LBL-13653
Preprint

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

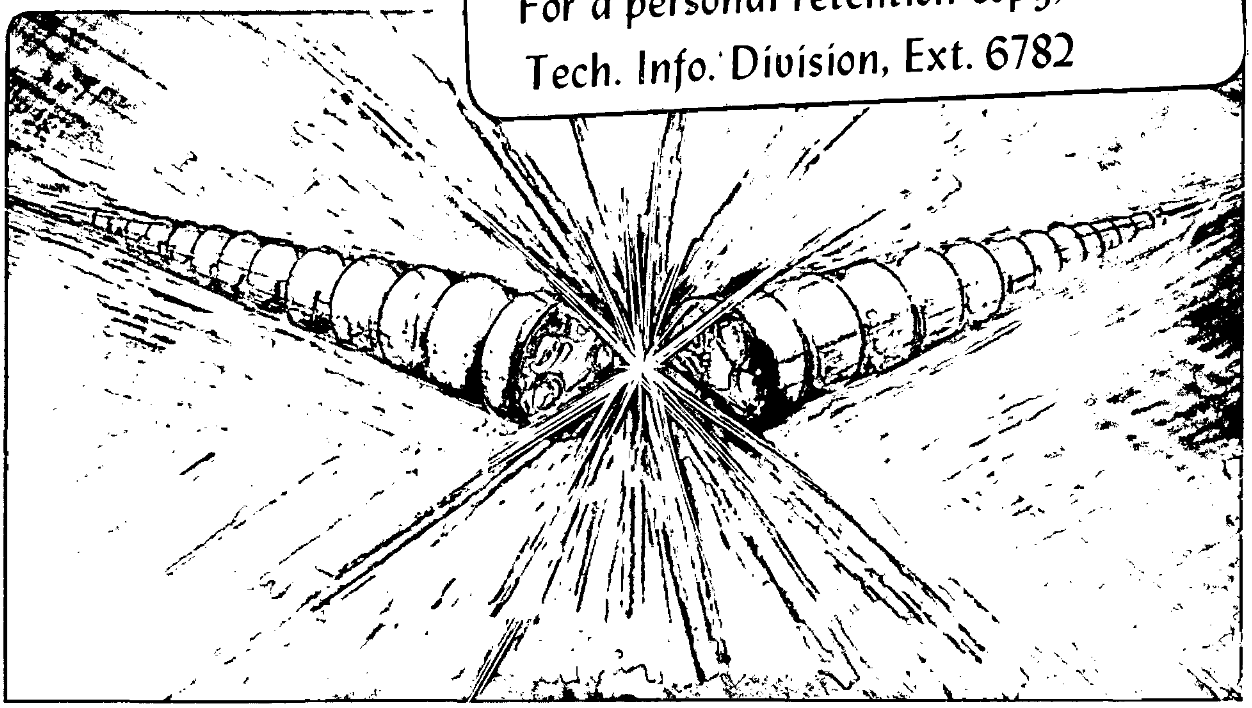
Accelerator & Fusion Research Division

Submitted to Physics Letters A

ALGEBRAIC STRUCTURE OF THE PLASMA QUASILINEAR EQUATIONS

Allan N. Kaufman and Philip J. Morrison

December 1981



Prepared for the U.S. Department of Energy under Contract W-7405-ENG-48

DISCLAIMER

Submitted to Physics Letters A LBL-13653

Algebraic Structure of the Plasma Quasilinear Equations*

Allan N. Kaufman

Lawrence Berkeley Laboratory
and
Physics Department
University of California
Berkeley, CA 94720

and

Philip J. Morrison **

Plasma Physics Laboratory
Princeton University
Princeton, NJ 08544

Abstract

The standard quasilinear equations of plasma physics are shown to possess an algebraic structure, although the system is dissipative. The energy functional yields the evolution equations and the conservation laws, in analogy to Hamiltonian systems.

---

* This work was supported by the Director, Office of Energy Research, Office of Fusion Energy, Applied Plasma Physics Division, of the U. S. Department of Energy under Contract No. W-7405-ENG-48 and DE-AC02-76-CH0373.

**Present address: Physics Department and Institute for Fusion Studies, University of Texas, Austin, Texas.

Our recent discovery [1] of a Hamiltonian structure for the Vlasov equation with Coulomb interaction (discovered independently by Gibbons [2]) has led us to search for an algebraic structure for the corresponding dissipative system, the quasilinear diffusion equations for an unstable plasma. By analogy to the Hamiltonian structure, we desire a bracket and an energy functional that yield the evolution equations and conservation laws. However, this bracket cannot be a Lie algebra, implying a Hamiltonian structure, since the quasilinear system possesses a Liapunov functional, the entropy, expressing irreversibility.

In the interests of simplicity and clarity, we deal here with the simplest case, a uniform unmagnetized plasma, with one species of resonant particles and one wave branch. The particle distribution in momentum space is $f(\vec{p})$, and the total particle energy functional is

$$H(f) = \int d^3p \;\; H(\vec{p})f(\vec{p}) \;, \tag{1}$$

where $H(\vec{p})$ is the single-particle energy. The wave action distribution in wave-vector space is $J(\vec{k})$, and the total wave energy functional is

$$H(J) = \int d^3k \;\; \omega(\vec{k})J(\vec{k}) \;, \tag{2}$$

where $\omega(\vec{k})$ is the wave dispersion relation. The total energy,

$$H(f,J) = H(f) + H(J) \;, \tag{3}$$

contains no interaction term. The resonant wave-particle interaction appears in the bracket, Eq. (5).

Consider now two observables, $A_1(f,J)$ and $A_2(f,J)$. We search for a bracket algebra: $\{A_1,A_2\} = A_3$, which is bilinear, antisymmetric, and operates on A_1 and A_2 with first functional derivatives. In addition, we demand that observables evolve in time as

$$\dot{A} = \{ A, H(f,J) \} \quad . \tag{4}$$

Since the quasilinear evolution equations are known, a short search yields the desired result:

$$\{A_1,A_2\} = \int d^3p \int d^3k \left(\frac{\delta A_1}{\delta f} \frac{\delta A_2}{\delta J} - \frac{\delta A_1}{\delta J} \frac{\delta A_2}{\delta f}\right) J(\vec{k}) \; Rf(\vec{p}); \tag{5}$$

with

$$R = \alpha(\vec{k}) \; \vec{k} \cdot \vec{\partial} \; \delta\left[\omega(\vec{k}) - \vec{k}\cdot\vec{\partial} H(\vec{p})\right] \vec{k} \cdot \vec{\partial} \; ; \tag{6}$$

$$\vec{\partial} = \partial/\partial\vec{p} \; ;$$

and $\alpha(\vec{k})$ is a coupling constant. The resonant wave-particle interaction resides in R. This bracket does not satisfy the Jacobi identity, and hence is not a Lie algebra.

Applying Eqs. (4) and (3) to $f(\vec{p})$, we obtain the diffusion equation:

$$\begin{aligned} \dot{f}(\vec{p}) &= \vec{\partial} \cdot \underline{D}(\vec{p}) \cdot \vec{\partial} f(\vec{p}) \; , \\ \underline{D}(\vec{p}) &= \int d^3k \; \vec{k}\vec{k} \; \alpha(\vec{k}) \; \delta\left[\omega(\vec{k}) - \vec{k}\cdot\vec{\partial} H(\vec{p})\right] J(\vec{k}) \quad . \end{aligned} \tag{7}$$

Applying Eq. (4) to $J(\vec{k})$, we obtain the linear growth equation:

$$\begin{aligned} \dot{J}(\vec{k}) &= 2\gamma(\vec{k}) J(\vec{k}), \\ 2\gamma(\vec{k}) &= \int d^3p \; \alpha(\vec{k}) \; \delta\left[\omega(\vec{k}) - \vec{k}\cdot\vec{\partial} H(\vec{p})\right] \vec{k}\cdot\vec{\partial} f(\vec{p}) \quad . \end{aligned} \tag{8}$$

These are the standard equations of quasilinear theory, with resonant interactions, and no refinements (such as resonance broadening).

The conservation laws should now follow directly from (4). Energy conservation,

$$\dot{H} = \{H, H\} = 0 , \tag{9}$$

is a trivial consequence of the antisymmetry of (5). For conservation of momentum

$$\vec{P}(f,J) = \int d^3p\ \vec{p}\ f(\vec{p}) + \int d^3k\ \vec{k}\ J(\vec{k}) , \tag{10}$$

we have

$$\dot{\vec{P}} = \int d^3p \int d^3k\ \left[\vec{p}\omega(\vec{k}) - \vec{k}H(\vec{p})\right] J(\vec{k})Rf(\vec{p}) ,$$

which vanishes, upon integration by parts. The Liapunov functional,

$$S(f) = -\int d^3p\ \ f(\vec{p})\ \ell n\ f(\vec{p}) , \tag{11}$$

evolves monotonically, as found from Eq. (14):

$$\dot{S} = \int d^3p \int d^3k\ \ \alpha\omega J f^{-1} (\vec{k} \cdot \vec{\partial} f)^2\ \delta(\omega - \vec{k} \cdot \vec{\partial} H) \geq 0 . \tag{12}$$

These results raise a number of questions for future investigation:

(1) How is the algebraic structure discussed here related to the underlying Lie structure of the Vlasov system, in particular to the fundamental work of Marsden and Weinstein [3]?

(2) How can this structure be modified to take into account resonance broadening and more recent improvements to quasilinear theory [4]?

(3) How can this structure be generalized to deal with nonuniform magnetized plasma, and with nonlinearities, such as the ponderomotive Hamiltonian [5]?

(4) Do similar algebraic structures exist for other dissipative systems, such as the Boltzmann equation?

References

1. P. J. Morrison, Phys. Letters 80A, 383 (1980).
2. J. Gibbons, Physica D 3D, 503 (1981).
3. J. Marsden and A. Weinstein, Physica D (in press).
4. J. Adam, G. Laval, and D. Pesme, Phys. Rev. Lett. 43, 1671 (1979).
5. J. Cary and A. Kaufman, Phys. Fluids 24, 1238 (1981); C. Grebogi, A. Kaufman, and R. Littlejohn, Phys. Rev. Lett. 43, 1668 (1979).

TECHNICAL INFORMATION DEPARTMENT
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720