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FOR FLUID DYNAMICS AND ELECTRODYNAMICS

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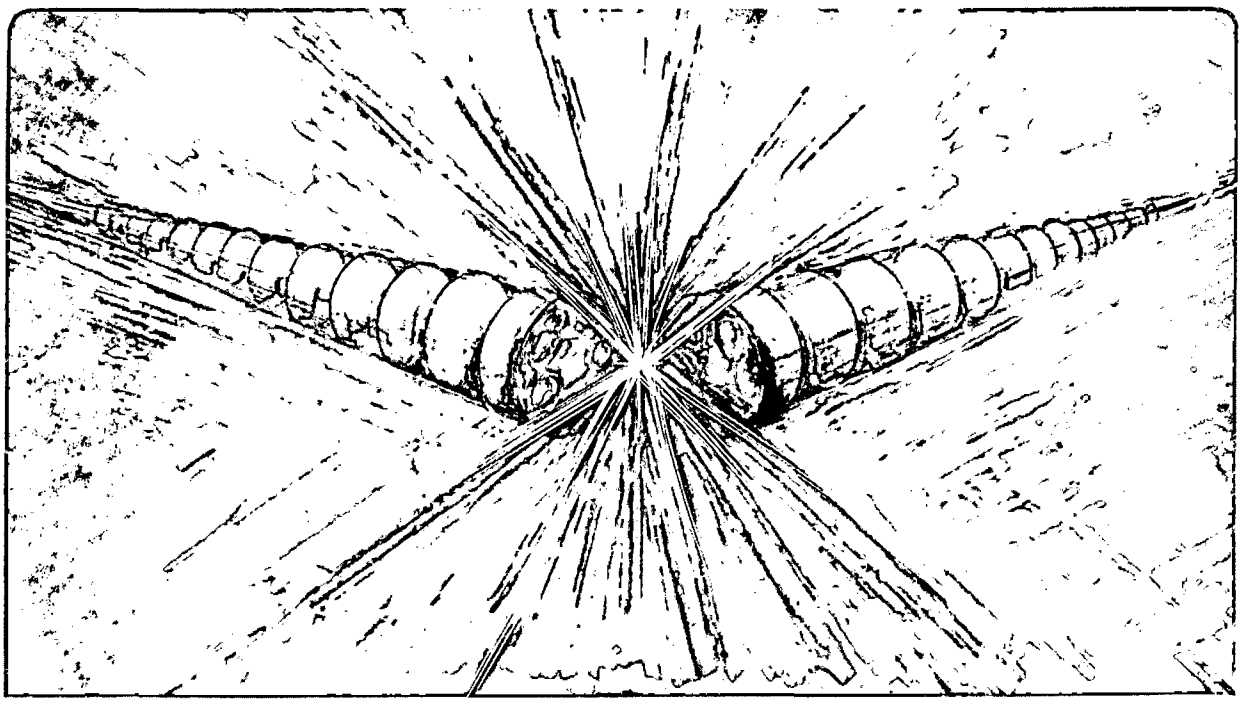

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Elementary Derivation of Poisson Structures for Fluid Dynamics and Electrodynamics

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Abstract

The canonical Poisson structure of the microscopic Lagrangian is used to deduce the noncanonical Poisson structure for the macroscopic Hamiltonian dynamics of a compressible neutral fluid and of fluid electrodynamics.

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I. INTRODUCTION

It has only recently become apparent that Hamiltonian structures exist for several common non-dissipative fluid models. This recognition had been delayed because of the near-universal belief among physicists that canonical variables were required for Hamiltonian formalisms. For example, in Zakharov's review (1), Hamiltonian fluid models were introduced in terms of canonical but unphysical fields.

Great advantages accrue when a model can be formulated in terms of a Hamiltonian with physical variables. This was demonstrated by Littlejohn (2) for the single-particle guiding-center problem. The current interest in fluid Hamiltonians was stimulated by Morrison and Greene's discovery of a Hamiltonian structure for MHD (3), followed by Morrison's formulation of such a structure for the Vlasov-Coulomb and Vlasov-Maxwell systems (4).

Morrison's approach (5) requires a considerable amount of ingenuity. An alternate approach, developed by Marsden and Weinstein (6), uses sophisticated group-theoretical techniques. A third method has now been introduced by Bialynicki-Birula and Hubbard (7), based on the corresponding microscopic model, which leads to the required Poisson brackets quite easily. They applied their method to the relativistic Vlasov-Maxwell system. In the present paper, their method is used to derive the known Poisson structures for non-relativistic compressible fluid dynamics (3) and electrodynamics (8). Our derivation is far more elementary than previous ones. As a slight generalization, curvilinear coordinates are employed.

II. FLUID DYNAMICS

Consider first a neutral fluid, one species for simplicity, composed of N interacting particles. The velocity-dependent part of the Lagrangian is

$$L = \sum_i \frac{1}{2} m\, g_{\mu\nu}(\underline{r}_i)\, v_i^\mu\, v_i^\nu \quad . \tag{1}$$

The canonical momenta,

$$p_\mu^i \equiv \partial L/\partial v_i^\mu = m\, g_{\mu\nu}(\underline{r}_i)\, v_i^\nu \quad , \tag{2}$$

satisfy the canonical Poisson brackets:

$$\{r_i^\mu,\, p_\nu^j\} = \delta_{ij}\, \delta_\nu^\mu \quad . \tag{3}$$

For a macroscopic model, introduce the mass and momentum densities:

$$\rho(\underline{x}) \equiv \sum_i m\delta(\underline{x}-\underline{r}_i); \quad g_\mu(\underline{x}) \equiv \sum_i p_\mu^i\, \delta(\underline{x}-\underline{r}_i) \quad . \tag{4}$$

Their brackets follow immediately from (3); for example,

$$\begin{aligned} \{\rho(\underline{x}),\, g_\mu(\underline{x}')\} &= \sum_i m\{\delta(\underline{x}-\underline{r}_i),\, p_\mu^i\}\, \delta(\underline{x}'-\underline{r}_i) \\ &= \sum_i m\, \delta(\underline{x}'-\underline{r}_i)(\partial/\partial r_i^\mu)\, \delta(\underline{x}-r_i) \\ &= -\,\partial_\mu\, \delta(\underline{x}-\underline{x}')\, \rho(\underline{x}') \;, \end{aligned} \tag{5}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$. Similarly,

$$\{g_\mu(\underline{x}),\, g_\nu(\underline{x}')\} = [-g_\mu(\underline{x}')\, \partial_\nu + g_\nu(\underline{x})\, \partial_\mu']\, \delta(\underline{x}-\underline{x}'). \tag{6}$$

Next we consider functionals of $\rho(\underline{x})$ and $g_\mu(\underline{x})$. For two such functionals F, G, we have (by the chain rule)

$$\{F,G\} \equiv \int d^3x \int d^3x' \, [F^\mu(\underline{x})\, G^\nu(\underline{x}')\, \{g_\mu(\underline{x}),\, g_\nu(\underline{x}')\}$$

$$+ [F_\rho(\underline{x})\, G^\mu(\underline{x}') - F^\mu(\underline{x}')\, G_\rho(\underline{x})]\, \{\rho(\underline{x}),\, g_\mu(\underline{x}')\}], \tag{7}$$

where $F^\mu(\underline{x}) \equiv \delta F/\delta g_\mu(\underline{x})$, $F_\rho(\underline{x}) \equiv \delta F/\delta\rho(\underline{x})$. Now substitute (5) and (6) into (7), obtaining

$$\{F,G\} = \int d^3x \, [- F^\mu(\partial_\mu G^\nu) g_\nu + G^\nu(\partial_\nu F^\mu) g_\mu$$

$$- F^\mu(\partial_\mu G_\rho)\rho + G^\mu(\partial_\mu F_\rho)\rho]. \tag{8}$$

To generalize to several (interacting) species, simply add species labels and sum over species. This is the result of Morrison and Greene (3).

To include an advected quantity σ_i, define its density

$$\sigma(\underline{x}) \equiv \sum_i \sigma_i \, \delta(\underline{x}-\underline{r}_i).$$

By (3), we have

$$\{\sigma(\underline{x}),\, g_\mu(\underline{x}')\} = - \partial_\mu \, \delta(\underline{x}-\underline{x}')\sigma(\underline{x}') \tag{9}$$

For functionals of ρ, g_μ, σ, we find an additional term for (8):

$$\int d^3x\ [-\ F^\mu(\partial_\mu\ G_\sigma)\sigma + G_\mu(\partial_\mu\ F_\sigma)\sigma]\ , \tag{10}$$

where $F_\sigma \equiv \delta F/\delta\sigma(\underline{x})$.

III. FLUID ELECTRODYNAMICS

To allow for interaction with the self-consistent electromagnetic field, we generalize (1) (for one species) to

$$L = \sum_i \frac{1}{2}\, m\ g_{\mu\nu}(\underline{r}_i)\ v_i^\mu\ v_i^\nu\ + \frac{1}{2}\int d^3x(\partial A_\mu/\partial t)(\partial A^\mu/\partial t)$$

$$+\ \sum_i ev_i^\mu\ A_\mu(\underline{r}_i), \tag{11}$$

where $E_\mu(\underline{x})=-\partial A_\mu/\partial t$ (in the radiation gauge). The particle canonical momenta are

$$\pi_\mu^i \equiv \partial L/\partial v_i^\mu = p_\mu^i + eA_\mu(\underline{r}_i), \tag{12}$$

while the canonical field conjugate to $\underline{A}$ is

$$\pi^\nu(\underline{x}) \equiv \partial L/\partial \dot{A}_\nu(\underline{x}) = \dot{A}^\nu(\underline{x}) = -\ E^\nu(\underline{x})\ . \tag{13}$$

The canonical brackets are

$$\{r_i^\mu,\ \pi_\nu^j\} = \delta_{ij}\ \delta_\nu^\mu, \quad \{A_\mu(\underline{x}),\ \pi^\nu(\underline{x}')\} = \delta_\mu^\nu\ \delta(\underline{x}-\underline{x}')\ . \tag{14}$$

It is now straightforward to calculate the brackets connecting $\underline{p}$, $\underline{E}$, $\underline{B}$:

$$\{p^i_\mu, p^i_\nu\} = \delta^{ij}\, e\, B_{\mu\nu}\,(\underline{r}_i), \tag{15}$$

where $B_{\mu\nu}(\underline{x}) = \partial_\mu A_\nu(\underline{x}) - \partial_\nu A_\mu(\underline{x}) = \epsilon_{\mu\nu\lambda}\, B^\lambda(\underline{x})$;

$$\{p^i_\mu, E^\nu(\underline{x})\} = \delta^\nu_\mu\, \delta(\underline{x}-\underline{r}_i); \tag{16}$$

$$\{E^\mu(\underline{x}), B^\nu(\underline{x}')\} = -\,\epsilon^{\mu\nu\lambda}\, \partial_\lambda\, \delta(\underline{x}-\underline{x}'). \tag{17}$$

Moving on to the kinetic momentum density $\underline{g}$, we find

$$\{g_\mu(\underline{x}), g_\nu(\underline{x}')\} = \delta(\underline{x}-\underline{x}')(e/m)\rho(\underline{x})\, B_{\mu\nu}(\underline{x}) + \text{Eq. (6)}; \tag{18}$$

$$\{g_\mu(\underline{x}), E^\nu(\underline{x}')\} = \delta(\underline{x}-\underline{x}')(e/m)\rho(\underline{x})\delta^\nu_\mu. \tag{19}$$

Finally, for functionals of $\underline{g}$, ρ, σ, $\underline{E}$, $\underline{B}$, we obtain

$$\begin{aligned}\{F, G\} = \int d^3x[&(\underline{G}\cdot\nabla\underline{F} - \underline{F}\cdot\nabla\underline{G})\cdot\underline{g} + (\underline{G}\cdot\nabla F_\rho - \underline{F}\cdot\nabla G_\rho)\rho \\ &+ (\underline{G}\cdot\nabla F_\sigma - \underline{F}\cdot\nabla G_\sigma)\,\sigma + (\underline{G}^B\cdot\nabla\times\underline{F}^E - \underline{F}^B\cdot\nabla\times\underline{G}^E) \\ &+ (e/m)\rho\,(\underline{F}\times\underline{G}\cdot\underline{B} + \underline{F}\cdot\underline{G}^E - \underline{G}\cdot\underline{F}^E),\end{aligned} \tag{20}$$

where $\underline{F} \equiv \delta F/\delta\underline{g}$, $\underline{F}^E \equiv \delta F/\delta\underline{E}$, $\underline{F}^B \equiv \delta F/\delta\underline{B}$.

To generalize to several species, simply sum (20) over species; there are no cross terms. This result agrees with that obtained by Spencer (8).

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