unitary, the operator $\tilde{P}(s, v)$ is not a true projection operator: $\tilde{P}(s, v)^2 \neq \tilde{P}(s, v)$ for $\vec{v} \neq 0$.

The covariant spin operators are Lorentz invariant spinor functions in the sense that

$$\Lambda \tilde{P}(L^{-1}s, L^{-1}v) = \tilde{P}(s, v) \, . \tag{2.25}$$

Here $\Lambda = \Lambda(A, B)$ and $L = L(A,B)$. This result follows directly from (2.12).

## 2.3 M Functions

Consider first a scattering process involving one spin-½ particle in the initial state and one spin-½ particle in the final state, and an arbitrary number of spinless particles. Let $p \equiv (p_a, t_a; p_b, t_b; , p_c, t_c; \ldots; p_d, t_d)$, where $p_a$ is the mathematical momentum-energy of the final spin-½ particle, $p_b$



is the mathematical momentum-energy of the initial spin-½ particle, and $p_c, \ldots, p_d$ are the mathematical momentum-energy vectors of the spinless particles. The mathematical momentum-energy vectors are equal to plus or minus the physical momentum-energy vectors for final and initial particles respectively. Thus $p_a = m_a v_a$ and $p_b = - m_b v_b$, where $v_a^0 > 0$ and $v_b^0 > 0$.

The $t_j$ are the mathematical type labels. They are related to the physical-type labels $t_j^{phys}$ by the relation $t_j = t_j^{phys}/\text{sign } p_j^0$, where $t_j$ and $-t_j$ label relative antiparticles. These type variables are sometimes suppressed.

According to quantum theory the probability for a scattering specified by $(p, s_a, s_b)$ is proportional to

$$\frac{1}{2} \text{Tr } \tilde{P}^r(s_a) \, S(p) \, \tilde{P}^r(s_b) \, S^\dagger(p) \; , \tag{2.26}$$

where $S(p)$ is the $S$ matrix. This can be written equivalently as

$$\frac{1}{2} \text{Tr } \tilde{P}(s_a, v_a) \, M(p) \, \tilde{P}(s_b, v_b) \, M^\dagger(p) \; , \tag{2.27}$$

where, as in §2.2,

$$\tilde{P}(s_a, v_a) = \sqrt{v_a \cdot \tilde{\sigma}} \, \tilde{P}^r(s_a) \, \sqrt{v_a \cdot \tilde{\sigma}} \tag{2.28a}$$

$$\tilde{P}(s_b, v_b) = \sqrt{v_b \cdot \tilde{\sigma}} \, \tilde{P}^r(s_b) \, \sqrt{v_b \cdot \tilde{\sigma}} \; , \tag{2.28b}$$

and

$$M(p) = \sqrt{v_a \cdot \sigma} \; S(p) \; \sqrt{v_b \cdot \sigma} \tag{2.29a}$$

$$M^\dagger(p) = \sqrt{v_b \cdot \sigma} \; S^\dagger(p) \; \sqrt{v_a \cdot \sigma} \; . \tag{2.29b}$$

The physical probability is assumed to be Lorentz invariant. This

physical invariance ensures that if the spin indices of $M(p)$ and $M^\dagger(p)$ are assigned spin-index type according to the rules

$$M(p) \to M_{\alpha\dot\beta}(p) \tag{2.30a}$$

and

$$M^\dagger(p) \to M^\dagger_{\alpha\dot\beta}(p), \tag{2.30b}$$

then the spinor functions $M(p)$ and $M^\dagger(p)$ are Lorentz invariant: for all proper (det $L = 1$) real Lorentz transformations

$$\Lambda M(L^{-1}(p)) = M(p) \tag{2.31a}$$

and

$$\Lambda M^\dagger(L^{-1}(p)) = M(p) , \tag{2.31b}$$

with

$$L^{-1}(p) \equiv (L^{-1}p_a, L^{-1}p_b, L^{-1}p_c, \ldots , L^{-1}p_d). \tag{2.31c}$$

These invariance properties entail that if $m^\mu(p)$ and $m^{\dagger\mu}(p)$ are defined by

$$M(p) \equiv m^\mu(p)\,\sigma_\mu \equiv m(p)\cdot\sigma \tag{2.32a}$$

and

$$M^\dagger(p) \equiv m^{\dagger\mu}(p)\sigma_\mu \equiv m^\dagger(p)\cdot\sigma \tag{2.32b}$$

then the quantities $m^\mu(p)$ and $m^{\dagger\mu}(p)$ are vector functions of the set of vectors $p$:

$$m^\mu(L(p)) = L^\mu{}_\nu m^\nu(p) = (Lm(p)) \tag{2.33a}$$

$$m^{\dagger\mu}(L(p)) = L^\mu{}_\nu m^{\dagger\nu}(p) = (Lm^\dagger(p))^\mu . \tag{2.33b}$$



Consequently, by virtue of (2.12), the spinor functions

$$\tilde M(p) \equiv m^\mu(p)\tilde\sigma_\mu = m(p)\cdot\tilde\sigma \tag{2.34a}$$

and

$$\tilde M^\dagger(p) \equiv m^{\dagger\mu}(p)\tilde\sigma_\mu = m^\dagger(p)\cdot\tilde\sigma \tag{2.34b}$$

are also Lorentz invariant spinor functions:

$$\Lambda\tilde M(L^{-1}(p)) = \tilde M(p) \tag{2.35a}$$

and

$$\Lambda\tilde M^\dagger(L^{-1}(p)) = \tilde M(p) . \tag{2.35b}$$

These simple transformation properties do not hold for the S-matrix $S(p)$.

The foregoing discussion can be immediately extended to processes in which there are $n$ initial spin $\frac{1}{2}$ particles, $n$ final spin $\frac{1}{2}$ particles, and $n'$ spinless particles. In this case the $M$ function can be written in the form (with type labels suppressed)

$$M(p_{a1},\alpha_1;\, p_{b1},\dot\beta_1;\, p_{a2},\, \alpha_2;\, p_{b2},\, \dot\beta_2;\, \ldots;\, p_{an},\, \alpha_n;\, p_{bn},\, \dot\beta_n;\, p_1,\ldots,p_{n'})$$

$$= m^{\mu_1\mu_2\cdots\mu_n}(p) \times \prod_{i=1}^{n} \sigma_{\mu_i\alpha_i\dot\beta_i} \tag{2.36}$$

where $m^{\mu_1\cdots\mu_n}(p)$ is a tensor function of the vectors

$$p = (p_{a1}, p_{b1}, \cdots, p_{an}, p_{bn}, p_1, \cdots, p_{n'}):$$

$$m^{\mu_1 \cdots \mu_n}(L(p)) = \left( \prod_{i=1}^{n} L^{\mu_i}{}_{\nu_i} \right) m^{\nu_1 \cdots \nu_n}(p). \qquad (2.37)$$

The way in which the $n$ initial spin-½ particles are associated with the $n$ final spin-½ particles is immaterial: (2.37) holds in any case.

## 2.4 Parity

Let $S(p)$ be written as $S(p) = S_+(p) + S_-(p)$, (2.38)

where

$$S_\pm(p) = \pm S_\pm(\tilde{p}) . \qquad (2.39)$$

Here

$$\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_N) ,$$

$$(\tilde{p}_i^\mu) = (p_L^o, -\vec{p}_i), \qquad (2.40a)$$

and

$$(p_i^\mu) = (p_i^o, \vec{p}_i) . \qquad (2.40b)$$

Let an intrinsic parity $\varepsilon_j$ be assigned to each particle $j$, and define the parity operator $\mathscr{P}$ by

$$\mathscr{P}(S(p)) = \prod_{j=1}^{N} \varepsilon_j \, S(\tilde{p}) . \qquad (2.41)$$



The product of $\varepsilon_i$'s for allowed processes must be + 1 or -1. Invariance under parity is then expressed by the equation

$$\mathscr{P}(S(p)) = S(p). \qquad (2.42)$$

If this equation is satisfied then $S_\varepsilon$ defined in (2.38) must be zero unless $\varepsilon = \prod_{j=1}^{N} \varepsilon_j$ .

Consider a process in which $n$ initial spin-½ particles, $i = 1, \ldots, n$, are scattered into $n$ final spin-½ particles. Let $p_{ai}$ and $p_{bi}$ denote the final and initial mathematical momentum-energies of the $i^{th}$ particle. Let $(p_1, \ldots, p_{n'})$ denote the momenta of $n'$ spinless particles that also participate in the reaction. Then as already mentioned, the $M$ matrix can be written

$$M(p_{a1}, t_{a1}; p_{b1}, t_{b1}; \ldots; p_{an}, t_{an}; p_{bn}, t_{bn}; p_1, t_1; \ldots; p_{n'}, t_{n'})$$

$$= m^{\mu_1 \ldots \mu_n}(p, t) \prod_{i=1}^{n} \sigma_{\mu_i}^{(i)} , \qquad (2.43)$$

where the matrix elements of $\sigma_{\mu_i}^{(i)}$ are $\sigma_{\mu_i \alpha_i \dot{\beta}_i}$ . The connection of $M(p)$ to $S(p)$ can be represented by the equation

$$M(p) = \left(\prod_i \sqrt{v_{ai} \cdot \sigma^{(i)}}\right) S(p) \left(\prod_i \sqrt{v_{bi} \cdot \sigma^{(i)}}\right)$$

$$\equiv \sqrt{V_a \cdot \sigma} \; S(p) \; \sqrt{V_b \cdot \sigma} . \qquad (2.44)$$

Define now

$$M_\pm(p) = \left(\prod_i \sqrt{v_{ai} \cdot \sigma^{(i)}}\right) S_\pm(p) \left(\prod_i \sqrt{v_{bi} \cdot \sigma^{(i)}}\right). \qquad (2.45)$$

Then

$$M_{\pm}(\tilde{p}) = \left(\prod_i \sqrt{\tilde{v}_{ai}\cdot\sigma^{(i)}}\right) S_{\pm}(\tilde{p}) \left(\prod_i \sqrt{\tilde{v}_{bi}\cdot\sigma^{(i)}}\right)$$

$$= \pm \left(\prod_i \sqrt{\tilde{v}_{ai}\cdot\sigma^{(i)}}\right) S_{\pm}(p) \left(\prod_i \sqrt{\tilde{v}_{bi}\cdot\sigma^{(i)}}\right)$$

$$= \pm \left(\prod_i v_{ai}\cdot\tilde{\sigma}^{(i)}\right) M_{\pm}(p) \left(\prod_i v_{bi}\cdot\tilde{\sigma}^{(i)}\right). \qquad (2.46)$$

This equation can be inverted to give

$$M_{\pm}(p) = \pm\left(\prod_i v_{ai}\cdot\sigma^{(i)}\right) M_{\pm}(\tilde{p}) \left(\prod_i v_{bi}\cdot\sigma^{(i)}\right). \qquad (2.47)$$

The parity transformation applied to the M functions is defined to be

$$\mathscr{P}(M(p)) \equiv \left(\prod_{j=1}^{N} \varepsilon_j\right) \left(\prod_i v_{ai}\cdot\sigma^{(i)}\right) M(\tilde{p}) \left(\prod_i v_{bi}\cdot\sigma^{(i)}\right). \qquad (2.48)$$

Then (2.48) and (2.47) ensure that the condition

$$\mathscr{P}(M(p)) = M(p) \qquad (2.49)$$

is equivalent to the condition that $M_{\varepsilon}$ be zero unless $\varepsilon = \prod_{j=1}^{N} \varepsilon_j$, which is equivalent to the parity invariance condition $\mathscr{P}(S(p)) = S(p)$.

For $n$ distinguishable spin-½ particles the no-scattering part of the $S$ matrix has the form

$$S_0(p) = \prod_{i=1}^{n} \left[(\sigma_0^{(i)})(2\pi)^3\delta^3(p_{ai} + p_{bi})\, 2\omega_i\right]. \qquad (2.50)$$



The corresponding $M$ function is

$$M_0(p) = \prod_{i=1}^{n} [v_i\cdot\sigma^{(i)}(2\pi)^3\delta^3(p_{ai} + p_{bi})2\omega_i]. \qquad (2.51)$$

In order that this no-scattering part be invariant under parity (for each particle $i$ separately) we must take $\varepsilon_{ai}\,\varepsilon_{bi} = 1$ for all $i$. But then (2.48) gives

$$\mathscr{P}(p_a\cdot\sigma) = (-p_b\cdot\sigma). \qquad (2.52)$$

This relationship, which stems from the condition that the no-scattering part be nonzero, is used later.

## 2.5 Crossing

Analysis of the pole singularity[15] shows that the analytic continuation of $M(p)$ along an appropriate path from an original region where $p_{ai}^0 > 0$ to a region where $p_{ai}^0 < 0$ gives the function that describes a process in which the final particle of type $t_{ai}$ is replaced by an initial particle of type $-t_{ai}$, i. e., by the antiparticle of the original particle of type $t_{ai}$. If the final particle $t_{ai}$ carries $q$ units of any conserved quantity out of the reaction then the antiparticle $-t_{ai}$ must carry $-q$ units into the reaction. This holds both for the total momentum-energy $p_{ai}$, for the components of spin, and for any quantity that is conserved by virtue of invariance under a $p$-independent transformation property. Consequently, the mathematical momentum-energy vector $p_{ai}$, the mathematical spin vector $s_{ai}$, and the mathematical type label $t_{ai}$ are equal, after the continuation, to minus their physical values:

$$p_{ai} = p_{ai}^{phys} / \text{sign } p_{ai}^{o} \tag{2.53a}$$

$$s_{ai} = s_{ai}^{phys} / \text{sign } p_{ai}^{o} \tag{2.53b}$$

$$t_{ai} = t_{ai}^{phys} / \text{sign } p_{ai}^{o} \quad . \tag{2.53c}$$

A similar argument gives

$$p_{bi} = p_{bi}^{phys} / \text{sign } p_{bi}^{o} \tag{2.53d}$$

$$s_{bi} = - s_{bi}^{phys} / \text{sign } p_{bi}^{o} \tag{2.53e}$$

$$t_{bi} = t_{bi}^{phys} / \text{sign } p_{bi}^{o} \, . \tag{2.53f}$$

The minus sign in (2.53e) arises from the fact that $s_{bi}$ characterizes the physical spin of the initial particle bi, not minus the physical spin. The $p_{bi}$ and $t_{bi}$ were defined originally to be minus the physical momentum-energy vector and minus the physical particle-type of the incoming particle (bi), and hence the equations for these are the same as those for $p_{ai}$ and $t_{ai}$.

The quantitites occurring in the transition probability formula

$$\text{Tr } \frac{1}{2} (v_a + s_a) \cdot \tilde{\sigma} \, M(p_a, t_a; p_b, t_b)$$

$$\times \frac{1}{2} (v_b + s_b) \cdot \tilde{\sigma} \quad M^{\dagger}(p_a, t_a; p_b, t_b) \tag{2.54}$$

are to be interpreted with the aid of (2.53). Thus, for example, if



$p_a^o$ and $p_b^o$ are both positive then the $s_a$ and $s_b$ in (2.54) are $s_a^{phys}$ and $-s_b^{phys}$, respectively, and the particle types $t_a$ and $t_b$ and $t_a^{phys}$ are $t_b^{phys}$. In this way we can use the same expression (2.54) in all the different channels.

The parity transformation $\mathscr{P}$ was defined to be

$$\mathscr{P}(M(p)) = \left( \prod_{j=1}^{N} \varepsilon_j \right) \left( \prod_{i=1}^{n} v_{ai} \cdot \sigma \right) M(\tilde{p}) \left( \prod_{L=1}^{n} v_{bi} \cdot \sigma \right) \quad . \tag{2.55}$$

In the original (direct) channel $(p_{ai}^o > 0, \; p_{bi}^o < 0)$ the parity invariance equation $\mathscr{P}(M(p)) = M(p)$ can be written as

$$M(p) = \left( \prod_{j=1}^{N} \varepsilon_j \right) \left( \prod_{i=1}^{n} \frac{p_{ai} \cdot \sigma}{m_{ai}} \right) M(\tilde{p}) \left( \prod_{i=1}^{n} \frac{-p_{bi} \cdot \sigma}{m_{bi}} \right)$$

$$= \left( \prod_{j=1}^{n'} \varepsilon_j \right) \left( \prod_{i=1}^{n} \frac{p_{ai} \cdot \sigma}{m_{ai}} \right) M(\tilde{p}) \left( \prod_{i=1}^{n} \frac{-p_{bi} \cdot \sigma}{m_{bi}} \right) , \tag{2.56}$$

where use has been made of the direct-channel result $\varepsilon_{ai} \varepsilon_{bi} = 1$, derived from forward scattering. (See (2.51)).

Analytic continuation to the crossed channel avoids all singularities of $M(p)$ and $M(\tilde{p})$. Thus equation (2.56) must hold in all channels, with the factor $\prod_{j=1}^{n'} \varepsilon_j$ from the spinless particles defined as in the original direct channel. This equation gives

$$M(p) = \left( \prod_{j=1}^{n'} \varepsilon_j \right) \left( \prod_{i=1}^{n} \text{sign } p_{ai}^o \right) \left( \prod_{i=1}^{n} \text{sign } p_{bi}^o \right)$$

$$\times \left( \prod_{i=1}^{n} v_{ai} \cdot \sigma \right) M(\tilde{p}) \left( \prod_{i=1}^{n} v_{bi} \cdot \sigma \right) \quad . \tag{2.57}$$

It will be shown presently that the parity transformation is defined in all channels by (2.55). Thus one can conclude that the $\varepsilon_j$ for the spinless particles is channel independent and that

$$\varepsilon_{ai}\,\varepsilon_{bi} = -\ \text{sign}\ p^o_{ai}\ \text{sign}\ p^o_{bi}\ . \qquad (2.58)$$

This means, in particular, that the intrinsic parity of each spin-½ particle must reverse under continuation to a crossed channel and that the intrinsic parity of a particle-antiparticle pair is $-(-1)^{\ell}$.

The product of the intrinsic parities of the particles of a parity conserving process is physically well defined: it is equal to the sign $\varepsilon$ in $S(\tilde{p}) = \varepsilon S(p)$, and hence to $(-1)^{\Sigma \ell_j}$. The argument leading to the equivalence of $\mathscr{P}(S(p) = S(p))$ to $\mathscr{P}(M(p)) = M(p)$, with $\mathscr{P}$ as defined in (2.55), was made explicitly in the direct channel. However, it holds equally well in all channels, provided the same factor $\prod_{j=1}^{N} \varepsilon_j$ occurs in both $\mathscr{P}(S(p))$ and $\mathscr{P}(M(p))$. Any extra sign or phase factor $e^{i\phi}$, that one might introduce into the connection between $S(p)$ and $M(p)$, in any given physical region, would be the same throughout that physical region and would drop out of (2.47), and hence not affect the argument that demonstrates the equivalence between $\mathscr{P}(S(p)) = S(p)$ and $\mathscr{P}(M(p)) = M(p)$, with $\mathscr{P}(M(p))$ defined as in (2.48) or (2.55). Thus this definition is applicable in all channels, and the result (2.58) on the intrinsic parities of spin ½ particles holds.[16]

## 2.6 Antiparticle Conjugation

Consider a process in which $p^o_{ai}$ and $p^o_{bi}$ are both positive, so that the two associated particles are both final particles.



Suppose that $t_{ai} = -t_{bi}$, so that these two final particles are relative antiparticles.

Consider now an original value of $(p_{ai}, p_{bi})$ and an analytic continuation that stays in the physical region of the process, but interchanges $p_{ai}$ and $p_{bi}$ leaving all other p's unchanged. Suppose we interchange also $s_a^{phys}$ and $s_b^{phys}$. Then the original process and the second one are physically the same except for the interchange $t_{ai} \leftrightarrow t_{bi}$, which is just $t_{ai} \leftrightarrow -t_{ai}$.

Suppose that the transition probabilities for these two processes were the same. Then the process would be invariant under the transformation $t_{ai} \leftrightarrow -t_{ai}$. Antiparticle conjugation invariance is invariance under the analogous change $t_{ai} \leftrightarrow -t_{ai}$ for all i.

If we keep only one particle-antiparticle pair, for notational simplicity, the antiparticle conjugation invariance condition described above is

$$\begin{aligned} &\mathrm{Tr}(v_a + s_a^{phys})\cdot\tilde{\sigma}\ M(p_a, p_b)\ (v_b - s_b^{phys})\cdot\tilde{\sigma}\ M^{\dagger}(p_a, p_b) \\ &= \mathrm{Tr}\ (v'_a + s'^{phys}_a)\cdot\tilde{\sigma}\ M(p'_a, p'_b)(v'_b - s'^{phys}_b)\cdot\tilde{\sigma}\ M^{\dagger}(p'_a, p'_b), \end{aligned} \qquad (2.59)$$

where (2.53) and (2.54) are used, and

$$\begin{aligned} p'_a &= p_b \qquad & p'^0_a > 0 \\ p'_b &= p_a \qquad & p'^0_b > 0 \\ s'^{phys}_a &= s^{phys}_b \\ s'^{phys}_b &= s^{phys}_a\ . \end{aligned} \qquad (2.60)$$

To see the consequences of this condition define

$$\mathscr{C} M(p_a, t; p_b, -t) = u_a \cdot \sigma \, \tilde{M}(p_b, t; p_a -t) \, u_b \cdot \sigma \tag{2.61}$$

where $u_a = p_a/m_a$ and $u_b = p_b/m_b$. Define also

$$M_{(\pm)} = \frac{1}{2} (1 \pm \mathscr{C}) M. \tag{2.62}$$

Then $M = M_{(+)} + M_{(-1)}$, and the property $(\mathscr{C})^2 = 1$ gives

$$\mathscr{C} M = M_{(+)} - M_{(-)} . \tag{2.63}$$

Hence if $M = M_{(+)}$ or $M_{(-)}$ then $M = \pm \mathscr{C} M$.

Insertion of this condition $M = \pm \mathscr{C} M$ into the LH side of (2.59) gives

$$\begin{aligned} &\mathrm{Tr}\, (v_a + s_a^{phys}) \cdot \tilde{\sigma} \, u_a \cdot \sigma \quad \tilde{M}(p_b, p_a) \, u_b \cdot \sigma \\ &\times (v_b - s_b^{phys}) \cdot \tilde{\sigma} \, u_b \cdot \sigma \quad \tilde{M}^\dagger (p_b, p_a) \, u_a \cdot \sigma \\ &= \mathrm{Tr}\, (v_a - s_a^{phys}) \cdot \sigma \quad \tilde{M} \quad (p_b, p_a) \\ &\times (v_b + s_b^{phys}) \cdot \sigma \quad \tilde{M}^\dagger (p_b, p_a) \\ &= \mathrm{Tr}\, (v_b + s_b^{phys}) \cdot \tilde{\sigma} \quad M(p_b, p_a) (v_a - s_a^{phys}) \cdot \tilde{\sigma} \\ &\times M^\dagger (p_b, p_a) , \end{aligned} \tag{2.64}$$



where in the second line the relations

$$u_a \cdot \sigma \quad u_a \cdot \tilde{\sigma} = 1 \tag{2.65a}$$

and

$$s_a \cdot \tilde{\sigma} \quad u_a \cdot \sigma = - u_a \cdot \tilde{\sigma} \quad s_a \cdot \sigma \text{ for } s_a \cdot u_a = 0 \tag{2.65b}$$

are used, and in the last line the equations

$$\sigma = C^{-1} \, \tilde{\sigma}^{Tr} C \tag{2.66a}$$

and

$$\tilde{\sigma} = C^{-1} \, \sigma^{Tr} C \tag{2.66b}$$

are used. Comparison of (2.64) to the RH side of (2.59), with the substitutions (2.60) made, shows that the condition $M = \pm \mathscr{C} M$ implies antiparticle conjugation invariance.

Notice that

$$\mathscr{C} p_a \cdot \sigma = p_a \cdot \sigma \tag{2.67a}$$

and

$$\mathscr{C} p_b \cdot \sigma = p_b \cdot \sigma . \tag{2.67b}$$

Thus both $p_a \cdot \sigma$ and $p_b \cdot \sigma$, and any superposition of them, are invariant under $\mathscr{C}$.

## 2.7 CPT Invariance

The physical transformation corresponding to CPT is

$$p_j^{phys} \to p_j^{phys} , \; s_j^{phys} \to - s_j^{phys} , \; t_j^{phys} \to - t_j^{phys} , \; \text{In} \leftrightarrow \text{Out}. \tag{2.68a}$$

The corresponding mathematical transformation is, by virtue of (2.53),

$$p_j \to -p_j, \quad s_j \to s_j, \quad t_j \to t_j. \tag{2.68b}$$

Thus CPT invariance is equivalent to invariance of transition probabilities under the transformation $p_j \to -p_j$ (all j).

Any Lorentz invariant spinor function $M(p)$ is invariant, up to a sign, under the transformation $p_j \to -p_j$ (all j). For the Lorentz invariance condition

$$\Lambda M(L^{-1}(p)) = M(p) \tag{2.69}$$

applied for the case $A = 1$, $B = -1$ gives, by virtue of (2.3) and (2.9),

$$M(-p) = (-1)^{N_d} M(p), \tag{2.70}$$

where $N_d$ is the number of dotted spinor indices (I mean here dotted two-valued spinor indices: Dotted (undotted) spinor indices for spin $\frac{n}{2}$ particles can be constructed trivially by combining $n + 2m$ dotted (undotted) two-valued spinor indices by means of the usual Clebsch-Gordan coeffieients. Thus a dotted spin $\frac{n}{2}$ spinor index contributes a term $n$ to $N_d$)

The matrix $B = -1$ can be continuously connected to $B = 1$ by the matrix

$$B(\psi) = \begin{pmatrix} e^{i\pi\psi} & 0 \\ 0 & e^{-i\pi\psi} \end{pmatrix} \tag{2.71}$$

which satisfies $B(0) = 1$ and $B(1) = -1$.



Since all Lorentz invariants are invariant under all real and complex Lorentz transformations the transformation $L(A,B) = L(1, B(\psi))$, with $0 \leqslant \psi \leqslant 1$, must generate complex values of the $p_j$, since no real mass-shell vector $p_j(\psi)$ can interpolate $p_j \to -p_j$.

The matrices $M(p)$ and $M^\dagger(p)$ have been assigned the transformation properties indicated by the indices $M_{\alpha\dot\beta}(p)$ and $M^\dagger_{\alpha\dot\beta}(p)$. For real $p$ the matrices $M(p)$ and $M^\dagger(p)$ are related by hermitian conjugation:

$$M^\dagger_{\alpha\dot\beta}(p) = (M_{\beta\dot\alpha}(p))^*. \tag{2.72a}$$

Thus if $M$ is transformed by a real Lorentz transformation to $AMB$ then $M^*$ is transformed to $A^*M^*B^*$ and $M^\dagger$ is transformed to $B^\dagger M^\dagger A^\dagger = AM^\dagger B$, as indicated by the indices on $M^\dagger_{\alpha\dot\beta}$.

For complex Lorentz transformations the condition $A = B^\dagger$ does not hold. However, (2.72a) is then inappropriate: the appropriate definition is

$$M^\dagger_{\alpha\dot\beta}(p) = M^*_{\dot\beta\alpha}(p) \equiv (M_{\beta\dot\alpha}(p^*))^*. \tag{2.72b}$$

This quantity is an analytic function of $p$, whereas the function on the RH side of (2.72a) is an analytic function of $p^*$. The function $M^\dagger$ defined in (2.72b) will continue to satisfy the Lorentz invariance condition

$$\left(\prod_{i=1}^{n} A^{(i)}\right) M^\dagger(L^{-1}(A,B)(p)) \left(\prod_{i=1}^{n} B^{(i)}\right) = M^\dagger(p) \tag{2.73}$$

for complex Lorentz transformations.[13] Thus in the formula for probabilities the factor $(-1)^{N_d}$ from (2.70) will be cancelled by the same factor $(-1)^{N_d}$ from

$$M^{\dagger}(-p) = (-1)^{N_d} M^{\dagger}(p) \quad . \tag{2.74}$$

Hence probabilities will be invariant under CPT.

## 2.8 Statistics

The order of writing the variables is important. If the variables in the set of arguments $p = (p_1, t_1; p_2, t_2; \ldots ; p_n, t_n)$ is such that all variables referring to initial particles stand to the right of all variables referring to final particles then one may write $p = (p_{fin}; p_{in})$. By convention

$$S(p_{fin}; p_{in}) \equiv \langle p_{fin} \mid S \mid \bar{p}_{in} \rangle, \tag{2.75}$$

where $\bar{p}_{in}$ is obtained from $p_{in}$ by reversing the signs of all energy vectors $p_j$ and all type variables $t_j$, and reversing the order of the variables. Thus if

$$p_{in} = (p_m, t_m; \ldots ; p_n, t_n) \tag{2.76a}$$

then

$$\bar{p}_{in} = (-p_n, -t_n; \ldots ; -p_m, -t_m). \tag{2.76b}$$

The diagram representing $\langle p_{fin} \mid S \mid \bar{p}_{in} \rangle$ is generally drawn by ordering the lines from top to bottom in the sequence in



which the corresponding arguments of $p_{fin}$ and $\bar{p}_{in}$ appear. The lines corresponding to $\bar{p}_{in}$ are on the right-hand side; those corresponding to $p_{fin}$ are on the left-hand side. The variables in (2.50) are in the order $(p_{1a}, p_{2a}, \ldots p_{2b}, p_{1b})$, so that each particle line goes straight through, without a change in order.

The functions $S(p)$ and $M(p)$ are assumed to be anti symmetric under the interchange of any two spin-½ particle variables $(p_i, t_i)$ and $(p_j, t_j)$. Analytic continuation $p_{ai} \to p_{bi}$ in (2.51) changes the sign of (2.51). This sign change is cancelled by the change of the order of variables required to bring the variables back into the form $(p_{fin}; p_{in})$. Thus (2.50) and (2.51) hold in all channels, for $p = (p_{fin}; p_{in})$, with the corresponding variables of $p_{fin}$ and $\bar{p}_{in}$ occurring in the same order.

With these conventions the relationship (2.45) between $M(p)$ and $S(p)$ holds in all channels.

Combinatoric factors $1/n!$ are discussed in Appendix A of Ref. 17.

# 3. MESONS

## 3.1 The Zero-Entropy Amplitudes

The basic building blocks of the topological expansion are the zero-entropy amplitudes. In the meson sector each zero-entropy amplitude is represented by a simple quark diagram D of the kind shown in Fig. 1, or by the equivalent quark graph G also shown there.

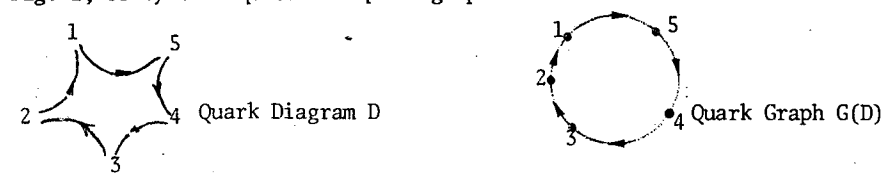


Figure 1 A zero-entropy quark diagram D and the equivalent quark graph $G = G(D)$.

The quark diagram D is converted to the equivalent quark graph $G = G(D)$ by simply connecting to a vertex the ends of the two quark lines at each opening of D. Thus each vertex of a meson quark graph G corresponds, at some level of approximation, to an initial or final particle of a scattering process. The zero-entropy amplitude corresponding to a process with n particles is represented, therefore, by a directed circular graph with n vertices. The n directed edges that connect these vertices all run in the same direction, as illustrated in Fig. 1.

The quark graphs are not abstract graphs, but are graphs placed on an oriented surface. The orientation of the boundary of the oriented circular disc bounded by the quark line is indicated by a second arrow, as shown in Figs. 2 and 3. The two graphs of Fig. 2 are equivalent to each other, and the two graphs of Fig. 3 are equivalent to each other. But those of Fig. 2 are not equivalent to those of Fig. 3.

Fig. 2. Two equivalent "ortho" graphs $G^o$

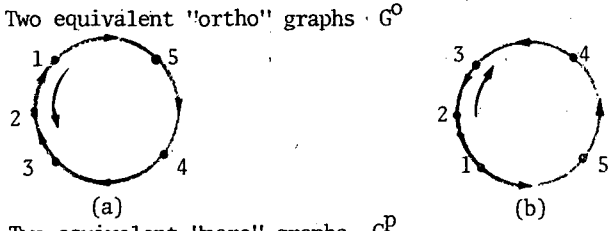


Fig. 3. Two equivalent "para" graphs $G^p$



The circular graphs in which the directions of all the quark lines agree with the direction of the boundary of the enclosed oriented disc , as in Fig. 2, are called "ortho" graphs. The circular graphs in which the directions of all the quark lines are opposite to the direction of the boundary of the oriented disc, as in Fig. 3, are called "para" graphs.

For each ortho or para graph G there is a corresponding amplitude. If G has n vertices then this amplitude has a set $(\mu) = (\mu_1, \ldots, \mu_n)$ of n vector indices. The amplitude corresponding to G has the form

$$A_{(\mu)}(G, p) = F_{(\mu)}(G, p) f(G, p) , \tag{3.1}$$

where $f(G, p)$ is a function of the scalar products of the mathematical momentum-energy vectors $p_j$ appearing in the set of arguments $p = (p_1, t_1; \ldots; p_n, t_n)$. For any ortho graph $G = G^o$ the function $F_{(\mu)}(G^o, p)$ is given explicitly by

$$F_{(\mu)}(G^o, p) = - \prod_{i=1}^{n} (2m_i^2)^{-\frac{1}{2}}$$

$$\times \ \mathrm{Tr}\, \tilde{\sigma}_{\mu_1} p_1 \cdot \sigma \ \tilde{\sigma}_{\mu_2} p_2 \cdot \sigma \cdots \ \tilde{\sigma}_{\mu_n} p_n \cdot \sigma \tag{3.2}$$

This factor $F_{(\mu)}(G^o, p)$ is minus the trace of a matrix formed from right to left by following the sense of the quark arrows in $G^o$ and replacing each vertex i by $\tilde{\sigma}_{\mu_i} / \sqrt{2}$ and each edge by the ortho quark "propagator" $p_{aj} \cdot \sigma / m_{aj} \equiv u_{aj} \cdot \sigma$ , where $p_{aj}$ is the mathematical momentum-energy vector associated with

the vertex that lies on the leading end of that quark edge.

If $G^p$ is the para graph obtained from $G^o$ by reversing the orientation of the disc then

$$A_{(\mu)}(G^p, p) = \mathscr{P}A_{(\mu)}(G^o, p)). \tag{3.3a}$$

Thus the function $A(G, p)$ is invariant under the parity operation, in the sense that if $\mathscr{P}G^o = G^p$ and $\mathscr{P}G^p = G^o$ then

$$\mathscr{P}A_{(\mu)}(\mathscr{P}G, p) = A_{(\mu)}(G, p). \tag{3.b}$$

The action of $\mathscr{P}$ on any $A$ is given by (2.48). Thus

$$\mathscr{P}f(G, p) = f(G, \tilde{p}), \tag{3.4a}$$

and, by virtue of (2.52),

$$F_{(\mu)}(G^p, p) = - \prod_{i=1}^{n} (2m_i^2)^{-\frac{1}{2}}$$

$$\times \operatorname{Tr}(-p_1 \cdot \sigma)\, \tilde{\sigma}_{\mu_1}(-p_2 \cdot \sigma)\, \tilde{\sigma}_{\mu_2} \cdots (-p_n \cdot \sigma)\, \tilde{\sigma}_{\mu_n} \tag{3.4}$$

This spinor part of the para amplitude is minus the trace of a matrix formed from right to left by following the sense of the quark arrows in $G^p$ and replacing each vertex $i$ of $G^p$ by $\tilde{\sigma}_{\mu_i} / \sqrt{2}$ and each edge by the para quark "propagator" $- p_{bj} \cdot \sigma / m_{bj} \equiv - u_{bj} \cdot \sigma$, where $p_{bj}$ is the mathematical momentum-energy



vector associated with the vertex that lies on the trailing end of the quark edge.

Notice that in both the ortho and para cases the orientation of the disc points from each edge to the vertex whose momentum appears in the propagator corresponding to that edge.

Each vertex $i$ is associated with a spin four-vector $s_i$. For a vector particle $s_i \cdot p_i = 0$ and $s_i^2 = -1$. For a pseudo scalar particle $s_i = u_i$, and $s_i^2 = 1$. The vector $(s_i)$ is the "wave function" of particle $i$ in spin space. The ortho and para amplitudes themselves are therefore

$$A(G^o, p, s) = - \left(\frac{i}{\sqrt{2}}\right)^n \operatorname{Tr}(s_1 \cdot \tilde{\sigma}\; u_1 \cdot \sigma \;\ldots\; s_n \cdot \tilde{\sigma}\; u_n \cdot \sigma)$$

$$\times f(G^o, p) \tag{3.5a}$$

and

$$A(G^p, p, s) = - \left(\frac{-i}{\sqrt{2}}\right)^n \operatorname{Tr}(u_1 \cdot \sigma\; s_1 \cdot \tilde{\sigma} \;\ldots\; u_n \cdot \sigma\; s_n \cdot \tilde{\sigma})$$

$$\times f(G^p, p)\ . \tag{3.5b}$$

## 3.2 Parity

Let $G^o$ and $G^p$ be ortho and para graphs related by disc reversal. Since $A(G^o)$ and $A(G^p)$ are related by $A(G^p) = \mathscr{P}A(G^o)$, the sum $A(G^p) + A(G^o)$ is invariant under parity. To see this explicitly use

$$u_i \cdot \sigma\; s_i \cdot \tilde{\sigma} = \pm s_i \cdot \sigma\; u_i \cdot \tilde{\sigma}$$

$$\begin{cases} + & \text{for spin } 0 \\ - & \text{for spin } 1 \end{cases} \tag{3.6}$$

to obtain

$$A(G^p, p, s) = -\left(\frac{-i}{\sqrt{2}}\right)^n (-1)^{\text{no. of spin 1's}}$$

$$\times \operatorname{Tr} s_1\cdot\sigma\, u_1\cdot\tilde{\sigma} \ldots s_n\cdot\sigma\, u_n\cdot\tilde{\sigma}$$

$$\times f^p(p)$$

$$= -\left(\frac{i}{\sqrt{2}}\right)^n (-1)^{\text{no. of spin 0's}}$$

$$\times \operatorname{Tr} s_1\cdot\sigma\, u_1\cdot\tilde{\sigma} \ldots s_n\cdot\sigma\, u_n\cdot\tilde{\sigma}$$

$$\times f^p(p)\,, \tag{3.7}$$

where $f^p(p) \equiv f(G^p,p)$ and $f^o(p) = f(G^o,p)$.

Any trace $\operatorname{Tr} a_1\cdot\tilde{\sigma}\, a_2\cdot\sigma\, a_3\cdot\tilde{\sigma} \ldots a_{2n}\cdot\sigma$ is a sum of a scalar part that is unchanged by $a_i \to \tilde{a}_i$ and a pseudoscalar part that changes sign. Since $f^p(p) = f^o(\tilde{p}) = f^o(p)$ the equations (3.5a) and (3.7) imply (with $A^o \equiv A(G^o)$, $A^p \equiv A(G^p)$) that

$$A^o + A^p = 2\text{x scalar part of } A^o \tag{3.8a}$$

if no. of spin zero's is even

$$A^o + A^p = 2\text{x pseudoscalar part of } A^o$$

if no. of spin zero's is odd. (3.8b)

This means that $A^o + A^p$ conserves parity, provided the spin-zero particles are identified as pseudoscalar particles and the spin-one particles are identified as vector particles.



## 3.3 Antiparticle Conjugation

The ortho and para propagators are $(p_{aj}\cdot\sigma)/m_{aj}$ and $(-p_{bj}\cdot\sigma)/m_{bj}$, respectively. According to (2.67) these forms are invariant under the antiparticle conjugation operation $\mathscr{C}$. This result suggests that the ortho and para amplitudes should be separately invariant under antiparticle conjugation. This invariance would, in fact, be strictly implied if the quarks could be considered separate entities, each with its own initial and final momenta $p_{bj}$ and $p_{aj}$. It was the analytic continuation $p_{aj} \leftrightarrow p_{bj}$ of these momenta into each other that was the basis of the discussion of antiparticle conjugation in §2.6. In that context antiparticle conjugation was equivalent (up to a sign) to reversing the directions of all the quark arrows. This reversal was accomplished by an equivalent analytic continuation. In that continuation the vector p in the propagator $p\cdot\sigma/m$ continues to be the momentum associated with fixed end of the quark line. Thus an ortho propagator is transformed into an ortho propagator, and para goes into para.

We therefore define antiparticle conjugation to be the operation of reversing the direction of each quark edge, with the ortho-para type left unchanged. Thus antiparticle conjugation interchanges the two graphs (a) and (b) of Fig. 4.

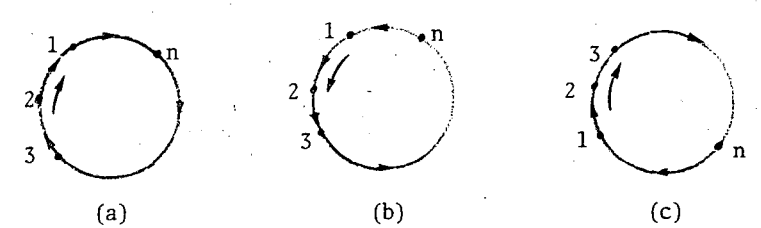


Fig. 4. Graphs (a) and (b) are related by antiparticle conjugation. Graph (c) is graph (b) turned over.

The scalar functions $f(p)$ are assumed to be unchanged by anti-particle conjugation. Thus the amplitudes associated with graphs (a) and (b) are

$$A_a^o = -\left(\frac{i}{\sqrt{2}}\right)^n (\mathrm{Tr}\, s_1\cdot\tilde{\sigma}\ u_1\cdot\sigma \ \ldots\ s_n\cdot\tilde{\sigma}\ u_n\cdot\sigma)\ f^o(p) \tag{3.9a}$$

and

$$A_b^o = -\left(\frac{i}{\sqrt{2}}\right)^n (\mathrm{Tr}\, s_n\cdot\tilde{\sigma}\ u_n\cdot\sigma \ \ldots\ s_1\cdot\tilde{\sigma}\ u_1\cdot\sigma)\ f^o(p)\ . \tag{3.9b}$$

Then use of (2.65) and (2.66) gives

$$A_b^o = (-1)^{\sum_i (\mathrm{Spin})_i}\ A_a^o \tag{3.10}$$

## 3.4 Isospin

Quark flavors have not yet been discussed. Introduction of the up and down quarks yields the $\pi, \rho, \eta,$ and $\omega$ mesons. To get the amplitude corresponding to a graph with these mesons as the external particles one includes for each vertex the isotopic spin factor $f_i$ defined in Fig. 5:

emits $\rho^+$ or $\pi^+$ $\quad f_i = -1$

emits $\rho^-$ or $\pi^-$ $\quad f_i = +1$

emits neutral meson $\quad f_i = \frac{1}{\sqrt{2}}$

Figure continued

emits neutral meson $\quad f_i = \begin{cases} -\frac{1}{\sqrt{2}} & \pi^0,\ \rho^0 \\ +\frac{1}{\sqrt{2}} & \eta^0,\ \omega^0 \end{cases}$

Figure 5. The isotopic spin factors. The full zero-entropy amplitude for any process involving a set of $n$ of these mesons is the sum of the amplitudes corresponding to all the ways in which the particles of the reaction can be identified with the vertices of ortho and para graphs with $n$ vertices.

G-parity is conserved for the ortho and para amplitudes separately. To see this note that for each ortho (para) graph contributing to a process there is another one in which the $u$ and $d$ quarks are interchanged and the cyclic order of the particles is reversed. The two associated ortho graphs are related as the two graphs (a) and (c) of Fig.4 apart from flavor labels. Since (c) is equivalent to (b) one obtains the factor (3.10) together with the isospin factors $f_i$ shown in Fig.5. These factors $f_i$ combine to give factors for the graphs (a) and (c) that differ by the factor $\prod_i (-1)^{(\mathrm{Isospin})_i}$. Thus the sum of the two contributions is

$$A_a^o + A_c^o = A_a^o\ (1 + (-1)^g) \tag{3.11a}$$

where

$$g = \sum_i (\mathrm{Spin})_i + (\mathrm{Isospin})_i\ . \tag{3.11b}$$

The factor $(-1)^g$ is G parity. Hence G parity is conserved for the ortho and para parts separately.

## 3.5 Products

The discontinuity formulas involve products of amplitudes represented by graphs of the kind shown in Fig. 6.

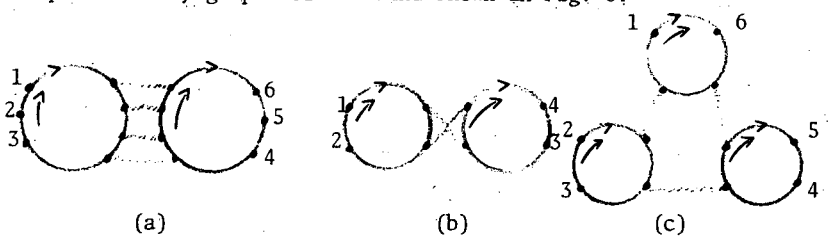


Figure 6. Diagrams representing products of amplitudes. The wiggly lines represent the intermediate mesons.

For each wiggly line there is a sum over the single pseudoscalar meson and the three vector mesons. When this sum is performed the spinor parts of these products are just the spinor parts of the functions associated with the diagrams of Fig. 7.

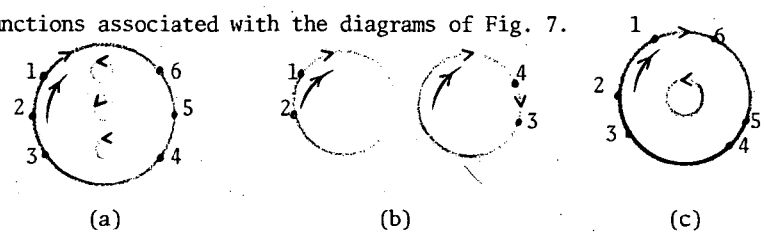


Figure 7. Alternative representation of the spinor parts of the products represented in Fig. 6. A circle with no vertices represents -Tr 1 = - 2.

In other words, the spinor parts satisfy the diagrammatic equations of Fig. 8.

Fig. 8. Spinor Identities



To obtain this result, and also a more general one, let the four orthogonal vectors $s_i$ associated with particle $i$ be labelled by $e \in (0, 1, 2, 3)$, with $s^u_{ie} = v^u_i$ for $e = 0$. To get the correct normalization we return to the level of the $S$ matrix. Then the four amplitudes $S_e$ are defined by

$$S_e = \mathrm{Tr}\, \frac{s^r_e \cdot \tilde{\sigma}}{\sqrt{2}}\, S$$

$$= \frac{1}{\sqrt{2}}\, s^r_e \cdot \tilde{\sigma}^{\dot{\alpha}\beta}\, S_{\beta\dot{\alpha}} \quad , \tag{3.12}$$

where the irrelevant indices on $S$ have been suppressed. The arguments of §2.3 then show that

$$S_e = \mathrm{Tr}\, \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}}\, M. \tag{3.13}$$

Consider therefore a product of the form

$$\sum_{e=0}^{3} S_e S'_e = \sum_{e=0}^{3} \left(\mathrm{Tr}\, \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}}\, M\right)\left(\mathrm{Tr}\, \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}}\, M'\right). \tag{3.14}$$

To evaluate it introduce into the second trace the identity

$$s_e \cdot \tilde{\sigma} = v \cdot \tilde{\sigma}\; s^e \cdot \sigma\; v \cdot \tilde{\sigma} \quad , \tag{3.15}$$

where $v$ is the velocity $\pm p/m$ of the relevant particle and $s^0 = s_0$, $s^1 = -s_1$, $s^2 = -s_2$, $s^3 = -s_3$. (Each $s_e$ is a four vector).

Thus

$$\sum_{e=0}^{3} S_e S'_e =$$

$$\frac{1}{2} \sum_{e=0}^{3} (\mathrm{Tr}\, M\, s_e \cdot \tilde{\sigma})(\mathrm{Tr}\, s^e \cdot \sigma\; v \cdot \tilde{\sigma}\; M'\; v \cdot \tilde{\sigma}). \quad (3.16)$$

Use of

$$\sum_{e=0}^{3} (s_e)^\mu (s^e)^\nu = g^{\mu\nu} \quad (3.17)$$

gives

$$\sum_{e=0}^{3} S_e S'_e =$$

$$\frac{1}{2} (\mathrm{Tr}\, M\, \tilde{\sigma}_\mu)\, g^{\mu\nu} (\mathrm{Tr}\, \sigma_\nu v \cdot \tilde{\sigma}\; M'\; v \cdot \tilde{\sigma}).$$

Use of

$$\frac{1}{2}\, \tilde{\sigma}_\mu^{\dot{\alpha}\beta}\, g^{\mu\nu}\, \sigma_{\nu\gamma\dot{\delta}} = \delta^{\dot{\alpha}}_{\dot{\delta}}\, \delta^{\beta}_{\gamma} \quad (3.18)$$

gives

$$\sum_{e=0}^{3} S_e S'_e = \sum_{e=0}^{3} (\mathrm{Tr}\, M \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}})(\mathrm{Tr} \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}} M')$$

$$= \mathrm{Tr}\, M\, v \cdot \tilde{\sigma}\; M'\; v \cdot \tilde{\sigma}\ . \quad (3.19)$$

This result says that summing over all four exchanged particles is equivalent to cutting the two quark lines at the vertices and reconnecting them in the way shown in Fig. 9, with a metric factor $v \cdot \tilde{\sigma}$ placed at each reconnection point.



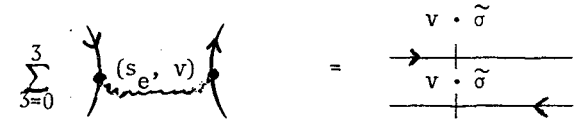


Figure 9. Diagrammatic representation of (3.19).

Consider one of the two reconnection points in Fig. 9 and its associated factor $v \cdot \tilde{\sigma}$ . This point connects two line segments that were originally parts of the two loops connected by the meson line shown on the left-hand side of Fig. 9, and also in Fig. 6. Each of these two line segments is associated with a propagator, in accordance with the formulas of (3.5). One of these two propagators is of the form $\pm v \cdot \sigma$ , and will cancel the factor $v \cdot \tilde{\sigma}$ associated with the reconnection point, up to a possible sign. Now the meson line connects two vertices. One is associated with an initial particle, the other with a final particle. Thus the signs of the corresponding vectors $u_i$ will be opposite. Consequently, the signs of the two factors $\pm v \cdot \sigma$ that cancel against the two factors, $v \cdot \tilde{\sigma}$ of Fig. 9 will be opposite, and the cancellation of the two factors $v \cdot \tilde{\sigma}$ will leave a residual minus sign. This minus sign cancels one associated with closed quark loops. (This closed-loop sign appears explicitly in front of the trace symbol in (3.5).) For the transformation between the two sides of Fig. 9 changes the number of quark loops by one.

The two factors of i associated with the quark wave functions $(is_i)$ of the two particles that are contracted out compensate in the case of the spin-one particle for the fact that the physical spin-vector $s_e$ occurring

in the derivation of (3.19) is related to the mathematical one occurring in (3.5) by $s^{phys} = s^{math}$ sign $u^0$, and in the case of the spin-zero particle by the fact that the physical vector $s_e = v$ occurring in the derivation of (3.19) is related to the mathematical vector $s_i = u_i$ occurring in (3.5) by this same relationship. Thus the relationship illustrated by Fig. 9 reduces to the simpler one illustrated by Fig. 8, from which follows the equality of the spin factors associated with the corresponding diagrams of Figs. 6 and 7.

An explanation of the signs and factors of i in (3.5) is in order. The minus sign in front of the trace comes from considering the quark wave function to be antisymmetric under the interchange of variables: an odd number of permulations is required to take the quark variables from their normal order (see Ref. 7) in which the propagators are $u_a \cdot \sigma = - u_b \cdot \sigma$ for forward scattering to the cyclic order associated with the closed loop. The imaginary unit is included in the wave function in order to allow the wave function to be folded into the basic unitarity equation for M functions, Eq. (7.1) of Ref. 7, without disrupting either the relative signs of the two terms on the left-hand side or the relationship $s^{phys} = s^{math}$ sign $u^0$.



## 3.6 Topological Classification

Each circular quark graph G corresponding to a zero-entropy ortho or para amplitude can be transformed by the rule illustrated in Fig. 10 into a particle graph $g = g(G)$ with one internal vertex.

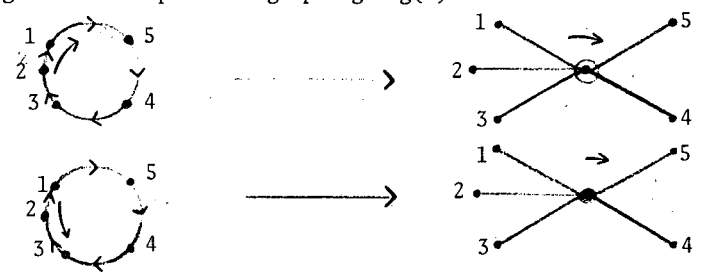


Figure 10. Transformation of circular ortho and para quark graphs G into the corresponding basic particle graphs g(G).

If G is a circular graph with n vertices then g(G) is a tree graph with n edges, n external vertices, and one internal vertex. This internal vertex of g(G) is classified as ortho or para according to whether G is ortho or para. These two kinds of internal vertices can be distinguished in the way illustrated in Fig. 10. The arrow near each internal vertex shows the direction of rotation of the quark line around that vertex. These graphs g are called basic particle graphs.

A product of basic particle graphs $g_1, g_2, \ldots$ is formed by identifying certain pairs of the external vertices, as illustrated in Fig. 11.

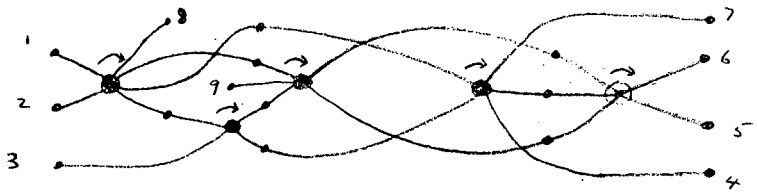


Figure 11. A product $g$ of 5 basic particle graphs $g_i$.

Each product graph $g$ has a well defined genus and boundary structure. These can be calculated by the Edmond's rule. One first draws all the orbits of $g$. An orbit of $g$ is a path in $g$ formed as follows: one picks any point $p$ on any edge of $g$ and a direction $d(p)$ at that point. Then one traces a path in $g$ by a moving point $p'$ that starts from $p$ in the direction $d(p)$. At each nontrivial vertex the moving point $p'$ shifts to the "next" line, with the order of the lines specified by the arrow that indicates the quark-line direction. The orbit is completed when the moving point $p'$ returns to the original point $p$ moving in the original direction $d(p)$.

Some of the orbits may pass through vertices that lie at the ends of single (external) edges. These vertices correspond to the "external particles" associated with the graph. An orbit that passes through at least one external-particle vertex is called a boundary. The boundary structure consists of the collection of boundaries, each identified by the sequence of external-particle vertices through which it passes. Each external-particle vertex appears on exactly one boundary. Graphs with only one boundary are called one-boundary graphs.

The number of different orbits of $g$--sometimes called faces



of $g$-- is denoted by $f(g)$. The numbers of edges and vertices of $g$ are denoted by $e(g)$ and $v(g)$, respectively. Then the genus of $g$-- sometimes called the handle number--is given by the Euler formula

$$h(g) = \frac{e(g) - v(g) - f(g) + 2c(g)}{2}, \qquad (3.20)$$

where $c(g)$ is the number of connected components of $g$. The graph of Fig. 11 has one orbit, which is the boundary (8, 6, 5, 9, 7, 4, 3, 2, 1), and its genus is two.

The zero-genus one-boundary graphs are the planar graphs. They are the graphs that can be drawn on a plane with no lines crossing and all external vertices identified with a single point at infinity.

An important characteristic of a graph $g$ is its Betti number $\beta(g)$, which is the number of independent closed loops that can be drawn in the graph. Its value is given by

$$\beta(g) = e(g) - v(g) + c(g). \qquad (3.21)$$

Let the number of boundaries of $g$ be $b(g)$. The orbits that are not boundaries are called windows, and their number is $w(g) = f(g) - b(g)$. The most important topological characteristic of $g$ is the topological index

$$\begin{aligned} \gamma(g) &= 2h(g) + b(g) - c(g) \\ &= \beta(g) - w(g). \end{aligned} \qquad (3.22)$$

This is the number of independent closed loops in $g$ minus the number of windows. For connected graphs with at least one boundary the topological index $\gamma(g)$ is zero if and only if the graph $g$ has zero-genus and exactly one boundary, i.e., if and only if $g$ is planar.

This topological index $\gamma(g)$ enjoys the following "entropy" property: if $g_1g_2$ is some connected product of two connected graphs $g_1$ and $g_2$ then

$$\gamma(g_1g_2) \geqslant \gamma(g_1) + \gamma(g_2) - 1. \tag{3.23}$$

To prove this let $n$ be the number of vertices at which $g_1$ and $g_2$ are joined. Then (3.21) gives

$$\beta(g_1g_2) = \beta(g_1) + \beta(g_2) - 1 + n. \tag{3.24}$$

On the other hand,

$$w(g_1g_2) = w(g_1) + w(g_2) + w'(g_1,g_2) \tag{3.25}$$

where $w'(g_1,g_2)$ is the number of windows of $g_1g_2$ that lie partly in $g_1$ and partly in $g_2$.

Each of these windows that lies partly in each subgraph must pass at least twice through the $n$ junction points. And each junction point lies exactly twice on the set of orbits. Thus one has the inequality

$$w'(g_1,g_2) \leqslant n, \tag{3.26}$$



which combines with (3.24) and (3.22) to give (3.23).

The entropy property (3.23) shows that the topological index $\gamma(g_1g_2)$ of a product graph $g_1g_2$ is greater than either component, provided one of them has $\gamma(g_i) > 1$. This means that the topological complexity, as measured by $\gamma(g)$, increases in general. The special case $\gamma(g_i) = 1$ allows the complexity to remain unchanged.

If one of the graphs has $\gamma(g_i) = 0$ then (3.23) would allow for a decrease in complexity. However, if $\gamma(g_1) = 0$ and the product $g_1g_2$ is such that at least one external vertex of $g_1$ is an external vertex also of the product graph $g_1g_2$ then the RH side of (3.26) can be replaced by $n - 1$, since then at least one boundary of $g_1g_2$ must pass twice through the set of junction points, and $\gamma(g_1g_2) \geqslant \gamma(g_1) + \gamma(g_2)$.

The graphs corresponding to physical-region singularities can always be constructed by taking successive products $g_1$, $g_2g_1$, $g_3g_2g_1$, ... so that the final external particles of each newly added graph are also final external particles of the new product graph.[18] If the product graphs are built in this way then the topological index $\gamma(g)$ can never decrease.

The product graphs $g$ are classified by their overall boundary structure and genus, and by their decomposition into ortho and para parts. This decomposition is made as follows: The trivial two-edge vertices at which two graphs are joined are called junction vertices. Each junction vertex that lies on a line joining an ortho vertex to a para vertex is cut. This cuts the graph into a set of graphs $g_i$ such that the internal vertices of each graph $g_i$ are all of the same kind, either ortho or para. Each of these graphs $g_i$ has a

boundary structure and genus. The complete topological classification of the graph $g$ is given by specifying the boundary structure and genus of each part $g_i$, and the set of pairs of external vertices of the graphs $g_i$ that are equated to form the junction vertices of $g$.

These specifications determine the overall boundary structure and genus of $g$ itself. However, these overall characteristics are nevertheless included, redundantly, in the complete topological specification of $g$.

Graphs $g$ that have the same topological specifications are said to lie in the same topological class. The zero-entropy graphs are the graphs $g$ with a single (ortho or para) part $g_i = g$ and topological index $\gamma(g) = 0$. The simplest of these are the basic graphs $g$ of the kind illustrated in Fig. 10.

The discussion of topological classification given above was made completely in terms of the particle graph $g$. It is sometimes useful to combine the particle graph $g(G)$ and the quark graph $G$ into a single quark-particle graph $\bar{g}(G)$, in the way illustrated in Fig. 12

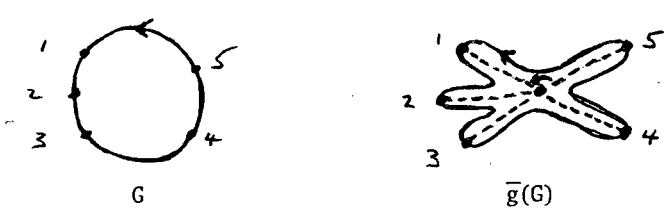


Figure 12. A graph $G$ and the quark-particle graph $\bar{g}(G)$ formed from $G$. The particle lines of $g(G)$ are drawn as dotted lines in $\bar{g}(G)$.



The orbits of $g(G)$ can be considered to be the independent closed loops on the quark lines of $\bar{g}(G)$. Those closed quark-line loops that pass through vertices are boundaries. Those that do not are windows. The number of vertices and edges that occurs in the Euler formula (3.20) for the genus is the number of vertices and edges of the particle graph $g(G)$, which is a subgraph of $\bar{g}(G)$.

## 3.7 Topological Expansion

Each physical-region singularity of the S matrix is associated with a Landau graph $g_L$. A formula for the discontinuity around the singularity associated with graph $g_L$ is obtained by replacing each vertex of $g_L$ by the corresponding scattering function[15,16]. This scattering function is specified by the set of edges incident upon the vertex to which it corresponds. These edges can be identified with the external edges of the particle graphs $g$ constructed above.

The topological expansion is the assumption that each scattering function can be expressed as a sum of terms, one corresponding to each of the different topological classes specified in the preceding subsection. This expansion is required to be compatible with the discontinuity formulas, in the sense that if the full expansion is introduced into each of the scattering functions that occur in any discontinuity equation, and the full equation is then decomposed into terms of different topological class then the terms of each class separately satisfy the equation: there is no cancellation among the terms in the equation that have different topological character. This assumption that the contributions to any discontinuity

equation corresponding to graphs of the same topological character should cancel among themselves has been discussed extensively before, in connection with the derivation of the discontinuity formulas[19,20,21].

### 3.8 The Zero-Entropy Functions

The validity of the topological expansion is assumed. Then the zero-entropy component of any discontinuity equation can be examined. Each scattering function is the sum of a pure ortho part plus a pure para part plus higher-order terms formed from products of ortho and para parts. These higher-order parts do not contribute to the zero-entropy component of the discontinuity equation. Thus the zero-entropy component separates into two parts, an ortho part and a para part, each of which must separately be satisfied, since each belongs, according to our classification scheme, to a separate topological class.

By virtue of the entropy property the zero-entropy terms can be formed only from zero-entropy factors. Thus the scattering function associated with each vertex of the Landau graph is replaced, in the ortho (para) zero-entropy component of the full discontinuity equation, simply by the zero-entropy ortho (para) amplitude. Consequently all the discontinuity equations for the zero-entropy ortho (para) amplitudes are identical to the discontinuity equations for the full scattering function with two exceptions: (1) the discontinuity is zero unless the Landau graph is planar; and (2) the full scattering amplitudes are replaced everywhere in the discontinuity equation by the corresponding ortho ( para ) amplitudes.

By virtue of the occurrence of only those singularities that correspond to planar Landau graphs the analytic structure of the ortho and para functions is much simpler than that of the full



scattering function. It is expected that these functions should have moving Regge poles but no Regge cuts. They should, in a first approximation, be similar to the Veneziano dual-resonance model functions,[22,23] with the addition of a spin-flavor structure, finite widths, and a planar singularity structure in momentum space.

### 3.9 Regge Recurrences

The property represented in (3.19) and Fig. 9 says the spinor factor in the zero entropy functions $A^o$ or $A^p$ has the pole-factorization property indicated in Fig. 13

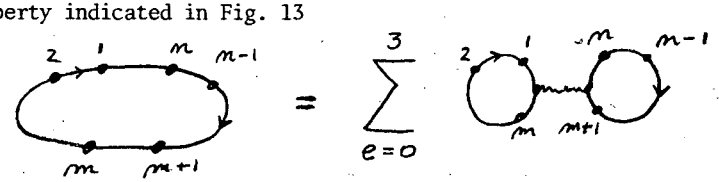


Figure 13. Pole factorization property.

Thus if $f^o(p)$ has a factorizable pole corresponding to a certain value $\ell \geqslant 1$ of angular momentum transferred between $(1, \ldots, m)$ and $(m + 1, \ldots, n)$ then the full function $A^o$ has factorizable poles corresponding to a set of four intermediate states, having total angular momentum values $J = \ell$ and $\ell + 1$, $\ell$, $\ell - 1$.

If $f^o(p)$ has Regge behavior of the kind exhibited by the Veneziano dual-resonance function[23], then for each factorizable pole of $f^o(p)$ corresponding to orbital angular momentum $\ell \geqslant 1$ there will be a quartet of factorizable poles of $A^o(p)$ corresponding to total angular momentum $\ell$ and $(\ell + 1, \ell, \ell - 1)$.

The function $f^o(p)$ is assumed to have a Regge pole with the lowest $\ell = 0$ pole identifiable with our external set of sixteen mesons $(\pi, \rho, \omega, \eta)$, which are assumed to be degenerate in the zero-entropy level. The higher values of $\ell$ will then generate recurrences

of the set $(\pi,\rho,\omega,\eta)$.

If the functions $f^{o}(p)$ and $f^{p}(p)$ are now generalized to represent the cases where the external particles are recurrences of the $\ell = 0$ mesons then one must include[24] for each external particle $i$ of angular momentum $\ell_i$ a set of $\ell_i$ vector indices, $\mu_1^{(i)}\ldots\mu_{\ell_i}^{(i)}$ that are such that $p_i \cdot f^{o}(p) = p_i \cdot f^{p}(p) = 0$ when the inner product is formed with any one of the indices $\mu_{\ell_j}^{(i)}$. Consequently, the earlier equation

$$f^{p}(p) = \mathscr{P}(f^{o}(p)) = f^{o}(\tilde{p}) = f^{o}(p) \tag{3.27a}$$

becomes replaced by

$$f^{p}(p) = \mathscr{P}(f^{o}(p)) = (-1)^{\sum_{i=1}^{n} \ell_i} f^{o}(\tilde{p}) \tag{3.27b}$$

When nonzero values of the $\ell_i$ are allowed there is also an extra factor of $(-1)^{\Sigma\ell_i}$ in the charge conjugation equation (3.10). This comes from a consideration of, for example, the two definitions of $\rho^+$ and $\rho^-$ implicit in Figs. 14a and 14b.

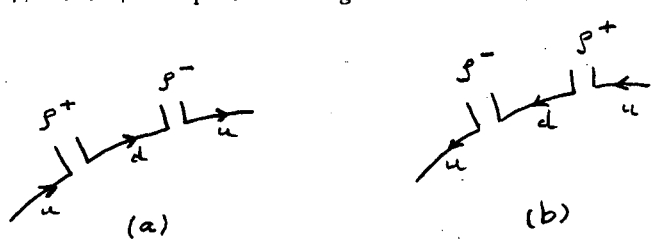


Figure 14. The normal quark structures of $\rho^+$ and $\rho^-$ is shown in (a), whereas (b) shows the definition induced by reversing the quark lines. If the quark wave function has angular momentum $\ell_i$ then the difference is represented by a factor $(-1)^{\ell_i}$.

In the discussion in §3.4 of isospin invariance there was no



change in the definitions of $\rho^+$ and $\rho^-$ of the kind shown in Fig. 14. However, the function $f^{o}(p)$ was changed due to a reversal of the order of the arguments. (See Fig. 4(c)) In the dual-resonance amplitude[7] this change induces a change $(-1)^{\Sigma\ell_i}$, and we assume that this property holds also for our function $f^{o}(p)$:

$$f^{o}(p_n,\ldots,p_1) = (-1)^{\Sigma\ell_i} f^{o}(p_1,\ldots,p_n). \tag{3.28}$$

The fact that one gets the same factor $(-1)^{\Sigma\ell_i}$ by either reversing the direction of the quark arrow, as in Fig. 4, (a) → (b), (or Fig.14, (a) → (b)) or by reversing the cyclic order of the vertices, as in Fig. 4, (a) → (c), means that the amplitude corresponding to a graph does not depend on how this graph is placed on the paper: the operation of turning over or reflecting a graph, as in Fig. 4, (b) → (c), does not alter the amplitude corresponding to it. Thus the equivalence of the two graphs of Fig. 2, or of Fig. 3, is maintained also for $\ell_i > 0$.

## 4. SUMMARY AND CONCLUSIONS

Spin can be incorporated into the meson topological expansion by doing the following four things:

(1) Adopt the M-Function formalism and associate the leading (resp. trailing) end of each quark line with a lower undotted (resp. dotted) index.

(2) Introduce into the topological expansion a "zero-entropy" level that lies below the "ordered" level of Chew and Rosenzweig[2].

(3) For each cyclically ordered set of n mesons (of fixed flavor content) form the various possible zero-entropy M functions. Each of these is a product of a scalar function $f_n(p_1, p_2, \cdots, p_n)$ of the scalar products of the momentum-energy vectors $p_j$ of the n mesons, times a spin factor. This spin factor is minus the product of n spin factors, one for each of n quark lines. The spin factor associated with any given quark line is either the ortho-quark factor $u_a \sigma_{\alpha\dot{\beta}}$ or the para-quark factor $-u_b \sigma_{\alpha\dot{\beta}}$, where $\alpha$ is the spinor index associated with the leading end of the quark line, $\dot{\beta}$ is the spinor index associated with the trailing end of the quark line, $u_a$ is the mathematical covariant velocity $p_a/m_a$ of the physical particle associated with the vertex lying on the leading end of the quark line, and $u_b$ is the mathematical covariant velocity $p_b/m_b$ of the physical particle associated with the vertex lying on the trailing end of the quark line. (The minus sign associated with the spin factor comes from the odd number of permutations of the quark variables required to take them from their normal order associated with the no-scattering part, for which



each spin factor is $v_a \cdot \sigma_{\alpha\dot{\beta}} = v_b \cdot \sigma_{\alpha\dot{\beta}}$, to the order in which the two indices $\alpha_j$ and $\dot{\beta}_j$ associated with physical particle j stand together and in the order $(\dot{\beta}_j, \alpha_j)$). (4) Institute invariance under the parity operation by adding together the $2^n$ zero-entropy functions that arise from the association of each of the n quark lines with, alternatively, either the ortho-function or the para-function.

The zero-entropy amplitudes discussed in the earlier sections are only those two special cases in which all of the n quark lines represent ortho-functions or all represent para-functions. A restriction to functions of these two special types would be unnatural: it would correspond to imposing parity invariance only globally for the entire process, rather than locally for the separate contribution from each individual quark line. All $2^n$ independent choices of the ortho-para character of each of the n quark lines should be included.

A quartet of amplitudes corresponding to one pseudo scalar meson and one three-component vector meson is obtained by folding the four wave functions $i\tilde{\sigma}_{\dot{\beta}_j\alpha_j} \cdot u_j$ and $i\tilde{\sigma}_{\dot{\beta}_j\alpha_j} \cdot s_{je}$ (e = 1, 2, 3) into the M function. Here $s_{j1}$, $s_{j2}$, and $s_{j3}$ are three vectors that satisfy $(s_{je})^2 = -1$ and $s_{je} \cdot v_j = 0$.

The theory obtained in this way from the two-component formalism can be directly transcribed into four-component notation by the methods of Ref. 7. This transcription is carried out in Appendix C, and the results are described here. The use of the four-component notation facilitates comparisons to earlier works.

For each cyclic ordering of n mesons (of fixed flavors) there are $2^n$ different zero-entropy amplitudes, one for each combination of choices of the ortho-para character of the n quark lines that cyclically

join the n vertices corresponding to the n mesons. The basic property of the zero-entropy amplitudes is that the spin-factors can be completely factored out of the associated discontinuity equations. Thus the nonlinear integral equations for the scalar factors $f_n(p)$ are the same for all $2^n$ zero-entropy functions. These equations are relatively simple, and should determine the coupling constants at the zero-entropy level. They also ensure that the same scalar factor $f_n(p)$ occurs in all $2^n$ zero-entropy functions. Hence the sum $\Sigma_n$ of the $2^n$ zero-entropy functions is simply the sum of the spin factors, times the common factor $f_n(p)$.

This sum $\Sigma_n$ of the $2^n$ zero-entropy functions is expressed in the 4-component formalism as

$$\Sigma_n = -\left(\mathrm{Tr}\,\Gamma_{1\mu_1}(u_1)\Gamma_{2\mu_2}(u_2)\ldots\Gamma_{n\mu_n}(u_n)\right)f_n(p_1,\ldots,p_n), \qquad (4.1)$$

where the pseudo-scalar particle is associated with the factor

$$\Gamma_{j5}(u_j) = i\gamma_5(1 + \gamma\cdot\mu_j), \qquad (4.2a)$$

and the three components of the vector particle are associated with the four factors

$$\Gamma_{j\mu}(u_j) = \gamma_\mu - (u_j)_\mu\gamma_\rho\, u_j^\rho - i\sigma_{\mu\rho}u_j^\rho. \qquad (4.2b)$$

Here $\mu$ and $\rho$ range over the set (0, 1, 2, 3), there is a sum over the repeated vector index $\rho$, and $u_j^\rho\Gamma_{j\rho}(u_j) = 0$.



The individual zero-entropy functions are obtained by choosing an ortho or para character for each quark line, and then inserting after each factor $\Gamma_{j\mu_j}(u_j)$ of (4.1) a factor of $\frac{1}{2}(1 + \gamma_5)$ or $\frac{1}{2}(1 - \gamma_5)$ according to whether the quark line that leads into vertex j has ortho or para character. These two operators project the onto states of opposite chirality.

Each quintet of factors $\Gamma_{j\mu}(u_j)$ in (4.2) defines a representation of the non-chiral group $(U(2) \times U(\tilde{2}))_{u_j}$. That is, in a rest frame of particle j, where $u_j$ is pure timelike and $\Gamma_{j0}(u_j) = 0$, one has

$$A_j^-\Gamma_{j\mu}(u_j)B_j^+ = \sum_\nu \Gamma_{j\nu}(u_j)\,\Lambda_\mu^\nu(A_j^-, B_j^+). \qquad (4.3)$$

Here the indices $\mu$ and $\nu$ range over the set (0, 1, 2, 3, 5), and $A_j^-$ and $B_j^+$ are

$$A_j^- = \exp i\,(1 - \beta u_j^0)\,\sigma\cdot a^- \qquad (4.4a)$$

and

$$B_j^+ = \exp i(1 + \beta u_j^0)\sigma\cdot b^+. \qquad (4.4b$$

where $a^-$ and $b^+$ are two real four-vectors, and the $\sigma_\mu$ are the $4\times 4$ matrices with two $2\times 2$ $\sigma_\mu$'s in the diagonal corners. The two other operators

$$A_j^+ = \exp i(1 + \beta u_j^0)\sigma\cdot a^+ \qquad (4.5a)$$

and

$$B_j^- = \exp i(1 - \beta u_j^0)\, \sigma \cdot b^- \tag{4.5b}$$

acting on the $\Gamma_{j\mu}(u_j)$'s from the left and right, respectively, act as unit operators. (See (C.72) and (C.48)). Hence (4.3) holds also if $A_j^-$ and $B_j^+$ are replaced by

$$A(a, a') = \exp(i\sigma \cdot a + i\beta\sigma \cdot a') \tag{4.6a}$$

and

$$B(b, b') = \exp(i\sigma \cdot b + i\beta\sigma \cdot b'), \tag{4.6b}$$

respectively.

The invariance of the $\Gamma_{j\mu}(u_j)$ under the transformations (4.5) is a consequence of the use of the 2-component M-function formalism, or, equivalently, of the Dirac equation for the corresponding four-component quantities. It eliminates the possible scalar and axial vector particles, and fixes the couplings of the pseudo-scalar and vector particles to be precisely those shown in (4.2).

The transformation properties shown in (4.3) and (4.4) do not in general entail corresponding invariance properties of the S-matrix. This is because the velocities $u_j$ of the various particles are generally different, and hence the transformations shown in (4.3) act, for different particles, in different frames of reference, which are related by Lorentz transformations.

One subgroup of the group generated by the transformations (4.6) is of particular interest. This is the subgroup $SU(2)_W$ formed by imposing the following restrictions on the coefficients in (4.6):



$$a_3 = -b_3, \tag{4.7a}$$

$$a_1' = -b_1', \tag{4.7b}$$

$$a_2' = -b_2', \tag{4.7c}$$

$$a_1 = a_2 = b_1 = b_2 = a_3' = b_3' = 0, \tag{4.7d}$$

and

$$a_0 = a_0' = b_0 = b_0' . \tag{4.7e}$$

These conditions entail that $B = A^{-1}$, and that the transformations corresponding to $(\delta a_1', \delta\alpha_2', \delta\alpha_3)$ are generated by the triad of generators

$$(W_1 = \beta\sigma_1;\ \ W_2 = \beta\sigma_2;\ W_3 = \sigma_3). \tag{4.8}$$

These three generators enjoy the same commutation realtions as the generators $(\sigma_1, \sigma_2, \sigma_3)$ of SU(2).

In the rest frame of particle j the quintet of factors $\Gamma_{j\mu}(u_j)$ occurring in (4.2) reduces to a quartet. If these four factors are identified with four factors $X_\mu$ in the following way

$$\gamma_3(1 + \beta u_j^0) = X_0, \tag{4.9a}$$

$$i\gamma_5(1 + \beta u_j^0) = X_3, \tag{4.9b}$$

$$-\gamma_1(1 + \beta u_j^0) = X_2, \tag{4.9c}$$

$$\gamma_2(1 + \beta u_j^0) = X_1, \tag{4.9d}$$

then, for all i, j, and k in the set (1, 2, 3),

$$[W_i, X_0] = 0 \tag{4.10a}$$

and

$$[W_i, X_j] = 2i\varepsilon_{ijk}X_k. \tag{4.10b}$$

Thus $X_0$ transforms as a W-spin singlet, and the set $(X_1, X_2, X_3)$ transforms as a W-spin triplet.

If all particles are at rest then the function $\Sigma_n$ defined in (4.1) is invariant under $SU(2)_W$. The Lorentz transformation that boosts $\Gamma_{j\mu}(u_j)$ from its rest frame form $\Gamma^0_{j\mu}(u_j)$ to its form $\Gamma_{j\mu}(u_j)$ in the standard coordinate system is given by

$$\Gamma_{j\mu}(u_j) = \begin{pmatrix} \sqrt{v_j\cdot\sigma} & 0 \\ 0 & \sqrt{v_j\cdot\tilde{\sigma}} \end{pmatrix} \Gamma_{j\mu}(u_j) \begin{pmatrix} \sqrt{v_j\cdot\tilde{\sigma}} & 0 \\ 0 & \sqrt{v_j\cdot\sigma} \end{pmatrix}. \tag{4.11}$$

If v lies in the 0 - 3 plane then the boost transformations appearing in (4.11) commute with the generators of $SU(2)_W$, since both $\gamma_3$ and $\gamma_0 = \beta$ are invariant under $SU(2)_W$. Thus if the three-velocities $\vec{v}_j$ of all n particles are directed along the third coordinate axis then the Lorentz transformations in (4.11) do not disrupt the invariance of (4.1) under $SU(2)_W$: the function $\Sigma_n$ remains invariant.

If quarks of three flavors are allowed and the 4 × 4 matrices $\Gamma_{j\mu_j}(u_j)$ are expanded to the corresponding 12 × 12 matrices $\Gamma'_{j\mu_j\alpha_j}(u_j)$, then these new 12 × 12 matrices will define a representation of



$SU(6)_W$, and the new function $\Sigma'_n$, formed as in (4.1) but with $\Gamma'_j$ in place of $\Gamma_j$, will be invariant under $SU(6)_W$.

A comparison with several earlier works may be helpful. Bardacki and Halpern[26] also introduce spin factors analogous to Chan-Paton factors, but arrive at a 16-particle multiplet in place of our 4-particle multiplet. They find in addition to P and V, the associated parity doublet partners S and A, which occur, moreover, as ghost (negative metric) particles. They find also a second set of eight particles (S', P', V', A') that couple differently to the quarks . These sixteen particles correspond to the sixteen independent matrices needed to span the space of 4 × 4 (Dirac) matrices.

The present work is based on the two-component formalism and consequently gives in place of the 16 particles of Bardakci and Halpern only four particles. Considered from the four-component viewpoint the two Dirac equations (C.48) reduce the multiplicity of particles by a factor of four: they reduce the 16 particles of Bardakci-Halpern to the four independent ones coupled in accordance with (4.2).

Bardakci and Halpern introduce the spinor solutions $U(p_j)$ and $\tilde{U}(p_j)$ of the Dirac equation, but their way of using these spinor solutions does not give them the crucial Dirac equations (C.48).

The present approach enforces the usual discontinuity equations, including pole factorization, at the zero-entropy level. But parity invariance is not maintained at that level. Parity invariance, and also $SU(6)_W$ invariance for spatially linear processes, is maintained for the sum $\Sigma_n$ of the $2^n$ zero-entropy functions, but pole-factorization does not hold for these sums.

The terms needed to restore pole-factorization at the physical level come from higher-order terms in the topological expansion. To obtain an approximate representation of a physical amplitude near a singularity one can insert into dispersion relations the discontinuity functions obtained from the planar amplitudes of Chew and Rosenzweig These amplitudes are built out of the zero-entropy functions. The aim of the construction of these zero-entropy amplitudes is not to obtained immediately a good approximation to the physical amplitudes. It is rather to define and determine the basic building blocks of the theory.

The results obtained here are formally similar to those obtained by Delbourgo et al.[27] from considerations of the group $\tilde{U}$ (12). They use a four-component spin formalism, but impose the Dirac (i.e., Bargmann-Wigner) equations in a way that yields results similar to (C.48). The principal difference is that they interprete their analogs of our functions $\Sigma_n$ as interaction terms of a local field theory, rather than as low-order amplitudes of a topological expansion. Thus the function $f_1(p_1, p_2, \ldots, p_n)$ would, in their approach, presumably be an undetermined constant whereas in the present approach it would be constrained ( and, it is hoped, determined ) by the nonlinear zero-entropy equations. It should also enjoy, for example, Regge asymptotic behavior. The full amplitude would be constructed in their theory by essentially a power series expansion, but in the present theory by including the remaining terms in the topological expansion. The crucial question is whether the self-consistent structure of the functions $f_n(p_1, \ldots, p_n)$ determined by the zero-entropy equations,



in conjunction with the topological expansion procedure, will eliminate the divergences of field theory associated with both renormalization and the divergence of the perturbation series expansion.

Before these questions can be addressed it is necessary to include baryons into the theory: Chew has found that the topological expansion scheme with mesons alone is not soluable, due to the minus sign associated with the closed loop. Inclusion of baryons (and baryonium) leads to a soluable system that gives a predicted ratio of the meson and hadronic coupling constants that is good agreement with experiment.[8] Preliminary results indicate that the overall magnitudes of the strong-interaction coupling constants, as determined by the nonlinear integral equations for the zero-entropy functions, agrees with experiment at least in order of magnitude.[9]

## APPENDIX A: ORIGIN OF DISCONTINUITY FORMULAS

Discontinuity formulas are derived in S matrix theory nominally from macrocausality and unitarity.[28] However, it was recognized long ago that unitarity is not essential. What is directly used in the derivations is not unitarity, but rather the property that the inverse of the S matrix possess the anti-normal analytic structure. This anti-normal analytic structure is the same as the normal analytic structure derived for the S matrix from macrocausality, except that the plus $i\varepsilon$ rule is replaced by the minus $i\varepsilon$ rule. It is the property that the singularities in the real region of definition be confined to the positive-$\alpha$ Landau surfaces, and that the function near these singularities be defined by the minus $i\varepsilon$ limiting procedure. That the inverse of S should possess this anti-normal analytic structure can be derived from unitarity and the fact that S possesses the normal analytic structure. But the property should hold regardless of whether S is unitary or not.

To see the essential point in the simplest way consider first the formal perturbative solution. Then the S matrix can be written in the form[29]

$$< p'|S|p > = < p'|p > - 2\pi i\delta(E_p - E_{p'})\lim_{\varepsilon\downarrow 0} < p'|T\ (E+i\varepsilon)|p > \tag{A.1}$$

where



$$T(E) = V + V\frac{1}{E-H_0}V + V\frac{1}{E-H_0}V\frac{1}{E-H_0}V + \ldots \tag{A.2}$$

If one defines $R^{\pm}$ by

$$-2\pi i\delta(E_p - E_{p'})\lim_{\varepsilon\downarrow 0}<p'|T(E_p \pm i\varepsilon)|p>\equiv<p'|R^{\pm}|p>, \tag{A.3}$$

then it is easy to verify, formally, that

$$R^+ - R^- = R^+R^- = R^-R^+. \tag{A.4}$$

Thus the definitions

$$S^{\pm} = I \pm R^{\pm} \tag{A.5}$$

allow one to write

$$S^+S^- = S^-S^+ = I. \tag{A.6}$$

Hence the operator $S^-$ defined by (A.5) is the inverse of $S \equiv S^+$. Consequently, the operators $R^+$ and $R^-$ defined by the plus $i\varepsilon$ and minus $i\varepsilon$ limiting procedures on the same function T(E) define formally the operators $(S - I)$ and $-(S^{-1} - I)$, respectively.

These relationships are usually derived from unitarity. But the above derivation does not depend on the Hermitian character of V: it goes through, formally at least, even if S is not unitary.

In S-matrix theory the anti-normal analytic structure of $S^{-1}$ is usually derived from unitarity and the normal analytic structure of S. However, it can be derived, alternatively, directly from the anti-macrocausality property of $S^{-1}$. This latter property is the same as the macrocausality property except that the sign of time is reversed, so that physical particles carry positive energy from later times to earlier times, rather than vice versa.[28,30]

If S satisfies macrocausality then $S^{-1}$, if it exists, should satisfy anti-macrocausality. This will not be proved here, but the following argument makes it very plausible.

Consider a normalized initial state $\phi$ that represents a system of incoming particles each of which is represented by an incident wave packet with fairly well defined momenta and trajectory region. The action of S takes $\phi$ to $S\phi = \psi$. And the action of $S^{-1}$ on $\psi$ takes it back to $\phi = S^{-1}\psi$. The two reciprocal processes $\phi \to \psi$ and $\psi \to \phi$ are thus closely related. If S is unitary then $\psi$ is normalized. If S is not unitary then $\psi$ need not be normalized. But in any case the action of $S^{-1}$ on $\psi$ is closely connected to the action of S on $\phi$: S constructs $\psi$ from $\phi$, and $S^{-1}$ reconstructs $\phi$ from $\psi$.

One can imagine $\psi$ to be decomposed into components $\psi_i$ corresponding to various combinations of outgoing particles with fairly well defined momenta and trajectory locations. Then the wave functions $\phi$ and $\psi_i$ can be wave functions of the kind used in the macrocausality arguments of reference 30. If S satisfies macrocausality then in various dilated situations of the kind discussed in that reference the dominant contributions to the process $\phi \to \psi$ will correspond to physical scattering



processes. If $S^{-1}$ should fail to satisfy anti-macrocausality then for some process $\phi \to \psi$ there would be contributions to $\psi \to \phi$ that do not corresponding to temporally reversed physical processes, yet do not fall off in the way demanded by macrocausality for the corresponding direct process that contributes to $\phi \to \psi$. But then the dominant contributions to $\phi \to \psi$ and $\psi \to \phi$ would not be temporal inverses of each other, and the close reciprocal connection of these two processes would have to be maintained via an intricate interplay of contributions that are not naturally related via temporal inversion.

Although such a situation is perhaps conceivable, it will almost surely not be achievable in situations having the complexity of relativistic particle physics. Thus I think it safe to assume, in the general S-matrix context, that $S^{-1}$, if it exists, should be anti-macrocausal regardless of whether the S matrix is unitary or not: the anti-macrocausality property of $S^{-1}$ is a more primitive and basic property than unitarity. From this anti-macrocausality property one can deduce immediately from the arguments of Ref. 30 the anti-normal analytic structure of $S^{-1}$ needed in the derivation of the discontinuity formulas.

Further insight into the connection between discontinuity equations and the inverse of S can be obtained by considering the S matrix from the point of view of the "in" and "out" parts of the wave functions in radial coordinates. Separating out the center-of-mass motion of an n particle system one is left with a function of various relative coordinates $(x_1, \ldots, x_{n-1})$. An alternative set of coordinates consists of $R = (r_1, \ldots, r_{n-1})$ and $\Omega = (\Omega_1, \ldots, \Omega_{n-1})$, where $(r_i, \Omega_i)$ are the radial and angular coordinates associated with

the relative motion of some pair of subsystems. If $\psi(R,\Omega)$ is a steady state solution and $K = (k_1, \ldots, k_{n-1})$ is a set of n - 1 scalars defining momentum magnitudes then one may define the asymptotic amplitude

$$A(K,\Omega) = \sum_{i=1}^{n-1}\left[\lim_{\varepsilon_i \downarrow 0} \int_0^\infty dr_i \frac{ik_i r_i}{(2\pi)^2} \times (\theta(r_i - \rho_i)\, e^{-i(k_i - i\varepsilon_i)(r_i - \rho_i)} + \theta(\rho_i - r_i) e^{-i(k_i + i\varepsilon_i)(r_i - \rho_i)})\right] \psi(R^\rho, \Omega) \qquad (A.7)$$

where $\rho_i \equiv \varepsilon_i^{-2}$, and $R^\rho \equiv (r_i - \rho_i, \ldots, r_{n-1} - \rho_{n-1})$.

This definition is such that if $\psi$ is a plane wave $\sum_{i=1}^{n-1} \exp i\, \vec{x}_i \cdot \vec{k}_i^0$, and the directions and magnitudes of the n - 1 three-vectors $\vec{k}_i^0$ are specified by the sets of coordinates $\Omega^0 = (\theta_1^0, \phi_1^0, \ldots \theta_{n-1}^0, \phi_{n-1}^0)$ and $K^0 = (k_1^0, \ldots, k_{n-1}^0)$, where $k_i^0 > 0$ for all i, then the corresponding $A(K,\Omega)$ is

$$A_0(K,\Omega;\, K^0,\Omega^0) = \delta(K, K^0)\delta(\Omega, \Omega^0) + \delta(K, \bar{K}^0)\delta(\Omega, \bar{\Omega}^0), \qquad (A.8)$$

where

$$\delta(K, K') = \prod_{i=1}^{n-1} \delta(k_i - k_i'), \qquad (A.9)$$

$$\delta(\Omega, \Omega') = \prod_{i=1}^{n-1} \delta(\cos\theta_i - \cos\theta_i')\delta(\phi_i - \phi_i'), \qquad (A.10)$$

$$\bar{K} = (-k_1, \ldots, -k_{n-1}), \qquad (A.11)$$



and

$$\bar{\Omega} = (\pi - \theta_1, \phi_1 + \pi, \ldots, \pi - \theta_{n-1}, \phi_{n-1} - \pi). \qquad (A.12)$$

If, on the other hand, $\psi$ represents the solution to the scattering problem with the incident or incoming state given by the same plane wave $\exp \sum_{i=1}^{n-1} i\, \vec{k}_i^0 \cdot \vec{r}_i$ that was used in (A.8), then the amplitude $A(\Omega, K)$ can be written as

$$A(K, \Omega;\, K^0, \Omega^0) = S(K, \Omega;\, K^0, \Omega^0) + \delta(K, \bar{K}^0)\delta(\Omega;\, \bar{\Omega}^0), \qquad (A.13)$$

where

$$\theta(K)S(K,\Omega;\, K^0, \Omega^0) = S(K, \Omega;\, K^0, \Omega^0). \qquad (A.14)$$

Here

$$\theta(K) = \prod_{i-1}^{n-1} \theta(k_i), \qquad (A.15)$$

and $\theta(k_i)$ is zero or one according to whether $k_i$ is negative or non-negative. Equation (A.14) expresses the condition that the incoming part of the asymptotic wave (i.e. the part having the behavior $(\exp - i\Sigma k_i r_i)$) is the same as that of the incident plane wave. Equation (A.13) defines the S-matrix in these variables.

The $k_i^0$ in $K^0$ all satisfy $k_i^0 > 0$. Substitution of the argument $(\bar{K}^0, \bar{\Omega}^0)$ for $(K^0, \Omega^0)$ in (A.13) gives

$$A(K,\Omega;\, \bar{K}^0, \bar{\Omega}^0) = S(K,\Omega;\, \bar{K}^0, \bar{\Omega}^0) + \delta(K, K^0)\delta(\Omega, \Omega^0). \qquad (A.16)$$

The condition, analogous to (A.14), that the two terms in (A.16) represent separately the asymptotic incoming and outgoing parts of $\psi$ is

$$\theta(\bar{K})S(K, \Omega; \bar{K}^0, \bar{\Omega}^0) = S(k, \Omega; \bar{K}^0, \bar{\Omega}^0). \tag{A.17}$$

where the $k_i^0$ in $K^0$ always satisfy $k_i^0 > 0$.

The wave functions $\psi$ corresponding to the asymptotic amplitudes defined in (A.13) and (A.16) represent solutions that have incoming and outgoing parts equal, respectively, to the incoming and outgoing parts of the plane wave whose asymptotic amplitude is given in (A.8).

We now invoke two general principles. The first is the superposition principle, which asserts that a linear superposition of solutions $\psi$ is a solution $\psi'$. The second is the causality principle that the incoming parts of a solution should determine uniquely the outgoing parts. Using the superposition principle one can form a solution $\psi'$ by taking a linear superposition over various values of $(K^0, \Omega^0)$ the solution $\psi$ that correspond to the amplitudes $A(K, \Omega; K^0, \Omega^0)$ of (A.13). The weight factor will be chosen to be $S(\bar{K}^0, \bar{\Omega}^0; \bar{K}_1^0, \bar{\Omega}_1^0)$, for some fixed $(\bar{K}_1^0, \bar{\Omega}_1^0)$. Thus the asymptotic amplitude corresponding to $\psi'$ will be



$$\begin{aligned} A'(K,\Omega) &= \\ &= \int dK^0 \int d\Omega^0 A(K, \Omega; K^0, \Omega^0) S(\bar{K}^0, \bar{\Omega}^0; \bar{K}_1^0; \bar{\Omega}_1^0) \\ &= \int dK^0 \int d\Omega^0 (S(K, \Omega; K^0, \Omega^0) + \delta(K, \bar{K}^0)\delta(\Omega, \bar{\Omega}^0) \\ &\quad \times S(\bar{K}^0, \bar{\Omega}^0; \bar{K}_1^0, \bar{\Omega}_1^0) \\ &= \int dK^0 \int d\Omega^0 S(K,\Omega; K^0, \Omega^0) S(\bar{K}^0, \bar{\Omega}^0; \bar{K}_1^0, \bar{\Omega}_1^0) \\ &\quad + S(K,\Omega; \bar{K}_1^0, \bar{\Omega}_1^0). \end{aligned} \tag{A.18}$$

The conditions (A.14) and (A.17) entail that the incoming part of $A'(K,\Omega)$ is the same as the incoming part of the solution whose asymptotic amplitude is given in (A.16), with $(K^0, \Omega^0)$ set equal to $(K_1^0, \Omega_1^0)$. Thus, by virtue of the causality principle, the asymptotic outgoing parts of these two solutions must also be the same:

$$\begin{aligned} &\int dK_1^0 \int d\Omega_1^0 \; S(K,\Omega; K_1^0, \Omega_1^0) S(\bar{K}_1^0, \bar{\Omega}_1^0; \bar{K}^0, \bar{\Omega}^0) \\ &\qquad = \delta(K, K_1^0)\, \delta(\Omega, \Omega_1^0). \end{aligned} \tag{A.19}$$

But then the introduction of the notation

$$S(\bar{K}^0, \bar{\Omega}^0; \bar{K}_1^0, \bar{\Omega}_1^0) \equiv \bar{S}(K^0, \Omega^0; K_1^0, \Omega_1^0) \tag{A.20}$$

allows one towrite (A.19) in (A.20) in the form

$$S\bar{S} = I. \tag{A.21}$$

Consequently the transformation of shifting to the barred coordinates takes S to its inverse. The transformation that takes $(K,\Omega)$ to $(\bar{K}, \bar{\Omega})$ reverses both the signs of all momentum magnitudes and the directions of all the vectors. Hence the associates sets of vectors $\vec{k}_i$ are transformed into themselves. Thus when the functions S and $S^{-1}$ are expressed in terms of these vector arguments one needs two separate coverings of the physical region, one for S and one for $S^{-1}$. On the other hand, the points $(K,\Omega)$ and $(\bar{K}, \bar{\Omega})$ lie in two separate regions of $(k,\Omega)$ space. These regions are joined at the point where all $k_i = 0$.

The formal manipulations given above suggest that there might be some sort of analytic connection between the functions $S(K, \Omega; K', \Omega')$ defined above in the two disjoint regions $k_i, k_i' > 0$ and $k_i, k_i < 0$. One may examine this question in specific models, and in particular in nonrelativistic models with real or complex (and local or nonlocal) potentials. If the potentials are short ranged, so that the singularities near the transition point where all $k_i = 0$ arise exclusively from the singularities in the propagators, then the singularity structure near the transition point should be correctly represented by perturbation theory.



Near the two-particle threshold in a theory with one kind of particle the propagator $(E - H_0)^{-1}$ becomes, when the overall center-of-mass motion is factored out, just a one-particle propagator. This propagator, as it occurs in the functions $\langle p'|R^{\pm}|p \rangle$ defined in (A.3), is[31]

$$-\frac{m}{2\pi} \frac{\exp \pm i\, k_p r}{r} \tag{A.22}$$

where

$$k_p = |(2m\, E_p)^{1/2}|. \tag{A.23}$$

Consequently the functions $\langle p|R^{+}|p' \rangle$ and $\langle p|R^{-}|p' \rangle$, when expressed in terms of the variables $(K,\Omega; K',\Omega')$, will be analytically connected, regardless of whether the potential is real or not. This analytic connection entails a corresponding connection between $S(K,\Omega; K',\Omega')$ and $S^{-1}(K,\Omega; K',\Omega') = S(\bar{K}, \bar{\Omega}'\ \bar{K}', \bar{\Omega}')$.

These questions can be discussed in greater depth within the context of various special models. However, the point of the above discussion is to note that very general considerations, which lie deeper than particular models, strongly indicate that the familiar analytic connection between S and $S^{-1}$ should be maintained independently of unitary.

## APPENDIX B: FAILURE OF UNITARITY FOR ORTHO AND PARA AMPLITUDES.

Let the M function be decomposed into its "unit" part plus the remainder:

$$M(P'; P'') = M_u(P'; P'') + M_r(P'; P'') \tag{B.1}$$

Then the basic discontinuity equation has the form[7]

$$M_r^+(P'; P'') - M_r^-(P'; P'')$$

$$= \int M_r^+(P'; P)\, V \cdot \tilde{\sigma}\, M_r^-(P; P'')dP, \tag{B.2}$$

where $V \cdot \tilde{\sigma}$ stands for the product of factors $v_i \cdot \tilde{\sigma}$ . Continuation around the leading threshold is supposed to take the connected part of $M_r^+(P'; P'') \equiv M_r(P'; P'')$ into the connected part of $M_r^-(P'; P'')$.

The basic topological assumption is that the separation of $M_r^+(P'; P'')$ and $M_r^-(P'; P'')$ into parts having different topological characters separates the discontinuity equation into parts having different topological characters. This entails that the ortho and para parts satisfy an equation of the form (B.2), but with only planar singularities.

The ortho and para parts of $M_r^+(P'; P'')$ and $M_r^-(P'; P'')$ have polynomial factors that are specified by the rules given in the text. These polynomial factors have no singularity at the the threshold. Thus they are the same for the ortho (resp. para) parts of $M_r^+(P'; P'')$ and $M_r^-(P'; P'')$. Moreover, they are the same also for the



ortho (resp. para) part of the right-hand side of (B.2). This consequence of our rules leads to an important simplification of the discontinuity equations: the polynomial spin factor is the same for each term, and hence can be factored out. Thus the discontinuity formula at the ortho or para level becomes, essentially, a discontinuity equation for the residual scalar function.

The polynomial spin factor in the ortho part of $M_r^+(P'; P'')$ is built of factors $u \cdot \sigma$, one for each quark line. For example, a quark line whose leading end lies at a vertex associated with a variable $p_i$ contributes a factor $u_i \cdot \sigma_{\alpha_i \dot{\alpha}_j}$, where $u_i = p_i/m_i$, and $\alpha_i$ and $\dot{\alpha}_j$ are spinor indices associated with the leading and trailing ends of the quark line, respectively.

The polynomial spin factor associated with $M_r^-(P'; P'')$ gets from this same quark line an identical factor $u_i \cdot \sigma_{\alpha_i \dot{\alpha}_j}$. However, the function $M_r^\dagger(P'; P'') = M_r^*(P''; P')$ gets a factor $-u_j \cdot \sigma_{\alpha_i \dot{\alpha}_j}$ , for real $u_j$. This is not equal to $u_i \cdot \sigma_{\alpha_i \dot{\alpha}_j}$, nor even proportional to it, since $u_i$ and $u_j$ are generally non parallel.

On the other hand, the contribution $u_i \cdot \sigma_{\alpha_i \dot{\alpha}_j}$ from any given quark line to the ortho amplitude $M_r(P'; P'')$ is the same as the contribution $u_i \cdot \sigma_{\alpha_i \dot{\alpha}_j}$ from this same quark line to the Hermitian conjugate of the <u>para</u> contribution to $M_r(P'; P'')$. Thus the sum of the ortho and para amplitudes satisfies

$$M_{r0}^- + M_{rP}^- = -\,(M_{r0}^\dagger + M_{rP}^\dagger). \tag{B.3}$$

Hence when the ortho and para parts are combined one recovers the familiar Hermitian analyticity relation, generally derived from (extended) unitarity.

Since Hermitian analyticity fails for ortho and para amplitudes one cannot expect unitarity to hold for them.



## Appendix C: Four-Component Formalism

This appendix transcribes the results obtained above in the two-component formalism into the more familiar four-component formalism.

The connection between the two-component and four-component formalisms is most easily expressed by using the Weyl representation for the four-by-four Dirac matrices. In this representation

$$\alpha_\mu = \begin{pmatrix} \tilde{\sigma}_\mu & 0 \\ 0 & \sigma_\mu \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\gamma_\mu = \beta\,\alpha_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

$$\sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \quad (i = 1, 2, 3) \tag{C.1}$$

The two-by-two charge conjugation matrix is

$$C = -\,i\,\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{C.2}$$

and gives

$$C\ \sigma_\mu C^{-1} = \tilde{\sigma}_\mu^{\,Tr} \tag{C.3}$$

where $\tilde{\sigma} = (\sigma_0, -\vec{\sigma})$. Thus

$$E = \begin{pmatrix} 0 & C \\ C^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix} \tag{C.4}$$

satisfies $E = E^{-1} = E^{Tr} = E^{\dagger} = E^{*}$, where Tr, dagger, and star represent transpose, Hermitian conjugation, and complex conjugation, respectively. The important properties of E are

$$E^{\dagger}\beta E = -\beta^{Tr} \tag{C.5}$$

$$E^{\dagger}\alpha_{\mu}E = \alpha_{\mu}^{Tr} \tag{C.6}$$

and

$$E^{\dagger}\sigma_i E = -\sigma_i^{Tr} . \tag{C.7}$$

The free-field operator for Dirac particles of type t is

$$\psi_{\alpha}(x) = \int \frac{d^4p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)\theta(p^0) \times$$

$$\sum_{\lambda=1,2} (U_{\alpha}(p,\lambda)e^{-ipx}a(p,\lambda,t) + V_{\alpha}(p,\lambda)e^{ipx}b^{\dagger}(p,\lambda,-t)) \tag{C.8}$$

where

$$V(p,\lambda) = EU^{*}(p,\lambda) \tag{C.9}$$

and

$$U(p,\lambda) = EV^{*}(p,\lambda) . \tag{C.10}$$



The charge conjugation operator $\mathcal{C}$ is defined so that

$$\mathcal{C}^{\dagger}\psi(x)\mathcal{C} = E\psi^{\dagger}(x) \equiv \psi^{c}(x) . \tag{C.11}$$

The interchange $\psi(x) \leftrightarrow \psi^{c}(x)$ is equivalent to the interchange $a \leftrightarrow b$.

Parity is represented by the operator $\mathcal{P}$, which satisfies

$$\mathcal{P}^{\dagger}\psi_i(x)\,\mathcal{P} = \varepsilon_i\beta\,\psi_i(\tilde{x}) \equiv \psi_i^{p}(x), \tag{C.12}$$

where $\varepsilon_i$ is the intrinsic parity associated with $\psi_i$.

The time reversal matrix $\Omega$ is

$$\Omega = \begin{pmatrix} C^{-1} & 0 \\ 0 & C \end{pmatrix} = \beta E , \tag{C.13}$$

and it satisfies

$$\Omega^{\dagger}\beta\Omega = \beta^{Tr} \tag{C.14}$$

$$\Omega^{\dagger}\alpha_i\Omega = -\alpha_i^{Tr} \tag{C.15}$$

and

$$\Omega^{\dagger}\sigma_i\Omega = -\sigma_i^{Tr} . \tag{C.16}$$

The Wigner complex conjugation operator K satisfies, for all states $\Psi$ and $\Phi$, and all complex numbers $\alpha$ and $\beta$,

$$< K\Psi|K\Phi > = <\Psi|\Phi>^* = <\Phi|\Psi> , \tag{C.17}$$

$$K(\alpha|\Psi > + \beta|\Phi>) = \alpha^*|K\Psi > + \beta^*|K\Phi > , \tag{C.18}$$

and

$$K^2 = 1. \tag{C.19}$$

Defining the operator $A^K$ by

$$< \Psi|A^K|\Psi > \equiv < K\Psi|A|K\Psi > \quad (\text{all } \Psi) \tag{C.20a}$$

one obtains from application of (C.17), (C.18), and (C.19) to

$$< \alpha\Psi + \beta\Phi|A^K|\alpha\Psi + \beta\Phi \; >, \tag{C.20b}$$

with $\alpha = 1$ and $\beta = 1$ and i, the result

$$< \Phi|A^K|\Psi > = < K\Psi|A|K\Phi > \tag{C.21}$$

for all $\Psi$ and $\Phi$. The definition of K is completed by taking

$$|Kx > = \; |x > \tag{C.22a}$$

and

$$|Kp > = |-p > \tag{C.22b}$$

in first quantized theory, and by taking

$$K\psi_{op}(x)K = \psi_{op}(x) \tag{C.23}$$



and

$$K|0 > = \; |0 > \tag{C.24}$$

in second quantized theory. That is, the ket $|x>$ , the field operator $\psi(x)$, and the vacuum are considered real. Thus in first quantized theory if

$$< x|\psi > \; = f(x) \tag{C.25}$$

then

$$< x| \; K\psi> = <Kx|\psi>^* = f^*(x) , \tag{C.26}$$

whereas in second quantized theory if

$$<0|\psi(x)|\Psi> = f(x) \tag{C.27}$$

then

$$\begin{aligned} &< 0|\psi(x)|K\Psi > \\ &\quad = < 0|\psi(x)K\Psi > \\ &\quad = < K0|K\psi(x)K\Psi >^* \\ &\quad = < 0|\psi \; (x)\Psi >^* \\ &\quad = \; f^*(x) . \end{aligned} \tag{C.28}$$

The time reveral operator is then

$$T = UKT \tag{C.29}$$

where

$$T^{\dagger}\psi(x)T = \psi(x^{t}), \tag{C.30}$$

with $x^{t} = (-t, x)$, and

$$U^{\dagger}\psi_{\alpha}(x)U = (\Omega\ \psi(x))_{\alpha}\,. \tag{C.31}$$

For any operator A one may define $A^{t}$ by

$$< T\psi|A|T\psi > = <\psi|A^{t}|\psi > \tag{C.32}$$

for all $\Psi$. Then arguments similar to those leading to (C.21) give

$$A^{t} = T^{\dagger}KU^{\dagger}A^{\dagger}UKT. \tag{C.33}$$

The current and spin operators

$$J_{\mu}(x, t) = \frac{1}{2}\,[\psi^{\dagger}(t, x),\ \alpha_{\mu}\psi(t, x)] \tag{C.34}$$

and

$$\sigma_{i}(x, t) = \frac{1}{2}\,[\psi^{\dagger}(t, x),\ \sigma_{i}\ \psi\ (t, x)] \tag{C.35}$$

then satisfy

$$J^{t}_{\mu}(t, x) = \tilde{J}_{\mu}(-t, x) \equiv (J_{0}(-t, x),\ -\vec{J}(-t, x)) \tag{C.36}$$

and

$$\sigma^{t}_{i}(t, x) = \tilde{\sigma}_{i}(-t, x) = -\sigma_{i}(-t, x)\,. \tag{C.37}$$



Thus the time reversal operation on the states generates the change in expectation values demanded by the physical meaning of the operation of time reversal.

Suppressing the dependence on all other particles one may write the S operator for the scattering of a Dirac particle as

$$S_{op} = \int \psi^{\dagger}(x')G(x';\ x)\psi(x)d^{4}x'd^{4}x. \tag{C.38}$$

The operators $a(p, \lambda, t)$ and $b(p, \lambda, t)$ are normalized by

$$< a(p, \lambda, t)a^{\dagger}(p',\lambda', t') >_{0} = <b(p,\lambda, t)b^{\dagger}(p', \lambda', t') >_{0}$$

$$= 2\omega(2\pi)^{3}\delta^{3}(\vec{p} - \vec{p}')\delta_{\lambda\lambda'}\delta_{tt'} \tag{C.39}$$

where $\omega \equiv (p^{2} + m^{2})^{1/2}$, and $\lambda$ and t are the spin and particle-type labels. Then the S matrix for the scattering of a Dirac particle of type t is

$$S(p',\lambda', t : p,\lambda, t) = < a(p',\lambda', t)\ S_{op}\ a^{\dagger}(p,\lambda, t) >_{0}$$

$$= U^{\dagger}(p',\lambda')G(p', -p)\ U(p,\lambda) \tag{C.40}$$

where the type label on G is suppressed.

The spinors in (C.40) are

$$U(p,\lambda) = \begin{pmatrix} (v\cdot\sigma)^{1/2}\ \phi_\lambda \\ (v\cdot\tilde{\sigma})^{1/2}\ \phi_\lambda \end{pmatrix}$$

$$U^\dagger(p,\lambda) = \begin{pmatrix} \phi_\lambda (v\cdot\sigma)^{1/2} \\ \phi_\lambda (v\cdot\tilde{\sigma})^{1/2} \end{pmatrix}$$

$$V(p,\lambda) = \begin{pmatrix} (v\cdot\sigma)^{1/2}\ \phi_\lambda^c \\ (-)(v\cdot\tilde{\sigma})^{1/2}\ \phi_\lambda^c \end{pmatrix}$$

and

$$V^\dagger(p,\lambda) = \begin{pmatrix} \phi_\lambda^c (v\cdot\sigma)^{1/2} \\ \phi_\lambda^c (v\cdot\sigma)^{1/2}(-1) \end{pmatrix} \qquad \text{(C.41)}$$

where $v = p/m$, $p_0 > 0$, $\phi^c = C\phi$, $\phi_{\lambda\alpha} = \delta_{\lambda\alpha}$, and $((v\cdot\sigma)^{1/2}\phi_\lambda)_\beta = \sum_\alpha (v\cdot\sigma_{\beta\dot{\alpha}})^{1/2}\phi_{\lambda\alpha}$, $(\phi_\lambda (v\cdot\tilde{\sigma})^{1/2})^\beta = \sum_\alpha \phi_{\lambda\alpha}(v\cdot\tilde{\sigma}^{\dot{\alpha}\beta})^{1/2}$ etc.

The M function for the scattering of the Dirac particle is defined by

$$M(p',t;-p,-t) = (v'\cdot\sigma)^{1/2}\, S(p',t:p,t)\,(v\cdot\sigma)^{1/2}. \qquad \text{(C.42)}$$

The insertion of (C.40) and (C.41) gives

$$M(p',t;-p,-t) = (v'\cdot\sigma)G_{UU}(p',-p)\,(v\cdot\sigma)$$

$$+ (v'\cdot\sigma)G_{UL}(p',-p) + G_{LU}(p',-p)v\cdot\sigma + G_{LL}(p',-p), \qquad \text{(C.43)}$$

where the four-by-four matrix G is written as

$$G = \begin{pmatrix} G_{UU} & G_{UL} \\ G_{LU} & G_{LL} \end{pmatrix} . \qquad \text{(C.44)}$$



In terms of the mathematical momentum-energy vectors k' and k, and the associated vectors $u' = k'/m$ and $u = k/m$, Eq. (C.43) can be written

$$M(k',t;k,-t) = (u'\cdot\sigma)G_{UU}(k',k)(-u\cdot\sigma)$$

$$+ (u'\cdot\sigma)G_{UL}(k',k)$$

$$+ G_{LU}(k',k)(-u\cdot\sigma)$$

$$+ G_{LL}(k',k) \quad . \qquad \text{(C.45)}$$

The four-by-four M function is defined by

$$M(k',k) = \begin{pmatrix} M_{UU}(k',k) & M_{UL}(k',k) \\ M_{LU}(k',k) & M_{LL}(k',k) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & u'\cdot\tilde{\sigma} \\ u'\cdot\sigma & 1 \end{pmatrix} G(k',k) \begin{pmatrix} 1 & (-u\cdot\sigma) \\ (-u\cdot\tilde{\sigma}) & 1 \end{pmatrix} \qquad \text{(C.46)}$$

where the type variables t are now suppressed. The original two-by-two M function is $M_{LL}(k',k)$. The other three two-by-two parts are trivially related to $M_{LL}(k',k)$. In particular, one has

$$M(k',k) = \begin{pmatrix} (u'\cdot\tilde{\sigma})M(k',k)(-u\cdot\tilde{\sigma}) & u'\cdot\tilde{\sigma}\ M(k',k) \\ M(k',k)(-u\cdot\tilde{\sigma}) & M(k',k) \end{pmatrix} \qquad \text{(C.47)}$$

where the two-by-two and four-by-four M functions are represented by the same symbol. These relationships, or (C.46), entail the

Dirac equations

$$(u'\cdot\gamma)(\beta M) = (\beta M) = (\beta M)(-u\cdot\gamma) . \qquad (C.48)$$

The two-by-two M-function given in (C.42) refers explicitly to the physical process involving an outgoing dirac particle of type t and an incoming Dirac particle of type t. The function G(x', x) in (C.38) describes also the three related processes in which the incoming particle is changed to an outgoing antiparticle or the outgoing particle is changed to an incoming antiparticle, or both. If, for example, the outgoing particle is changed to an incoming antiparticle then (C.40) is replace by

$$\begin{aligned} S(:p',\lambda',-t;\ p,\ \lambda,\ t) &= \langle S_{op} b^{\dagger}(p',\lambda',\ -t) a^{\dagger}(p,\lambda,t) \rangle_0 \\ &= -\ (V^{\dagger}(p',\lambda'))G(-p',\ -p)U(p,\lambda)) \\ &= \phi^c_{\lambda'}\ [(v'\cdot\sigma)^{1/2}(-1)G_{UL}(-p',\ -p)(v\cdot\sigma)^{1/2} \\ &\quad + (v'\cdot\sigma)^{1/2}(-1)G_{UL}(-p',\ -p)(v\cdot\tilde{\sigma})^{1/2} \\ &\quad + (v'\cdot\tilde{\sigma})^{1/2}G_{LU}(-p',\ -p)(v\cdot\sigma)^{1/2} \\ &\quad + (v'\cdot\tilde{\sigma})^{1/2}G_{LL}(-p',\ -p)(v\cdot\tilde{\sigma})^{1/2}]\phi_{\lambda} . \end{aligned} \qquad (C.40')$$



The charge conjugation operator in $\phi^c$ arises from the convention whereby $\lambda'$ is to be contracted onto the right-hand index of the spin operator if the particle is an initial particle but onto the left-hand index of the spin operator if the particle is a final particle. The convention for M functions is that the contraction rule is independent of whether the particle involved is initial or final, but that the sign of the mathematical rest-frame spin vector $s_r$ occurring in the spin operator $s_r\cdot\sigma$ is tied to the sign of the associated energy component in the manner specified in (2.53). Then (C.3) allows the c on $\phi^c$ in (C.40') to be absorbed into the definition of the spin operators. The S matrix corresponding to the M function is thus the quantity in the brackets in the last line of (C.40'). It is converted to the M function of the process, namely to M(-p', t; -p, -t), by multiplying it by the factors $(v'\cdot\sigma)^{1/2}$ and $(v\cdot\sigma)^{1/2}$, just as in (C.42). This gives

$$\begin{aligned} M(-p',\ t;\ -p,\ -t) = & \\ & -\ v'\cdot\sigma\ G_{UU}(-p',\ -p)\ v\cdot\sigma \\ & -\ v'\cdot\sigma\ G_{UL}(-p',\ -p) \\ & +\ G_{LU}(-p',\ -p)\ v\cdot\sigma \\ & +\ G_{LL}(-p',\ -p)\ . \end{aligned} \qquad (C.43')$$

For initial particles v = - u. Thus

this equation is a special case of (C.45). Thus the two different processes are described by the one function $M(k', t; k, -t)$ defined in (C.45), evaluated in different regions in $(k', k)$ space. The function $G(k', k)$ in these different regions is obtained from the Fourier transformation of the single function $G(x', x)$.

The third case is that in which the original incoming particle is changed to an outgoing particle. Then (C.40) is replaced by

$$
\begin{aligned}
S(p',\lambda', t; p,\lambda, -t:) &= \langle a(p',\lambda', t) b(p,\lambda,-t) S_{op} \rangle_0 \\
&= - U^{+}(p',\lambda') G(p', p) V(p,\lambda) \\
&= \phi_{\lambda'} [ (v'\cdot\sigma)^{1/2} G_{UU}(p', p) (v\cdot\sigma)^{1/2} (-1) \\
&\quad + (v'\cdot\sigma)^{1/2} G_{UL}(p', p) \\
&\quad + (v'\cdot\tilde{\sigma})^{1/2} G_{LU}(p', p) (v\cdot\sigma)^{1/2} (-1) \\
&\quad + (v'\cdot\tilde{\sigma})^{1/2} G_{LL}(p', p) (v\cdot\tilde{\sigma})^{1/2} ] \phi^{c}_{\lambda} . \qquad (C.40'')
\end{aligned}
$$

The S matrix corresponding to this M function is the quantity in the bracket in the last line of (C.40''). It is converted to the M function by multiplying it by $(v'\cdot\sigma)^{1/2}$ and $(v\cdot\sigma)^{1/2}$, just as in (C.42). This gives



$$
\begin{aligned}
M(p', t; p, -t) &= \\
&= v'\cdot\sigma \; G_{UU}(p', p)(-v\cdot\sigma) \\
&\quad + v'\cdot\sigma \; G_{UL}(p', p) + G_{LU}(p', p)(-v\cdot\sigma) + G_{LL} \qquad (C.43'')
\end{aligned}
$$

Since both particles are final one has $u = v$ and $u' = v'$. Thus this is also a special case of the function defined in (C.45).

The fourth and final case is similar. The order in which the arguments of the M function are placed is the same as the order of the operators that create or annihilate the corresponding particles from the vacuum. (See (C. 40), (C. 40'), and (C.40''), and the corresponding equations (C.43), (C.43'), and (C.43'').) Then the fourth case gives

$$
\begin{aligned}
-M(p, -t; -p', t) &= M(-p', t; p, -t) \\
&= (-v'\cdot\sigma) \; G_{UU}(-p', p)(-v\cdot\sigma) + (-v'\cdot\sigma) \; G_{UL}(-p', p) \\
&\quad + G_{LU}(-p', p)(-v\cdot\sigma) + G_{LL}(-p', p), \qquad (C.43'')
\end{aligned}
$$

where the antisymmetry of M under the interchange of spin-$\frac{1}{2}$ variables is used. The second part of (C.43'') is equivalent to (C. 45), and hence the one function $M(k', t; k, \bar{t})$ describes all four processes.

The formulas given above refer to a single spin-$\frac{1}{2}$ particle. But they immediately generalize to the case of n spin-$\frac{1}{2}$ particles: one treats each such particle in the manner shown above.

The parity transformation P defined in (C.12) induces in $G(k', k)$ the change

$$
G(k', k) \rightarrow \epsilon(\pi\otimes\beta_i) G(\tilde{k}', \tilde{k}) ( \prod_i \otimes \beta_i ), \qquad (C.49)
$$

where $\varepsilon$ is the product of intrinsic parities at the fields, and i runs over the particle-antiparticle pairs. A little algebra shows that this transformation on G induces the same transformation on the M function defined in (C.46):

$$M(k', k) \to \varepsilon(\prod_i \otimes \beta_i) M(\tilde{k}', \tilde{k})(\prod_i \otimes \beta_i) \ . \qquad (C.50)$$



Consideration of the no-scattering part entails that the product of the intrinsic parities of $\psi_i(x)$ and $\psi_i^\dagger(x)$ is unity. Thus the intrinsic parities of the spin-$\frac{1}{2}$ fields drop out of $\varepsilon$, as in (2.56). Then (2.58) allows (2.55) to be written as

$$PM(k', t; k, -t) = \varepsilon(u'\cdot\sigma)M(\tilde{k}', t; \tilde{k}, -t)(-u\cdot\sigma), \qquad (C.51)$$

where $\varepsilon$ is the product of the intrinsic parities of the scalar particles. The parity transformation (C.49) on the function G occurring in the expression (C.45) for the two-by-two M function induces the transformation $M \to PM$, with $PM$ defined by (C.51).

The antiparticle conjugation operation $\psi(x) \leftrightarrow \psi^c(x)$ defined in (C.11) replaces $S_{op}$ by

$$\begin{aligned} S_{op}^c &= \int ((\psi^c(x'))^\dagger \, G(x' \; x) \psi^c(x) d^4x' d^4x \\ &= \int (E\psi^\dagger(x'))^\dagger \, G(x', x) \; E\psi^\dagger(x) \; d^4x' d^4x \\ &= \int \psi^\dagger(x)(-E^\dagger G(x', x)E)^{Tr} \psi(x) \; d^4x' d^4x . \end{aligned} \qquad (C.52)$$

Comparison to (C.38) shows that the antiparticle conjugation operation is equivalent to the operation

$$\begin{aligned} G(x', x) &\leftrightarrow G^c(x', x) \\ &= (-E^\dagger G(x, x')E)^{Tr} \\ &= - E \, G^{Tr}(x, x')E , \end{aligned} \qquad (C.53)$$

where Tr means transpose in spin space.

The momentum-energy space version of (C.53) is obtained by replacing x and x' by k and k', respectively. This transformation on G(k', k) induces on the four-by-four M function defined in (C.46) the transformation

$$M(k', t; k', -t) \leftrightarrow M^{C}(k', t' k, -t)$$

$$= \begin{pmatrix} 1 & u'\cdot\hat{\sigma} \\ u'\cdot\sigma & 1 \end{pmatrix} \begin{pmatrix} -E\, G^{Tr}(k, k')E \end{pmatrix} \begin{pmatrix} 1 & -u\cdot\sigma \\ -u\cdot\tilde{\sigma} & 1 \end{pmatrix}$$

$$= -\, E\, M^{Tr}(k, t; k', -t)E\ . \qquad \text{(C.54)}$$

It induces on the two-by-two M function defined in (C.45) the transformation $M(k', t; k, -t) \leftrightarrow C\, M(k', t' k, -t)$, where $CM$ is defined by (2.61).

The parity (or antiparticle conjugation) operation acting on M converts it to the M function that describes processes in a conceivable world in which the amplitude for any process P is equal to the amplitude that the parity inverse (or antiparticle conjugate) of P has in the actual physical world. The analogous time reversal operation on the four-by-four M function is obtained by making the substitution $S_{op} \leftrightarrow S^{t}_{op}$, defined by (C.38) and (C.33):



$$S_{op} \leftrightarrow S^{t}_{op}$$

$$= T^{\dagger}KU^{\dagger}S^{\dagger}_{op}\ UKT$$

$$= TKU^{\dagger} \int \psi^{\dagger}(x)\ G^{\dagger}(x', x)\ \psi(x')\ d^4x'd^4x\ UKT$$

$$= \int K(U\psi^{\dagger}(x^{t})U)\ G^{\dagger}(x', x)\ U^{\dagger}\psi(x'^{t})UK\ d^4x'd^4x$$

$$= \int K\psi^{\dagger}(x^{t})\Omega^{Tr}G^{\dagger}(x', x)\Omega\psi(x'^{t})K\ d^4x'd^4x$$

$$= \int K\psi^{\dagger}(x')\Omega^{Tr}G^{\dagger}(x^{t}, x'^{t})\Omega\psi(x)K\ d^4x'd^4x$$

$$= \int \psi^{\dagger}\ (x')\Omega^{\dagger}\ G^{Tr}(x^{t}, x'^{t})\Omega\ \psi(x)\ d^4x'd^4x\ . \qquad \text{(C.55)}$$

Thus the time-reversal operation on the states or fields is equivalent to the operation on G

$$G(x', x) \leftrightarrow G^{t}(x', x)$$

$$= \Omega^{\dagger}G^{Tr}(x^{t}, x'^{t})\Omega\ , \qquad \text{(C.56)}$$

or equivalently

$$G(k', k) \leftrightarrow G^{t}(k', k)$$

$$= \Omega^{\dagger}G^{Tr}(-\tilde{k}, -\tilde{k})\Omega\ . \qquad \text{(C.57)}$$

This transformation induces in the four-by-four M function defined by (C.46) the analogous transformation

$$M(k', k) \leftrightarrow M^{t}(k', k) = \Omega^{\dagger} M^{Tr}(-\tilde{k}, -\tilde{k}')\Omega. \qquad (C.58)$$

The transformation (C.57) induces in the two-by-two M function defined by (C.45) the transformation

$$M(k', t; k, -t) \leftrightarrow \mathcal{T} M(k', t; k, -t)$$

$$= \tilde{M}(-\tilde{k}, t; -\tilde{k}', -t) \qquad (C.59)$$

In a model where all particles are constructed from spin-$\frac{1}{2}$ particles the intrinsic parity factor $\varepsilon$ is unity. Then the product of parity inversion, antiparticle conjugation, and time reversal on the four-by-four M function gives, by virtue of (C.50), (C.54), (C.58), and (C.13),

$$M(k', t; k, -t) \rightarrow \mathcal{TCP}\, M(k', t; k, -t)$$

$$= -\, M(-k', t; -k, -t) \qquad (C.60)$$

The same result holds for the two-by-two submatrix $M_{LL} = M$.

The two-by two M function has one dotted and one undotted index. Thus, by virtue of (2.70), the change of the sign of all of its vector arguments changes its sign. Thus the two-by-two M function is transformed into precisely itself by the product of the $\mathcal{T}$, $\mathcal{C}$, and $\mathcal{P}$



transformations.

The formula (C.60) refers to a situation involving only one spin-$\frac{1}{2}$ particle. For the case of n such particles the variables k', k, and t are 4n-vectors, or n-vectors, and the minus sign in front of M should be $(-1)^n$. Thus the product of the transformations $\mathcal{T}$, $\mathcal{C}$, and $\mathcal{P}$ again leaves M invariant.

In carrying out the calculations whose results are summarized above it is helpful to recall that the two-by-two function $\tilde{M}$ can be expressed as $C^{-1} M^{Tr} C$. If M is a product of Pauli matrices then $\tilde{M}$ is obtained from M by transposing the order of these matrices and replacing each $\sigma_i$ (i = 1, 2, 3) by $-\sigma_i$.

The Lorentz transformation properties of the two-by-two M function is indicated by assigning spinor index types according to the rule

$$M(K', t; k, -t) = M_{LL}(k', t; k, -t) \rightarrow$$

$$M_{\alpha\dot{\beta}}(k', t; k, -t) = M(k', \alpha, t; k, \dot{\beta}, -t) \qquad (C.61a)$$

The transformation properties of several other two-by-two functions are indicated by the following assignments:

$$\begin{aligned}
\tilde{M}(k, t; k', -t) &\rightarrow \tilde{M}^{\dot{\alpha}\beta}(k, t; k', -t) \\
M(\tilde{k}', t; \tilde{k}, -t) &\rightarrow M^{\dot{\alpha}\beta}(\tilde{k}', t; \tilde{k}, -t) \\
M_{UU}(k', t; k, -t) &\rightarrow M_{UU}^{\dot{\alpha}\beta}(k', t; k, -t) \\
M_{UL}(k', t; -k, -t) &\rightarrow M_{UL\dot{\beta}}^{\dot{\alpha}}(k', t; k, -t) \\
M_{LU}(k', t; -k, t) &\rightarrow M_{LU\alpha}{}^{\beta}(k', t; k, -t)\,.
\end{aligned} \qquad (C.61B)$$

In these equations one may interprete the variables k' and k as 4n vectors, the variables t as n-vectors, and the variables $\alpha$ and $\beta$ as n-spinors: e.g., $\alpha \doteq (\alpha_1, \ldots \alpha_n)$. Our convention is that the spinor $\alpha$ goes with k' and the spinor $\beta$ goes with k. The quantities on the right-hand sides of (C.61) are invariant under the simultaneous action of the spinor transformation (2.9) and $(k', k) \to (L^{-1}(\Lambda)k', L^{-1}(\Lambda)k)$.

Comparison of (C.61) to (C.47) shows that one may interprete $(u'\cdot\tilde{\sigma})$ and $(-u\cdot\tilde{\sigma})$ acting on the left and right, respectively, as operators that simultaneously raise and dot (or undot) the indices $\alpha$ and $\dot{\beta}$, respectively. Then the subscripts U and L on the two-by-two functions M(k', t; k, -t) can be dropped and the four-by-four M function written as

$$M(k', t; k, -t) = \begin{pmatrix} M^{\dot{\alpha}\beta}(k', t; k, -t) & M^{\dot{\alpha}}{}_{\dot{\beta}}(k', t; k, -t) \\ M_{\alpha}{}^{\beta}(k', t; k, -t) & M_{\alpha\dot{\beta}}(k', t; k, -t) \end{pmatrix} \tag{C.62}$$

In formulas (3.5a) and (3.5b) for the ortho and para amplitudes corresponding to quark closed loops the (quark) M functions are just the products $\prod_i (u_i'\cdot\sigma)$ and $\prod_i (-u_i\cdot\sigma)$ respectively. In both cases the individual two-by-two M functions are $M_{LL}$ and their spinor index types are as in $M_{\alpha\dot{\beta}}$. However, it is possible to use different choices of index type, provided one makes compensating changes in the spin operators that occur in (3.5). One convenient choice is to use $M^{\dot{\alpha}}{}_{\dot{\beta}}$ for the ortho propagators and $M_{\alpha}{}^{\beta}$ for the para propagators. Comparison of (C.62) to (C.47) shows that the ortho propagator is then $\delta^{\dot{\alpha}}{}_{\dot{\beta}}$ and the para progator is $\delta_{\alpha}{}^{\beta}$.



The propagators can be considered the analogs of the functions G of field theory. Thus we write

$$G^0 = \begin{pmatrix} 0 & \delta^{\dot{\alpha}}{}_{\dot{\beta}} \\ 0 & 0 \end{pmatrix} \tag{C.63a}$$

and

$$G^p = \begin{pmatrix} 0 & 0 \\ \delta_{\alpha}{}^{\beta} & 0 \end{pmatrix} . \tag{C.63b}$$

The associated spin factors are

$$F_i^0 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ (s_i\cdot\tilde{\sigma}\, u_i\cdot\sigma)^{\dot{\alpha}}{}_{\dot{\beta}} & 0 \end{pmatrix} \tag{C.64a}$$

and

$$F_i^p = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & (i\, u_i\cdot\sigma\, s_i\cdot\tilde{\sigma})_{\alpha}{}^{\beta} \\ 0 & 0 \end{pmatrix} \tag{C.64b}$$

Then one may write, in place of (3.5),

$$A(G^0, k, s) = -\left(\mathrm{Tr} \prod_i (F_i^0 G^0)\right) f(k). \tag{C.65a}$$

and

$$A(G^p, k, s) = -\left(\mathrm{Tr} \prod_i (F_i^p G^p)\right) f(k). \tag{C.65b}$$

The sum of the two amplitudes is then

$$A^0 + A^p = -\left(\mathrm{Tr} \prod_i (F_i' G)\right) f(k) \tag{C.66}$$

where

$$G = G^0 + G^p \text{ and } F_i' = F_i^0 + F_i^p.$$

Equation (C.40) shows that the four-by-four matrix G occurs in the form $U^{\dagger}GU$. The matrix $\beta G$ occurs in $\bar{U}\beta GU$, where $\bar{U} = U^{\dagger}\beta$. Correspondingly, (C.66) may be written in the alternative form

$$A^0 + A^p = -(\mathrm{Tr}\prod_i (F_i'\beta)(\beta G)f(k) \tag{C.67}$$

where

$$\beta G = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix} \tag{C.68}$$

and

$$F_i'\beta = i\begin{pmatrix} (-u_i\cdot\sigma)(\tilde{s}_i\cdot\sigma)_\beta{}^\alpha & 0 \\ 0 & (\tilde{s}_i\cdot\sigma)(\tilde{u}_i\cdot\sigma)^{\dot\beta}{}_{\dot\alpha} \end{pmatrix} \tag{C.69}$$

Then the contributions $A^0$ and $A^p$ come from the lower left-hand and upper right-hand two-by-two sectors, respectively.

If the meson corresponding to vertex i is a spin-zero meson then $s_i = u_i$, and (C.67) becomes

$$F_i'\beta = i\begin{pmatrix} (-u_i\cdot\sigma)(u_i\cdot\tilde{\sigma}) & \\ & (u_i\cdot\tilde{\sigma})(u_i\cdot\sigma) \end{pmatrix}$$

$$= i\gamma_5 \,. \tag{C.70a}$$

On the other hand, if the meson associated with vertex i is a spin-one meson then $u\cdot s = 0$ and

$$F_i'\beta = i\begin{pmatrix} (s_i\cdot\sigma)(u_i\cdot\tilde{\sigma}) & 0 \\ 0 & (s_i\cdot\tilde{\sigma})(u_i\cdot\sigma) \end{pmatrix} \tag{C.70b}$$

$$= s^\mu\sigma_{\mu\rho}u^\rho$$

where



$$\sigma_{\mu\rho} = \frac{i}{2}(\gamma_\mu\gamma_\rho - \gamma_\nu\gamma_\rho). \tag{C.71}$$

Thus the coupling of the spin-zero particle to the ortho and para quarks are via the matrices $i\gamma_5(1+\gamma_5)/2$ and $i\gamma_5(1-\gamma_5)/2$ respectively. The couplings of the spin-one particle to the ortho and para quarks are via the matrices $s^\mu\sigma_{\mu\nu}u^\nu(1+\gamma_5)/2$ and $s^\mu\sigma_{\mu\nu}u^\nu(1-\gamma_5)/2$, respectively.

This separation of the ortho and para contributions into the two orthogonal parts of spin space means that at the zero-entropy level the ortho and para quarks are, in effect, distinct entities: they are represented by orthogonal states. The rules shown in Fig. 8 mean that there are also, in effect, two different kinds of mesons, one composed of a quark and antiquark of ortho type the other composed of a quark and antiquark of para type.

The close analogy at the zero-entropy level between the ortho-para types and the flavor types suggests that one should allow, at the zero-entropy level, also the mesons built from an ortho quark and a para antiquark, or from a para quark and an ortho antiquark. The coupling of these two new types of mesons will be obtained by filling the two empty spaces in the coupling matrix $F_i\beta$ of (C.69). Indeed, if one goes back to the two-component formalism, and follows the normal and natural practice of imposing parity invariance on each quark propagator individually, rather than on the process as whole, then the function $F_i\beta$ becomes

$$\Gamma_i = F_i\beta = i\begin{pmatrix} ((-u_i\cdot\sigma)(s_i\cdot\tilde{\sigma}))_\beta{}^\alpha & ((-u_i\cdot\sigma)s_i\cdot\tilde{\sigma}(u_i\cdot\sigma))_{\beta\dot\alpha} \\ s_i\cdot\tilde{\sigma}^{\dot\beta\alpha} & ((s_i\cdot\tilde{\sigma})(u_i\cdot\sigma))^{\dot\beta}{}_{\dot\alpha} \end{pmatrix} \tag{C.72}$$

For the case of a spin-zero meson, where $s_i = u_i$, the two new terms reduce to

$$F''_{i\beta} = i \begin{pmatrix} 0 & -u_i\cdot\sigma \\ u_i\cdot\tilde{\sigma} & 0 \end{pmatrix} = i\gamma_5(\gamma_\mu\cdot u_i^\mu). \tag{C.73a}$$

For the case of a spin-one meson, where $u_i\ s_i = 0$, the two new terms reduce to

$$F''_{i\beta} = i \begin{pmatrix} 0 & s_i\cdot\sigma \\ s_i\cdot\tilde{\sigma} & 0 \end{pmatrix} = i\gamma_\mu\cdot s_i^\mu \quad . \tag{C.73b}$$

The explicitly appearing spin vector $s^\mu$ in (C.70b) and (C.73b) can be eliminated, since the index $\mu$ against which it is contracted can play an equivalent role. Recall that the summation over the three physical spin-one states is represented, as in (3.19), by

$$\sum_{e=1}^{3} s_e^{\text{phys}\ \mu}\ s_e^{\text{phys}\ \nu}$$

$$= (-\ g^{\mu\nu} + v^\mu v^\nu)$$

$$= \sum_{e=1}^{3} (i\ s_e^{\text{math}\,\mu})(\ is_e^{\text{math}\ \nu}) \tag{C.74}$$

Thus the vectors $is_e^{\text{math}\ \mu}$ in (C.70b) and (C.73b) can be omitted and the index $\mu$ of $\sigma_{\mu\rho}$ or $\gamma_\mu$ contracted directly onto the metric tensor $(-\ g^{\mu\nu} + v^\mu v^\nu)$.

The result can now be compared to the results of Bardakci and Halpern,[26] who use the standard four-component formalism. The complete coupling of the spin-zero meson is via the factor



$$\Gamma_j = i\gamma_5(1 + \gamma\cdot\mu_j), \tag{C.75a}$$

whereas the complete coupling of the spin-one meson is via the factor

$$\Gamma_j = (\gamma_\mu - i\ \sigma_{\mu\rho}u_j^\rho). \tag{C.75b}$$

where $u_j = k_j/m = v_j \operatorname{sign} u_j^0$, and $v_j$ is the four-velocity of the meson j.

The sum of all of the zero-entropy amplitudes corresponding to a fixed cyclic order of the meson variables is the common scalar factor f(k) times the trace of the cyclically ordered product of factors $\Gamma_i$:

$$A(k_1, \ldots k_n) = -\mathrm{Tr}(\Gamma_1\Gamma_2, \ldots\Gamma_n)f_n(k_1, \ldots, k_n), \tag{C.76}$$

These coupling are the coupling associated with positive metric pseudoscalar and vector mesons. These mesons are the mesons that are the basic particles of the ordered Hilbert space, and thus of the physical Hilbert space.

The factorization property does not hold for the sum of zero-entropy amplitudes discussed above. It holds rather for the individual zero-entropy amplitudes. An individual zero-entropy amplitude is obtained by assigning to each quark line segment of the closed loop of, say, Fig. 1 an ortho or para label, and inserting an associated factor of $(1 + \gamma_5)/2$ or $(1 - \gamma_5)/2$, respectively, between the corresponding factors $\Gamma_i$ and $\Gamma_{i+1}$ of the trace in (C.76).

Notice that when a meson is coupled to a zero-entropy function only one or the other of the two terms of (C.75a) or (C.75b) will contribute, and this term will be the same at the vertices lying on the two ends of the meson connection line, by virtue of the identity represented in Fig. 8, and the two similar identities associated with the two other ortho-para type mesons. Thus the only coupling matrices that enter are those associated with positive metric pseudo-scalar and vector mesons.

The present theory thus resolves simultaneously four serious difficulties that have long plagued this kind of approach. These problems are[6,26] first the apparent necessity for a doubling of the pseudo-scalar and vector particles; second, the apparent necessity for a parity doublet partner of each of the above mentioned particles; third the apparent demand that each of these parity doublet partners have the wrong metric (i.e., be a ghost), and fourth the lack of any rationale for the empirically observed $SU(6)_W$ symmetry of vertices. This latter symmetry emerges automatically in the present theory for the amplitudes formed as the sum of zero-entropy amplitudes, provided all momentum vectors $\vec{p}_j$ are parallel to the third coordinate axis.



## ACKNOWLEDGEMENT

This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract W-7405-ENG-48.

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This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

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*LAWRENCE BERKELEY LABORATORY*
*UNIVERSITY OF CALIFORNIA*
*BERKELEY, CALIFORNIA 94720*