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March 1982

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OVERVIEW OF THE FINITE ELEMENT METHOD IN GROUNDWATER HYDROLOGY

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ABSTRACT

Since its introduction into the groundwater literature during the mid 1960's, the finite element method has developed into a very powerful numerical tool for analyzing a variety of groundwater flow problems. Applications of the method cover flow in multi-aquifer systems, flow with a free surface, saturated-unsaturated flow, land subsidence, fractured-porous systems, and large groundwater basins under steady or nonsteady conditions. The method derives its power from the fact that it uses a very general technique for the evaluation of spatial gradients in any direction at any point within the flow domain. This advantage is complemented in the method by an integral statement of the conservation equation at the point of interest. The algorithms stemming from this approach permit relatively simple geometric inputs, even when the problem of interest has complex geometries. From a conceptual perspective there is reason to suspect that alternate formulations of the finite element method may be possible in which the weighted integration technique is dispensed with in favor of an explicit definition of the subdomains of integration. The flexibility of existing finite element algorithms may be enhanced by having options for inputting preprocessed geometric inputs in addition to nodal point coordinates and element lists. Direct formulation of the finite element equations from conservation integrals may provide an alternative that deserves attention. With the advent of mini computers, the finite element method promises to become an every day tool for the practising engineer during the 1980's.

INTRODUCTION

The power of the finite element method was first revealed early in the 1960's in the field of structural engineering. It was soon recognized that the method was sufficiently general to

permit application to a much wider class of problems. The earliest attempts to apply the method in groundwater hydrology commenced during the mid 1960's and by the early 1970's the method was well established as a valuable tool for solving groundwater flow problems in complex domains. Indeed, the impact of the method in the field of quantified hydrogeology can rightly be gauged from the very organization of the present international conference which convenes every two years and continues to attract papers with many new results and novel applications to water resource problems. It is appropriate in a conference such as this for one to stop for a moment and review the finite element method, identifying its strengths as well as its weaknesses. The purpose of this paper is to attempt such an overview in so far as it relates to problems in groundwater hydrology.

Over the past decade literature on the finite element method in groundwater hydrology and related disciplines has grown at a phenomenal rate. It is therefore not realistic to attempt a comprehensive review of all the important papers that have been published on the subject. Instead we shall take the more limited approach of presenting an outline of the more important applications that have been made and to analyze some of the conceptual features of the method as they are currently understood. In developing this approach we shall rely on physical arguments rather than drawing upon the theories of mathematical operations.

A majority of groundwater problems relate to the movement of water under isothermal conditions in the subsurface. More recently, the phenomenon of chemical transport in groundwater systems has assumed a major role of practical importance. In the present work, we shall restrict our attention to the diffusion process, which not only governs the movement of water in variably saturated deformable media but also treats the transport of solutes by real or apparent diffusive (or dispersive) mechanisms. We shall not concern ourselves with the convective or advective transport of solutes by flowing water.

SOME KEY APPLICATIONS IN GROUNDWATER HYDROLOGY

In as much as stress analysis problems deal with systems under equilibrium, it was natural that the earliest application of the method to groundwater flow problems related to steady state seepage. Zienkiewicz and his various colleagues first applied the finite element method to steady state heat flow (Zienkiewicz and Cheung, 1965), and then to steady state seepage problems in anisotropic and heterogeneous media (Zienkiewicz et al, 1966). Taylor and Brown (1967) followed quickly with applications to seepage with a free surface. At the same time, the method was being extended to the transient diffusion problem based on physical considerations by Wilson and Nickell (1966), who successfully simulated transient conduction in concrete structures. Borrowing from their

work, Javandel and Witherspoon (1968, 1969) were able to extend the finite element methods to problems of transient flow of fluids in porous media. Unlike the physical approach taken by Wilson and Nickell (1966), Javandel and Witherspoon used the mathematically formal technique of Gurtin (1965) of employing variational principles to develop the finite element equations. They applied the method to the problem of transient flow of water to a well in multi-layered aquifer systems.

The phenomenon of transient seepage in deformable materials is of fundamental importance in the field of geotechnical engineering, which shares many field problems with hydrogeology. Sandhu and Wilson (1970) addressed themselves to the problem of fluid flow in saturated, deformable porous media and successfully applied the finite element method to the simulation of land subsidence due to fluid withdrawal. The work of Sandhu and Wilson paved the way for the coupled stress-strain and fluid flow analysis of saturated soil masses, which forms the basis for a wide class of problems in the field of soil mechanics. Later, during the mid 1970's other workers solved the subsidence problem using the finite element method, notable among them being Gambolati et al, (1974) and Lewis and Schrefler (1978).

Contemporaneous with this work in porous systems, an increasing interest on the problems of fluid flow in fractured rocks also developed, both by hydrogeologists and by geotechnical engineers. It was therefore natural that researchers sought to apply the newly found tool to the study of problems related to fluid flow in fractures. Wilson and Witherspoon (1970), who investigated the laminar flow of water in fractured porous rocks, applied the finite element method to simulate steady flow of water in a network of rigid fractures, representing fractures as line elements. Noorishad et al, (1971) extended Wilson's work to solve the problem of steady flow in a network of deformable fractures using the finite element method. Noorishad and Witherspoon (1981) have extended this approach even further and can now handle coupled thermo-hydro-mechanical transient behavior for non-isothermal fluid flow in fractured rocks.

The problem of groundwater flow with a free surface has interested civil engineers for nearly a hundred years since the early investigations of Dupuit, Boussinesq and Forschheimer during the late nineteenth century. The free surface problem is characterized by the fact that one segment of the boundary of the flow region, namely, the free surface, is either unknown (in the case of the steady state problem) or changes with time (in the case of the transient problem). The free surface problem is especially cumbersome to investigate using conventional numerical techniques such as the finite difference technique. Indeed, it was soon discovered that the very nature of the finite element method makes it admirably suited to manipulate problems in which the geometry needs to be changed frequently. Neuman and Witherspoon (1970) demonstrated that by using the finite element techniques in conjunction with an

iterative, deformable mesh, one could efficiently solve steady free surface problems. Neuman and Witherspoon (1971) and Ehlers (1971) extended the method to non-steady problems. Later, France (1974) handled three dimensional free surface problems using the finite element method. Gureghian and Youngs (1975) also used the method to calculate steady state water table heights in drained soils.

The next important class of problems that needed attention in subsurface fluid flow involved consideration of the effects of variable water saturation, that is, the phenomenon of saturated-unsaturated flow, which is of importance in civil engineering, hydrogeology and soil physics. Neuman (1973) was among the earliest workers to apply the finite element method to analyse fluid flow in saturated-unsaturated porous media. Later, Neuman et al,(1975) extended the method to include such other features as water uptake by roots, seepage faces, and an evaporation boundary. The same problem was also solved by other workers using the finite element method and these include, Reeves and Duguid (1975), Segol (1976), Narasimhan et al, (1978), Frind and Verge (1978), and so on.

An ingeneous concept was proposed by Barenblatt et al (1960) to simulate the flow of water in fractured porous media. Noting that in a fractured porous medium the diffusivities of the fractures and the porous matrix may vary widely, Barenblatt et al reasoned that the fractures could all be lumped into one continuum while the porous materials could all be lumped into another continuum. The two continua could then interact by introducing appropriately defined coupling terms. Although this approach has certain limitations, it has provided an economical way of analysing certain field problems too difficult to simulate otherwise. This concept is familiarly known as the "double porosity" approach in the petroleum engineering literature. Duguid and Lee (1977) modified Barenblatt's concept slightly and successfully incorporated the double porosity approach into a finite element model.

From a phenomenological point of view, the investigations cited above more or less cover the spectrum of problems that are commonly encountered in groundwater hydrology. We will not attempt here to cite specific references that deal with the application of the method to site-specific problems.

DEVELOPMENTS RELATED TO METHODOLOGY

Although the early work of Wilson and Nickell (1966) used a physically based approach to the finite element method, a desire for mathematical formalism and rigor soon established itself. Accordingly, variational calculus was introduced to formulate the finite element equations. The variational principle for the steady state problem originally proposed by Euler formed the basis for solving the elliptic partial differential equation using finite elements. It also provided the starting point for deriving extended variational principles for

other steady state problems such as those involving the free surface (Neuman and Witherspoon, 1971). The variational approach, however, initially led to one difficulty, namely that a true variational principle is defined only for a system under equilibrium. To overcome this shortcoming, Gurtin (1965) introduced the novel concept of defining a variational principle in the Laplace domain, after initially eliminating the time dimension from the partial differential equation using the Laplace transform. The variational principle in such a transformed domain involves convolution integrals. These variational principles can be minimized using the Rayleigh-Ritz technique to arrive at the discretized finite element equations.

Meanwhile, yet another technique was discovered that facilitated formulation of the finite element equations: this was the method of weighted volume integration, proposed at the turn of the century by the Russian engineer, Galerkin (1915), to solve problems related to the vibration of elastic plates. Finlayson and Scriven (1967) have compared the variational technique with the weighted integration technique. They concluded that it is easier to formulate the finite element equations using the Galerkin technique than by the variational technique, although the latter has certain advantages in providing some fundamental insights into the nature of the physical problem. It is probably reasonable to state that at present the Galerkin approach is the most widely used to formulate finite element equations for fluid flow in subsurface systems.

At the root of the finite element methodology is the concept that one uses a set of non-collinear points to define the spatial variation of a given piecewise-continuous function. In the simplest and primitive case, the number of points required is three in two dimensions and four in three dimensions. These considerations lead to the simple "triangular element" and the "tetrahedral element" in two and three dimensions, respectively. Both of these simple elements permit only first degree approximations to the variation of the desired spatial variable, just as is the case with classical finite differences. Ergatoudis et al (1968) correctly reasoned that by increasing the number of non-collinear points used to define the "finite element", one could introduce higher order approximations to define the spatial variability of the function of interest. Also by taking advantage of the role of the weighting function in the volume integration process, they made the higher order approximation serve the added purpose of handling complex geometrical shapes within the elemental regions. This contribution introduced the now well-known, isoparametric elements. An added flexibility to the isoparametric elements was proposed by Pinder and Frind (1972) who devised a technique for combining different orders of approximations along different sides of the finite element; the so called mixed isoparametric elements. Unlike the triangular and tetrahedral elements, the higher order elements necessitated the numerical evaluation of the

integrals before setting up the global matrix for final solution. The efficiency and accuracy of integration in this regard was greatly facilitated by the use of Gaussian quadrature (Zienkiewicz, 1971).

In its simplest form, the finite element method assumes that the permeability of the material inside an element is constant. Thus, material discontinuities coincide with interfaces of finite elements. In order to improve the accuracy of handling non-linear problems (problems where permeability changes with potential), Pinder et al, (1973) have devised the technique of functional coefficients.

A key task in the application of the finite element method is that of setting up a matrix of algebraic equations to be solved simultaneously for the various unknowns. Even with the fast computers that are increasingly becoming available, the solution of very large matrices is still not an easy process. Matrix solution techniques can be broadly divided into two classes: direct and iterative solvers. Perhaps the most widely used variant of the former is the Gaussian elimination scheme. The chief difficulty with the direct solver techniques when applied to large problems is that of computer storage. In a poorly ordered, large, sparse matrix, considerable time and space are wasted in storing and operating on the zeroes. Hence, many workers continue to develop improved direct solution techniques to go with the finite element algorithms. Among these one should mention the LU decomposition scheme of Cholesky (see Weaver, 1967) and the frontal solution algorithm of Irons (1970). In order to handle very large, sparse matrices, Gupta and Tanji (1976) developed an algorithm in which they operated on portions of the matrix at a time. They chose to store and retrieve all the intermediate information from auxiliary devices such as tapes and discs.

As opposed to direct solvers, iterative methods have the advantage of storing and operating only on non-zero quantities so that the storage requirements are always minimal. For this reason iterative schemes are continuing to be developed and applied to large problems involving the finite element method. Three such techniques that have enjoyed wide usage since the mid 50's are the Alternating Direction Implicit Procedure (Peaceman and Rachford, 1955), the Line Successive Over Relaxation Scheme (Young, 1962) and the Strongly Implicit Procedure (Stone, 1968).

There are several shortcomings associated with the iterative techniques. The first of these is of course the need to provide a good initial guess for the solution and the need to establish an acceptable estimate of error tolerance. Additionally, the iterative scheme may fail to converge or may converge extremely slowly, if at all, should the matrix be too stiff or should it lack diagonal dominance. When one seeks a solution

for a large heterogeneous flow region using the finite element method, it may become very difficult to avoid these difficulties. To lessen the problems arising from storage requirements associated with large sparse matrices, Narasimhan et al, (1978) borrowed the concept of mixing explicit and implicit methods of solution during a time step. This was first developed by Edwards (1969) for his integral finite difference scheme. In a stiff matrix, the time constants associated with nodal points are widely variable; nodes with small time constants react very quickly to perturbations while others react more slowly. It is therefore possible during a given time step, to use the forward differencing approximation to decouple those groups of nodes that can be solved explicitly from those that can be solved implicitly to assure stability. In other words, the mixed explicit-implicit scheme provides a means of replacing a large matrix with a few small submatrices, leading to smaller storage requirements and faster execution times.

CURRENT STATUS

There is little doubt that the finite element method has developed into a valuable tool for investigating complex groundwater systems. In this regard, its strengths include the ability to handle complex geometries, heterogeneities and anisotropy. Additional factors are the ease of input organization and the generality of the formulation. Many computer algorithms currently exist that incorporate the finite element method to solve a variety of groundwater problems. Some of these are general purpose codes while others have been written for a specific class of problems. Clearing houses such as the Holcomb Research Institute of Butler University in Indiana help to distribute these programs or their documentation to interested users.

However, there still exist differences of opinion about the advantages of the finite element scheme over other numerical schemes such as the classical finite differences or the more recent integral finite difference method (Narasimhan and Witherspoon, 1976). Claims of improved accuracy and speed are often made and disputed. Because of wide variations in: (a) the programming efficiencies of individual codes, and (b) the nature of the subcomponents of the different algorithms, it is often extremely difficult to prove or disprove these claims. Familiarity or personal preferences often dictate the approaches taken by many users.

Perhaps it is more relevant to attempt to synthesize the various methods rather than evaluating their individual superiorities. Such a synthesis may lead to a successful blending of the particular advantages of each method and thus generate a more powerful tool not hitherto available. The next section of this paper will attempt such a synthesis.

A DISCUSSION OF THE FINITE ELEMENT METHOD

The physics of the transient diffusion problem requires, in a macroscopic sense, that intensive quantities such as potential be defined as averages over suitably defined domains around each point of interest in the flow region. Secondly, the physics also requires that gradients of potential be evaluated at any surface segment of interest in order that fluid flux across that segment may be calculated.

The finite element method seeks to satisfy the first requirement by associating with each nodal point, a subdomain which is some weighted fraction of the subregion formed by the union of all finite elements that contain the given nodal point as a corner point. Potentials and changes in potentials are averages integrated over these subdomains. The second requirement, namely, that of evaluating gradients of potential, leads to a technique which gave the finite element method its special identity. In this technique one first defines a "finite element", a small region formed by a set of non-collinear points, within which the potential function of interest is assumed to be piecewise continuous and is approximated in terms of the "point" values at the corner points. Once this is done, it is easy to compute the gradient of potential at any point and in any direction within the finite element by simply differentiating the piecewise continuous, approximating function at that point.

This method of gradient evaluation is obviously far more general and powerful than the classical finite differencing technique. First, the finite element approach can provide gradients in more than one direction, an ability which is vital for handling general non-isotropic conditions. Secondly, this approach can provide higher order approximations to the spatial variation of the potential function by merely increasing the number of non-collinear points that define the finite element. Both these features can only be attained using finite differences with a great deal of difficulty, if at all.

The two requirements mentioned above, that of averaging potentials over subdomains and that of evaluating gradients at interfaces, are mutually independent in nature. The uniqueness of the finite element concept is inherent in the latter requirement but not in the former. Indeed, it is possible to have other approaches such as integrated finite differences in which potentials are defined as averages over finite subdomains so that complex geometries can be handled. Yet, the finite element method, as it is commonly understood, is treated as being synonymous with defining the subdomain by a process of weighted volume integration. How relevant is it to uniquely associate the finite element method with weighted volume integration? Is it possible to use finite element gradients with alternate ways of defining the subdomain over which the accumulation term is evaluated in space?

Let us return to the problem of the physical basis for the concept of fluid movement. In order to evaluate the average change in potential over any domain in space bounded by a closed surface, we first evaluate the net accumulation of mass by summing up the fluxes crossing the bounding surface. This accumulation is then divided by the capacity of the domain to obtain the change in potential. A fundamental task therefore is to define the geometry of the subdomain over which the accumulation is evaluated by surface integration. The surface integral, thus, is primitive and is of fundamental importance. In practice, however, one could replace the surface integral by an equivalent volume integral, provided that the equivalence can be established at all times. Indeed one can show (e.g. Narasimhan and Witherspoon, 1976) that, in so far as triangular elements in two dimensions are concerned, the weighted volume integral of the finite element method is exactly equivalent to an appropriate surface integral and this equivalence is in dependent of time in the transient problem. This reasoning can be also extended to tetrahedral elements in three dimensions.

An important consequence of the weighted volume integration approach is that a geometrical description of the subdomain associated with a nodal point is only implied in the integral statement. The location or the nature of the bounding surface is never explicitly stated. While one can give a reasonable physical interpretation to such an approach for triangular and tetrahedral elements, such an interpretation becomes extremely tenuous when applied to higher order elements. If such a bounding surface exists, does it change with time? In the case of higher order elements, what can one say about the equivalence between surface integrals and their corresponding volume integrals? The equivalence is not a factor in steady state problems in which the accumulation is zero over any region surrounding a nodal point. However, the transient problem is of special interest because one must evaluate volume averages for the time derivative. There is a suspicion that if one cannot assure the exact equivalence of surface and volume integrals at all times, then, the weighted volume procedure can in fact prove to be less than desirable. It may be of interest to explore a technique in which gradients are evaluated using the finite element method, while integration of fluxes is carried out over surfaces of explicitly defined subdomains.

One of the most attactive features of the finite element method is the ease with which the geometric information to define the discretized problem can be provided as input. To solve the diffusion problem in general, three types of geometric information must be provided a priori. These are: (a) a set of one dimensional data (lengths between discrete points) to enable the evaluation of gradients of potential; (b) a set of two dimensional data (areas of interfaces between adjoining volume elements) to evaluate fluxes across surface segments; and (c) a set of three dimensional data (bulk volumes of elements) to

calculate the "capacity" of each element.

The need to calculate these quantities beforehand can be bypassed by merely providing as input, the coordinates of the nodal points, supplemented by a node list identifying the corner points of each finite element. From these input data, lengths, areas and volumes are implicitly generated through the process of weighted integration. For many practical problems of interest, this ability to avoid calculating areas and volumes beforehand is extremely convenient and desirable.

Although one should take full advantage of this remarkable feature of the finite element method, it is worthwhile to consider whether or not the flexibility of the method can be enhanced by having an option to input preprocessed geometric data directly. One of the drawbacks inherent in choosing to work with nodal point coordinates is that the domain of volume integration must have a relatively simple shape (triangle, square, tetrahedron, cube) either in the real domain or in the transformed domain. Not only does this force one to write the algorithms for specific coordinate systems, but also the chances for errors of integration are increased because all actual geometric shapes must be approximated in terms of predetermined elemental shapes. When complex geometries are involved, such errors can be minimized by preprocessing the required data as accurately as one may wish.

Another situation in which the preprocessing option would be beneficial is the problem of providing for mesh refinement in certain locations. There often are situations in which a single large volume element may have to communicate with several smaller neighboring ones. Such a situation arises when a wellbore or a fracture element may communicate with several elements in the porous medium. Although one can develop the conductances between the different sized finite elements using published procedures, (e.g. Narasimhan et al, 1978), it is far simpler to handle the situation using preprocessed geometric inputs. We believe the finite element method can benefit from the added flexibility of having an option to provide geometric input information as needed.

A question that is occasionally raised for the finite element method concerns the manner in which the time derivative is handled in the discretized equations. Depending on how one chooses to handle this derivative, one obtains either a "distributed" capacity matrix or a "lumped" capacity matrix. Let us attempt to analyze this question from a conceptual perspective.

The equation of motion, namely, the Darcy-Buckingham law, relates gradient of potential to fluid flux under steady-state conditions. In order to make use of this law, the potential is assumed to be a discrete piecewise continuous function over

each finite element such that the potentials at nodal points are functions only of time while the linear coefficients, which help to express the potential within an element in terms of those at the corner points, are only functions of space. Furthermore, the value at a given nodal point is an appropriate average over a simply connected subdomain immediately surrounding the point. It follows, therefore, that the approximating function in the finite element method is to be restricted solely to the spatial domain and cannot be used in the time domain. Hence, the accumulation term evaluated for a given point using the weighted volume integration procedure can only be converted into an average change in potential at that point by dividing it (the accumulation) by the capacity of the simply connected subdomain associated with that point. This manner of treating the time derivative in which the accumulation at each point is converted to a single average time derivative using a single capacity term of the point in question, gives rise to the so called "lumped" capacity matrix. From the basic physical postulates that govern the problem, it appears that the lumped capacity matrix is the most appropriate way of handling the time derivative.

However, a different situation arises when one derives the finite element equations by minimizing variational principles or by integrating the partial differential equation according to the method of weighted residuals. In the differential equation, one treats the dependent variable, the potential, to be a continuous function of space and time. In formulating finite element equations from the differential equation, one starts by approximating the dependent variable as a piecewise continuous function over a finite element and substitutes this approximating function both in the spatial derivative and in the time derivative. Then, one proceeds to integrate the differential equation over the union of finite elements by appropriately weighting the integrals with respect to each point of interest. Under these conditions, the accumulation at a point is converted into a sum of time derivatives, not only at the point of accumulation but also at its neighbors. The capacity matrix, thus, includes the capacity of the accumulation point as well as that of its neighbors. Such an operation is often referred to as a "distributed" capacity matrix. It is known that in certain problems (e.g. saturated-unsaturated flow, Neuman, 1973) better solutions are obtained by associating all accumulation at a point with a single node rather than distributing it over several nodes.

Which of the two procedures is correct; lumping or distributing? One way of judging would be to apply the procedures to many typical problems with known analytical solutions and observing which of the two methods provides a better comparison with analytic solutions. This trial and error approach, however, is not very satisfying since one cannot generalize the conclusions to other problems for which no analytical solutions

are available. Therefore a choice must be made on conceptual grounds, utilizing the fundamental postulates of the diffusion process and the theoretical foundations of integration.

The concept of a "point" quantity is easily suited for setting up a differential equation and obtaining analytical solutions to it. However, if we seek to obtain a solution by a process of integration, the notion of a value at a point is of little use. Instead, the concept of an average value over a subdomain becomes critical. Therefore the question arises, how appropriate is it to substitute the approximating function in the time derivative before integration is carried out? It appears that the formulation of the finite element method has been very much restricted to one particular perspective. In as much as the numerical problem is ultimately one of integration, there must be an independent way of formulating the equations, apart from that of starting with the differential equation. One such attempt was made by Narasimhan (1978) who developed the necessary equations directly by defining the domains of integration and the operators of averaging. His analysis suggests that the "lumped" capacity matrix is logically more consistent with the physics than the "distributed" capacity matrix.

The foregoing discussion points out the need for a more critical analysis of the foundations of the finite element method from different perspectives than has hitherto been done. The modern digital computer is primarily an integrating device. The power of these devices is likely to be far better utilized by directly formulating groundwater problems as integral systems rather than relying heavily on the integration of differential equations as is done now. A basic change in perspective is clearly needed.

If the 1970's proved the decade of large digital computers, the 1980's promise to belong to the desk-top, mini-computers. The qualifier "mini" remarkably misleads one about the power of these devices. The capacities of the first generation mini-computers already exceed that of the large computers of only a decade ago. With such tools available, solutions to complex problems will come within the everyday reach of practising engineers and hydrologists. Efforts must already be underway to modify existing finite element programs to the desk top computer. The finite element method will thus find more and more applications in problems of groundwater hydrology in the new decade.

CONCLUDING REMARKS

Since its introduction some twenty years ago, the finite element method has come to be recognized as a powerful tool of analysis in the field of groundwater hydrology. The method has been used to solve problems related to large groundwater basins, multiple aquifer systems, flow to wells, flow with a

free surface and a seepage face, flow in the unsaturated zone, water uptake by plant roots and so on. In order to satisfy the needs of efficiency and rigor, the methodology of the scheme has also continued to improve. New concepts such as isoparametric elements and functional coefficients were introduced. Advantage has also been taken of better quadrature schemes and efficient matrix solvers. The method will continue to be a highly desired tool for quantitative analysis in groundwater hydrology.

How the finite element method can be further improved is a question of more than passing interest. We have attempted to answer this question from a conceptual perspective. The uniqueness of the finite element notion lies in its ability to provide a means of evaluating gradients of potential in any direction at a given location. It is this ability that sets the finite element method distinctly apart from the finite difference method. The integral aspect of the finite element method, that is, expressing potentials and changes in potentials as averages over small subdomains through a process of weighted integration, is not necessarily unique to the method. It is possible to have integral methods (such as the integrated finite difference method) which do not use finite element gradients. It will be of interest to change the traditional formulations and evaluate the advantage of the finite element method, independent of weighted volume integration. Perhaps the flexibility of the method can be enhanced by providing options to input geometric data in a form other than that of a set of nodal point coordinates. It will also be of interest to formulate the finite element method directly from the theory of integration by defining domains of integration and operators of averaging. This should provide an alternate perspective not available in the traditional approach of integrating point equations.

The mini-computer revolution is now here. The finite element method will soon become an everyday tool, well within the reach of the practising engineer and hydrologist, and will greatly enhance their ability to solve complex problems in groundwater hydrology.

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