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Ghassan G. Batrouni

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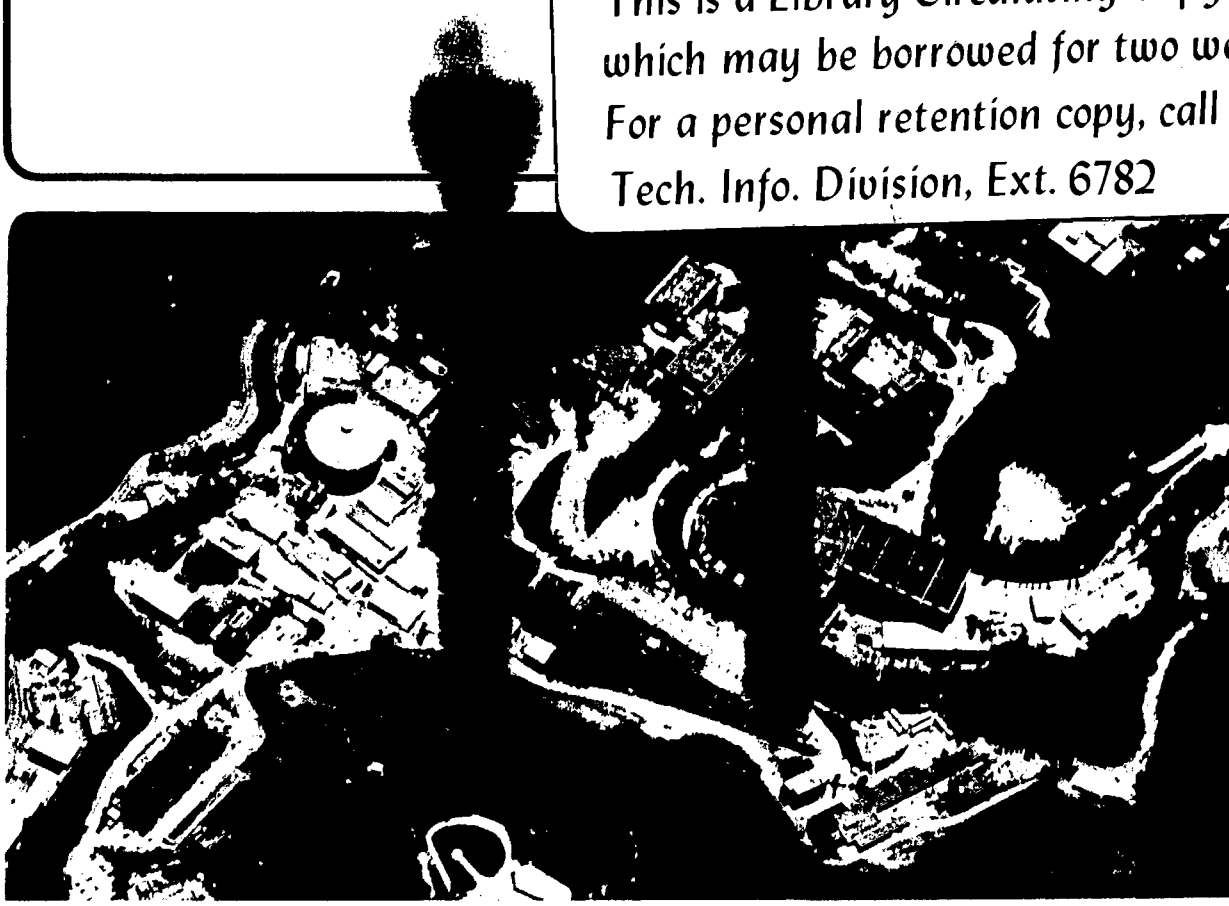

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GAUGE INVARIANT MEAN-PLAQUETTE METHOD FOR LATTICE GAUGE THEORIES*

Ghassan G. Batrouni
Lawrence Berkeley Laboratory
and
Department of Physics
University of California
Berkeley, California 94720

ABSTRACT

We present a new gauge-invariant mean-plaquette method for lattice gauge theories. This is used to calculate plaquette energies and critical couplings for lattice $QED_{4,5}$, Z(2), SU(2) and SO(3). The two main features of this method is that it is gauge invariant and it does not predict a first order phase transition for all theories. In particular, for lattice QCD_4, the method gives a smooth cross-over region in excellent agreement with Monte Carlo data.

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I. INTRODUCTION

The purpose of this paper is to present a new gauge-invariant mean-plaquette calculation method for lattice gauge theories. The starting point of this method is our plaquette formulation of lattice gauge theories[1,2], which is an application of Halpern's field strength and dual potential formulation in the continuum.[3]

The method is quite straight forward for all lattice gauge theories, Abelian and non-Abelian. The strong coupling plaquette energies it gives are in excellent agreement with Monte Carlo data, for all theories we examined, up to the critical point if there is one. For lattice $QCD_4(SU(2))$, which is known to have a continuous crossover instead of a phase transition, our mean-plaquette results are in remarkable agreement with Monte Carlo data from $\beta = 0$ to $\beta = 5$. Our calculation tracks the crossover with high accuracy and does not predict a phase transition. Abelian theories that we examined have phase transitions and their critical couplings are predicted quite accurately by our method. However, we have not yet succeeded in solving our mean-plaquette equations in the weak coupling phases of the Abelian theories. These difficulties will be discussed in more detail.

Our most complete and impressive results are for lattice QCD_4. We will, however, illustrate our method for lattice QED_4 first because it is easiest to understand for Abelian theories, and because we want to motivate some short cuts we will take when doing the calculation for the non-Abelian theories.

It is worth noting, that in low order, the method yields equations that are simple enough to solve on a programmable calculator. All calculations in this paper were done with an HP29C calculator. We should mention, however, that numerical results, especially for the non-Abelian theories, can be easily improved by using a computer. We did not do that here because our purpose is to explain the method and show how it works. Our low order calculation is quite good enough for that.

The paper is organized as follows. In Section II we obtain the mean-plaquette equation for lattice QED_4. In Section III we show how to find approximate solutions to this equation in the strong coupling phase. We also discuss our difficulties with the weak coupling phase. In Section IV we present our results for lattice QED_5 and Z(2) gauge theory in 4 dimensions. We obtain the approximate mean plaquette equation for lattice QCD_4 (SU(2)) in Section V and find approximate solutions in Section VI. In Section VII we present the results for SO(3) lattice gauge theory and compare it with QCD_4. Conclusions and final remarks are in Section VIII.

II. MEAN-PLAQUETTE FOR LATTICE QED_4

In this section we will show, in detail, how to obtain the mean-plaquette equation for lattice QED_4.

Consider the expectation value of a plaquette $P_{\alpha\beta}(r_0)$

$$\langle P_{\alpha\beta}(r_0)\rangle = Z^{-1}\int DU_\mu P_{\alpha\beta}(r_0)e^{\frac{\beta}{4}\Sigma[P_{\mu\nu}(r)+P^+_{\mu\nu}(r)]} \qquad \text{(II.1)}$$

Where $P_{\mu\nu}(r)$ is the product of the four link variables, U_μ, forming the plaquette. Z is the partition function. Using the notation and results of Reference 1 we change the variable of integration to plaquettes

$$\langle P_{\alpha\beta}(r_o)\rangle = Z^{-1}\int DP_{\mu\nu}[\prod_{r,\nu}\delta(e^{i\Delta_\mu\tilde{\theta}_{\mu\nu}(r)}-1)]P_{\alpha\beta}(r_0)e^{\frac{\beta}{4}\Sigma[P_{\mu\nu}(r)+P^+_{\mu\nu}(r)]} \qquad \text{(II.2)}$$

where $P_{\mu\nu}(r) \equiv \exp i\ \theta_{\mu\nu}(r)$. The δ function of the lattice Bianchi identity is periodic and can, therefore, be expanded in a Fourier series giving[1]

$$\langle P_{\alpha\beta}(r_0)\rangle = Z^{-1}\sum_{\{n_\nu(\rho)\}}\int DP_{\mu\nu}P_{\alpha\beta}(r_0)e^{\frac{\beta}{4}\Sigma[P_{\mu\nu}(r)+P^+_{\mu\nu}(r)]+i\Sigma_{\mu<\nu}\tilde{n}_{\mu\nu}(\rho)\theta_{\mu\nu}(r)}$$

$$= Z^{-1}\sum_{\{n_\nu(\rho)\}}\int DP_{\mu\nu}P_{\alpha\beta}(r_0)e^{\frac{\beta}{2}\Sigma_{\mu<\nu}[P_{\mu\nu}(r)+P^+_{\mu\nu}(r)]}\prod_{\substack{r\\ \mu<\nu}}[P_{\mu\nu}(r)]^{\tilde{n}_{\mu\nu}(\rho)} \qquad \text{(II.3)}$$

where

$$\sum_{\{n_\nu(\rho)\}} \equiv \prod_{\rho,\nu}\sum_{n_\nu(\rho)=-\infty}^{+\infty} \qquad \text{(II.4a)}$$

$$\tilde{n}_{\mu\nu}(\rho) = \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}[\Delta_\rho n_\sigma(\rho) - \Delta_\sigma n_\rho(\rho)] \qquad \text{(II.4b)}$$

And as discussed in Reference 1, $n_\mu(\rho)$ is the dual potential, it lies along a link in the dual lattice. ρ and r are, respectively, the coordinates on the dual and original lattices, and since they are in one to one correspondence, a product over r also means a product over ρ.

Now we make the mean-plaquette approximation. Make the following substitution for all plaquettes except $P_{\alpha\beta}(r_0)$

$$P_{\mu\nu}(r),\ P^{+}_{\mu\nu}(r) = P^{-1}_{\mu\nu}(r) \rightarrow \langle P\rangle \qquad \text{(II.5a)}$$

which means

$$[P_{\mu\nu}(r)]^n \rightarrow \langle P\rangle^{|n|} \qquad \text{(II.5b)}$$

We dropped the indices from $\langle P\rangle$ because all plaquettes have the same expectation value. Equation (II.3) becomes

$$\langle P\rangle = Z^{-1} \sum_{\{n_\nu(\rho)\}} \langle P\rangle^{\sum'_{\mu<\nu}|\tilde{n}_{\mu\nu}(\rho)|} \int dP_{\alpha\beta}(r_0)[P_{\alpha\beta}(r_0)]^{1+\tilde{n}_{\alpha\beta}(\rho_0)} e^{\frac{\beta}{2}[P_{\alpha\beta}(r_0)+P^{+}_{\alpha\beta}(r_0)]} \qquad \text{(II.6)}$$

where Σ' means the summation does not include $\tilde{n}_{\alpha\beta}(\rho_0)$. Doing the integral finally gives

$$\langle P\rangle = \frac{\sum_{\{n_\nu(\rho)\}} \langle P\rangle^{\sum'_{\mu<\nu}|\tilde{n}_{\mu\nu}(\rho)|}\, I_{(1+\tilde{n}_{\alpha\beta}(\rho_0))}(\beta)}{\sum_{\{n_\nu(\rho)\}} \langle P\rangle^{\sum'_{\mu<\nu}|\tilde{n}_{\mu\nu}(\rho)|}\, I_{\tilde{n}_{\alpha\beta}(\rho_0)}(\beta)} \qquad \text{(II.7)}$$

where $I_n(\beta)$ is a modified Bessel function, and $\tilde{n}_{\alpha\beta}(\rho_0)$ is the plaquette dual to $P_{\alpha\beta}(r_0)$.

Equation (II.7) is the mean plaquette self consistency equation for the plaquette energy in lattice QED_4. It is obviously invariant under gauge transformations of the dual link variable $n_\mu(\rho)$. Moreover, the mean-plaquette substitution (II.5) is invariant under gauge transformations of the original variables. So, this mean-plaquette method preserves the gauge invariance of the theory and its dual.

III. APPROXIMATE SOLUTION OF THE MEAN-PLAQUETTE EQUATION FOR QED_4

We do not know how to solve Equation (II.7) because there is a very large number of dual links being summed over. However, we can make approximations that will simplify (II.7) in the strong coupling phase. We will make some comments concerning the weak coupling phase at the end of this section.

To motivate the strong coupling ansatz that we will use for the dual potential, $n_\mu(\rho)$, we first recall some of our results from Reference 1. In that paper, we showed that the strong coupling expansion is an expansion towards restoring the lattice Bianchi identity. In other words, in the strong coupling phase, only small fluctuations of the dual potential are important. This means that in this phase, the plaquettes are very weakly correlated, and that a given plaquette is more strongly correlated to plaquettes that share a cube with it than to others that do not.

For these reasons we use the following ansatz

$$n_\mu(\rho) = \Delta_\mu \Lambda(\rho) \tag{III.1}$$

for all dual links except the four that make the dual plaquette $\tilde{n}_{\alpha\beta}(\rho_0)$. $\Lambda(\rho)$ is an integer scalar field. Equation (III.1) says that all, but four, of the dual links are pure gauge. Looking at Equation (II.3) we see that this ansatz has the effect of ignoring the lattice Bianchi identity everywhere except for the four cubes that share the plaquette $P_{\alpha\beta}(r_0)$. The ansatz (III.1) makes all $\tilde{n}_{\mu\nu} = 0$ except $\tilde{n}_{\alpha\beta}(\rho_0)$ and those dual plaquettes that share a link

with it. Equation (II.7) becomes

$$\langle P\rangle = Z^{-1} \sum_{\{\Lambda(\rho)\}} \sum_{\{n_\nu(\rho)\}} \langle P\rangle^{5} \sum_{\mu,\rho}{}' \, |n_\mu(\rho)-\Delta_\mu\Lambda(\rho)| \; I_{(\beta)}(1+\tilde{n}_{\alpha\beta}(\rho_0)) \tag{III.2}$$

where here $\sum_{\{n_\nu(\rho)\}}$ is a summation over the values of the four nongauge dual links that form the plaquette $\tilde{n}_{\alpha\beta}(\rho_0)$. Also, Σ' includes only these four dual links. The 5 in the exponent of $\langle P\rangle$ comes from the fact that each of the surviving "live" dual links that form $\tilde{n}_{\alpha\beta}(\rho_0)$ belongs to five other dual plaquettes. The summation over $\Lambda(\rho)$ is just a sum over all gauge equivalent classes and will only result in a multiplicative infinity that cancels between numerator and denominator. So we only need to do the calucalation for one gauge class and we do it for $\Delta_\mu\Lambda(\rho) = 0$. This finally gives

$$\langle P\rangle = \frac{\sum_{\{n_{1,2,3,4}\}} \langle P\rangle^{5} \, I_{(\beta)}(1+n)^{\sum_{i=1}^{4}|n_i|}}{\sum_{\{n_{1,2,3,4}\}} \langle P\rangle^{5} \, I_n(\beta)^{\sum_{i=1}^{4}|n_i|}} \tag{III.3}$$

where we called the four live dual links n_1, n_2, n_3, n_4 and

$$\tilde{n}_{\alpha\beta}(\rho_0) \equiv n = n_1 + n_2 - n_3 - n_4 \tag{III.4}$$

Equation (III.3) generates a polynomial in $\langle P\rangle$ which can be solved numerically. This polynomial is of infinite order, but we can truncate it at a suitably high power because $\langle P\rangle$ is always less than 1.

Notice that the leading term given by (III.3), all $n_i = 0$, is just the leading term in the strong coupling expansion

$$\langle P \rangle = \frac{I_1(\beta)}{I_0(\beta)} \qquad \text{(III.5)}$$

This is plotted as curve 1 in Figure 1. This term is known to agree with Monte Carlo data at large coupling but starts to deviate appreciably at $\beta \approx 0.5$.

If we include the first nonleading term, one of the four $n_i = \pm 1$, we get the polynomial

$$\langle P \rangle - \frac{I_1(\beta)}{I_0(\beta)} - 4\langle P \rangle^5 \left(1 + \frac{I_2(\beta)}{I_0(\beta)}\right) + \frac{8I_1(\beta)}{I_0(\beta)} \langle P \rangle^6 = 0 \qquad \text{(III.6)}$$

This can be easily solved numerically. The solution is plotted as curve 2 in Figure 1. It is very interesting to note that already, at this very low order, agreement between our calculation and Monte Carlo has improved dramatically. Moreover, equation (III.6) has no roots less than one for $\beta > 0.93$*. We interpret this as an indication of a phase transition at $\beta_c = 0.93$. We will have more to say about this interpretation later.

Moreover, writing (III.6) in the form

$$\langle P \rangle = \frac{I_1(\beta) + 4\langle P \rangle^5 (I_0(\beta) + I_2(\beta))}{I_0(\beta) + 8I_1(\beta)\langle P \rangle^5}$$

* Notice that $\langle P \rangle = 1$ is always a solution of (III.3) for any β. For this reason we do not attach any phyical significance to it. The same is true for the mean-plaquette equations of all the theories we examined, Abelian and non-Abelian.

it is easy to see that solving it by iteration will immediately yield the first 4 terms in the strong coupling expansion plus a number of higher order terms. Although these higher order are present in the strong coupling expansion, they are not complete. To generate a more complete set of strong coupling expansion terms, one needs to keep higher powers of $\langle P \rangle$ in the numerator and denominator.

So, already at the level of the first nonleading term, our method yields remarkably accurate results for the strong coupling phase. To estimate the importance of higher order terms we do the following. We find the expression for the next higher order term, for example the next term that contributes in (III.6) is

$$-\left\{22\frac{I_1(\beta)}{I_0(\beta)} + 10\frac{I_3(\beta)}{I_0(\beta)}\right\}\langle P \rangle^{10} + \left\{12 + 20\frac{I_2(\beta)}{I_0(\beta)}\right\}\langle P \rangle^{11} \qquad \text{(III.7)}$$

Then we use the solution of (III.6) to evaluate (III.7) at the coupling of interest. For example, at $\beta = 0.9$, the value of (III.7) is 0.4% of the value of $\langle P \rangle$ as given by (III.6), and so we do not expect it to change the value of $\langle P \rangle$ by much. Indeed, if we include this term in (III.6) and solve for $\langle P \rangle$, we find its value to be shifted up by 1.2%.

Using this method, we find that the combined values of the terms of order 20 and 21 is 0.003% of $\langle P \rangle$ at $\beta = 0.9$. This is very small and the contributions of higher order terms are even smaller and can be ignored. When the solution of the polynomial of order 21 generated by (III.3) is plotted in Figure 1, it is indistinguishable from curve 2, but stops at $\beta_c = 0.91$.

So the mean-plaquette curve in Figure 1 stops at $\beta_c = 0.91$ because the polynomial that we solve has no roots less than one for $\beta > 0.91$. This is interpreted as being due to the complete invalidity of the strong coupling ansatz (III.1) for $\beta > 0.91$. In other words, the strong coupling vacuum, as approximated by the ansatz (III.1), is no longer the vacuum of the theory for $\beta > 0.91$. The theory has now a different vacuum that needs to be approximated by a different ansatz. So, a phase transition occurs at $\beta_c = 0.91$. Notice also, that in terms of monopoles, (III.1) says that the strong coupling vacuum is a condensate of these monopoles with small fluctuations in their density.[1]

A suitable weak coupling ansatz for the dual potential, n_ν, has to take into account the fact that for $\beta > \beta_c$, <u>all fluctuations</u> of this potential are important. This can be done by changing the summation over n_ν in (II.7) into an integration by using the Poisson sum formula

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx\ f(x) e^{2\pi i m x} \qquad \text{(III.8)}$$

However, we could not do the integrals exactly. Work on an approximation scheme is underway.

Another question that needs to be settled is the order of the phase transition. At the moment it is not clear how to determine that. We feel however, that the weak coupling calculation might give an indication of how to answer this question.

The results obtained for the strong coupling phase are in excellent agreement with Monte Carlo. There are two ways to improve

this agreement near the critical point. The first is to keep higher order terms in the polynomial generated by (III.3). The second way, which would require the use of computers, is to keep more "live" dual links. In (III.1) we kept live only the four dual links that formed the plaquette dual to $P_{\alpha\beta}(r_0)$. We should be able to do better by keeping live, in addition to that dual plaquette, all other dual plaquettes that share a link with it. This will give 64 dual links to sum over. It has the effect of taking into account longer correlation lengths which become important near the critical coupling. Another way of saying this is that keeping more live links takes into account larger fluctuations in the density of the monopole condensate that forms the strong coupling vacuum. Such fluctuations are important near the critical coupling.

IV. OTHER ABELIAN LATTICE GAUGE THEORIES

The mean plaquette method can be applied, in exactly the same way as for lattice QED_4, to other Abelian lattice gauge theories. For example, the analog of (II.7) for lattice QED_5 is derived in exactly the same way as (II.7)(but here there are ten types of Bianchi identities) and has the same form. The difference is that the dual variable in QED_4 was a link n_ν, and for 5 dimensions it is a dual plaquette $n_{\mu\nu}$. Also, $\tilde{n}_{\mu\nu}(\rho)$ in (II.7) is replaced by

$$\tilde{n}_{\mu\nu}(\rho) = \frac{1}{2}\,\varepsilon_{\alpha\beta\gamma\mu\nu}\Delta_\alpha n_{\beta\gamma}(\rho) \qquad \text{(IV.1)}$$

where the indices take the vales 1,2,3,4,5. So, $\tilde{n}_{\mu\nu}(\rho)$ is made from the six dual plaquettes that form a 3 dimensional cube.

The reasoning that led from (II.7) to (III.3) was as follows. The only dual variables (links in QED_4) that are kept live, are those that form the geometrical object (plaquette in QED_4) which is dual to $P_{\alpha\beta}(r_0)$. Applying the same reasoning to QED_5 we find that we keep live only the six dual plaquettes that form the cube dual to $P_{\alpha\beta}(r_0)$. All other dual plaquettes are put equal to pure gauge. And since each live plaquette is shared by 5 cubes, other than the cube appearing in the order of the Bessel function, the analog of Equation (III.3) for QED_5 is

$$\langle P\rangle = \frac{\sum\limits_{\{n_{1,2,3,4,5,6}\}=-\infty}^{+\infty} \langle P\rangle^{5\sum_{i=1}^{6}|n_i|}\, I_{(1+n)}(\beta)}{\sum\limits_{\{n_{1,2,3,4,5,6}\}=-\infty}^{\infty} \langle P\rangle^{5\sum_{i=1}^{6}|n_i|}\, I_n(\beta)} \qquad \text{(IV.2)}$$

where n_i are the six live dual plaquette variables that form the cube n

$$n = n_1 + n_2 + n_3 - n_4 - n_5 - n_6$$

It is straight forward to convince one's self that in d dimensions Equation (IV.2) still holds but where instead of 6 live dual variables, there are 2(d-2) of them. And

$$n = \sum_{i=1}^{d-2} n_i - \sum_{i=d-1}^{2(d-2)} n_i$$

In particular, for d = 2 there are no dual variables and Equation (IV.2) yields the exact solution as it should since there are no Bianchi identities.

For Z(2) lattice gauge theory in four dimensions, the analog of (III.3) is

$$\langle P\rangle = \frac{\sum\limits_{\{n_{1,2,3,4}\}=0}^{1} \langle P\rangle^{5\sum_{i=1}^{4}|n_i|}\, e^{\tilde{\beta}\cos\pi(n-1)}}{\sum\limits_{\{n_{1,2,3,4}\}=0}^{1} \langle P\rangle^{5\sum_{i=1}^{4}|n_i|}\, e^{\tilde{\beta}\cos\pi n}} \qquad \text{(IV.3)}$$

where

$$n = n_1 + n_2 - n_3 - n_4$$

$$\tilde{\beta} = \frac{1}{2}\,\ln\left(\frac{1+e^{-2\beta}}{1-e^{-2\beta}}\right) \qquad \text{(IV.4)}$$

The solutions to (IV.2) and (IV.3) are plotted in Figures 2 and 3 respectively. We have in these theories the same problems in the weak coupling phase as we do in QED_4, and the same discussion presented in

the previous section holds here too. However, Z(2) lattice gauge theory in four dimensions is self dual, and this enables us to calculate the weak coupling plaquette energy by using

$$\langle P\rangle(\beta) = \cosh 2\tilde{\beta} - \langle P\rangle(\tilde{\beta}) \sinh 2\tilde{\beta} \qquad \text{(IV.5)}$$

where $\tilde{\beta}$ is given by (IV.4).

V. MEAN-PLAQUETTE FOR LATTICE QCD_4

In this section we will apply to lattice QCD_4 the same mean-plaquette method we applied to Abelian lattice gauge theories and derive the analog of Equation (III.3). Because the Bianchi identities for lattice QCD_4 are very complicated, we will avoid deriving the analog of Equation (II.7) by using the Abelian results to motivate some short cuts that we will take.

The plaquette energy for lattice QCD_4 is given by

$$\frac{\langle \mathrm{Tr}P_{\alpha\beta}(r_1)\rangle}{2} = Z^{-1}\int DU_\mu \frac{\mathrm{Tr}P_{\alpha\beta}(r_1)}{2} e^{\frac{\beta}{4}\sum_{\mu<\nu}\mathrm{Tr}\{P_{\mu\nu}(r)+P^+_{\mu\nu}(r)\}}$$

$$\equiv \langle P\rangle \qquad \text{(V.1)}$$

where $P_{\mu\nu}(r)$ is the product of link variables around a plaquette. As we showed for QCD_3 in Reference 2, we can change the variable of integration in (V.1) from links to plaquettes.

$$\langle P\rangle = Z^{-1}\int DP_{\mu\nu}[\prod_c \delta(P_c - 1)] \frac{\mathrm{Tr}P_{\alpha\beta}(r_1)}{2} e^{\frac{\beta}{4}\sum_{\mu<\nu}\mathrm{Tr}\{P_{\mu\nu}(r)+P^+_{\mu\nu}(r)\}} \qquad \text{(V.2)}$$

Where, following the notation of Reference 2, $\delta(P_c - 1)$ is the lattice Bianchi identity associated with a given cube, and $\prod_c$ is a product over all cubes in the lattice. Here, there are four types of lattice Bianchi identities because there are four types of three dimensional cubes. These Bianchi identities are derived in exactly the same way as in the appendix of Reference 1.

As we saw for QCD_3,[2] the non-Abelian lattice Bianchi identities are nonlocal and somewhat messy to write down and work with. However,

here we do not need to write them out in detail. From the results of the mean-plaquette calculation for Abelian theories, we expect only the four Bianchi identities associated with the 4 cubes that share $P_{\alpha\beta}(r_1)$, to be important in a mean-plaquette calculation for QCD_4. This should be true at least in the strong coupling region.

Dropping all the Bianchi identities, except the four mentioned above, Equation (V.2) becomes

$$\langle P\rangle \approx Z^{-1}\int DP_{\mu\nu}\left[\prod_{c=1}^{4}\delta(P_c-1)\right]\frac{\mathrm{Tr}P_{\alpha\beta}(r_1)}{2}\, e^{\frac{\beta}{4}\sum_{\mu<\nu}\mathrm{Tr}\{P_{\mu\nu}(r)+P^{+}_{\mu\nu}(r)\}} \tag{V.3}$$

where c = 1 to 4 labels the four cubes that share $P_{\alpha\beta}(r_1)$. We emphasize, that Equation (V.3) is expected to give accurate results for the whole strong coupling region after we make the mean-plaquette approximation not before. $\delta(P_c-1)$ can be character expanded[2]

$$\delta(P_c-1) = \sum_{J_c}(2J_c+1)\,\chi_{J_c}(P_c) \tag{V.4}$$

$$J_c = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}\ldots$$

where $\chi_{J_c}(P)$ is the trace of P in the $(2J_c+1)$ dimensional representation. Moreover, because we have only 4 non-Abelian lattice Bianchi identities that share one plaquette, we can easily "Abelianize" them.[2] Then, each P_c is just the product of the six gauge invariant plaquettes that form the cube. To fix ideas, let $P_{\alpha\beta}(r_1)$ be $P_{12}(r_1)$, then the Abelianized Bianchi identity for the (xyz) cube sitting on top of $P_{12}(r_1)$ is given by[2]

$$\delta[P_{12}(r_1)P_{23}(r_1)P_{13}(r_1+\hat{2})P^{+}_{12}(r_1+\hat{3})P^{+}_{23}(r_1+\hat{1})P^{+}_{13}(r_1)-1]$$

$$=\sum_{J_c=0,\frac{1}{2}..}^{\infty}(2J_c+1)\,\chi_{J_c}[P_{12}(r_1)P_{23}(r_1)P_{13}(r_1+\hat{2})P^{+}_{12}(r_1+\hat{3})P^{+}_{23}(r_1+\hat{1})P^{+}_{13}(r_1)] \tag{V.5}$$

where $\hat{1}$, $\hat{2}$, and $\hat{3}$ are unit vectors in the 1, 2 and 3 directions. Using the Gross and Witten trick,[4] the right hand side of Equation (V.5) becomes

$$\sum_{J_c=0,\frac{1}{2}..}^{\infty}(2J_c+1)^{-4}\chi_{J_c}(P_{12}(r_1))\chi_{J_c}(P_{23}(r_1))\chi_{J_c}(P_{13}(r_1+\hat{2}))\chi_{J_c}(P^{+}_{12}(r_1+\hat{3}))$$

$$\times\,\chi_{J_c}(P^{+}_{23}(r_1+\hat{1}))\,\chi_{J_c}(P^{+}_{13}(r_1))$$

$$=\sum_{J_c=0,\frac{1}{2}...}^{\infty}(2J_c+1)^{-4}U_{2J_c}\left(\frac{\mathrm{Tr}P_{12}(r_1)}{2}\right)U_{2J_c}\left(\frac{\mathrm{Tr}P_{23}(r_1)}{2}\right)U_{2J_c}\left(\frac{\mathrm{Tr}P_{13}(r_1+\hat{2})}{2}\right)$$

$$\times\,U_{2J_c}\left(\frac{\mathrm{Tr}P^{+}_{12}(r_1+\hat{3})}{2}\right)U_{2J_c}\left(\frac{\mathrm{Tr}P^{+}_{23}(r_1+\hat{1})}{2}\right)U_{2J_c}\left(\frac{\mathrm{Tr}P^{+}_{13}(r_1)}{2}\right) \tag{V.6}$$

where we used

$$\chi_J(P) = U_{2J}\left(\frac{\mathrm{Tr}P}{2}\right) \tag{V.7}$$

and $U_n(x)$ is the Chebyshev polynomial of the second kind and order n. The other three Bianchi identities are treated similarly. Substituting (V. 4, 5, 6) in (V.3), and making the mean-plaquette approximation

$$\frac{1}{2}\,\mathrm{Tr}P_{\mu\nu}(r) \longrightarrow \langle P\rangle \tag{V.8}$$

for all plaquettes except $P_{12}(r_1)$ gives

$$< P > = Z^{-1} \sum_{\{J_{1,2,3,4}\}} [\prod_{i=1}^{4}(2J_i+1)^{-4} U^5_{2J_i}(< P >)] \int dP_{12}(r_1) [\prod_{i=1}^{4} U_{2J_i}(\frac{TrP_{12}(r_1)}{2})$$

$$\times \frac{TrP_{12}(r_1)}{2} \quad e^{\frac{\beta}{4} Tr[P_{12}(r_1) + P^+_{12}(r_1)]} \qquad (V.9)$$

For each configuration of the J_i's, the integral over $P_{12}(r_1)$ is easy to do. Equation (V.9) is the approximate mean-plaquette equation for lattice QCD_4. It is the analog of Equation (III.3) for lattice QED_4. It can be solved numerically by truncating the summation at high enough values of the J_i's. This will be done in the next section.

VI. APPROXIMATE SOLUTIONS FOR LATTICE QCD_4

As for Abelian theories, the leading term, all $J_i = 0$, of Equation (V.9) is just the leading term of the strong coupling expansion

$$< P > = \frac{I_2(\beta)}{I_1(\beta)} \qquad (VI.1)$$

This is plotted as curve 1 in Figure 4. As is well known, this is accurate only in the very large coupling limit.

If we include the first nonleading term, one of the $J_i = \frac{1}{2}$, we obtain the polynomial*

$$<P> - \frac{I_2(\beta)}{I_1(\beta)} - 4<P>^5(1 + \frac{3I_3(\beta)}{I_1(\beta)}) + 16 \frac{I_2(\beta)}{I_1(\beta)} <P>^6 = 0 \qquad (VI.2)$$

Note the similarity of (VI.2) to (III.6). This similarity disappears when higher order terms are included. Equation (VI.2) is plotted as curve 2 in Figure 4. That figure shows that agreement with Monte Carlo improved markedly upon the inclusion of the first nonleading term. Moreover, already at this very low order, curve 2 exhibits a (mild) crossover region at about the same coupling where the Monte Carlo data exhibits such a behaviour.

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* Actually, for β less than about 1, the contribution from the (1,0,0,0) configuration of J_i is larger than that from $(\frac{1}{2},0,0,0)$, but both are small enough to be unimportant. Nonleading terms, however, become important for $\beta \gtrsim 1$, and in this region the $(\frac{1}{2},0,0,0)$ term is the larger. That is why it appears in (VI.2) as the first nonleading term.

The importance of higher order terms is estimated in the same way as for the Abelian theories. Using that method, we find that at $\beta \approx 2$, the values of J_i that give the dominant contributions are (0,0,0,0), $(\frac{1}{2},0,0,0)$, $(\frac{1}{2}, \frac{1}{2}, 0,0,)$, $(\frac{3}{2},0,0,0)$, and $(\frac{3}{2}, \frac{1}{2},0,0)$. Contributions from other configurations of J_i may be ignored up to $\beta \approx 2$. Notice that we are not attempting to find all the roots of (V.9) but are verifying that (V.9) has a solution that agrees very well with Monte Carlo data.

Solving Equation (V.9) keeping only the above mentioned terms yields curve 3 in Figure 4. This curve is in excellent agreement with the Monte Carlo data points. It tracks the crossover with very high accuracy, and even up to $\beta = 5$, the discrepancy between the mean plaquette and the Monte Carlo curves is only 5%.

One of the most gratifying results of this calculation is the obvious absence of a phase transtion.

The fact that we kept only a few terms from the expansions of the lattice Bianchi identities, and the excellent agreement between the mean-plaquette curve and Monte Carlo results, tell us that the correlation length is quite short even up to $\beta = 5$.

Moreover, the excellent agreement with Monte Carlo indicates that we were justified in dropping the terms that we dropped. To improve agreement in the $\beta > 2.8$ region, one must keep some more higher order terms. This is straight forward but requires the use of a computer.

VII. MEAN-PLAQUETTE FOR SO(3) LATTICE GAUGE THEORY

The plaquette energy for SO(3) lattice gauge theory is

$$\frac{\langle \chi_1(P_{\alpha\beta}(r_1))\rangle}{3} = Z^{-1}\int DP_{\mu\nu}[\prod_c \delta(P_c - 1)]\,\frac{\chi_1(P_{\alpha\beta}(r_1))}{3}\, e^{\frac{\beta}{3}\sum_{\mu<\nu}\chi_1(P_{\mu\nu}(r))} \tag{VII.1}$$

where $\chi_1(P)$ is the trace of P in the three dimensional representation. Going through the same steps as for QCD_4, with the mean-plaquette substitution being

$$\frac{1}{3}\,\chi_1(P_{\mu\nu}(r)) \rightarrow \langle P\rangle \tag{VII.2}$$

gives

$$\langle P\rangle = Z^{-1}\sum_{\{J_{1,2,3,4}\}}\Big[\prod_{i=1}^{4}(2J_i+1)^{-4}U^5_{2J_i}\{U_2(\langle P\rangle),U_0(\langle P\rangle)\}\Big]\int dP_{\alpha\beta}(r_1)$$
$$\times\Big[\prod_{i=1}^{4}U_{2J_i}\Big(\frac{\mathrm{Tr}P_{\alpha\beta}(r_1)}{2}\Big)\Big]\frac{1}{3}\,U_2\Big(\frac{\mathrm{Tr}P_{\alpha\beta}(r_1)}{2}\Big)\,e^{\frac{\beta}{3}\chi_1(P_{\alpha\beta}(r_1))} \tag{VII.3}$$

$J_i = 0,1,2,3,\ldots$

where $U_{2J_i}\{U_2(\langle P\rangle), U_0(\langle P\rangle)\}$ means that we must express U_{2j_i} in terms of U_2 and U_0 (which can always be done because J_i takes only integer values), <u>then</u> put $U_2(\langle P\rangle) = 3\langle P\rangle$ (by Equations (V.7), (VII.2)). Unlike QCD_4, the J_i here take only integer values because the SO(3) action is even under $P_{\mu\nu}(r) \rightarrow - P_{\mu\nu}(r)$ while $\chi_{(\frac{2n+1}{2})}(P_{\mu\nu}(r))$ is odd and thus the integrals vanish.

Again, by explicit calculation we find that for $\beta < 2.5$, the dominant J_i configurations and (0,0,0,0),(1,0,0,0),(1,1,0,0),(3,0,0,0),(1,1,1,0), (1,3,0,0). Taking only these into account, Equation (VII.3) is solved

numerically yielding Figure 5. That figure shows that the mean-plaquette calculation predicts a first order phase transition at $\beta_c \approx 2.5$, in excellent agreement with Monte Carlo simulations.

Agreement between mean-plaquette and Monte Carlo is not so good in the weak coupling phase. The reason is that the terms that were dropped for $\beta < 2.5$, because they were small, become important for $\beta > 2.5$ and must therefore be kept. Therefore, to improve agreement between the mean-plaquette calculation and Monte Carlo, for SO(3) and QCD_4 at weak coupling, more terms should be kept when solving (V.9) or (VII.3). These terms are of interest in the strong coupling region as well where, although their effect is small, it is not zero. They may change the results by about 3%. We may also need to keep a few more Bianchi identities, according to the same prescription as for lattice QED_4, in order to get better agreement in the weak coupling region especially for SO(3).

The fact that keeping only a few terms from the expansion of the lattice Bianchi identity did give the plaquette energy for the weak coupling phase of SO(3) (althought not accurately, in the order we worked with), means that the fluctuations of the dual potential (J_i) are still not very large. In other words the plaquettes are still quite disordered and the correlation length rather short.

An interesting question is then why does QCD_4 have a crossover while SO(3) has a phase transition? The only difference between Equations (V.9) and (VII.3), other than the value of the integrals, is that in (V.9) all values of J_i contribute whereas in (VII.3) only integer values do. As mentioned before, the absence of half odd

integer values of J_i for SO(3) is due to the inability of this action to distinguish between $P_{\alpha\beta}(r)$ and $-P_{\alpha\beta}(r)$. On the other hand, the QCD_4 action can distinguish these two cases and therefore has an extra Z(2) degree of freedom that SO(3) lacks. The effect of this Z(2) degree of freedom in QCD_4 is to allow half odd integer values of J_i to contribute in (V.9). Moreover, as pointed out in Section VI, the nonzero values of J_i that give the dominant contributions to $\langle P \rangle$ for $\beta \approx 2$ (i.e. near the crossover) are half odd integers. This indicates that the Z(2) degree of freedom is playing an important role at the crossover. This seems to agree with the idea that the Z(2) degree of freedom smoothes out the phase transition and makes it a continuous crossover.

VIII. CONCLUSIONS

Finally we present our physical interpretation of this method and the reasons it seems to work better than previous mean-field schemes.

It is clear that topological excitations play an important role in the phase structure of lattice gauge theories. Usual mean-field methods are insensitive to fluctuations of these excitations and it is, perhaps, for this reason that, in a low number of dimensions, these methods give the wrong critical behaviour.

The mean-plaquette method presented here, automatically takes into account the existence of the topological excitations and the fluctuations in their density. This is easiest to illustrate with the Abelian theories. As mentioned in Section (III), the strong coupling ansatz (III.1) says that the topological excitations form a condensate with small fluctuations in its density. As we keep contributions of larger values of n_i in (III.3), we are in fact calculating, perturbatively, the effect of larger fluctuations in the density of this condensate - i.e. longer correlation lengths. And since the difference in the density of the topological excitations, between the strong and weak coupling phases, is nonperturbative, the strong coupling ansatz (III.1) cannot give the weak coupling results. The same interpretation applies to non-Abelian theories, only here the fluctuations in the density of the topological excitations seem to be always perturbative.

This is the extent of our understanding of why the method works. It is not complete, and more work is needed to understand it more fully.

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FIGURE CAPTIONS

Figure 1. QED_4 plaquette energy. The dots are Monte Carlo data points[5,6]. Curve 1 is the leading term of the strong coupling expansion. Curve 2 is the solution to Equation (III.6) and is indistinguishable from the solution to the polynomial of order 21 generated by (III.3). β_c(mean-plaquette) = 0.91, β_c(Monte Carlo) = 1.

Figure 2. QED_5 plaquette energy. The points are Monte Carlo data points,[5] and the solid curve is the mean-plaquette result. β_c(mean plaquette) = 0.79 and β_c(Monte Carlo) = 0.74

Figure 3. Plaquette energy of Z(2) in 4 dimensions. The dote are Monte Carlo data,[7] and the solid curve is the mean plaquette result. β_c(mean plaquette) $\approx$ 0.4247, β_c(exact) = 0.4407.

Figure 4. QCD_4 plaquette energy. The dots are Monte Carlo data.[6] Curve 1 is the leading term of the strong coupling expansion (VI.1). Curve 2 is the solution of (VI.2) and curve 3 is explained below Equation (VI.2).

Figure 5. Plaquette energy for SO(3). The dots are Monte Carlo data.[6] The solid lines are the mean plaquette results. β_c(mean-plaquette) = 2.5, β_c(Monte Carlo) = 2.48.

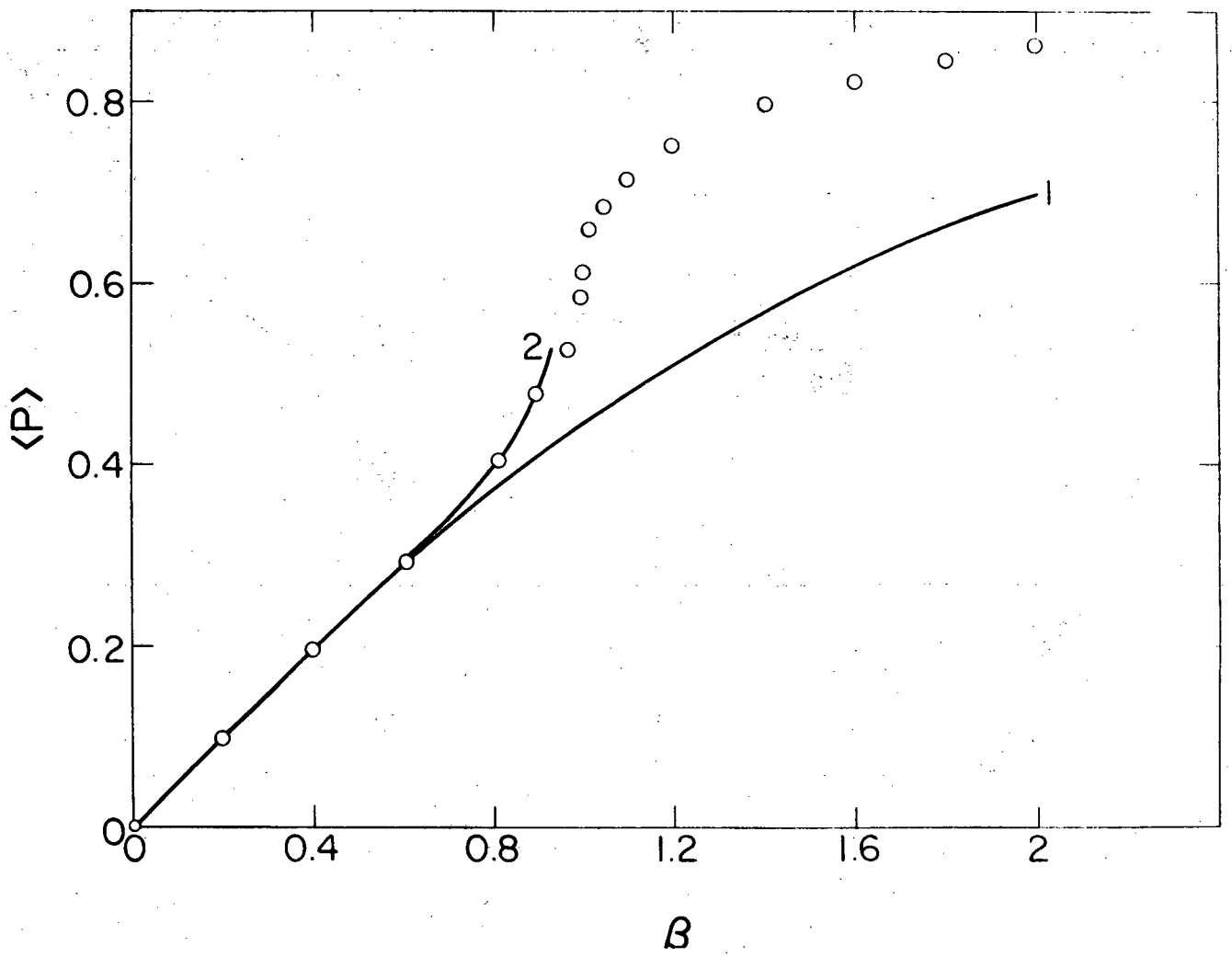

Figure 1

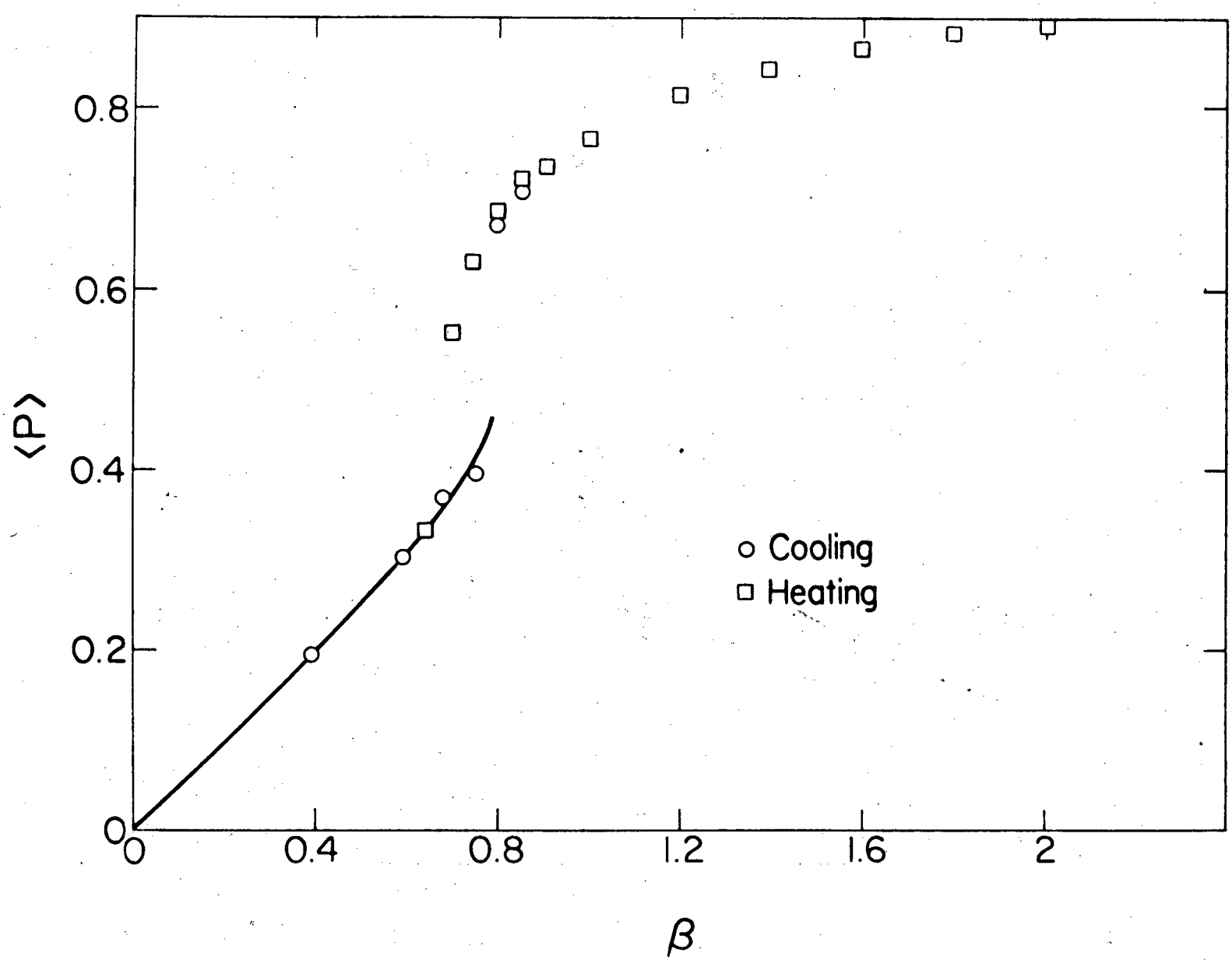

Figure 2

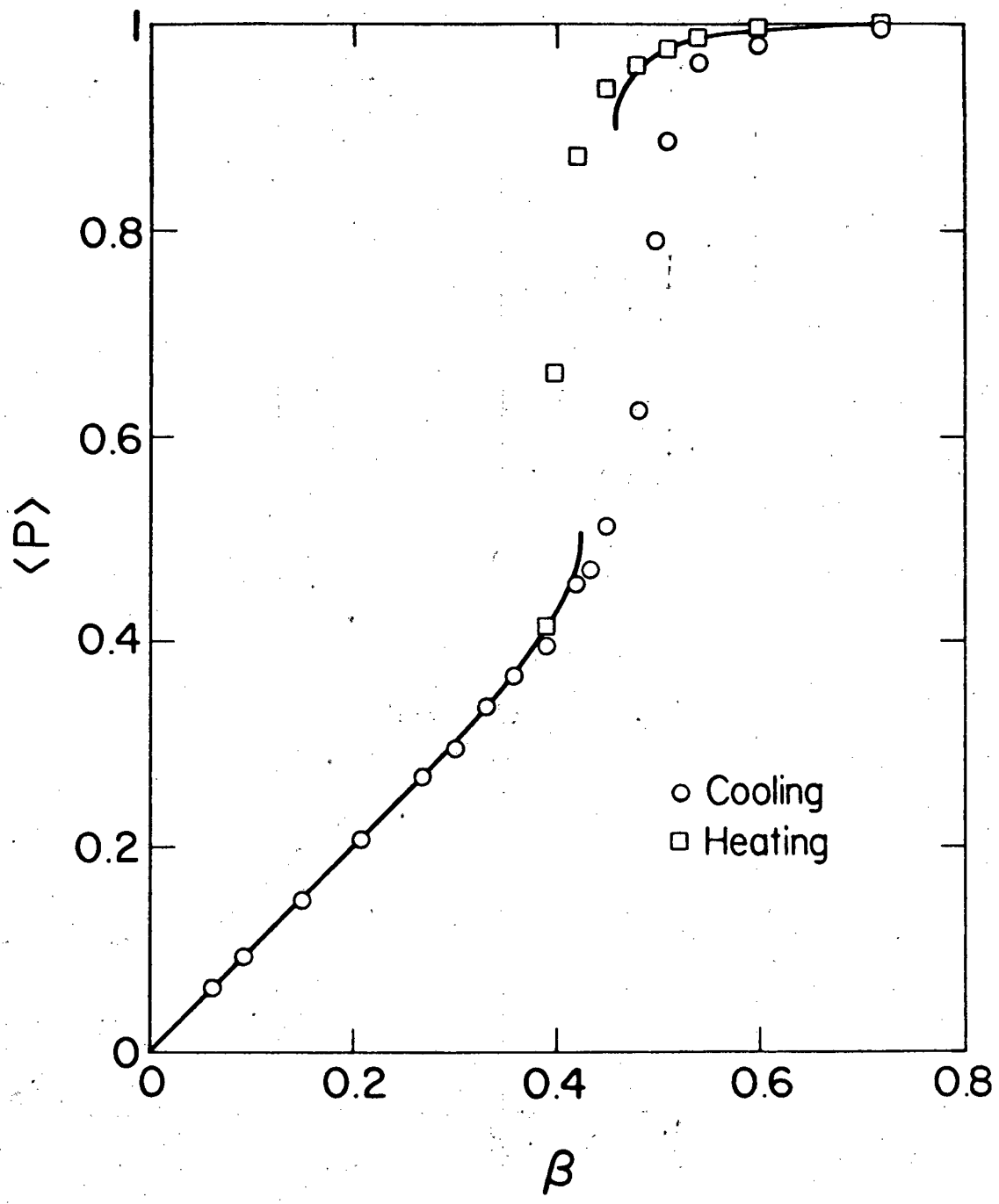

Figure 3

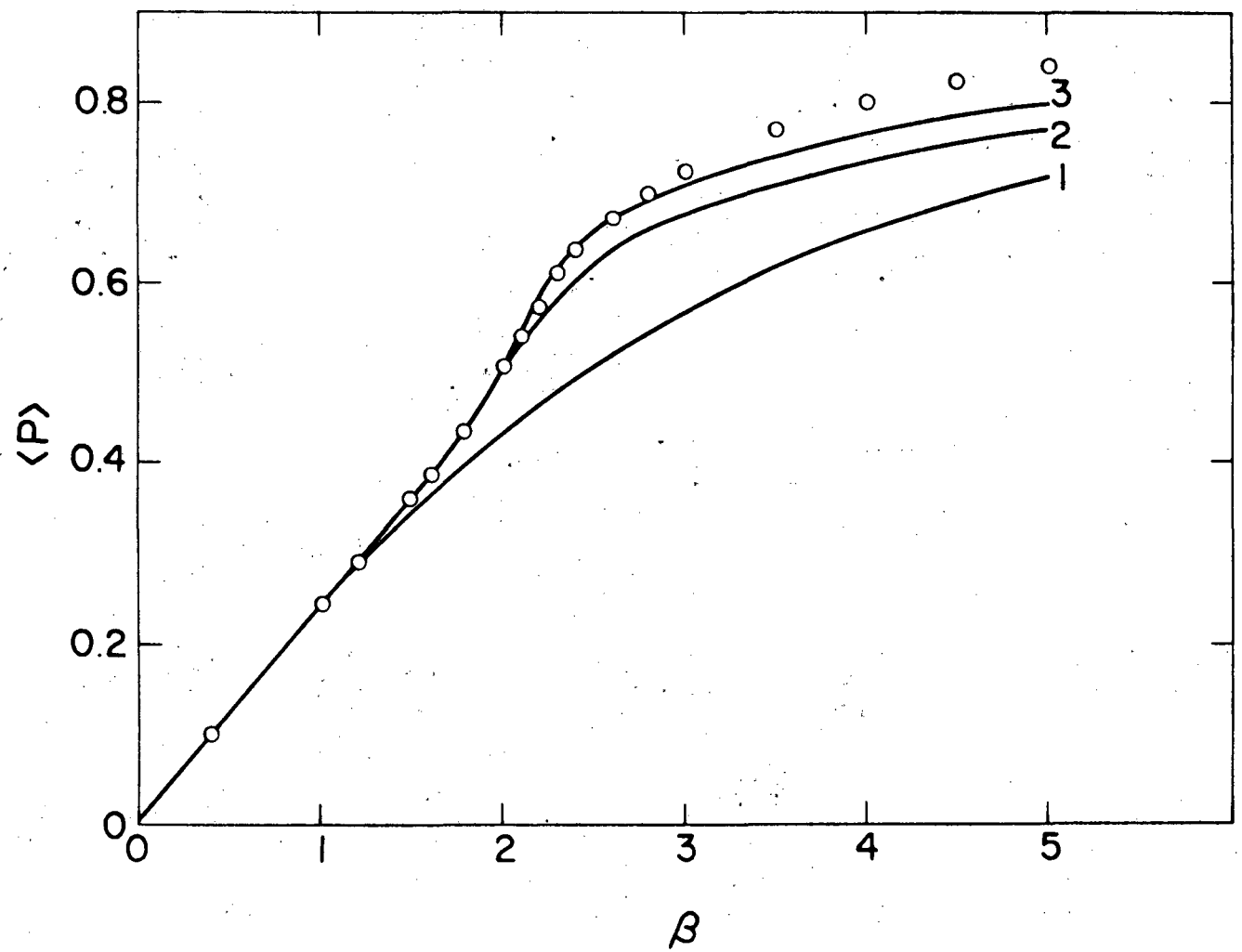

Figure 4

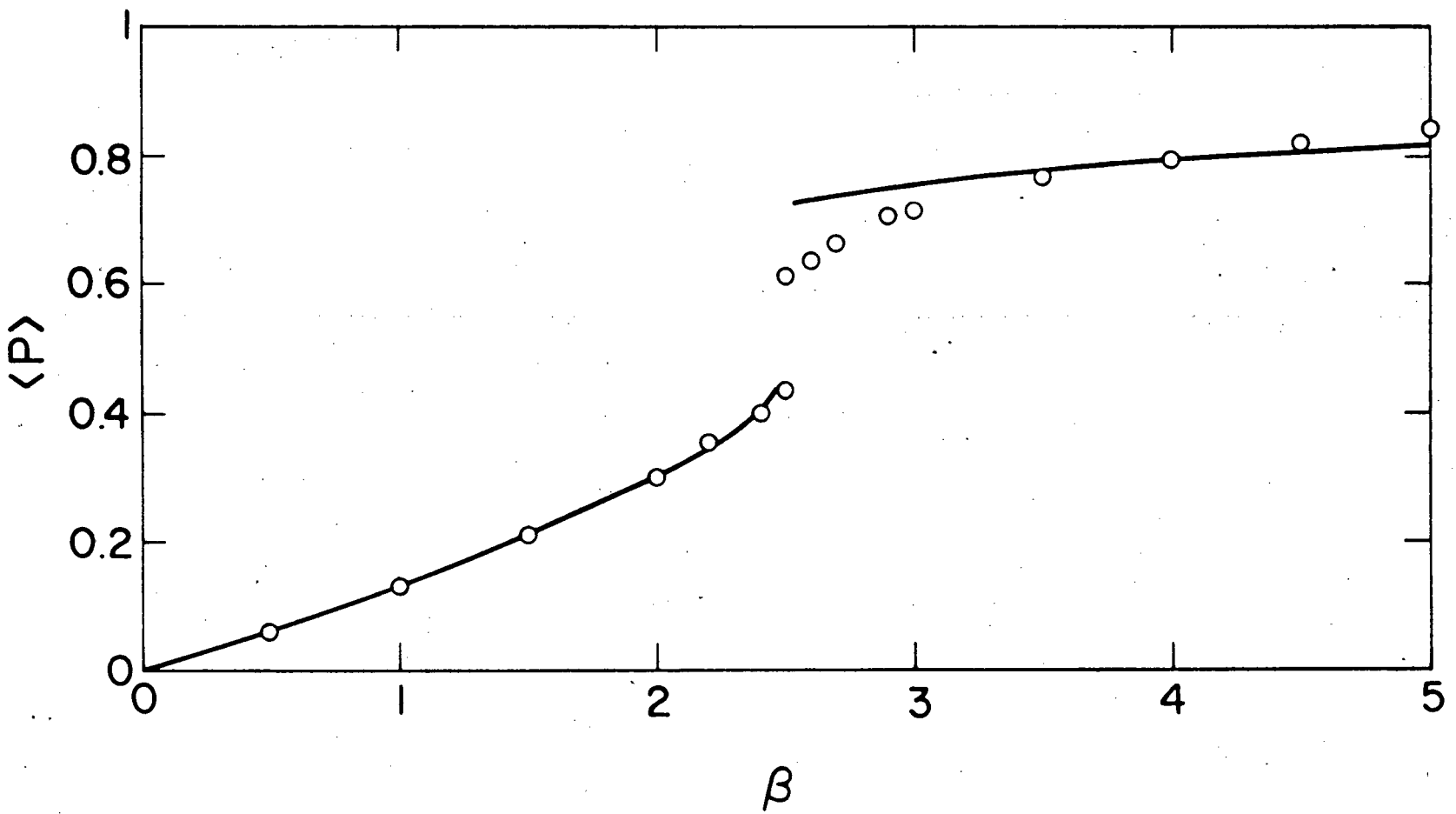

Figure 5

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