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Inseparability of Fermions in a Model of Cyclic Symmetric Field Theory

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Abstract:

In a field theory model of two scalar fields and two massless fermion fields with internal cyclic symmetry Z(3) there exists an algebraic relation between the fermion fields that makes them inseparable in both 3 + 1 and 1 + 1 dimensions.

Solutions of the field equations in 1 + 1 dimensions show bound states for the fermions in interaction with the solitons generated by the scalar fields.

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1. DEFINITION OF THE MODEL

The variables of the model considered are two classical scalar fields ϕ_1 and ϕ_2 and two fermion fields ψ_1 and ψ_2 defined on the space-time manifold.

We construct the matrix fields

$$\phi = \begin{pmatrix} \phi_1 & -\phi_2 \\ -\phi_2 & -\phi_1 \end{pmatrix} = \sigma_3\phi_1 - \sigma_1\phi_2 \quad , \tag{1}$$

where σ_1, σ_2, σ_3 are the Pauli matrices and

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \; ; \; \bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2), \; \bar{\psi}_k = \psi_k^+\gamma_0, \; (k = 1,2) \quad . \tag{2}$$

The Lagrangian density of our model is a trace of 2 x 2 matrices

$$\begin{aligned} \mathcal{L} = & \frac{1}{4}\,\mathrm{Tr}(\partial_\mu\phi)^2 - \frac{1}{2}\,\mathrm{Tr}[\lambda\phi^4 - \alpha(\sigma_3\phi)^3 - \beta\phi^2 - \delta] \\ & + i\bar{\psi}\gamma_\mu\overleftrightarrow{\partial}_\mu\psi - m\bar{\psi}\psi + g\bar{\psi}\phi\psi \end{aligned} \tag{3}$$

with the partial derivatives $\partial_\mu = \partial/\partial x_\mu$, $\overleftrightarrow{\partial}_\mu = \frac{1}{2}(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)$ and the anticommuting gamma matrices $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$.

The coupling constants $\lambda > 0$, $\alpha < 0$, $\beta > 0$ determine the strength of the self-interaction of the scalar fields, and the constant δ is defined to set the potential density in the vacuum state to zero V(vacuum) = 0. m is the mass of the fermion fields, and g is the coupling constant of the Yukawa interaction between scalar and fermion fields.

In 3 + 1 dimensions $\mu,\nu = 0,1,2,3$: γ_μ are the Dirac-gamma matrices and the metric tensor is $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. In 1 + 1 dimensions $\mu,\nu = 0,1$ and $\gamma_0 = -\sigma_2$, $\gamma_1 = i\sigma_1$ (Ref. 1), $g_{\mu\nu} = \mathrm{diag}\,(1,-1)$.

The Lagrangian (3) is invariant under the transformations of the Poincaré group and has the internal symmetry Z(3), the cyclic group of order three:

$$\phi \to R^{+}\phi R,\ \psi \to R^{2}\psi, \bar{\psi} \to \bar{\psi}\left(R^{+}\right)^{2} \quad , \tag{4}$$

with the representation

$$R = \exp(-\frac{i}{2}\theta\sigma_2),\ \theta = \frac{2\pi}{3}n,\ n = 0,1,2 \quad . \tag{5}$$

R^3 is the unit matrix which guarantees the invariance of the interaction term $\bar{\psi}\phi\psi$.

The field equations are coupled differential equations

$$\partial_\mu^2\phi_1 = -4\lambda(\phi_1^2 + \phi_2^2)\phi_1 + 3\alpha(\phi_1^2 - \phi_2^2) + 2\beta\phi_1 - g(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2) \tag{6}$$

$$\partial_\mu^2\phi_2 = -4\lambda(\phi_1^2 + \phi_2^2)\phi_2 - 6\alpha\phi_1\phi_2 + 2\beta\phi_2 - g(-\bar{\psi}_1\psi_2 - \bar{\psi}_2\psi_1) \tag{7}$$

$$i\gamma_\mu\partial_\mu\psi_1 - m\psi_1 = -g(\psi_1\phi_1 - \psi_2\phi_2) \tag{8}$$

$$i\gamma_\mu\partial_\mu\psi_2 - m\psi_2 = -g(-\psi_2\phi_1 - \psi_1\phi_2) \quad . \tag{9}$$

2. INSEPARABILITY OF FERMIONS

From the linear structure of the equations (8) and (9) and the symmetry of the problem, we find as a main result a connection between the fermion fields independent of the form of the fields ϕ_1 and ϕ_2.

Theorem: For massless m = 0 fermion fields, any pair of solutions ψ_1 and ψ_2 of the system of equations (8) and (9) are related algebraically:

$$\psi_2(t,x) = A\psi_1(t,x) \tag{10}$$

with $A = i\gamma_5$.

Remark: We choose the γ_5 matrix to be $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, $\gamma_5^2 = +1$, $\{\gamma_5,\gamma_\mu\} = 0$: $\mu = 0,1,2,3$ in 3 + 1 dimensions: and $\gamma_5 = \gamma_0\gamma_1 = -\sigma_3$, $\gamma_5^2 = +1$, $\{\gamma_5,\gamma_\mu\} = 0$, $\mu = 0,1$ in 1 + 1 dimensions.

Proof: Substitute ψ_2 from (10) into Eqs. (8) and (9). Set m = 0. Then:

$$i\gamma_\mu\partial_\mu\psi_1 = -g(\psi_1\phi_1 - A\psi_1\phi_2) \tag{11}$$

$$i\gamma_\mu\partial_\mu A\psi_1 = -g(-A\psi_1\phi_1 - \psi_1\phi_2) \quad . \tag{12}$$

Equation (12) multiplied from the left by A is identical to (11) because $A\gamma_\mu A = +\gamma_\mu$ and $A^2 = -1$ on both four and two dimensional space-time manifolds. This makes the fermions inseparable.

If one component becomes zero the other vanishes too or the presence of one component implies the presence of the second component.

3. SOLUTIONS IN 1 + 1 DIMENSIONS

Approximate solutions of the field equations are found in 1 + 1 dimensions in an iterative way:

(i) First we solve exactly the system Eqs. (6), (7) without any fermions present $\psi_1 = \psi_2 = 0$,

(ii) then we solve exactly the equation of motion (8) and (9) of the fermions in the scalar field, and

(iii) calculate in a linear approximation the correction of the scalar fields due to the presence of fermions. And last we

(iv) check that the new scalar inserted into the field fermionic equations (8) and (9) does not change the fermionic currents but only adds a complex phase to the fermionic fields.

(i) Solutions of the scalar field equations $\psi_1 = \psi_2 = 0$

Because of the cyclic symmetry the potential

$$V(\phi_1,\phi_2) = \lambda(\phi_1^2 + \phi_2^2)^2 - \alpha(\phi_1^3 - 3\phi_1\phi_2^2) - \beta(\phi_1^2 + \phi_2^2) - \delta \tag{13}$$

has three minima or vacuum states located at $\Omega_1 = (\psi_V, 0)$, $\Omega_2 = (-\frac{1}{2}\phi_V, \frac{\sqrt{3}}{2}\phi_V)$ and $\Omega_3 = (-\frac{1}{2}\phi_V, \frac{-\sqrt{3}}{2}\phi_V)$ such that the potential and its first derivatives vanish in all Ω's. $V(\Omega_1) = V(\Omega_2) = V(\Omega_3) = 0$, $(\partial V/\partial\phi_1)(\Omega_1) = 0$ (defines ϕ_V).

Using the trajectory method of Rajaraman[2] for coupled nonlinear differential equations in one dimension we find time-independent soliton solutions.[3,4] When $\alpha = \frac{2}{3}\lambda\phi_V$, $\beta = \lambda\phi_V^2$, and $\delta = -\frac{2}{3}\lambda\phi_V^4$, a linear trajectory connecting the minima Ω_1 and Ω_2 -- $\phi_1 + \sqrt{3}\phi_2 = \phi_V$ -- decouples the equations and leads to the solutions:

$$\phi_1(x) = \frac{1}{4}\,\phi_V[1 + 3\tanh(ax)] \quad , \tag{14}$$

$$\phi_2(x) = \frac{\sqrt{3}}{4}\,\phi_V[1 - \tanh(ax)] \quad , \tag{15}$$

with $\quad a = \phi_V\,\sqrt{3\lambda/2}$.

The fields are combinations of constants and kinks of the single field φ^4-theory.[5] The asymptotic values are $\lim_{x\to+\infty}(\phi_1,\phi_2) = \Omega_1$ and $\lim_{x\to-\infty}(\phi_1,\phi_2) = \Omega_2$.

The choice of the coupling constants α and β implies a linear trajectory in the sense that the saddle points of the potential surface $V(\phi_1,\phi_2)$ lie on the straight lines connecting the minima Ω pairwise. The tunneling of the solutions occurs along these lines through the saddle points.

Two more pairs of solutions are obtained from (14) and (15) by rotations of 120° in ϕ space.

The energy density of the soliton configurations is

$$T_{00}(x) = \frac{9}{8}\lambda\phi_V^4\,[\cosh(ax)]^{-4} \tag{16}$$

and the total energy or mass is

$$M^\circ = \int_{-\infty}^{\infty} T_{00}(x)dx = \phi_V^3\,\sqrt{3\lambda/2} \quad . \tag{17}$$

(ii) <u>The motion of fermions in the scalar soliton field</u>

Due to the theorem Eq. (10) it is sufficient to solve Eq. (11) for ψ_1.

$$i\left(\gamma_0 \frac{\partial}{\partial t} - \gamma_1 \frac{\partial}{\partial x}\right)\psi_1(t,x) = -g[\psi_1(t,x)\phi_1(x) + i\sigma_3\psi_1(t,x)\phi_2(x)] \qquad (18)$$

Because the variables t and x separate, we assume a stationary time dependence for ψ_1

$$\psi_1(t,x) = NU \exp\,[-iEt - H(x)] \quad , \qquad (19)$$

where E is the energy, N the normalization factor, U a two component spinor, and H(x) a positive definite function.[6]

Equation (18) splits into two parts: the time dependent but x-independent part

$$i\gamma_0\partial_0\psi_1 = -g\,\frac{1}{4}\,\phi_V(1 + i\sigma_3\,\sqrt{3})\psi_1 \qquad (20)$$

and the x-dependent part of the equation

$$-i\gamma_1\partial_1\psi_1 = -g\,\frac{\sqrt{3}}{4}\,\phi_V\,\tanh(ax)(\sqrt{3} - i\sigma_3)\psi_1 \quad . \qquad (21)$$

Equation (20) determines the energy:

$$E = (\text{sign } E)\,\frac{1}{2}\,g\phi_V \qquad (22)$$

(sign E = ±1) and the spinor U (by choice the upper component is one $u_1 = 1$):

$$U = \begin{pmatrix} 1 \\ -(\text{sign } E)\,\exp\,(-i\frac{\pi}{6}) \end{pmatrix} \quad . \qquad (23)$$

Equation (21) determines H(x):

$$H(x) = -\,(\text{sign } E)(2\lambda)^{-1/2}\,g\,\ln\,[\cosh(ax)] \quad . \qquad (24)$$

For the normalizability of the fermionic fields H(x) must be a positive definite function, i.e. sign E = -1.

Then $\psi_2 = -i\sigma_3\psi_1$ and the fermionic currents are

$$j_1(x) = \psi_1\psi_1 - \psi_2\psi_2 = 4N^2 \sin\left(\frac{\pi}{6}\right) \exp[-2H(x)] \ , \tag{25}$$

and

$$j_2(x) = -\bar{\psi}_1\psi_2 - \bar{\psi}_2\psi_1 = 4N^2 \cos\left(\frac{\pi}{6}\right) \exp[-2H(x)] \ . \tag{26}$$

The normalization constant N is defined by:

$$\int_{-\infty}^{\infty} \psi_1^+\psi_1 dx = 1 \quad \text{, which also implies} \quad \int_{-\infty}^{\infty} j_1(x)dx = 1 \ . \tag{27}$$

We obtain an exact expression[7] for N^2:

$$\begin{aligned} N^2 &= (3\lambda/8)^{1/2}\phi_V\Gamma\left[\frac{1}{2} + (2\lambda)^{-1/2}g\right] \Gamma^{-1}(\tfrac{1}{2})\Gamma^{-1}[(2\lambda)^{-1/2}g] \\ &= (3\lambda/8)^{1/2}\phi_V B^{-1}\left[\tfrac{1}{2}, g(2\lambda)^{-1/2}\right] \ . \end{aligned} \tag{28}$$

where B is the beta function.

The kinetic energy of the fermion fields is zero because $\bar{U}\gamma_1 U = 0$.

The energy contribution comes from their interaction with the scalar soliton field. The energy density is

$$T_{00}^{int}(x) = -\ 2gN^2\phi_V\ [\cosh(ax)]^{-g\sqrt{2/\lambda}} \tag{29}$$

and the total energy

$$\int_{-\infty}^{\infty} T_{00}^{int}(x)dx = -g\phi_V \tag{30}$$

is twice the energy E of Eq. (22).

(iii) <u>Feedback of the fermions onto the scalar fields</u>

We expand the scalar fields Eq. (14) and Eq. (15) by φ_1 and φ_2. The new fields are

$$\tilde{\phi}_1(x) = \phi_1(x) + \varphi_1(x) \quad , \tag{31}$$

$$\tilde{\phi}_2(x) = \phi_2(x) + \varphi_2(x) \quad . \tag{32}$$

The equations for the corrective fields are:

$$\begin{aligned} -\frac{d^2}{dx^2}\varphi_1 = {} & d_1\varphi_1 + d_2\varphi_1^2 + d_3\varphi_1^3 + d_4\varphi_2 + d_5\varphi_1\varphi_2 \\ & + d_6\varphi_2^2 + d_7\varphi_1\varphi_2^2 - gj_1(x) \quad , \end{aligned} \tag{33}$$

$$\begin{aligned} -\frac{d^2}{dx^2}\varphi_2 = {} & e_1\varphi_2 + e_2\varphi_2^2 + e_3\varphi_2^3 + e_4\varphi_1 + e_5\varphi_1\varphi_2 + e_6\varphi_1^2 \\ & + e_7\varphi_2\varphi_1^2 - gj_2(x) \quad , \end{aligned} \tag{34}$$

with the expansion coefficients d and e:

$$d_1 = -12\lambda\phi_1^2 - 4\lambda\phi_2^2 + 6\alpha\phi_1 + 2\beta \tag{35a}$$

$$d_2 = -12\lambda\phi_1 + 3\alpha \tag{35b}$$

$$d_3 = -4\lambda \tag{35c}$$

$$d_4 = -8\lambda\phi_1\phi_2 - 6\alpha\phi_2 \tag{35d}$$

$$d_5 = -8\lambda\phi_2 \tag{35e}$$

$$d_6 = -4\lambda\phi_1 - 3\alpha \tag{35f}$$

$$d_7 = -4\lambda \tag{35g}$$

and

$$e_1 = -4\lambda\phi_1^2 - 12\lambda\phi_2^2 - 6\alpha\phi_1 + 2\beta \tag{36a}$$

$$e_2 = -12\lambda\phi_2 \tag{36b}$$

$$e_3 = -4\lambda \tag{36c}$$

$$e_4 = -8\lambda\phi_1\phi_2 - 6\alpha\phi_2 \tag{36d}$$

$$e_5 = -8\lambda\phi_1 - 6\alpha \tag{36e}$$

$$e_6 = -4\lambda\phi_2 \tag{36f}$$

$$e_7 = -4\lambda \quad . \tag{36g}$$

Because the currents are proportional to each other $j_2 = \sqrt{3}j_1$ [Eqs. (25) and (26)], we make the ansatz:

$$\varphi_2 = \sqrt{3}\varphi_1 \equiv \sqrt{3}\varphi \tag{37}$$

and reduce the problem (33) and (34) to a single corrective field φ_1 that we henceforth denote by φ.

The equations for φ_1 and φ_2 become:

$$-\frac{d^2}{dx^2}\varphi = (d_1 + \sqrt{3}d_4)\varphi + (d_2 + \sqrt{3}d_5 + 3d_6)\varphi^2 + (d_3 + 3d_7)\varphi^3 - gj_1 \tag{38}$$

$$-\frac{d^2}{dx^2}\varphi = \left(e_1 + \frac{e_4}{\sqrt{3}}\right)\varphi + \left(\sqrt{3}e_2 + e_5 + \frac{e_6}{\sqrt{3}}\right)\varphi^2 + (3e_3 + e_7)\varphi^3 - gj_1 . \tag{39}$$

The coefficients of the odd powers of φ on the righthand side are equal;

$$d_1 + \sqrt{3}d_4 = e_1 + \frac{e_4}{\sqrt{3}} = -4\lambda\phi_V^2 \quad , \tag{40}$$

and

$$d_3 + 3d_7 = 3e_3 + e_7 = -16\lambda \quad , \tag{41}$$

but the coefficients of the quadratic terms differ:

$$d_2 + \sqrt{3}d_5 + 3d_6 = -16\lambda\phi_V - 12\lambda\phi_V \tanh(ax) \quad , \tag{42}$$

$$\sqrt{3}e_2 + e_5 + \frac{e_6}{\sqrt{3}} = -16\lambda\phi_V + 4\lambda\phi_V \tanh(ax) \quad . \tag{43}$$

We solve for φ in a linear approximation when we can use the Greens function of the hyperbolic differential equation in one dimension:

$$\left(-\frac{d^2}{dx^2} + 4\lambda\phi_V^2\right)\varphi(x) = -gj_1(x) \quad , \tag{44}$$

which has the solution:

$$\varphi(x) = -g(4\lambda^{1/2}\phi_V)^{-1} \int_{-\infty}^{\infty} dx' \exp\left[-2\lambda^{1/2}\phi_V |x-x'|\right] j_1(x') \quad , \tag{45}$$

that has negative values for all x and vanishes at infinity.

Because $\int_{-\infty}^{\infty} j_1(x)dx = 1$, φ is bounded from below by $-g/(4\lambda^{1/2}\phi_V)$.

The trajectory in ϕ-space is no longer a straight line as in the absence of the fermions but is tilted toward the center of the coordinate system $\phi_1 = 0$, $\phi_2 = 0$: $\tilde{\phi}_1 + \sqrt{3}\tilde{\phi}_2 = \phi_V + 4\varphi(x)$ $(\varphi < 0)$.

(iv) Reevaluation of the fermion fields

As the correction φ of the scalar fields is a function of space coordinate, it will not affect the energy E or spinor part U but adds to the Eq. (21).

The correction h(x) to H(x): $H(x) \to H(x) + h(x)$ satisfies the linear differential equation:

$$-\frac{d}{dx} h(x) = -2ig\varphi(x) \quad . \tag{46}$$

h(x) is therefore imaginary ($h^* = -h$), which adds only a complex phase to the fields ψ and leaves the currents and the normalization constant N unchanged. The linear approximation is thus consistent with the fermion equations.

4. THE MASS SPECTRUM

The total energy of the composite system of scalar and fermion particles determined from the fields $\tilde{\phi}_1$, $\tilde{\phi}_2$, Eq. (31),(32) and the currents j_1, j_2, Eqs. (25),(26) is exact up to orders of $O(\varphi^2)$.

To illustrate the mass spectrum we calculate the masses for the particular values of the parameters in the model $\lambda = 1$, $\phi_V = 2.6717$, $g = 5$ when the mass of a system made only of scalar fields Eq. (17) is 22.31 and the mass of the composite system of scalar and fermion particles is 17.03 in units of λ.

We have a two level spectrum. The ground-state with two fermions present has always a smaller mass M than the state without fermions M°: $M \leqslant M°$ when $g \geqslant 0$. The equality of M and M° is obtained in the limit $g \to 0$.

CONCLUSIONS

In a classical field theory model of two scalar fields and two fermion fields with an internal cyclic symmetry Z(3) induced by the cubic self-interaction of the scalar fields we observe that the massless fermions are inseparable, i.e., the solutions of the field equations in 3 + 1 and 1 + 1 dimensions are related algebraically.

We illustrate the structure of the model in 1 + 1 dimensions, where we obtain a two level mass-spectrum of bound states, the mass of the composite system of bosons and fermions being lower than the mass of the system made of bosons alone.

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