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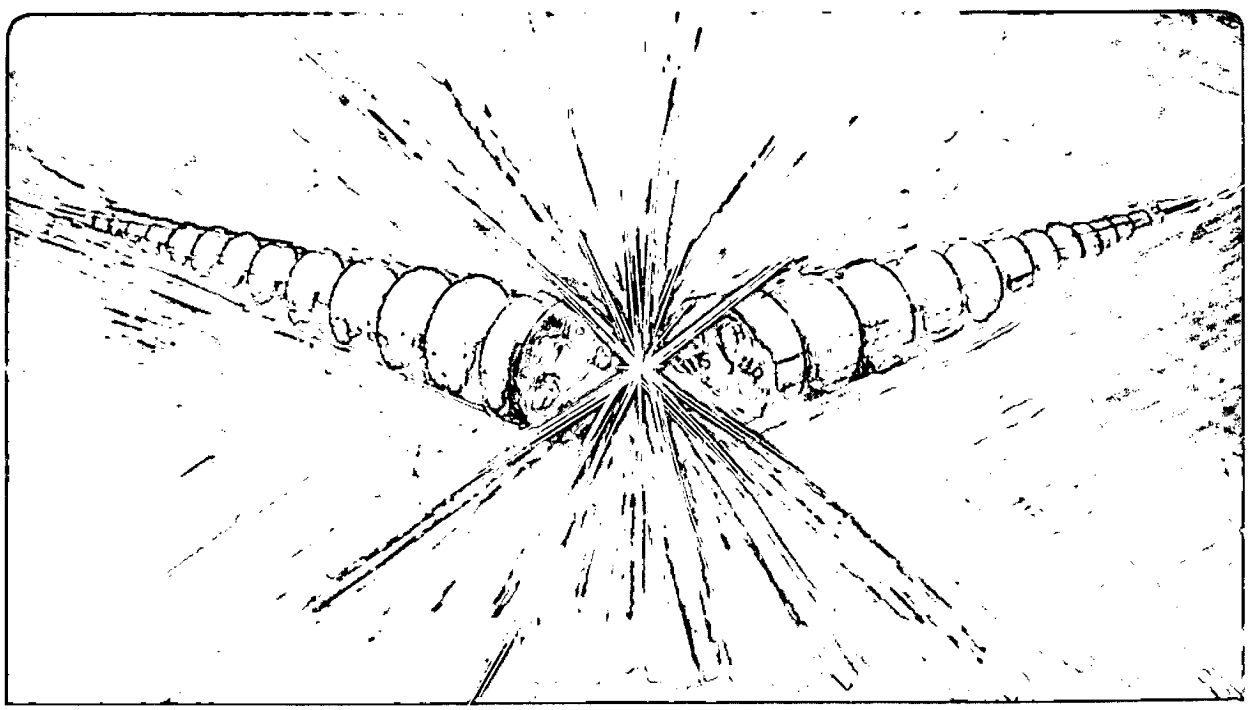

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The Hamiltonian Structure of Multi-Species Fluid Electrodynamics

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ABSTRACT

The phase space for multi-species fluid electrodynamics is the function space of fluid variables and Maxwell field variables. The Poisson bracket on phase functionals is constructed as a Lie algebra product following general methods of infinite dimensional symplectic geometry.

Recently, there has been a considerable amount of interest in the underlying Hamiltonian structures of the three standard models of non-dissipative plasma physics. Morrison and Greene (1) and Holm and Kupershmidt (2) have considered ideal magnetohydrodynamics, while Morrison (3), Weinstein and Morrison (4), and Marsden and Weinstein (5) have treated the Maxwell-Vlasov system, the special relativistic generalization of which has been studied by Bialynicki-Birula and Hubbard (6). We present here the Hamiltonian structure for multi-species fluid electrodynamics (7).

One conclusion that has emerged from this activity is that it is advantageous to derive the appropriate Poisson bracket from geometric considerations which will insure that it is bilinear, skew-symmetric, and satisfies the Jacobi identity. Our work follows the approach of Marsden and Weinstein in that the main ingredients in the construction of the Poisson structure for functionals of the fluid variables and the Maxwell field variables are the symplectic manifold structure of the co-adjoint orbits of a Lie group (8), and the reduction of phase spaces with symmetry (9), (10).

We define the physical system of charged fluids under consideration. Label fluid species with subscript s; each is composed of particles of mass m_s and charge q_s. Let $a_s=q_s/m_s$; $q_s=0$ is allowed. Our treatment holds for an arbitrary number of species, but two (oppositely charged) species is the situation most commonly discussed. Then, in terms of the fluid velocities $\underline{u}_s$, mass densities ρ_s, specific entropies σ_s, electric field $\underline{E}$, and magnetic field $\underline{B}$, the equations of ideal multi-fluid dynamics, in rationalized units, are

$$\dot{\underline{E}} = \nabla \times \underline{B} - \sum_s a_s \rho_s \underline{u}_s \qquad \dot{\underline{B}} = -\nabla \times \underline{E} \tag{1a}$$

$$\nabla \cdot \underline{E} = \sum_s a_s \rho_s + \rho_{ext} \qquad \nabla \cdot \underline{B} = 0 \tag{1b}$$

$$\dot{\rho}_s = -\nabla\cdot(\rho_s \underline{u}_s) \tag{2a}$$

$$\dot{\sigma}_s = -\underline{u}_s\cdot\nabla\sigma_s \tag{2b}$$

$$\dot{\underline{u}}_s = -(\underline{u}_s\cdot\nabla)\underline{u}_s + a_s(\underline{E} + \underline{u}_s \times \underline{B}) - \rho_s^{-1}\nabla P_s \tag{2c}$$

where the specific internal energy $U_s(\rho_s, \sigma_s)$, expressed as an equation of state, yields the (partial) pressure P_s according to

$$P_s = \rho_s^2\, \partial U_s/\partial\rho_s \quad . \tag{3}$$

Eqs. (1) are the Maxwell equations, including an external time-independent charge density ρ_{ext}, and eqs. (2) and (3) are the laws of compressible ideal fluid dynamics. We neglect heat flow, and therefore express entropy convection by the adiabatic equation (2b).

It is natural in our construction to replace the velocity field variable $\underline{u}_s$ with the momentum density $\underline{M}_s \equiv \rho_s\, \underline{u}_s$. Then phase space consists of the set of dynamical variables $(\underline{M}_s, \rho_s, \sigma_s, \underline{E}, \underline{B})$, while the energy of the system is

$$H(\underline{M}_s, \rho_s, \sigma_s, \underline{E}, \underline{B}) = \sum_s \int\left(\tfrac{1}{2}\rho_s^{-1}|\underline{M}_s|^2 + \rho_s U_s\right)d^3x + \int\left(\tfrac{1}{2}|\underline{E}|^2 + \tfrac{1}{2}|\underline{B}|^2\right)d^3x. \tag{4}$$

(We shall sometimes use the notation of writing, e.g., $\underline{M}_s \equiv (\underline{M}_1, \ldots, \underline{M}_k)$ for k species. Whether this is the case or whether $\underline{M}_s$ refers to the single species s will always be clear from the context.)

We wish to express eqs. (1a) and (2) as Hamiltonian evolution equations

$$\dot{Z} = \{Z, H\} \quad , \tag{5}$$

where Z represents one of the dynamical variables, and the Hamiltonian H is given by the energy, eq. (4). Because we are abandoning canonical coordinates in favor of these physical variables, the bracket in eq. (5) is not expected to have the form of a standard Poisson bracket. Here, we present only a sketch of the construction of the Poisson bracket; a detailed exposition will be presented in a later publication.

We regard the system defined by eqs. (1), (2), (3), and (4) as the coupling of the vacuum Maxwell equations to ordinary ideal fluid dynamics. Therefore, we shall briefly review the Hamiltonian structures of these theories.

The equations of motion for a single fluid species composed of uncharged particles are eqs. (2) and (3), with s=1 and a_s=0. The Hamiltonian is the first integral in eq. (4). The Poisson bracket for this has been given by Morrison and Greene, and rederived by Marsden and Weinstein (private communication) from the symplectic structure on co-adjoint orbits of a certain Lie group G. In order to motivate a guess of what G might be, we reason as follows. The dual to the Lie algebra g of G, denoted g^*, must contain the phase space variables $(\underline{M}, \rho, \sigma)$. This is achieved here by noting that momentum densities are just vector field densities on $\mathbb{R}^3$, the set of which we write as $\mathcal{X}^*$, and that mass and entropy densities are

scalar field densities on $\mathbb{R}^3$, written $\mathcal{F}^*$. Thus the fluid phase space is, with x denoting the direct product,

$$\{(\underline{M}, \rho, \sigma)\} = g^* = \mathcal{X}^* \times \mathcal{F}^* \times \mathcal{F}^* \quad .$$

The duality between g^* and g is via the L^2 pairing, "multiply and integrate". It is therefore immediate that

$$g = \mathcal{X} \times \mathcal{F} \times \mathcal{F}$$

where $\mathcal{X}$ denotes vector fields and $\mathcal{F}$ denotes scalar fields on $\mathbb{R}^3$. Now $\mathcal{X}$, which may be thought of as containing the velocity fields on $\mathbb{R}^3$, is the set of generators of displacements of the fluid. Hence the part of G which corresponds to the factor $\mathcal{X}$ in g is the group of diffeomorphisms on $\mathbb{R}^3$, denoted $\mathcal{D}$ (see (11) for a discussion of $\mathcal{D}$ as a Lie group). The group product is composition, since that is how two successive displacements of a fluid element combine to yield a single displacement. On the other hand, $\mathcal{F}$ is a vector space, so that the parts of G corresponding to these factors in g are $\mathcal{F}$ itself.

Hence G may be regarded as sets of triples of parameters labeling positions, densities, and entropies of fluid elements. These are physically related in the following way. Let η denote a diffeomorphism representing the change in position of fluid elements from the set of their initial positions $\{x_0\}$ to their positions $\{x\}$ at some later time. Then for the values of the density and entropy of the fluid element at x, we have, respectively, $\rho(x) = \rho\circ\eta(x_0)$ and $\sigma(x) = \sigma\circ\eta(x_0)$. In other words, we keep track of these quantities by composition with η. Hence the group $\mathcal{D}$ acts on $\mathcal{F} \times \mathcal{F}$ according to

$$\eta \bullet (f_1, f_2) = (f_1\circ\eta, f_2\circ\eta), \text{ where } (f_1, f_2) \in \mathcal{F} \times \mathcal{F}.$$

Finally, then, we conjecture that

$$G = \mathcal{D} \odot (\mathcal{F} \times \mathcal{F}) \quad .$$

The semi-direct product structure, indicated by $\odot$, is specified by the action above.

Then, by constructing explicitly the Kostant-Kirillov-Souriau symplectic structure on co-adjoint orbits of G, one obtains Morrison and Greene's Poisson structure on all of g^*. Suppressing the species label and using a dynamical variable as a subscript to denote the functional derivative with respect to that variable, it is

$$\{F,G\}(\underline{M}, \rho, \sigma) = -\int \left[F_{\underline{M}} \cdot (\nabla G_{\underline{M}}) - G_{\underline{M}} \cdot (\nabla F_{\underline{M}}) \right] \cdot \underline{M}\, d^3x \tag{6}$$
$$- \int \left[F_{\underline{M}} \cdot (\nabla G_\rho) - G_{\underline{M}} \cdot (\nabla F_\rho) \right] \rho\, d^3x - \int \left[F_{\underline{M}} \cdot (\nabla G_\sigma) - G_{\underline{M}} \cdot (\nabla F_\sigma) \right] \sigma\, d^3x \quad .$$

The equations of motion for an ideal fluid now follow from eq. (5).

The vacuum Maxwell equations have been treated as a Hamiltonian system by Pauli (12) and Born and Infeld (13), but the discussion of Marsden and Weinstein (5) from the viewpoint of the reduction procedure has additional noteworthy aspects. The phase space is the cotangent bundle $T^*\mathcal{A}$ to the group $\mathcal{A}$ of vector potential fields $\underline{A}$ on $\mathbb{R}^3$. For $(\underline{A},\underline{Y}) \in T^*\mathcal{A}$, we define the electric and magnetic fields $\underline{E} = -\underline{Y}$, $\underline{B} = \nabla \times \underline{A}$. Then the canonical symplectic structure on $T^*\mathcal{A}$ yields the following bracket of two functionals F, G of $\underline{E}$ and $\underline{B}$:

$$\{F, G\}(\underline{E}, \underline{B}) = \int \left[F_{\underline{E}} \cdot (\nabla \times G_{\underline{B}}) - G_{\underline{E}} \cdot (\nabla \times F_{\underline{B}}) \right] d^3x \ . \tag{7}$$

This, together with the Maxwell Hamiltonian (the last term of eq. (4)), yields the evolution equations (1a) in the vacuum case. To obtain the vacuum versions of eqs. (1b) one uses the symmetry of the theory under the gauge transformation

$$\underline{A} \longmapsto \underline{A} + \nabla \Lambda \quad , \tag{8}$$

$\Lambda \in F$. We denote the gauge group $\{\Lambda\}$ by G. The Maxwell momentum map found in the reduction is $J_M(\underline{A}, \underline{Y}) = -\nabla \cdot \underline{Y}$.

We now consider the combined system of charged species plus the Maxwell equations. The Hamiltonian, written in terms of the canonical momenta $\underline{N}_s = \underline{M}_s + a_s \rho_s \underline{A}$ and the other pre-reduction variables, is

$$H(\underline{N}_s, \rho_s, \sigma_s, \underline{A}, \underline{Y}) = \sum_s \int \left[\tfrac{1}{2}\rho_s^{-1} |\underline{N}_s - a_s \rho_s \underline{A}|^2 + \rho_s U_s \right] d^3x + \tfrac{1}{2}\int \left[|\nabla \times \underline{A}|^2 + |\underline{Y}|^2 \right] d^3x \ . \tag{9}$$

To apply reduction to the full phase space $P = \{(\underline{N}_s, \rho_s, \sigma_s, \underline{A}, \underline{Y})\} \equiv g_s^* \times T^*A$, the action of G on T^*A, eq. (8), must be extended to an action Φ of G on all of P. We require that $H \circ \Phi = H$, and that Φ preserves Poisson brackets of functionals F, G on P, i.e., $\{F \circ \Phi, G \circ \Phi\} = \{F, G\} \circ \Phi$. It is obvious from eq. (9) that the action

$$\Phi_\Lambda(\underline{N}_s, \rho_s, \sigma_s, \underline{A}, \underline{Y}) = (\underline{N}_s + a_s \rho_s \nabla \Lambda, \rho_s, \sigma_s, \underline{A} + \nabla \Lambda, \underline{Y}) \tag{10}$$

satisfies the first requirement, and one can show that it also satisfies the second.

In order now to obtain the momentum map $J: P \longrightarrow G^*$ for Φ, it suffices to calculate $J_s: g_s^* \longrightarrow G^*$, the momentum map on $g_s^* \equiv \{(\underline{N}_s, \rho_s, \sigma_s)\}$. We first define a map $\alpha_s: G \longrightarrow F(P_s)$, from the Lie algebra of G to functions on P_s. Letting $\Lambda[g_s^*]$ be the infinitesimal generator of the action on g_s^* corresponding to $\Lambda \in G$, and letting $X[H]$ denote the Hamiltonian vector field on co-adjoint orbits in g_s^* with Hamiltonian function $H: P_s \rightarrow \mathbb{R}$, α_s is given by the relation $X[\alpha_s(\Lambda)] \equiv \Lambda[g_s^*]$. J_s is then defined by $J_s(\underline{N}_s, \rho_s, \sigma_s)(\Lambda) = \alpha_s(\Lambda)(\underline{N}_s, \rho_s, \sigma_s)$; one finds $J_s(\underline{N}_s, \rho_s, \sigma_s) = -a_s \rho_s$.

Therefore, the momentum map on P, obtained by summing

$$J(\underline{N}_s, \rho_s, \sigma_s, \underline{A}, \underline{Y}) = J_M(\underline{A}, \underline{Y}) + \sum_s J_s(\underline{N}_s, \rho_s, \sigma_s), \tag{11}$$

is $J(\underline{N}_s, \rho_s, \sigma_s, \underline{A}, \underline{Y}) = -\nabla \cdot \underline{Y} - \sum_s a_s \rho_s$.

With $\underline{E} = -\underline{Y}$, reduction at the external charge density ρ_{ext} then specifies that the dynamics takes place on the level set of constant external charge

$$J^{-1}(\rho_{ext}) = \{(\underline{N}_s, \rho_s, \sigma_s, \underline{A}, \underline{Y}) \in P \mid \nabla \cdot \underline{E} = \rho_{ext} + \sum_s a_s \rho_s, \ \underline{E} = -\underline{Y}\} \ .$$

Coordinates on the reduced phase space $J^{-1}(\rho_{ext})/G$ are now given by:

Proposition 1:

$$J^{-1}(\rho_{ext})/G = \{(\underline{M}_s, \rho_s, \sigma_s, \underline{B}, \underline{E}) | \nabla \cdot \underline{E} = \rho_{ext} + \sum_s a_s \rho_s, \nabla \cdot \underline{B} = 0\} .$$

Proof: To elements $(\underline{N}_s, \rho_s, \sigma_s, \underline{A}, \underline{Y})$ of P, associate quintuples $(\underline{M}_s, \rho_s, \sigma_s, \underline{B}, \underline{E})$, where $\underline{M}_s = \underline{N}_s - a_s \rho_s \underline{A}$, $\underline{B} = \nabla \times \underline{A}$, and $\underline{E} = -\underline{Y}$. Then the proposition follows from the momentum map constructed above, and from a simple verification that two elements of $J^{-1}(\rho_{ext})$ are associated to the same quintuple if and only if they are related by the gauge transformation eq. (10).

It remains now to compute the Poisson structure on $J^{-1}(\rho_{ext})/G$.

Proposition 2: For two functionals F,G of the field variables $(\underline{M}_s, \rho_s, \sigma_s, \underline{E}, \underline{B})$, the Poisson bracket is given by

$$\{F,G\}(\underline{M}_s, \rho_s, \sigma_s, \underline{E}, \underline{B}) = \sum_s \{F,G\}(\underline{M}_s, \rho_s, \sigma_s) + \{F,G\}(\underline{E}, \underline{B}) \qquad (12)$$
$$+ \sum_s \int (F_{\underline{M}} \cdot G_{\underline{E}} - G_{\underline{M}} \cdot F_{\underline{E}} + \underline{B} \cdot F_{\underline{M}} \times G_{\underline{M}}) a_s \rho_s \, d^3x$$

where the first and second terms are defined in eqs. (6) and (7), and species subscripts have been suppressed in the functional derivatives.

Proof: Given F and G, define $\overline{F}$ on P according to $\overline{F}(\underline{N}_s, \rho_s, \sigma_s, \underline{A}, \underline{Y}) = F(\underline{M}_s, \rho_s, \sigma_s, \underline{B}, \underline{E})$. Define $\overline{G}$ similarly. Then $\{F,G\}$ is found by computing $\{\overline{F},\overline{G}\}$ as the sum of eq. (6), written for unreduced variables, and the canonical bracket on T^*A, and by expressing the result in terms of the variables on $J^{-1}(\rho_{ext})/G$.

We observe that the first term of eq. (12) involves only the fluid variables and that the second is purely electromagnetic, while the third provides the coupling of the fluids to the electric and magnetic fields. Bilinearly, skew symmetry, and the Jacobi identity all follow for eq. (12) by the methods used in its derivation. In addition it is readily verified that the correct evolution equations for the phase space variables, in the form of eq. (5), follow from eqs. (12) and (4). Additional body forces, such as gravity, can easily be incorporated into eq. (2c) by the inclusion of an appropriate term in the Hamiltonian. Finally, eqs. (1b), rather then being postulated separately as initial conditions, follow from the gauge invariance of electromagnetism.

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REFERENCES

1. P. Morrison and J. Green, Phys. Rev. Lett. 45, 790 (1980).
2. D. Holm and B. Kupershmidt, preprint.
3. P. Morrison, Phys. Lett. 80A, 383 (1980).
4. A. Weinstein and P. Morrison, Phys. Lett. 86A, 235 (1981).
5. J. Marsden and A. Weinstein, Physica 4D (in press).
6. I. Bialynicki-Birula and J. Hubbard, submitted to Phys. Rev. A.
7. R. Spencer and A. Kaufman, submitted to Phys. Rev. A (Rapid Comm.).
8. R. Abraham and J. Marsden, Foundations of Mechanics (W. A. Benjamin Co., Reading, Mass, 1978).
9. J. Marsden and A. Weinstein, Rep. Math. Phys 5, 121 (1974).
10. J. Marsden, A. Weinstein, R. Schmid, and R. Spencer, Hamiltonian Systems and Symmetry Groups with Applications to Plasma Physics (in preparation).
11. J. Marsden, D. Ebin, and A. Fischer, Proceedings of the thirteenth biennial seminar of the Canadian Mathematical Congress, ed. J. Vanstone, pg. 135 (1972).
12. W. Pauli, General Principles of Quantum Mechanics (1933), (reprinted in English translation by Springer-Verlag, New York, New York, 1981).
13. M. Born and L. Infeld, Proc. Roy. Soc. A 150, 141 (1935).

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