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John David Crawford and John R. Cary

February 1982

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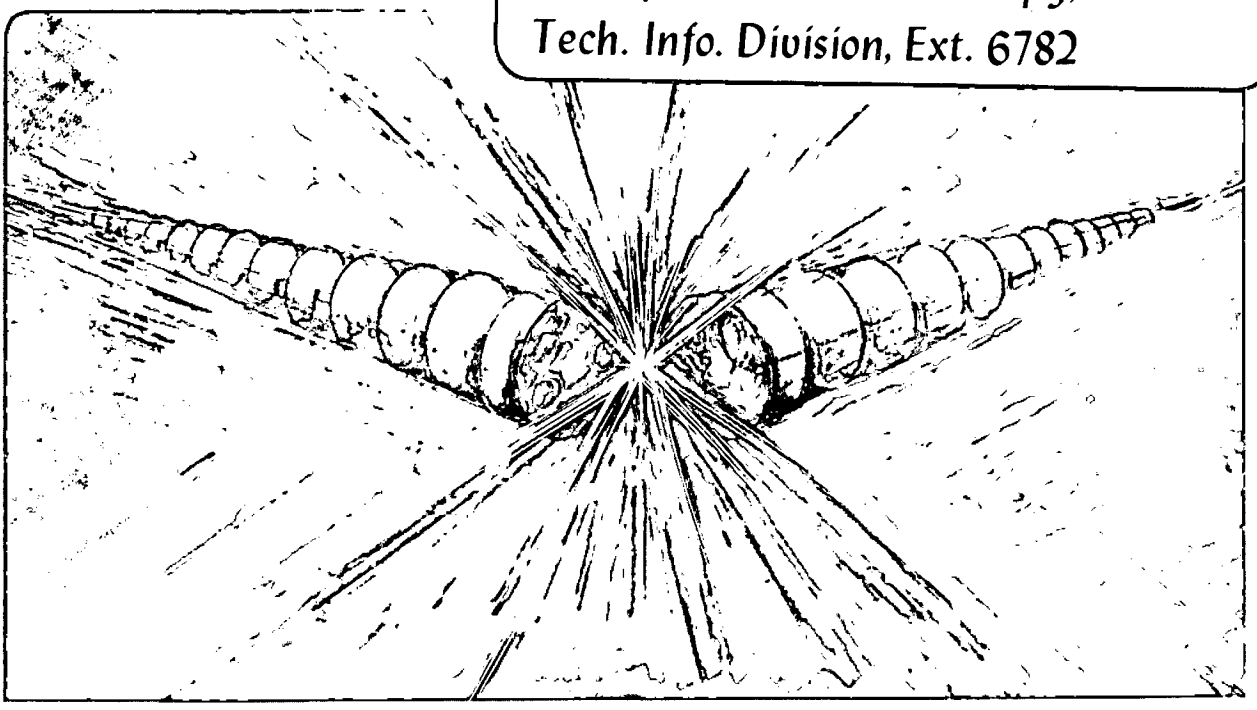

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Decay of Correlations in a Chaotic Measure Preserving Transformation*

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Abstract

For a chaotic, area preserving map on the torus, we study the decay of correlations in detail. Taking as observables the square-integrable functions, we find examples of decay rates which are algebraic, exponential, and faster than exponential. For correlations that decay exponentially the rate is sensitive to the choice of function. The implications for numerical experiments of this nonuniformity in the decay are discussed.

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I. Introduction

In the theory of dynamical systems, the properties of correlation functions have been studied with a variety of motivations: to detect a transition from ordered to chaotic motion, to test whether a system relaxes to equilibrium, and to determine the transport coefficients for dynamical variables. In the absence of rigorous results, it is widely assumed that for systems which are "sufficiently" chaotic, the decay of an autocorrelation function, Eq. (3), denoted $C_F(t)$, will be exponential

$$C_F(t) \sim A(F)\, e^{-\gamma t} \qquad t > 0 \tag{1}$$

and the decay rate, γ, should be determined by the underlying dynamics in a way that is insensitive to the particular observable, F, e.g., γ would be given by the metric entropy or perhaps an average of the Lyapunov exponent.

There are rigorous results for certain classes of chaotic maps which reinforce this belief. For diffeomorphisms satisfying Smale's Axiom A, Ruelle proved the bound,

$$\left| C_F(t) \right| \leq A(F)\, e^{-\gamma t}$$

for functions which are continuously differentiable.[1] For Anosov diffeomorphisms, or C-systems, Sinai has established exponential decay for correlation functions of C^1 functions.[2]

On the other hand, there are growing indications that an uncritical acceptance of (1) is naive. We briefly mention a few contributions that encourage a skeptical attitude toward results which depend on simplifying, non-rigorous assumptions about correlation functions. Casati, Valz-Gris, and Guarneri have criticized the theoretical basis of efforts to predict chaotic transitions from the behavior of correlation functions.[3] Niwa found algebraic decay of correlations in a highly idealized, one dimensional, hard point gas.[4] Recent numerical work on the standard map has revised the conclusion that it exhibits a simple exponential decay of correlations.[5] There are new results by Bunimovich and Sinai on the periodic Lorentz gas which establish that the velocity autocorrelation function decays as $\exp(-n^{\gamma})$ where $0 < \gamma \leq 1$ and n plays the role of time; however Casati, et. al., report that Sinai has found this conclusion depends on the periodicity of the gas.[6,3] When the periodicity is broken the decay becomes algebraic.

In this paper we study a measure preserving transformation chosen from the class of dynamical systems studied by Ruelle and Sinai. The decay of the autocorrelation function for any square-integrable function is analyzed in detail, and we find a range of behaviors. The decay rate can be faster than exponential, exponential, or algebraic depending on the function selected. Moreover, even within the class of functions which decay exponentially, the rate is not uniform; it depends in an essential way on the characteristics of the function. These results serve to refine the expectations created by the work of Ruelle and Sinai.

Finally, we discuss when one expects to find the behavior illustrated by our example and what the implications are for numerical studies of correlation functions.

2. The Autocorrelation Function

We consider an area preserving map which, though lacking a clear physical interpretation, has the virtue of allowing a complete analysis. Our example is an automorphism of the torus, denoted by T,

$$(I', \theta') = T(I, \theta) \quad \text{where}$$

$$\left.\begin{aligned} I' &= I + \theta \\ \theta' &= I + 2\theta \end{aligned}\right\} \quad \text{mod } 2\pi \tag{2}$$

whose ergodic properties have been studied in detail.[7] Equation (2) is known to be mixing with positive entropy, and exhibits a uniform sensitive dependence on initial conditions. In the terminology of dynamical systems, it is a C-system and satisfies Axiom A.[8]

For a function on the torus, $F = F(I,\theta)$, the autocorrelation function of F, $C_F(t)$, is defined by

$$C_F(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} F(T^{j+t}(I,\theta))^* F(T^j(I,\theta)) - \left| \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} F(T^j(I,\theta)) \right|^2 \tag{3}$$

where $T^n(I,\theta)$ is the nth iteration of (2) from an initial condition of (I,θ). Since our system is ergodic relative to the Lebesgue measure, $d\mu = dI\ d\theta/(2\pi)^2$, the autocorrelation function may be expressed in terms of phase space averages. We will restrict ourselves to choices for $F(I,\theta)$ which are square integrable and have zero mean, then (3) becomes,

$$C_F(t) = \int d\mu\ F(T^t(I,\theta))^*\ F(I,\theta)$$

where (4)

$$\int d\mu\,|F|^2 < \infty \qquad \text{and} \qquad \int d\mu\ F = 0.$$

With the Fourier expansion of F,

$$F(I,\theta) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f_{mn}\ \phi_{mn}\ (I,\theta)$$

where $\phi_{mn}\ (I,\theta) = \exp\ i(mI+n\theta)$, the autocorrelation function (4) is written,

$$C_F(t) = \sum_{i,j,m,n=-\infty}^{\infty} f^*_{mn}\ f_{ij} \int d\mu\ \phi^*_{mn}(T^t(I,\theta))\phi_{ij}(I,\theta) \qquad (5)$$

The t-dependence of $\phi_{mn}\ (T^t(I,\theta))$ in (5) is simple since we have the relation,

$$\phi_{mn}(T(I,\theta)) = \phi_{m'n'}\ (I,\theta)$$

where

$$\begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} \tag{6}$$

Hence if M denotes the above matrix that acts on the Fourier indices, and $\underset{\sim}{r} = (m,n)$ then

$$\phi_{\underset{\sim}{r}} (T^t(I,\theta)) = \phi_{M^t \underset{\sim}{r}}(I,\theta) \tag{7}$$

The eigenvalues of M, $\lambda_{\pm} = \frac{3 \pm \sqrt{5}}{2}$, define expanding and contracting directions in the m–n plane and $\ln\lambda_+$ corresponds to the (positive) Lyapunov exponent[7] for (2). From the definition (6) we find,

$$M^t = \begin{cases} \begin{pmatrix} S_{2t-1} & S_{2t} \\ S_{2t} & S_{2t+1} \end{pmatrix} & \text{for } t \geq 1 \\ \begin{pmatrix} S_{-2t+1} & -S_{-2t} \\ -S_{-2t} & S_{-2t-1} \end{pmatrix} & \text{for } t \leq -1 \end{cases} \tag{8}$$

where $S_1 = 1$, $S_2 = 1$, and $S_j = S_{j-1} + S_{j-2}$ for $j \geq 3$ are the Fibonacci numbers. The elements of this sequence grow geometrically; more precisely there exist numbers $N_2 > N_1 > 0$ such that

$$N_1 \lambda_+^t \leq S_{2t-1}, S_{2t}, S_{2t+1} \leq N_2 \lambda_+^t \tag{9}$$

for $t \geq 1$. (This property is easily established by diagonalizing M^t and relating $S_{2t-1}, S_{2t}, S_{2t+1}$ to λ_+^t.)

Equations (7) and (8) show that under the action of M, the Fourier basis, ϕ_{mn}, breaks up into a countable number of distinct orbits. For $(m,n)=(0,0)$ the orbit consists of a single element, but all other orbits contain an infinite number of elements which lie on hyperbolas defined by $m^2+mn-n^2=a$ where a is an integer. This orbit structure is partially shown in Fig. 1.

The analysis of (5) is greatly simplified if we re-express $F(I,\theta)$ in terms of a basis which respects this orbit structure. To this end we define the orbit basis,

$$\psi_{t,\underset{\sim}{r}}(I,\theta) \equiv \phi_{M^t\underset{\sim}{r}}(I,\theta) \tag{10}$$

where $-\infty < t < \infty$ and $\underset{\sim}{r}$ is assumed to run over a subset of the Fourier basis which includes a single element of each orbit. Let this subset be denoted by $\mathcal{F}$. This reorganization of the Fourier basis has the property

$$\psi_{t,\underset{\sim}{r}}(T^n(I,\phi)) = \psi_{t+n,\underset{\sim}{r}}(I,\phi) \tag{11}$$

We now determine a specific choice for $\mathcal{F}$ to be used in our calculation of $C_F(t)$. Consider first the Fourier basis elements in the right hand quadrant of Fig. 1 which contains the m-axis. Under one iteration of (6) a point on the m-axis is mapped to a point on the 45° line e.g., $(m,0) \succ (m,m)$. If a point lies between the m-axis and the 45° line ((m,n) such that $m > 0$, $n > 0$, and $n/m < 1$) then on one forward iteration of (6) it crosses the 45° line and on one backward iteration

of (6) it is mapped below the m-axis. These observations imply that each orbit in this quadrant has one and only one point lying in the wedge described by $m \geq 1$ and $n = 0,1,2\ldots, m-1$. For this quadrant if these wedge points are included in $\mathcal{J}$ then each orbit is counted and counted only once. Similar considerations apply to the other three quadrants, and provide us with a convenient choice for $\mathcal{J}$.

$$\mathcal{J} = \bigcup_{i=1} \mathcal{J}_i \text{ where}$$

$$\begin{aligned}
\mathcal{J}_1 &= \{(m,n) \mid m \geq 1, \quad n = 0,1,2,\ldots,m-1\} \\
\mathcal{J}_2 &= \{(m,n) \mid m \leq -1, \quad n = 0,-1,-2,\ldots,1-|m|\} \\
\mathcal{J}_3 &= \{(m,n) \mid n \geq 1, \quad m=0,-1,-2,\ldots,-n+1\} \\
\mathcal{J}_4 &= \{(m,n) \mid n \leq -1, \quad m=0,1,2,\ldots,|n|-1\}
\end{aligned}$$

Knowing $\mathcal{J}$ allows us to expand $F(I,\theta)$ in the new basis,

$$F(I,\theta) = \sum_{\alpha=-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \{f_{\alpha,(m,n)}\, \psi_{\alpha,(m,n)} + f_{\alpha,(-m,-n)}\, \psi_{\alpha,(-m,-n)}$$

$$+ f_{\alpha,(-n,m)}\, \psi_{\alpha,(-n,m)} + f_{\alpha,(n,-m)}\, \psi_{\alpha,(n,-m)}\}$$

and $C_F(t)$ becomes

$$C_F(t) = \sum_{\alpha=-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \{f^*_{\alpha,(m,n)}\, f_{\alpha+t,(m,n)} + f^*_{\alpha,(-m,-n)}\, f_{\alpha+t,(-m,-n)}$$

$$+ f^*_{\alpha,(-n,m)}\, f_{\alpha+t,(-n,m)} + f^*_{\alpha,(n,-m)}\, f_{\alpha+t,(n,-m)}\} \qquad (12)$$

The decay of $C_F(t)$ is controlled by the dependence of the $f_{\alpha,(m,n)}$ on the α index. Since these coefficients are only required to satisfy

$$\sum_{\alpha=-\infty}^{\infty}\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\left\{\left|f_{\alpha,(m,n)}\right|^2+\left|f_{\alpha,(-m,-n)}\right|^2+\left|f_{\alpha,(-n,m)}\right|^2+\left|f_{\alpha,(n,-m)}\right|^2\right\}<\infty$$

from (4), the decay of $C_F(t)$ can vary widely. We will discuss some of the possibilities in the next section.

3. Decay of the Autocorrelation Function

Because (2) is a mixing transformation one knows without detailed calculation that for any square integrable function, the autocorrelation function must decay to zero in the limit $t \to \infty$. The "mechanism" of this decay is the hyperbolic orbit structure of Fig. 1. With our representation for $C_F(t)$ in (12), it is possible to compute the rate of decay for any such function once its expansion coefficients in the orbit basis are known. We now examine some specific classes of functions.

(A) Let $F(I,\theta)$ be independent of θ, then $f_{\alpha,(m,n)} = 0$ if $\alpha \neq 0$ or $n \neq 0$. Thus, correlations vanish at the first time step,

$$C_F(t) = 0 \quad \text{for} \quad t > 0$$

The same behavior is found if F is independent of I; this is the result of Cary and Meiss for the sawtooth map.[9] It implies that one may rigorously compute a finite diffusion coefficient.

(B) Let $F(I,\theta)$ be sufficiently smooth such that all partial derivatives up to and including r^{th} order are square integrable, e.g., for all non-negative integers a,b such that $a + b \leq r$,

$$\int d\mu \left| \frac{\partial^{a+b} F(I,\theta)}{\partial I^a \, \partial\theta^b} \right|^2 < \infty \tag{13}$$

To understand what this implies for $f_{\alpha,(m,n)}$ we note from (8) and (10),

$$\frac{\partial^{a+b}}{\partial I^a \, \partial\theta^b} \psi_{\alpha,(m,n)}(I,\theta) = (i)^{a+b} (A_\alpha)^a (B_\alpha)^b \psi_{\alpha,(m,n)}(I,\theta)$$

where

$$A_\alpha(m,n) = \begin{cases} S_{2\alpha-1} m + S_{2\alpha} n & \alpha \geq 1 \\ m & \alpha = 0 \\ S_{-2\alpha+1} m - S_{-2\alpha} n & \alpha \leq -1 \end{cases} \tag{14}$$

$$B_\alpha(m,n) = \begin{cases} S_{2\alpha} m + S_{2\alpha+1} n & \alpha \geq 1 \\ n & \alpha = 0 \\ -S_{-2\alpha} m + S_{-2\alpha+1} n & \alpha \leq -1 \end{cases}$$

In terms of A_α and B_α, (13) becomes

$$\sum_{\alpha=-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \{A_\alpha(m,n)^{2a} B_\alpha(m,n)^{2b} \left| f_{\alpha,(m,n)} \right|^2 + (m \to -m,\ n \to -n)$$

$$+ (m \to -n,\ n \to m) + (m \to n,\ n \to -m)\} < \infty \tag{15}$$

The behavior of A_α and B_α is easily estimated from (9) and (14),

$$\left.\begin{array}{ll} \alpha \geq 1, & N_1 \lambda_+^{\alpha} (m+n) \\ \alpha \leq -1, & N_1 \lambda_+^{|\alpha|} (m-R_2 n) \end{array}\right\} \leq \left| A_\alpha(m,n) \right| \leq \left\{\begin{array}{ll} N_2 \lambda_+^{\alpha} (m+n), & \alpha \geq 1 \\ N_2 \lambda_+^{|\alpha|} (m-R_2 n), & \alpha \leq -1 \end{array}\right.$$

$$\left.\begin{array}{ll} \alpha \geq 1, & N_1 \lambda_+^{\alpha} (m+n) \\ \alpha \leq -1, & N_1 \lambda_+^{|\alpha|} (m-R_1 n) \end{array}\right\} \leq \left| B_\alpha (m,n) \right| \leq \left\{\begin{array}{ll} N_2 \lambda_+^{\alpha} (m+n), & \alpha \geq 1 \\ N_2 \lambda_+^{|\alpha|} (m-R_1 n), & \alpha \leq -1 \end{array}\right. \tag{16}$$

where $R_1 \equiv S_{2|\alpha|-1}/S_{2|\alpha|}$ and $R_2 \equiv S_{2|\alpha|}/S_{2|\alpha|+1}$ are positive numbers less than or equal to unity. $A_\alpha(m,n)$, $B_\alpha(m,n)$ grow exponentially in α, and linearly in m and n. Knowing this essentially fixes the behavior of the expansion coefficients $f_{\alpha,(m,n)}$. Consider the first set of terms in (15), let

$$C_1(a,b) = \sum_{\alpha=-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} (A_\alpha(m,n))^{2a} \, (B_\alpha(m,n))^{2b} \left| f_{\alpha,(m,n)} \right|^2$$

then (16) implies

$$\Lambda(a,b,N_1) \leq C_1(a,b) \leq \Lambda(a,b,N_2)$$

with

$$\Lambda(a,b,\ N) = \sum_{\alpha=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} (N\ \lambda_+^{\alpha}(m+n))^{2(a+b)} \left| f_{\alpha,(m,n)} \right|^2$$

$$\sum_{\alpha=-\infty}^{-1} \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} (N\ \lambda_+^{|\alpha|})^{\ 2(a+b)} (m-R_2 n)^{2a}\ (m-R_1 n)^{2b} \left| f_{\alpha,(m,n)} \right|^2$$

$$\sum_{m=1}^{\infty} \sum_{n=0}^{m-1} (m)^{2a}\ (n)^{2b} \left| f_{0,(m,n)} \right|^2$$

For $C_1(a,b)$ to be finite, Λ must be finite and this imposes the strongest conditions on $f_{\alpha,(m,n)}$ when $a+b=r$. Assume for $f_{\alpha,(m,n)}$ the bound,

$$\left| f_{\alpha,(m,n)} \right|^2 \leq \frac{\delta}{\lambda_+^2\ |\alpha|^{\ r+\varepsilon_1} (m+n)^{2r+\varepsilon_2}} \tag{17}$$

with $\delta > 0$. Then (for $a+b=r$) the first set of terms in Λ,

$$\sum_{\alpha=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} (N\lambda_+^{\alpha}\ (m+n))^{\ 2r} \left| f_{\alpha,(m,n)} \right|^2$$

will converge for $\varepsilon_1 > 1$ and $\varepsilon_2 > 2$ and diverge if $\varepsilon_1 \leq 1$ or $\varepsilon_2 \leq 2$. For $\varepsilon_1 > 1$, $\varepsilon_2 > 2$ the other two groups of terms in Λ will also converge. The crucial point of the estimate in (17) is that for Λ to be

finite $\left|f_{\alpha,(m,n)}\right|^2$ must decay in α <u>faster</u> than $\lambda_+^{-2|\alpha| r}$; it is the decay in α which governs the decay of $C_F(t)$ with t.

An analysis of the remaining three sets of terms in (15) leads to the same conclusion regarding the decay of $\left|f_{\alpha,(m,n)}\right|^2$ in α, thus we simply adopt the bound in (17) as a typical behavior and compute the decay of $C_F(t)$ which results. With (12) and (17) it is straightforward to derive the bound,

$$\left|C_F(t)\right| \leq M(F)e^{-(r \ell n \lambda_+)t} \qquad t \geq 0 \tag{18}$$

where $M(F) > 0$ is a constant which depends on the specific function.

The decay of $C_F(t)$ is determined by the Lyapunov exponent of (2) and the degree of smoothness assumed for F; a generalization of our argument shows that if F is infinitely differentiable ($r \to \infty$) the decay is faster than exponential.

(C) Let $F(I,\theta)$ be a characteristic function for a rectangular subset of the phase space.

$$F(I,\theta) = \begin{cases} 1 \text{ if } 0 < I_1 \leq I \leq I_2 < 2\pi \\ \qquad 0 < \theta_1 \leq I \leq \theta_2 < 2\pi \\ \\ 0 \quad \text{otherwise} \end{cases}$$

Although $F(I,\theta)$ is not differentiable, in the sense that its partial derivatives are not square-integrable, $C_F(t)$ decays exponentially. This calculation is sketched in the Appendix; the result is

$$\left| C_F(t) \right| \leq M(F)e^{-t(2\ln\lambda+)}, \quad M(F) > 0 \tag{19}$$

(D) Finally, there are functions such as

$$F(I,\theta) = \sum_{\substack{\alpha=-\infty \\ \alpha\neq 0}}^{\infty} \frac{1}{\alpha} \psi_{\alpha,(m,n)}$$

which decay algebraically; for this example we have,

$$C_F(t) = 2/t^2$$

This latter class of functions (and certainly this particular example) are peculiar in that they are everywhere continuous and nowhere differentiable.

4. Discussion

It is clear that a conjecture such as (1) oversimplifies the actual situation for our example. It ignores the existance of a wide range of decay behaviors, and it assumes the decay rate is insensitive to the choice for $F(I,\theta)$. In our example, even for those correlations which decay exponentially, we do not find for $C_F(t)$ a natural factorization into a decaying exponential which depends only on the map and an amplitude which depends only on the function. Instead, for the function in class B of the previous section, the rate of decay is fixed by the smoothness of the function and the Lyapunov exponent of the map.

In retrospect it is also clear that the property of the orbit basis expressed in (11) was crucial to our analysis. The existence of the orbit basis in this problem is not an accident; it reflects the fact that a C-system has countable Lebesgue spectrum.[7] Any dynamical system with Lebesgue spectrum will admit an orthonormal basis for the square-integrable functions such that the action of the dynamics on that basis is simply a shift on one index. For our problem (11) is the concrete realization of this.

Given a basis which satisfies (11), it is easy to choose expansion coefficients in that basis such that the resulting autocorrelation function will decay algebraically or exponentially or faster than exponentially. Thus we may anticipate a range of behaviors, comparable to that found in the previous section, whenever our dynamical system has Lebesgue spectrum. Examples of systems which have Lebesgue spectrum include Bernoulli shifts, automorphisms of the torus, geodesic flow on manifolds with negative curvature, and certain models of the hard sphere gas.[7]

A practical consequence of our discussion is the recognition that numerical studies, which use correlation functions as a tool to study chaotic dynamics, have an inherent and non-trivial element of ambiguity unless one can determine how the characteristics of the chosen function affect the decay. In our work the orbit basis could be found explicitly and the influence of $F(I,\theta)$ on the decay could be estimated. In general we cannot expect to be so fortunate, and it may not be easy to separate the features of the decay which reflect the equations of motion from the features which reflect the choice of observable.

In physical terms, even in systems with highly chaotic dynamics, the approach to equilibrium may be nonuniform with correlations in different observables decaying on widely disparate time scales.

Appendix

We outline here the derivation of (19). We assume

$$0 < \Delta I = I_2 - I_1 < 2\pi$$
$$0 < \Delta\theta = \theta_2 - \theta_1 < 2\pi$$

If either ΔI or $\Delta\theta$ is 2π, then case A applies.

The expansion coefficients of $F(I,\theta)$ are found from

$$f_{\alpha(m,n)} = \int d\mu \; F(I,\theta) \; \psi^{*}_{\alpha,(m,n)}$$

and the following bounds are easily computed,

$$\left| f_{0,(0,n)} \right| \leq \frac{2\,\Delta I}{(2\pi)^2 \; |n|} \qquad n \neq 0$$

$$\left| f_{0,(m,0)} \right| \leq \frac{2\,\Delta\theta}{(2\pi)^2 \; |m|} \qquad m \neq 0$$

$$\left| f_{0,(m,n)} \right| \leq \frac{4}{(2\pi)^2 \; |mn|} \qquad m,n \neq 0$$

$$\left| f_{\alpha,(m,n)} \right| \leq \frac{4}{(2\pi)^2 \; |A_\alpha(m,n) \; B_\alpha \; (m,n)|} \qquad \alpha \neq 0$$

We can get a rather crude estimate of $|C_F(t)|$ by replacing (12) with

$$|C_F(t)| \leq \sum_{\alpha=-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \left\{ \left| f_{\alpha,(m,n)} \quad f_{\alpha+t,(m,n)} \right| + (m \to -m, n \to -n) \right.$$

$$\left. + (m \to -n,\ n \to m) + (m \to n,\ n \to -m) \right\}$$

Plugging in the bounds on the coefficients yields

$$|C_F(t)| \leq \frac{16}{(2\pi)^4} \sum_{m=1}^{\infty} \left[\frac{\Delta\theta}{A_t(m,0)\ B_t(m,0)} + \frac{\Delta I}{A_t(0,m) B_t(0,m)} \right]$$

$$+ \frac{32}{(2\pi)^4} \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} \frac{1}{mn} \left[\frac{1}{A_t(m,n)\ B_t(m,n)} + \frac{1}{|A_t(-n,m)\ B_t(-n,m)|} \right]$$

$$+ \frac{32}{(2\pi)^4} \sum_{\substack{\alpha=-\infty \\ \alpha \neq 0}}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \left[\frac{1}{|A_\alpha(m,n)\ B_\alpha(m,n)\ A_{\alpha+t}(m,n)\ B_{\alpha+t}(m,n)|} \right.$$

$$\left. + \frac{1}{|A_\alpha(-n,m)\ B_\alpha(-n,m) A_{\alpha+t}(-n,m) B_{\alpha+t}(-n,m)|} \right]$$

Finally by applying the lower bounds on A_α and B_α given in (16) it is straightforward to show

$$|C_F(t)| < M(F)\ e^{-t(2\ln\lambda_+)}$$

where

$$M(F) = \frac{1}{\pi^4 N_1^2} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} \left[\frac{4(m^2 + n^2)}{mn(m^2-n^2)} + \delta_{n,1} \frac{(\Delta\theta + \Delta I)}{m^3} \right] \right.$$

$$\left. +2 \sum_{\substack{\alpha=-\infty \\ \alpha\neq 0}}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \lambda_+^{-4|\alpha|} \left[\frac{(m+n)^4 + (m-n)^4}{(m^2-n^2)^4} \right] \right\}$$

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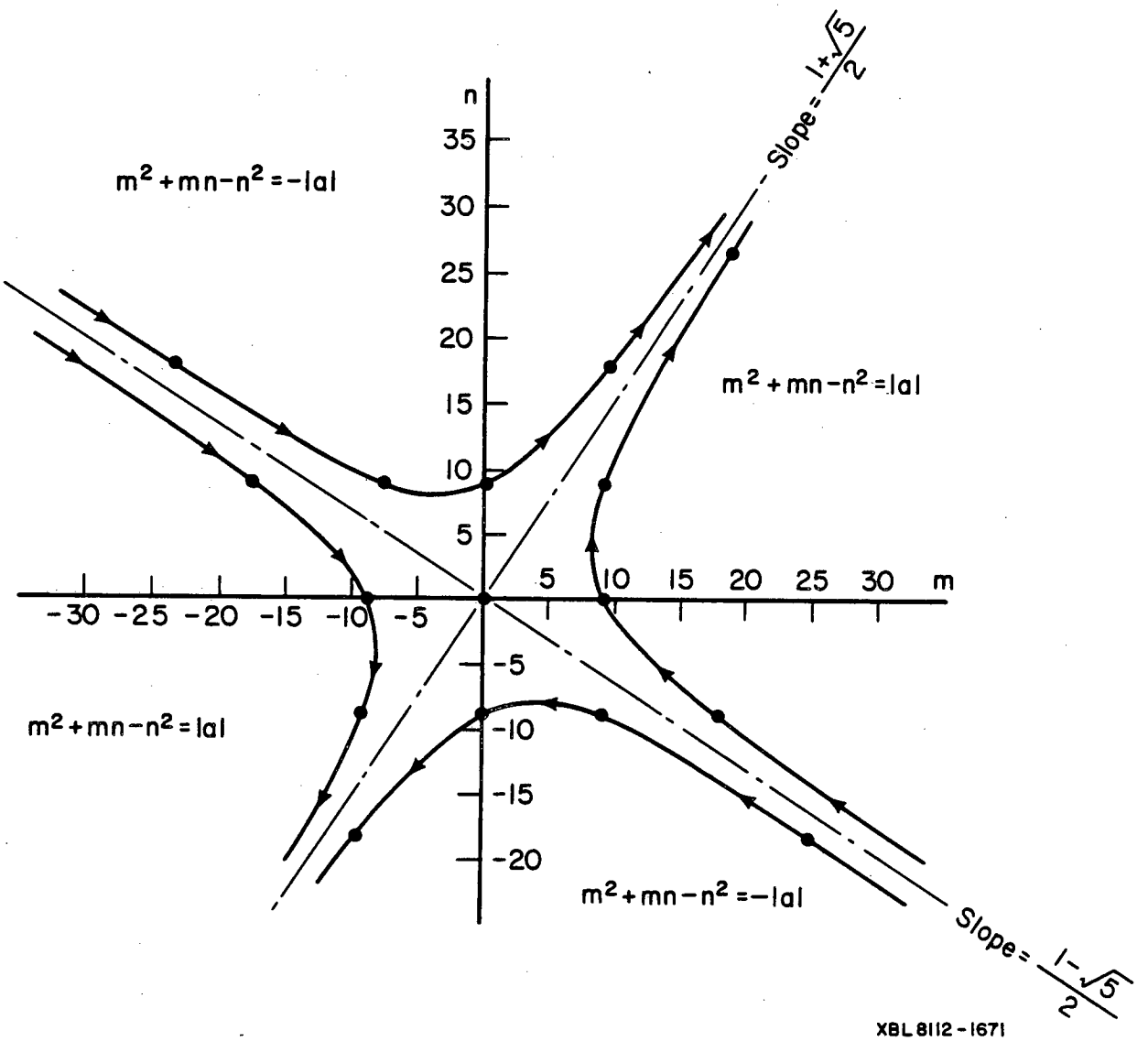

Figure 1. Evolution of the basis vectors, $\phi_{m,n}$, under composition with the mapping. Except for the constant function, $\phi_{0,0}$, at the origin, all vectors are mapped along hyperbolas. Shown are orbits corresponding to $|a| = 81$.

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