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ON THE DESTRUCTION OF MAGNETIC SURFACES IN TOROIDAL SYSTEMS

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ABSTRACT

The behavior of the field lines in a torus is analogous to the motion of a non-linear oscillator. If ε, small and positive, is the perturbation parameter, we consider a toroidal system in which terms of higher order than ε are assumed to give nonobservable contributions. For large non-linearity ($x \gg \varepsilon^{1/2}$) we found that two sets of resonances are sufficient to explain the destruction of the magnetic surfaces in the toroidal system. Resonances that transform the unperturbed surfaces into a structure of magnetic islands we call primary resonances, and the secondary resonances transform the bound-state-like contours of a given island into similar structures of secondary magnetic islands. To every primary island we attach two types of stochasticity, external, due to the overlapping of primary resonances and, internal, due to the overlapping of secondary resonances. Depending on the resonance and the system, the destruction of the magnetic surfaces occurs by either or both processes. In the immediate neighborhoods of the elliptic singularities the perturbation is not canonical and leads to evolution, creating either regions that are forbidden to the field lines or regions of localized stochasticity. The sizes of these regions are functions of the perturbation and the primary resonance parameters, increasing for

smaller non-linearity. For small non-linearity ($x \lesssim \varepsilon^{1/2}$) the field lines may escape the resonance domain into regions of fast changing perturbation, causing instabilities.

For the levitron, we calculated the theoretical perturbations for which some of the primary resonances are completely destroyed. These theoretical values are given in Table II and are in good agreement with the numerical results. A typical example of destruction by internal overlapping is shown in Fig. 2; we note the appearance, in Fig. 2 part c, of secondary magnetic islands and contours.

1. Introduction

Rosembluth, et al.[1] have shown that if resonances overlap, rapid destruction of their island structure occurs. Filonenko, et al.[2] found that in the stellerator, destruction of the magnetic surfaces near the separatrix occurs before the overlapping of primary resonances. It is shown here that in a pertrubed torus the magnetic surfaces at the separatrix are always destroyed, if not by external, by internal overlapping.

We present a general study of formation and destruction of magnetic surfaces in toroidal systems. If r, ϕ and z are the toroidal coordinates, the role of time is played by the z-coordinate while magnetic islands are observed in the toroidal cross section perpendicular to the z-axis. The magnetic field is taken as the sum of a "stationary" field, (stationary in the sense that it is not a function of z), and a small non-stationary perturbation. The unperturbed system is defined to include all stationary contributions. The behavior of the field lines in the perturbed system is shown to follow non-linear oscillating equations where rotational transform plays the role of frequency.

In Section 2, the linearized equations in various regions of the primary magnetic island are derived. For various order of magnitude of the non-linearity coefficients various cases were obtained and are tabulated in Table I.

In Section 3 we study the structure of magnetic island in the limit of large non-linearity. The magnetic islands are shown to contain closed contours centered at each elliptic singularity. Each elliptic singularity, therefore, mark the position of a local magnetic axis for the island. The magnetic contours of adjacent islands of the same

primary resonance are connected through a common contour referred to as a local separatrix. These contours have slow characteristic frequencies and interact with the non-stationary part of the pertrubation. The resonant island contours are referred to as secondary resonances. The behavior of the field lines near the elliptic singularities is treated in Section 3.4.

In Section 4 the behavior of the field line in the limit of small non-linearity is examined. In this limit it is shown that the field lines may escape the resonance zone into regions of fast changing perturbation causing instabilities.

In Section 5 we evaluate the critical perturbations for the destructions of the magnetic surfaces.

We have published[3] a résumé of this work at its early stage. All conclusions drawn in that résumé are also drawn here. However, the critical perturbations, carefully determined in the present analysis, are different from the previously reported perturbations and are in better agreement with the numerical calculations.

2. The Field Line Equations

We consider the magnetic field in the toroidal system of coordinates r, ϕ, z:

$$\vec{B}(r,\phi,z) = \vec{B}^{o}(r,\phi) + \varepsilon\, \vec{B}^{1}(r,\phi,z) \tag{2.1}$$

where $\vec{B}^{o}(r,\phi)$ represent the unperturbed field and is "stationary" in the sense that it is not a function of z, $\varepsilon \geqslant 0$ is a small parameter and $\vec{B}^{1}(r,\phi,z)$ is a non-stationary perturbation field that is periodic with respect to z. The field line equations are:

$$\frac{dr}{B_r} = \frac{r d\phi}{B_\phi} = \frac{dz}{B_z} \tag{2.2}$$

where B_r, B_ϕ and B_z are the field components in the toroidal system expressed as functions of r, ϕ and z.

2.1. The Unperturbed System

We define the unperturbed system to include all stationary terms. In Eq. (2.1) we let $\varepsilon = 0$ and define t "time" by:

$$dt \equiv \frac{dz}{B_z^{o}} \quad . \tag{2.3}$$

We write the unperturbed field line equations from Eq. (2.2) in the following form:

$$\frac{d\frac{r^2}{2}}{dt} = r\, B_r^{o} \tag{2.4)a}$$

$$\frac{d\phi}{dt} = \frac{B_\phi^{o}}{r} \quad . \tag{2.4)b}$$

Using the divergence theorem $\vec{\nabla} \cdot \vec{B}^{o} = 0$ and Eqs. (2.4)a,b we derived the first integral:

$$H\left(\frac{r^2}{2},\phi\right) = \int^{r} B_\phi^o(r',\phi)\, dr' \tag{2.5}$$

where:

$$\frac{d\frac{r^2}{2}}{dt} = -\frac{\partial H}{\partial \phi} \tag{2.6a}$$

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial \frac{r^2}{2}} \quad . \tag{2.6b}$$

The unperturbed magnetic surfaces are represented by surfaces of constant H. Equations (2.6)a,b are similar to Hamilton's equations where H is the hamiltonian, $\frac{r^2}{2}$ and ϕ are the canonical variables and t plays the role of time.

We normalize the major radius of the torus to one and conveniently let $B_z^o(0,\phi) = 1$. By varying z from zero to 2π the field line does not necessarily return to its original position after having gone around the torus. The problem we are concerned with is a perturbation around a situation in which exact flux surfaces exist. We introduce the action and angle variables (I,θ) corresponding to the canonical variables $(\frac{r^2}{2},\phi)$:

$$I = I(H) = \frac{1}{2\pi}\oint \frac{r^2}{2}\, d\phi \tag{2.7a}$$

$$\theta(\phi,I) = \frac{\partial S(\phi,I)}{\partial I} \tag{2.7b}$$

where

$$S(\phi,I) = \int^{\phi} \frac{r^2}{2}\, d\phi \tag{2.7c}$$

and the frequency:

$$\nu = \nu(I) = \frac{dH}{dI} \quad . \tag{2.8}$$

We have chosen the definition for the flux in Eq. (2.7)a in order to give 2π changes for θ every time ϕ changes by 2π. In the action angle representation the field line equation of the unperturbed system are:

$$\frac{dI}{dt} = 0 \tag{2.9)a}$$

$$\frac{d\theta}{dt} = \nu \quad . \tag{2.9)b}$$

The rotational transform $\frac{R}{2\pi}$, measured in number of field line rotations about the toroidal axis per rotation about the major axis of the torus is defined by:

$$\frac{R}{2\pi} = \frac{\delta\phi}{\delta z} \tag{2.10}$$

The rotational transform is null at the separatrix and reverses signs beyond the separatrix. In the interval of interest, between the central magnetic axis and the separatrix, we consider $R/2\pi$ to be positive, this can always be achieved by properly orienting the toroidal coordinates with respect to the field lines. (Although we assumed the rotational transform to be positive, for convenience, the problem with negative transform can be treated in a similar fashion.) Let T be the change in the variable t, averaged with respect to ϕ and corresponding to a variation in z equal to 2π. From Eq. (2.3) we get:

$$T = 2\pi(1 - \langle \delta B_z^o \rangle_\phi) \tag{2.11)a}$$

where

$$\langle \delta B_z^o \rangle_\phi = \frac{1}{2\pi} \int_0^{2\pi} d\phi [B_z^o(r,\phi) - B_z^o(0,\phi)] \tag{2.11)b}$$

Near the central magnetic axis $\langle \delta B_z^o \rangle_\phi$ is small and T is approximately equal to 2π.

In accordance with Eq. (2.9)b:

$$\nu = \frac{d\theta}{dt} = \frac{\delta\theta}{\delta\phi} \quad \frac{\delta\phi}{\delta z} \quad \frac{dz}{dt} \quad ,$$

since $\delta\phi = 2\pi$ when $\delta\theta = 2\pi$ changes and $\frac{dz}{dt} = \frac{2\pi}{T}$ we get:

$$\nu = \frac{R}{2\pi}\,\Omega = \frac{R}{2\pi}\,(1 + \langle \delta B_z^o \rangle_\phi) \tag{2.12)a}$$

where higher order terms in $\langle \delta B_z^o \rangle \phi$ are neglected and

$$\Omega = \frac{2\pi}{T} \quad . \tag{2.12)b}$$

The meaning of Ω becomes clear from $dz = \Omega dt$. Equation (2.12)a relate the frequency ν to the rotational transform $R/2\pi$. These two quantities are approximately equal in the immediate neighorhood of the central magnetic axis. In Ref. 3 we approximated the problem by letting Ω equal to 1. In the present paper, we do not make this approximation.

2.2. The Perturbed System

In this section we determine how Eqs. (2.9)a,b are affected by the perturbation. We let $\frac{dr}{dt} = \frac{dr^o}{dt} + \frac{\varepsilon dr^1}{dt}$ and $\frac{d\phi}{dt} = \frac{d\phi^o}{dt} + \frac{\varepsilon d\phi^1}{dt}$ and linearize Eq. (2.2) to obtain:

$$\frac{dr^o}{dt} = B_r^o \tag{2.13)a}$$

$$\frac{d\phi^o}{dt} = \frac{B_\phi^o}{r} \tag{2.13)b}$$

and

$$\frac{dr^1}{dt} = B_r^1 + \frac{Br^o}{B_z^o} \quad B_z^1 \tag{2.14)a}$$

$$\frac{d\phi^1}{dt} = \frac{1}{r}\left(B_\phi^1 + \frac{B_\phi^o}{B_z^o} B_z^1\right) . \qquad (2.14)b$$

The perturbations in the action and angle variables due to perturbing $\frac{dr}{dt}$ and $\frac{d\phi}{dt}$ (or equivalently the magnetic field) are given by:

$$\frac{dI}{dt} = \frac{dI}{dH}\left[\frac{\partial H}{\partial r}\frac{dr}{dt} + \frac{\partial H}{\partial \phi}\frac{d\phi}{dt}\right] \qquad (2.15)a$$

$$\frac{d\theta}{dt} = \left.\frac{\delta\theta}{\delta\phi}\right)^o \frac{d\phi}{dt} \qquad (2.15)b$$

Using Eqs. (2.6)a,b we get:

$$\frac{dI}{dt} = -\frac{\varepsilon r}{\nu}\left[\frac{dr^o}{dt}\frac{d\phi^1}{dt} - \frac{d\phi^o}{dt}\frac{dr^1}{dt}\right] \qquad (2.16)a$$

$$\frac{d\theta}{dt} = \nu\left[1 + \varepsilon\left(\frac{d\phi^1}{dt}\right) / \left(\frac{d\phi^o}{dt}\right)\right] \qquad (2.16)b$$

By substituting Eqs. (2.13)a,b and Eqs. (2.14)a,b into Eqs. (2.16)a,b we get:

$$\frac{dI}{dt} = \frac{\varepsilon}{\nu}\,\gamma\left(\frac{r^2}{2},\phi,z\right) \qquad (2.17)a$$

$$\frac{d\theta}{dt} = \nu\left[1 + \varepsilon\,\pi\left(\frac{r^2}{2},\phi,z\right)\right] \qquad (2.17)b$$

where r, ϕ and z are the coordinates of the unperturbed system and

$$\gamma\left(\frac{r^2}{2},\phi,z\right) = B_\phi^o B_r^1 - B_\phi^1 B_r^o \qquad (2.18)a$$

$$\pi\left(\frac{r^2}{2},\phi,z\right) = \frac{B_\phi^1}{B_\phi^o} + \frac{B_z^1}{B_z^o} \qquad (2.18)b$$

From Eqs. (2.3), (2.5) and (2.7)a,b,c z can be obtained as a function of I, θ and t on one hand and r and ϕ as functions of I and θ on the other. We introduce the functions:

$$\Gamma(I,\theta,t) \equiv \gamma\left[\frac{r^2}{2}(I,\theta)\ ;\ \phi(I,\theta)\ ;\ z(I,\theta,t)\right] \tag{2.19)a}$$

$$\Pi(I,\theta,t) = \pi\left[\frac{r^2}{2}(I,\theta)\ ;\ \phi(I,\theta)\ ;\ z(I,\theta,t)\right] \tag{2.19)b}$$

and write Eqs. (2.17)a,b in the consistent form:

$$\frac{dI}{dt} = \frac{\varepsilon}{\nu}\,\Gamma(I,\theta,t) \tag{2.20)a}$$

$$\frac{d\theta}{dt} = \nu + \varepsilon\nu\,\Pi(I,\theta,t) \tag{2.20)b}$$

Equations of the form of Eqs. (2.20)a,b were derived for the straight stellerator field[2] and for the levitron.[4]

For an exact flux surface of a rotational transform equal to $R/2\pi$ the field lines are conserved under the transformation: $\phi \to \phi + 2\pi$ $z \to z + \frac{2\pi}{R}\ 2\pi$. This corresponds to a change in θ equal to 2π and a change in t equal to $\frac{2\pi}{\nu}$. We conclude that the function $\Gamma(I,\theta,t)$ and $\Pi(I,\theta,t)$ are periodic with respect to θ of period 2π and with respect to t of period $\frac{2\pi}{\nu}$.

2.3. The Linearized Equations

Let $\{\nu_i\}$ represent the set of resonant frequencies. From Eq. (2.8) $\nu_i = \nu(I = I_i)$. To $\{\nu_i\}$ corresponds a set of values, $\{I_i\}$, for the action. From Eq. (2.7)a $\{I_i\}$ are the unperturbed fluxes characterized by the transforms $\frac{1}{2\pi}\{R_i\} = \frac{1}{2\pi}\left\{\frac{\nu_i}{\Omega_i}\right\}$.

In the domain of a given resonance ($|\Delta I| \ll I_i$), we let

$$I = I_i + \Delta I \tag{2.21}$$

and Taylor expand $\nu(I)$:

$$\nu(I) = \nu_i + \frac{d\nu_i}{dI}\ \Delta I + \ldots$$

where $\frac{d\nu_i}{dI} = \frac{d\nu}{dI}\ (I = I_i)$.

Let x_i define the non-linearity coefficients:

$$x_i = \frac{I_i}{\nu_i}\ \left|\frac{d\nu_i}{dI}\right| , \tag{2.22}$$

and substitute in the Taylor expansion keeping only the first two terms to get.

$$\left|\frac{\Delta\nu}{\nu_i}\right| = x_i\ \left|\frac{\Delta I}{I_i}\right| \tag{2.23}$$

where $\Delta\nu = \nu - \nu_i$.

Let $\overline{\Delta\nu} = \sqrt{\langle (\Delta\nu)^2 \rangle}$ and $\overline{\Delta I} = \sqrt{\langle (\Delta I)^2 \rangle}$ where the average is taken over a complete period of the slow variable $\Delta\theta = \theta - \nu_i t$ and consider the three following approximations:

(i):

$$\frac{\overline{\Delta\nu}}{\nu_i} >> \varepsilon \quad . \tag{2.24)a}$$

From Eq. (2.23) this condition is equivalent to $\overline{\Delta I} >> \frac{\varepsilon I_i}{x_i}$ and since $\frac{\overline{\Delta I}}{I_i} << 1$ then:

$$\frac{\varepsilon I_i}{x_i} << \overline{\Delta I} << I_i \tag{2.24)b}$$

defines the domain of case (i).

In this domain the second term on the right-hand side of Eq. (2.20)b is much smaller than the first and can be dropped . Furthermore we linearize Eqs. (2.20)a,b to obtain:

$$\frac{d\Delta I}{dt} = \frac{\varepsilon}{\nu_i} \Gamma(I_i, \Delta\theta + \nu_i t, t) \tag{2.25)a}$$

$$\frac{d\Delta\theta}{dt} = \frac{d\nu_i}{dI} \Delta I \quad . \tag{2.25)b}$$

These are the magnetic island equations.

(ii)

$$\frac{\overline{\Delta\nu}}{\nu} \sim \varepsilon \tag{2.26)a}$$

which is equivalent to:

$$\overline{\Delta I} \sim \frac{\varepsilon I_i}{x_i} \tag{2.26)b}$$

the linearized equations are:

$$\frac{d\Delta I}{dt} = \frac{\varepsilon}{\nu_i} \Gamma(I_i, \Delta\theta + \nu_i t, t) \tag{2.27)a}$$

$$\frac{d\Delta\theta}{dt} = \frac{d\nu_i}{dI} \Delta I + \varepsilon\nu_i \Pi(I_i, \Delta\theta + \nu_i t, t) \tag{2.27)b}$$

(iii)

$$\frac{\overline{\Delta\nu}}{\nu_i} << \varepsilon \quad , \tag{2.28)a}$$

and in terms of $\overline{\Delta I}$

$$\overline{\Delta I} << \frac{\varepsilon I_i}{x_i} \quad . \tag{2.28)b}$$

In this case the first term on the right hand side of Eq. (2.20)b is much smaller than the second and can be dropped. After linearizing we get:

$$\frac{d\Delta I}{dt} = \frac{\varepsilon}{\nu_i} \Gamma(I_i, \Delta\theta + \nu_i t, t) \tag{2.29)a}$$

$$\frac{d\Delta\theta}{dt} = \varepsilon\nu_i \Pi(I_i, \Delta\theta + \nu_i t, t) \quad . \tag{2.29)b}$$

Of the three above approximations only Eqs. (2.25)a,b may possess bounded solutions. In Section 3 we solve these equations and show that ΔI is of the order of $\varepsilon^{1/2} I_i$. Let us define the resonance domain by $\overline{\Delta I} \lesssim \varepsilon^{1/2} I_i$. In other words, a field line is said to fall in the domain of a resonant surface if it is separated from it by ΔI smaller or of the order of magnitude of $\varepsilon^{1/2} I_i$.

In the action angle plane, the domains of validity of the approximations described above are dependent on the non-linearity coefficients x_i. By using Eqs. (2.24)b, (2.26)b and (2.28)b we determine these domains for various orders of magnitudes of x_i. The results are given in Table I. The various cases are:

<u>For $x_i \gg \varepsilon^{1/2}$</u>

Case I:	Eqs. (2.25)a,b	$\varepsilon^{1/2} I_i \simeq \overline{\Delta I} \ll I_i$
Case II':	Eqs. (2.27)a,b	$\varepsilon I_i \simeq \overline{\Delta I} \ll \varepsilon^{1/2} I_i$
Case III":	Eqs. (2.29)a,b	$0 < \overline{\Delta I} \ll \varepsilon I_i$

<u>For $x_i \simeq \varepsilon^{1/2}$</u>

Case II:	Eqs. (2.27)a,b	$\varepsilon^{1/2} I_i \simeq \overline{\Delta I} \ll I_i$
Case III':	Eqs. (2.29)a,b	$0 < \overline{\Delta I} \ll \varepsilon^{1/2} I_i$

<u>For $x_i \ll \varepsilon^{1/2}$</u>

Case III:	Eqs. (2.29)a,b	$0 < \overline{\Delta I} \ll I_i$

3. The Structure of Magnetic Islands

For large non-linearity coefficient $x_i \gg \varepsilon^{1/2}$ the magnetic islands are shown to contain closed countours centered at each elliptic singularity. Each elliptic singularity, therefore, mark the position of a local magnetic axis for the island. The magnetic courtours of adjacent islands of the same primary resonance are connected through a common countour, referred to as local separatrix. These countours have slow characteristic frequencies and are preturbed by the presence of other resonances in the system. Resonant island countours are referred to as secondary resonances. The behavior of the field lines near the elliptic singularities is treated in Section 3.4.

3.1. The Island Countours

In this section we solve Eqs. (2.25)a,b. First we derive an averaging method that determines the contribution to a given resonance due to all the resonant harmonics.

To simplify the problem we separate Γ into a symmetrical and an autisymetrical parts:

$$\Gamma(I_i,\theta,t) = \Gamma^S(I_i,\theta,t) + \Gamma^A(I_i,\theta,t) \tag{3.1}$$

where

$$\Gamma^S(I_i,\theta,t) = \frac{1}{2}\,\Gamma(I_i,\theta,t) + \frac{1}{2}\,\Gamma(I_i,-\theta,-t) \tag{3.2a}$$

$$\Gamma^A(I_i,\theta,t) = \frac{1}{2}\,\Gamma(I_i,\theta,t) - \frac{1}{2}\,\Gamma(I_i,-\theta,-t) \quad . \tag{3.2b}$$

We expand Γ^S and Γ^A in Fourier series and rearrange terms to get:

$$\Gamma^S(I_i,\theta,t) = 2\sum_{m=0}^{\infty}\sum_{\ell=1}^{\infty}[\gamma^S_{m,\ell}(I_i)\cos(m\theta + \ell\Omega_i t) + \gamma^S_{m,-\ell}(I_i)\cos(m\theta - \ell\Omega_i t)] \tag{3.3a}$$

$$\Gamma^A(I_i,\theta,t) = 2\sum_{m=0}^{\infty}\sum_{\ell=1}^{\infty}[\gamma^A_{m,\ell}(I_i)\ \sin(m\theta + \ell\Omega_i t) + \gamma^A_{m,-\ell}\ (I_i)\ \sin(m\theta - \ell\Omega_i t)] \tag{3.3)b}$$

where we have made use of the relations $\gamma^S_{m,\ell} = \gamma^S_{-m,-\ell}$ and $\gamma^A_{m,\ell} = -\gamma^A_{-m,-\ell}$ and assumed that only terms that are function of t are present in the perturbation i.e., $\gamma_{m,0} \equiv 0$. (In fact it was indicated in Section 2.1 that all stationary terms must be included in the unperturbed system representation.)

We let $\theta = \Delta\theta + \nu_i t$ and average Eqs. (3.3)a,b over a complete period of t; the only non null contributions come from secular terms where

$$m\nu_i - \ell\Omega_i = 0 \tag{3.4}$$

In other words, $\frac{\nu_i}{\Omega_i}$ is rational. Let ℓ_i, m_i be the lowest integer to satisfy Eq. (3.4), ℓ_i and m_i depend on ν_i/Ω_i and therefore characterize the resonance. Since $\frac{\nu_i}{\Omega_i} = \frac{R_i}{2\pi} = \frac{\ell_i}{m_i}$ the rotational transform of the resonant surface is rational. Depending on the perturbation, a subgroup of the exact flux surfaces are excited; we call them primary resonances.

To a given primary resonance of $\frac{R_i}{2\pi} = \frac{\ell_i}{m_i}$ there are contributions from all the resonant harmonics characterized by $(m,\ell) = (pm_i, p\ell_i)$ where $p \geqslant 1$. In the vicinity of this resonance by averaging Eqs. (2.25)a,b over fast oscillations we obtain:

$$\frac{d\Delta I}{dt} = \frac{2\varepsilon}{\nu_i}\sum_{p=1}^{\infty}[\gamma^S_{pm_i,-p\ell_i}(I_i)\ \cos pu - \gamma^A_{pm_i,-p\ell_i}\ (I_i)\ \sin pu] \tag{3.5}$$

where $u = -m_i\Delta\theta \equiv -m_i(\theta - \nu_i t) = -m_i\theta + \ell_i\Omega_i t$.

In the appendix we evaluated the right-hand side of Eq. (3.5) and got:

$$\frac{d\Delta I}{dt} = \frac{\varepsilon}{\nu_i} \; f_{\nu_i}(u) \tag{3.6)a}$$

$$\frac{du}{dt} = \mu \Delta I \tag{3.6)b}$$

where $\mu = - m_i \frac{d\nu_i}{dI}$ and

$$f_{\nu_i}(u) = \frac{1}{T_i} \int_0^{T_i} dt \; \Gamma(I_i, - \frac{u}{m_i} + \nu_i t, t) \quad . \tag{3.7}$$

For simplicity, the indice i will be dropped from most equations up to the end of Section 3.2. Equations (3.6)a,b have the first integral:

$$K(\Delta I, u) = \frac{\mu}{2}(\Delta I)^2 + \frac{\varepsilon}{\nu} V(u) \tag{3.8}$$

where

$$V(u) = - \int_0^u du' \; f_\nu(u') \tag{3.9}$$

Surfaces of constant K represent the island countour equations.

Let us introduce a parameter a_j characterizing the a-countour in the j-island by the initial condition $\Delta I(u = a_j) = 0$. In our notation, by changing j we change from one island of a given resonance to another island of the same resonance where by changing a we change from one countour to another within the j-island. The island countour characterized by a_j is given by:

$$\Delta I = \sigma \sqrt{\frac{2\varepsilon}{\nu}} \left\{ \frac{1}{\mu} \left[V(a_j) - V(u) \right] \right\}^{1/2} \tag{3.10)a}$$

for all u's satisfying the condition:

$$[V(a_j) - V(u)]/\mu \geqslant 0 \quad . \tag{3.10)b}$$

σ = sgn ΔI and is equal to ± 1. σ will change sign with the variation δu (as viewed from the central magnetic axis position), thus forming symetrical countours by reflection with respect to the $I = I_i$ unperturbed surface.

It is easy to show that $\mu V(u = k\pi) = 0$ for $k = 0,1...$ Therefore between $\frac{u}{m_i} = 0$ and $\frac{u}{m_i} = 2\pi$ the function $\mu V(u)$ has $2m_i$ zeros. Taking $\mu V(u)$ to be finite and continueous function of u, it follows that $\mu V(u)$ has m_i minima and m_i maxima. Let $\{\alpha_j\}_1^{m_i}$ be the set of minima and $\{\beta_j\}_1^{m_i}$ be the set of maxima and arrange the u = 0 axis to obtain

$$0 \leqslant \alpha_j \leqslant a_j \leqslant \beta_j < \alpha_{j+1} \leqslant a_{j+1} \leqslant \beta_{j+1} \ ; \text{ all } j \quad . \tag{3.11}$$

For clarification the reader is advised to examin Fig. 1. In order to obtain the maximum excursion in the action we express, as a first step, the action excursion expression in Eq. (3.10)a in terms of variables of the j-island. Let u_j represent the variable u restricted to vary between β_j and β_{j+1} and let $u_j' = u_j - \alpha_j$. Equation (3.10)a expressed in terms of u_j' is:

$$\Delta I(u_j') = \sigma \sqrt{\frac{2\varepsilon}{\nu}} \left\{ \frac{1}{\mu} \left[V'(a_j') - V'(u_j') \right] \right\}^{1/2} \tag{3.12)a}$$

where

$$V'(u_j') = - \int_{\alpha_j}^{\alpha_j + u_j'} du' \, f_\nu(u') \quad , \tag{3.12)b}$$

and $a_j' = a_j - \alpha_j$.

The maximum excursion in the action for the a_j countour is obtained by maximizing $|\Delta I(u_j', a_j')|$ with respect to u_j' and the maximum excursion in the action is obtained by maximizing the result with respect to a_j'. Let ΔI_M be this maximum and let $\beta_j' = \beta_j - \alpha_j$, then

$$\Delta I_M = |\Delta I(0,\beta_j')| = \sqrt{\frac{2\varepsilon}{\nu}} \left\{\frac{V'(\beta_j')}{\mu}\right\}^{1/2} . \tag{3.13}$$

Since α_j is a minimum of $\frac{V(u)}{\mu}$ one can easily show that $\frac{V'(a_j')}{\mu} \geqslant 0$ and therefore $\frac{V'(\beta_j')}{\mu} \geqslant 0$, necessary condition for Eq. (3.13). Another important property is that the function $\frac{V'(\beta_j')}{\mu}$ does not depend on the j-island of a given resonance and is function of the perturbation at the resonance, say $\sqrt{|V'(\beta_j')|} = F_i(P)$. Where P symbolizes the dependence on the perturbation at the primary resonance.

Let W_i define the resonance width; ($W_i = \mathrm{Max}|\Delta\nu_i|$). From Eq. (2.23):

$$W_i = \left|\frac{d\nu_i}{dI}\right| \Delta I_M \tag{3.14}$$

therefore:

$$W_i = \sqrt{\frac{2\varepsilon x_i}{m_i I_i}} \; F_i(P) \tag{3.15}$$

3.2. The Island Contour Oscillations

Consider the field line cross section that is located at $t = t_1$ in the j-island. At "time" $t = t_1 + \frac{2\pi}{\Omega_i \ell_i}$ it will appear in the j + 1 island. Therefore, a given island is transformed into itself in intervals of $\frac{2\pi}{\nu_i}$. We therefore introduce the "new time" for the island oscillations defined by:

$$\tau = \nu_i t \tag{3.16}$$

Since t is "time" we define sgn δt to be positive. From Eq. (3.6) sgn $\delta\, u_j' = \sigma$ sgn μ. If we call $\sigma' =$ sgn μ and introduce the variables:

$$w_a = \sigma' u' \tag{3.17)a}$$

$$v_a = \Delta I(u' = \sigma' w_a) \tag{3.17)b}$$

where j is understood. Equations (3.6)a,b in terms of w_a and v_a are:

$$\frac{dv_a}{d\tau} = \frac{\varepsilon}{\nu^2} f_\nu(\alpha + \sigma' w_a) \tag{3.18)a}$$

$$\frac{d}{d\tau} w_a = \frac{|\mu|}{\nu} v_a \tag{3.18)b}$$

and can be derived from the hamiltonian:

$$K'(v_a, w_a) = \frac{|\mu|}{2\nu} v_a^2 + \frac{\varepsilon\sigma'}{\nu^2} V(\sigma' w_a) \tag{3.19}$$

It is obvious from Eq. (3.18)b that sgn δw_a is the same as sgn v_a and the island countour oscillations for the a-countour occur for w_a variations between $-a'$ and $+a'$ and therefore are of the libration type.

Here we introduce the new action angle variables (η_a, J_a) corresponding to the pair of variables (w_a, v_a):

$$J_a = \frac{1}{4a'} \oint v_a dw_a \tag{3.20)a}$$

$$\eta_a = \frac{\partial S'(J_a, w_a)}{\partial J_a} \tag{3.20)b}$$

where

$$S'(J_a, w_a) = \int^{w_a} v_a dw_a \tag{3.20)c}$$

In this representation the island countour equations are given by:

$$\frac{dJ_a}{d\tau} = 0 \tag{3.21)a}$$

$$\frac{d\eta_a}{d\tau} = \omega_a \tag{3.21)b}$$

where

$$\omega_a = \frac{dK'}{dJ_a} \tag{3.22}$$

J_a is a function of a' and is positive except for $J_a(a' = 0) = 0$. ω_a is also a function of a' and is positive within the island, except for $\omega_a(a' = \beta') = 0$. We therefore refer to the elliptic singularity at $a' = 0$ as the local magnetic axis of the j-island and the island countour at $a' = \beta'$ as the local separatrix. The local separatrix is common for all the islands of a given primary resonance.

In the limit of small oscillation i.e., a' small, the hamiltonian (Eq. (3.19)) reduces to:

$$K'(v_a,w_a) = \frac{|\mu|}{2\nu} v_a^2 + \frac{c}{2\nu} w_a^2 \tag{3.23)a}$$

where

$$c = -\frac{\varepsilon\sigma'}{\nu} \frac{df}{du} (u = \alpha) \quad . \tag{3.23)b}$$

since α is a minimum of $\sigma' V(u)$ then $\sigma' \frac{df}{du} (u = \alpha) < 0$, therefore c is positive. In the limit of small oscillations:

$$K' = \frac{c}{2\nu} a'^2 \quad , \tag{3.24}$$

$$J_a = \frac{\pi}{4} a' \sqrt{\frac{c}{|\mu|}} \tag{3.25}$$

and

$$\omega_a = \frac{\sqrt{|\mu|\, c}}{\nu} \frac{4}{\pi} a' \tag{3.26}$$

It is important to note that the small oscillations limit is possible only for large non-linearities i.e., large x, otherwise the field lines in the small oscillations region will be governed by Eqs. (2.27)a,b, treated in Section 3.4.

Before we close this section we introduce the non-linearity coefficients X_a, associated with the slow frequency ω_a of the island countours:

$$X_a = \frac{J_a}{\omega_a} \left| \frac{d\omega_a}{dJ} \quad (J = J_a) \right| \tag{3.27}$$

In the small oscillations region $X_a = 1$ (refer to Eqs. (3.25) and (3.26)).

3.3. <u>The Island Countour Perturbation and Secondary Resonances</u>

The oscillations examined in the previous section are subject to a "non-stationary perturbation" related to the presence of other resonances in the system. To simplify our notations we drop the indice a and let dots represent derivative with respect to τ. The island perturbation equations are determined by substracting Eq. (3.18)a,b from Eq. (2.25)a,b, we get:

$$\dot{v} = \frac{\varepsilon}{\nu_i^2} \left[\Gamma\left(I_i, -\frac{\sigma' w}{m_i} + \tau - \frac{\alpha}{m_i} ; \frac{\tau}{\nu_i}\right) - f_{\nu_i}(\sigma' w + \alpha) \right] \tag{3.28a}$$

$$\dot{w} = \frac{|\mu|}{\nu_i} v \tag{3.28b}$$

The resulting action and angle perturbations are:

$$\dot{J} = \frac{1}{\omega}\left[\dot{v}\,\frac{\partial K}{\partial v} + \dot{w}\,\frac{\partial K}{\partial w}\right] \tag{3.29)a}$$

$$\dot{\eta} = \dot{w}\,\frac{d\eta}{dw} \tag{3.29)b}$$

since $\dot{v}$ and $\dot{w}$ are of the order of ε and $\varepsilon^{1/2}$ respectively, the highest contributions to Eqs. (3.29)a,b corresponds to evaluating the derivatives $\frac{\partial K}{\partial v}$, $\frac{\partial K}{\partial w}$ and $\frac{\partial \eta}{\partial w}$ on the unperturbed island countour. $\frac{\partial K}{\partial v}$ and $\frac{\partial K}{\partial w}$ are, therefore, obtained from Eqs. (3.18)a,b and Hamilton's equations while:

$$\frac{d\eta}{dw} = \frac{d\eta^{\circ}/d\tau}{dw^{\circ}/d\tau} = \frac{\omega\nu_i}{|\mu|v} \tag{3.30}$$

If we substitute for $\frac{\partial K}{\partial v}$, $\frac{\partial K}{\partial w}$ and $\frac{d\eta}{dw}$ in Eqs. (3.29)a,b we get:

$$\dot{J} = \frac{1}{\omega}\sqrt{\frac{2|\mu|\varepsilon^3}{\nu_i^7}}\;\Gamma'(J,\eta,\tau) \tag{3.31)a}$$

$$\dot{\eta} = \omega(1 + O(\varepsilon^{1/2})) \tag{3.31)b}$$

where

$$\Gamma'(J,\eta,\tau) = \sigma\left\{\sigma' V_i'(a') - \sigma' V_i'[\sigma' w(J,\eta)]\right\}^{1/2}\;\Gamma\left[I_i;\tau - \frac{\alpha}{m_i} - \frac{\sigma' w(J,\eta)}{m_i}\;;\;\frac{\tau}{\nu_i}\right]. \tag{3.32}$$

For an obvious reason we have dropped all stationary terms. The function $w(J,\eta)$ is determined from Eqs. (3.20)a,b, and for most cases is very difficult to determine explicitely. For libration type motion one important property of w is its periodicity in η.[5] Taking this property into consideration we have shown that the function $\Gamma(J,\eta,\tau)$ is a periodic function of η and τ.

Equations (3.31)a,b for J and η are similar to Eqs. (2.25)a,b for I and θ. And, similarly, they are valid in the region defined by:

$$\frac{\varepsilon J_K}{X_K} \ll \overline{\Delta J} \ll J_K \quad . \tag{3.33}$$

In order to explain the terms in Eq. (3.33) we need to introduce the concept of secondary resonances. Those are the excited island countours whose slow transforms are rational. Let $\{\omega_K\}$ represent the set of secondary resonances frequencies to which, from Eq. (3.22), correspond $\{J_K\}$ for the action. $\Delta J = J - J_K$ and $\overline{\Delta J} = \sqrt{\langle (\Delta J)^2 \rangle}$ where the average is taking over a complete period of $\Delta\eta = \eta - \omega_K \tau$.

Let p_K and q_K be the prime integers satisfying $\frac{\omega_K}{\Omega_K'} = \frac{p_K}{q_K}$, $(\Omega_K' \simeq \nu_i \Omega_i)$, and let $\xi = -q_K \Delta\eta$. In the domain defined by Eq. (3.33) we average Eqs. (3.31)a,b with respect to τ and get:

$$\Delta\dot{J} = \frac{1}{\omega_K} \sqrt{\frac{2|\mu|\varepsilon^3}{\nu_i^7}}\; f_{iK}'(\xi) \tag{3.34)a}$$

$$\dot{\xi} = M\Delta J \tag{3.34)b}$$

where $f_{iK}'(\xi) = \langle \Gamma'(J_K;\ \omega_K\tau - \frac{\xi}{q_K};\tau) \rangle_\tau$ and $M = -\, q_K \frac{d\omega}{dJ}\ (J = J_K)$.
Equations (3.34)a,b can be derived from the hamiltonian:

$$h(\Delta J, \xi) = \frac{M}{2}(\Delta J)^2 - \frac{1}{\omega_K} \sqrt{\frac{2|\mu|\varepsilon^3}{\nu_i^7}} \int_0^{\xi} f_{iK}'(\xi')\, d\xi' \tag{3.35}$$

where surfaces of constant h represent the secondary island countours. The maximum excursion in the action, ΔJ_M, is equal to the maximum of $|\Delta J|$ with respect to ξ and h. The secondary resonance width given by:

$$\Delta_{iK} = \left|\frac{d\omega}{dJ}\,(J = J_K)\right| \Delta J_M \quad , \tag{3.36}$$

is equal to:

$$\Delta_{iK} = \left(\frac{2\varepsilon m_i x_i}{\nu_i^6 I_i}\right)^{1/4} \left(\frac{2\varepsilon X_K}{q_K J_K}\right)^{1/2} F'_{iK}(P') \tag{3.37}$$

where $F'_{iK}(P')$ depends on the perturbation at the secondary resonance, symbolized by P'.

3.4. Behavior of the Field Lines Near the Elliptic Singularities in the Limit of Large Non-Linearity Coefficients

The objective of this section is to solve for Cases III" and II'.

Case III":

If we apply the method of averaging derived in Section 3.1 to Eqs. (2.29)a,b in the vicinity of a given resonance, $\frac{\nu_i}{\Omega_i} = \frac{\ell_i}{m_i}$, we get:

$$\frac{d\Delta I}{dt} = \frac{\varepsilon}{\nu_i} f_{\nu_i}(u_j) \tag{3.38)a}$$

$$\frac{du_j}{dt} = -\,\varepsilon \ell_i \Omega_i g_{\nu_i}(u_j) \tag{3.38)b}$$

where $f_{\nu_i}(u_j)$ was defined in Eq. (3.7) and $g_{\nu_i}(u_j)$ is defined similarly by:

$$g_{\nu_i}(u_j) = \frac{1}{T_i}\int_0^{T_i} dt\,\Pi(I_i, -\frac{u_j}{m_i} + \nu_i t, t) \quad . \tag{3.39}$$

Near the elliptic singularity $u'_j = u_j - \alpha_j$ is small, we expand $f_{\nu_i}(u_j)$ and $g_{\nu_i}(u_j)$ in terms of u'_j:

$$f_{\nu_i}(u_j) = 0 + \frac{u_j - \alpha_j}{1!}\,\frac{df_{\nu_i}}{du_j}(\alpha_j) + \ldots \tag{3.40)a}$$

$$g_{\nu_i}(u_j) = g_{\nu_i}(\alpha_j) + \frac{u_j-\alpha_j}{1!} \frac{dg_{\nu_i}}{du_j}(\alpha_j) + \ldots \qquad (3.40)b$$

($f_{\nu_i}(\alpha_j) = 0$ because, by definition, α_j are minima of $V(u_j)$).

For Case III" where $\Delta I \ll \varepsilon I_i$, we are practically at the elliptic singularity and, thus, only the zeroth order terms in Eqs. (3.40)b need to be considered. By substituting into Eqs(3.38)a,b we obtain:

$$\frac{d\Delta I}{dt} = -\varepsilon\mu A\, u_j' \qquad (3.41)a$$

$$\frac{du_j'}{dt} = -\varepsilon\mu A' \qquad (3.41)b$$

where

$$A = -\frac{1}{\mu\nu_i} \frac{df_{\nu_i}}{du_j}(\alpha_j) \qquad (3.42)a$$

and

$$A' = \frac{\ell_i}{\mu}\Omega_i g_{\nu_i}(\alpha_j) \quad . \qquad (3.42)b$$

Since α_j is a minimum of $\frac{V(u_j)}{\mu}$ then

$$A = -\frac{1}{\mu\nu_i} \frac{df_{\nu_i}}{du_j}(\alpha_j) = \frac{1}{\mu\nu_i} \frac{d^2V}{du_j^2}(\alpha_j)$$

is positive or null. If $g_{\nu_i}(\alpha_j)$ is equal to zero then $A' = 0$, in this case one must keep the first order term in Eq. (3.40)b, and $\mu\Delta I$, this leads to replacing Eq. (3.41)b by:

$$\frac{du_j'}{dt} = \mu\Delta I - \varepsilon\mu B u_j'$$

where

$$B' = \frac{\ell_i}{\mu} \Omega_i \frac{dg_{\nu_i}}{du_j} (\alpha_j) \quad . \tag{3.42)c}$$

The solution of Eqs. (3.41)a,b subject to the initial value condition $\Delta I(u_j' = a_j') = 0$ is given by:

$$\Delta I = \int_{u_j'}^{a_j'} \frac{A}{A'+\delta} \; u_j' du_j' \tag{3.43}$$

where δ is introduced to avoid the apparent singularity at $A' = 0$. For $A' = 0$ δ should be replaced by

$$\frac{\mu}{\varepsilon} \Delta I - B' u_j' \; ;$$

thus Eq. (3.43) becomes an integral equation of the type discussed in Case II' below. For $A' \neq 0$

$$\Delta I = \frac{A}{A'} \; \frac{a_j'^2 - u_j'^2}{2} \tag{3.44}$$

which represent a parabolic branch in the action angle plane.

In terms of an initial "time", t_o, the initial conditions are:

$$u'_j(t=t_o) = a'_j \tag{3.45)a}$$

$$\Delta I(t=t_o) = 0 \tag{3.45)b}$$

For $A' \neq 0$ the solution of Eqs. (3.41)a,b subject to the conditions in Eq. (3.45)a,b are:

$$u'_j(t) = a'_j - \varepsilon\mu A'(t-t_o) \tag{3.46)a}$$

$$\Delta I'(t) = a'_j \; (t - t_o) - \varepsilon\mu A' \frac{(t - t_o)^2}{2} \tag{3.46)b}$$

in the limit of large t the quantities $|u_j'|$ and $|\Delta I'|$ become large, and the field lines will escape from the domain of case III" to the domain of case II'.

Case II':

Equations (2.27)a,b averaged over fast oscillations in the domain of the resonance $\frac{\nu_i}{\Omega_i} = \frac{\ell_i}{m_i}$ are:

$$\frac{d\Delta I}{dt} = \frac{\varepsilon}{\nu_i} f_{\nu_i}(u_j) \tag{3.47)a}$$

$$\frac{du_j}{dt} = \mu\Delta I - \varepsilon\ell_i\Omega_i g_{\nu_i}(u_j) \tag{3.47)b}$$

In the domain of II' we keep the zero and first order terms in the expansions (Eq. (3.40)a,b). By substituting into Eqs. (3.45)a,b we obtain

$$\frac{d\Delta I'}{dt} = -\varepsilon\mu A u_j' \tag{3.48)a}$$

$$\frac{du_j'}{dt} = \mu\Delta I' - \varepsilon\mu B' u_j' \tag{3.48)b}$$

where $\Delta I' = \Delta I - \varepsilon A'$ and A, A' and B' are given from Eq. (3.42)a,b,c. By considering exponential solutions for the system of Eqs. (3.48)a,b of the form $\exp(\lambda t)$, we obtain the characteristic equation:

$$\lambda^2 + \varepsilon\mu B'\lambda + \varepsilon\mu^2 A = 0 \quad . \tag{3.49}$$

Since ε is small and positive and $A \geqslant 0$ then:

$$4\varepsilon\mu^2 A - \varepsilon^2\mu^2 B'^2 > 0 \ , \tag{3.50}$$

and therefore

$$\lambda = \frac{\varepsilon\mu B'}{2} \pm i\lambda_1 \quad , \tag{3.51)a}$$

where

$$\lambda_1^2 = \varepsilon\mu^2 A - \frac{\varepsilon^2\mu^2 B'^2}{4} \quad . \tag{3.51)b}$$

From Eqs. (3.51)a,b the solutions of Eqs. (3.48)a,b, subject to the initial conditions in Eqs. (3.45)a,b are easily obtained:

$$u_j'(t) = a_j' e^{\frac{-\varepsilon\mu B'}{2} t} \left(\cos\lambda_1 t - \frac{\varepsilon\mu B'}{2\lambda_1} \sin\lambda_1 t\right) \tag{3.52)a}$$

$$\Delta I'(t) = \frac{-\varepsilon\mu A a_j'}{\lambda_1} e^{\frac{-\varepsilon\mu B'}{2} t} \sin\lambda_1 t \tag{3.52)b}$$

For $\mu B' > 0$, Eqs. (3.52)a,b give evolution of the field lines from the domain of II' to the domain of III"; if $A' = 0$ this evolution continues to zero e.g., to the position of the elliptic singularity. If A' is different from zero we have shown above that there is also evolution of the field lines from the domain of III" to the domain of II". The combined effect, therefore, confines the field lines of the perturbed system to the domain of III" and II', but also destroys the structure of the surfaces in these domains, thus producing regions of localized stochasticity.

For $\mu B' < 0$ there is evolution from the domain of II' to the domain of I. The field lines located in the immediate neighborhoods of the elliptic singularities in the unperturbed system are quickly drawn out of these neighborhoods when the perturbation is turned on. In the perturbed system these regions are, therefore, forbidden to the field lines.

In the domain of I, magnetic contours are closed and, therefore, the limiting contours of the regions described above are closed. Their sizes are function of the perturbation and the primary resonance parameters; from Section 2.3 they are expected to increase for smaller non-linearity. Numerical verifications of the existence of these regions are found in the literature. We mention the numerical calculations of M. Hénon and C. Heiles.[6]

4. Behavior of the Field Lines in the Limit of Small Non-Linearity Coefficients

There are three cases to be considered (refer to Table I): For $x \simeq \varepsilon^{1/2}$ case III' and II, and for $x << \varepsilon^{1/2}$ case III.

The field line perturbation equations in the domain of case III', namely $0 < \overline{\Delta I} << \varepsilon^{1/2} I_i$, are given by Eqs. (3.38)a,b. A solution of these equations in this domain can be obtained by expanding $f_{\nu_i}(u_j)$ and $g_{\nu_i}(u_j)$ in term of u_j'. The method is similar to that employed in Section 3.4 for case III" except this time the first order term in Eq. (3.40)b is kept, thus adding a perturbation to the parabolic trajectories described by Eq. (3.44). Similarly to the behavior described in Section 3.4, there is evolution of the field lines from the domain of III' to the domain of II, or to zero, depending on the perturbation.

The domain of case II is given by: $\varepsilon^{1/2} I_i \simeq \overline{\Delta I} << I_i$, and the field line perturbation equations in this domain are given by Eqs. (3.47)a,b. In the domain of II, the expansions described in Eqs. (3.40)a,b are not valid and, therefore, it is difficult to obtain an analytical solution of Eqs. (3.47)a,b in this domain. However, it can be safely stated that there will be no stable behavior of the field lines in this domain due to the noncanonical nature of the perturbation.

For $x \simeq \varepsilon^{1/2}$ the domains of III' and II cover all of the resonance domain. Therefore, the field lines may escape the resonance domain to regions of fast changing perturbation outside that domain causing instabilities.

For $x \ll \varepsilon^{1/2}$ the behavior of the field lines, in all of the resonance domain, is described by Eq. (3.38)a,b. Again, due to the noncanonical nature of the perturbation, the behavior of the field lines is unstable and may lead to evolution from the resonance domain to regions of fast changing perturbation outside that domain, causing instabilities.

5. Destruction of the Magnetic Surfaces by the Overlapping of Resonances

It is well confirmed that a strong instability with random-like behavior occurs when resonances overlap.[1]

The overlapping of two neighboring resonances will occur if their separation is smaller or equal to the sum of their widths. There is at least one overlapping of resonances below a given frequency ν_i if the sum of the widths of all resonances with frequencies smaller or equal to ν_i is greater or equal to ν_i. We take this as a definition for partial sotchasticity below ν_i, it is equivalent to:

$$\nu_i \leqslant \sum_{\nu_j \leqslant \nu_i} W_j \quad . \tag{5.1}$$

Although the frequencies of all the resonances below ν_i range from zero to ν_i their number is infinite. We let $\{j\}_i^\infty$ represent the set of resonances below ν_i ordered in such a way that $\nu_{j \geqslant i} \leqslant \nu_i$ and $\nu_\infty = 0$. Equation (5.1) is therefore the same as:

$$\nu_i \leqslant \sum_{j=i}^{\infty} W_j \quad , \tag{5.2}$$

where W_j is the width of the j-resonance.

The limit of total stochasticity below ν_i is obtained if all the resonances of frequencies smaller or equal to ν_i satisfies the criteriom of partial stochasticity. Equation (5.2) should give an underestimate of the critical perturbation for ν_i, an overestimate is obtained from the limit of total stochasticity below ν_i. The critical limit is somewhere in between. We introduce $C_i \leqslant 1$ to be a positive constant

characteristic of the resonance ν_i such that

$$\nu_i = C_i \sum_{j=i}^{\infty} W_j \tag{5.3}$$

defines the critical perturbation for the i-resonance. (C_i is constant in the sense that it is not explicitely dependent on the resonance parameters; however, C_i varies from one resonance to the other and is dependent on the system.)

The widths of primary resonances are given by Eq. (3.15), if ε_i is the critical perturbation for the i-resonance, from Eq. (5.3) we get:

$$C_i G_i \varepsilon_i^{1/2} = \left(\frac{1}{2x_i}\right)^{1/2} \tag{5.4}$$

where,

$$G_i = \frac{1}{\nu_i} \sum_{j=i}^{\infty} \left(\frac{x_j}{x_i} \frac{1}{I_j} \frac{1}{m_j}\right)^{1/2} F_j(P) \tag{5.5}$$

$F_j(P)$ having the dimension of $(m_j I_j)^{1/2} \nu_j$, we arranged Eq. (5.5) so that G_i is dimensionless. G_i is like a structure factor characterizing the perturbed system below ν_i.

Let us call external overlapping the overlapping of primary resonances and internal overlapping the overlapping of secondary resonances, Eq. (5.4) thus describes the critical perturbation by external overlapping. In Section 3.4 we have shown that zones of unstable behavior covers the immediate domains of the elliptic singularities. The limit of each of these zones is a closed contour situated in the domain of Case I. Let ω_s represent the frequency of the secondary resonance

closest to this limiting contour. Thus, the total destruction of the ν_i-resonance by internal overlapping is obtained by the critical perturbation below the ω_s-secondary-resonance. Let ε_s represent the critical perturbation by internal overlapping, from Eqs. (3.37) and (5.3) we get:

$$C_s' G_s' \varepsilon_s^{3/4} = \left(\frac{\nu_i^6 I_i}{2m_i x_i}\right)^{1/4} \left(\frac{1}{2X_s}\right)^{1/2} \tag{5.6}$$

where C_s'($\leqslant 1$ and positive) is defined, similarly to C_i, for the secondary resonances, and

$$G_s' = \frac{1}{\omega_s} \sum_{K=s}^{\infty} \left(\frac{X_K}{X_s} \frac{1}{J_K} \frac{1}{q_K}\right)^{1/2} F_{iK}'(P') \ . \tag{5.7}$$

Our definition of G_s' for the secondary resonances (Eq. (5.7)) is similar to that of G_i for the primary resonances (Eq. (5.5)). G_s' is also like a structure factor characterizing the secondary system below ω_s.

For large x_i we can assume the ω_s-resonance to be in the small oscillations region, treated at the end of Section 3.2. From Eqs. (3.25) and (3.26).

$$X_s = 1 \ . \tag{5.8}$$

From Eqs. (5.8) and (5.6) the limit of internal stochasticity for the ν_i-resonance is obtained:

$$C_s' G_s' \varepsilon_s^{3/4} = \left(\frac{\nu_i^6 I_i}{8 m_i x_i}\right)^{1/4} \tag{5.9}$$

Equation (5.4) for the external stochasticity and Eq. (5.9) for the internal stochasticity give the dependence of the critical perturbations by these two processes on the primary resonance parameters. For a given

resonance, the physical critical perturbation is the smallest of the two above mentioned critical perturbations. From the present analysis it is possible, in an experimental situation, to determine the primary resonance widths explicitly. The values of the critical perturbations, by external overlapping, are, therefore, obtained by directly checking the overlapping of neighboring primary resonances. However, the secondary resonance widths are difficult to determine explicitly from the present analysis.

Secondary resonances are related to the presence of more than one primary resonance in the system. For systems where many primary resonances are possible it is, therefore, logical to assume that the secondary systems are similar to the primary system.[7] This leads us to assume that

$$C_i G_i = e C_s' G_s' \tag{5.10}$$

where e is a constant. If we take the ratio of Eqs. (5.9) and (5.4) and use Eq. (5.10) we get:

$$\varepsilon_s^{3/4} = e \left(\frac{\nu_i^6 I_i x_i}{2m_i} \right)^{1/4} \varepsilon_i^{1/2} \tag{5.11}$$

Equation (5.11) relates the internal critical perturbation to the external critical pertrubation . Thus, from the values of ε_i and the primary resonance parameters, the values of ε_s are determined.

For the stellerator[2], where stochasticity occurred near the separatrix for values smaller than the external stochasticity limit ε_i, Eq. (5.11) shows that this had occurred by internal overlapping.

In general, formula (5.11) asserts that independently of how small is the perturbation, the magnetic surfaces are always destroyed at the separatrix.

For the levitron* we calculated ε_i by directly checking the overlapping of neighboring primary resonances and determined ε_s from Eq. (5.11), where we have taken $e \simeq 1$ and $\Omega_i \equiv 1$. The results are tabulated in Table II. (For the levitron ε is a tilt angle;[4] in Table I it is given in degrees.) We conclude that for $\nu_i \leqslant 2$ the destruction is caused by internal overlapping. In Table II quantities in parenthesis are the theoretical limits for destruction. ε_c are the numerically measured tilts for which the resonances are completely destroyed.[8] There is a very good agreement between theoretical and numerical values of the critical perturbations. In Fig. 2, we show a typical example of destruction by internal overlapping in levitrons; the secondary magnetic islands and contours appear in part C.

Before we close this section we would like to comment on Eq. (5.10) which says that $C_i G_i$ is proportional to $C_s' G_s'$. It can be argued that the limiting secondary resonance parameters are functions of the primary resonance parameters and the perturbation at the primary resonance. On the other hand, the summation in Eq. (5.5) over the indice

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*The basic features of an average minimum B levitron, are represented by a simplified model which consists of a single filamentary conducting circular loop located at the center of the torus, a straight filamentary conductor along the vertical axis and a uniform vertical magnetic field. The location of the separatrix surface is determined by the ratio of the uniform vertical field to the loop field. The distance from the loop to the separatrix approximates the minor radius of the torus and thus determines the system's aspect ratio.

j transforms the dependence of F_j on the perturbation at ν_j into a dependence of G_i on the primary system below ν_i. Similarly G'_s is dependent on the secondary system below ω_s. We have argued before that for systems with many resonances the secondary systems are similar to the primary system. Thus, there is an approximate similarity between the primary system below ν_i and the secondary system below ω_s. This similarity and the dependence of G_i and G'_s on the primary resonance parameters only, suggest that G_i and G'_s can be related. Although this does not prove the proportionality of C_iG_i and $C'_sG'_s$ it does give a strong support in favor of it. The proportionality constant, e, is in general dependent on the system; from Table II, taking e equal to one seems to be a good approximation for the levitron under consideration.

6. Conclusions

It has been established that if resonances overlap, a rapid destruction of their island structure occurs.[1]

1. Thus, if primary resonances overlap, a rapid destruction of their flux surfaces is expected.

2. For large non-linearity ($x \gg \varepsilon^{1/2}$), the field lines are trapped in an effective potential well in the primary resonance domain forming families of closed contours at the elliptic singularities. We refer to the resonant island contours as secondary resonances. Another possible phenomenon of destruction is the overlapping of secondary resonances. Depending on the primary resonance parameters (and the system) destruction may occur by either or both phenomena.

3. The excursion in the action for the primary magnetic island increases as $\varepsilon^{1/2}$ while for the secondary island it increases as $\varepsilon^{3/4}$. Internal overlapping proceeds almost orderly from the local separatrix to the elliptic singularity. Therefore, for islands that are most affected by internal overlapping the observed primary excursion should increase at a lower rate than $\varepsilon^{1/2}$ due to the successive disappearance of outer contours destroyed by secondary resonance overlapping. This is in agreement with the numerical observation by Freis et al.,[4] where the 1 and $\frac{1}{2}$ resonance widths increase as $\varepsilon^{1/2}$ until breadup while the $\frac{3}{2}$, 2, $\frac{5}{2}$ and 3 resonance widths increase as $\varepsilon^{3/2}$.

4. In the immediate neighborhoods of the elliptic singularities the perturbation is not canonical and leads to evolution, thus, creating either regions of localized stochasticity or regions that are forbidden to the field lines, depending on the perturbation and the system. The sizes of these regions are functions of the pertrubation and the primary

resonance parameters. They increase for smaller non-linearity.

5. For small non-linearity ($x \lesssim \varepsilon^{1/2}$) the field lines may escape the resonance zone into regions of fast changing perturbation causing instabilities.

6. It is known since Poincaré[9] that a hierarchy of resonances are generated in a non-linear oscillating system. In a system where terms of the first order in ε give the highest order observable contribution, we found that two sets of resonances are sufficient to explain the formation and destruction of the magnetic surfaces. In general, if ε^n, where n is a positive integer, gives the highest order observable contribution then 2n sets of resonances are sufficient.

7. Equation (5.4) for the external stochasticity and Eq. (5.9) for the internal stochasticity give the dependence of the critical perturbations by these two processes on the primary resonance parameters. For a given resonance these relations give the dependence of the critical perturbation on the non-linearity coefficient.

8. From Eq. (5.11), as ν_j approaches zero, the region of internal stochasticity extends over all the x-ε plane thus, independently of how small is $\varepsilon > 0$, the flux surfaces are always destroyed near the separatrix.

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Appendix

The Averaging Method

To evaluate the right-hand side of Eq. (3.5) we consider the Fourier coefficients:

$$\gamma^S_{pm_i,-p\ell_i}(I_i) = \frac{1}{2\pi T_i}\int_0^{T_i} dt \int_0^{2\pi} d\theta\Gamma^S(I_i,\theta,t)\cos(pm_i\theta - p\ell_i\Omega_i t)$$

$$= \frac{1}{2\pi T_i}\int_0^{T_i} dt \int_{\ell_i\Omega_i t}^{\ell_i\Omega_i t-2\pi m_i}\left(-\frac{du}{m_i}\right)\Gamma^S(I_i,-\frac{u}{m_i}+\nu_i t,t)\cos pu \tag{A.1}$$

where $u = -m_i\Delta\theta = -m_i\theta + \ell_i\Omega_i t$.

Since $\Gamma^S(I_i,\theta,t)$ is periodic in θ and t then:

$$\int_{\ell_i\Omega_i t}^{\ell_i\Omega_i t-2\pi m_i} du\Gamma^S(I_i,-\frac{u}{m_i}+\nu_i t,t) = \int_0^{-2\pi m_i} du\Gamma^S(I_i,-\frac{u}{m_i}+\nu_i t,t) \ . \tag{A.2}$$

From Eqs. (A-1) and (A-2):

$$\gamma^S_{pm_i,-p\ell_i}(I_i) = \frac{1}{2\pi m_i}\int_{-2\pi m_i}^{0} du\, f^S_{\nu_i}(u)\cos pu \tag{A.3}$$

where

$$f^S_{\nu_i}(u) = \frac{1}{T_i}\int_0^{T_i} dt\Gamma^S(I_i,-\frac{u}{m_i}+\nu_i t,t) \ . \tag{A.4}$$

Similarly

$$\gamma^A_{pm_i,-p\ell_i}(I_i) = -\frac{1}{2\pi m_i}\int_{-2\pi m_i}^{0} du\, f^A_{\nu_i}(u)\sin pu \tag{A.5}$$

where

$$f^A_{\nu_i}(u) = \frac{1}{T_i}\int_0^{T_i} dt\Gamma^A(I_i,-\frac{u}{m_i}+\nu_i t,t) \ . \tag{A.6}$$

If we substitute from Eqs. (A.3) and (A.5) into Eq. (3.5) we get:

$$\frac{d\Delta I}{dt} = \frac{2\varepsilon}{\nu_i} \frac{1}{2\pi m_i} \int_{-2\pi m_i}^{0} du' \, f_{\nu_i}^{S}(u') \sum_{p=1}^{\infty} \cos pu \cos pu' + f_{\nu_i}^{A}(u') \sum_{p=1}^{\infty} \sin pu \sin pu' \tag{A.7}$$

In order to evaluate Eq. (A.7) we express $\sum_{p=1}^{\infty} \cos pu \cos pu'$ and $\sum_{p=1}^{\infty} \sin pu \sin pu'$ in terms of delta distributions, consider x in the one dimensional real space, R^1, then:

$$\sum_{k=-\infty}^{+\infty} e^{2\pi ikx} = \sum_{n=-\infty}^{+\infty} \delta(x - n)$$

where n is an integer. If x varies only in an interval of R^1, say $[x_1, x_2]$, $x_1 < x_2$, and if $n_1 = \mathrm{Ent}(x_1) + 1$ and $n_2 = \mathrm{Ent}(x_2)$ then:

$$\sum_{k=-\infty}^{+\infty} e^{2\pi ikx} = \sum_{n=n_1}^{n_2} \delta(x - n)$$

It follows that:

$$\sum_{k=1}^{\infty} \cos 2\pi kx = -\frac{1}{2} + \frac{1}{2} \sum_{n=n_1}^{n_2} \delta(x - n) \quad . \tag{A.8}$$

By using

$$\sum_{p=1}^{\infty} \cos pu \cos pu' = \frac{1}{2} \sum_{p=1}^{\infty} [\cos p(u + u') + \cos p(u - u')]$$

$$\sum_{p=1}^{\infty} \sin pu \sin pu' = \frac{1}{2} \sum_{p=1}^{\infty} [-\cos p(u + u') + \cos p(u - u')]$$

and noting that the interval of variation of u+u' and u-u' is $[0, 2\pi m_i]$ we get:

$$\sum_{p=1}^{\infty} \cos pu \cos pu' = -\frac{1}{2} + \frac{\pi}{2} \sum_{n=0}^{m_i} \delta(u + u' - 2\pi n) + \sum_{n=0}^{+m_i} \delta(u - u' - 2\pi n) \tag{A.9)a}$$

$$\sum_{p=1}^{\infty} \sin pu \sin pu' = \frac{\pi}{2} \sum_{n=0}^{m_i} \delta(u + u' - 2\pi n) + \sum_{n=0}^{+m_i} \delta(u - u' - 2\pi n) \tag{A.9)b}$$

By using $f^S_{\nu_i}(u) = f^S_{\nu_i}(-u)$ and $f^A_{\nu_i}(u) = -f^A_{\nu_i}(-u)$ after substituting Eq. (A-9)a and Eq. (A-9)b into Eq. (A-7) we get:

$$\frac{d\Delta I}{dt} = \frac{2\varepsilon}{\nu_i} \frac{2}{4m_i} \sum_{n=0}^{m_i} f_{\nu_i}(u - 2\pi n) \tag{A.10}$$

where

$$f_{\nu_i}(x) = f^S_{\nu_i}(x) + f^A_{\nu_i}(x) \tag{A.11}$$

From the definition of $f^{S,A}_{\nu_i}(u)$ and the periodicity of $\Gamma(I,\theta,t)$ with respect to θ and t it is straightforward to show that:

$$f_{\nu_i}(u) = f_{\nu_i}(u \pm 2\pi n) \; ; \; n = 0,1,2, \ldots \tag{A.12}$$

By substituting Eq. (A-12) into Eq. (A-10) we get:

$$\frac{d\Delta I}{dt} = \frac{\varepsilon}{\nu_i} f_{\nu_i}(u) \; . \tag{A.13}$$

Equation (A-13) can also be derived directly by averaging Eq. (2.20)a over fast oscillating terms in t. This shows the equivalence of the present analysis and the averaging technique often used in the literature.

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Figure and Table Captions

Fig. 1. Structure of the $\left(\frac{\nu_i}{\Omega_i} = \frac{1}{4}\right)$ - primary resonance in the action angle plane. The various angles shown are defined in the text.

Fig. 2. (a) A closed primary contour of the π resonance at $1+r\cos\phi \simeq 0.51$ and part of the 2π primary resonance at $1+r\cos\phi \simeq 0.61$. (b) The perturbation is doubled and the contour is heavily distorted. (c) The perturbation is doubled again, the contour is completely destroyed and secondary magnetic islands and contours appear. This is a typical example of destruction by internal overlapping. The 2π primary resonance island after reaching its maximum flux in part (c) is seen partially destroyed in part (d). This figure is taken from Ref. 3.

Table I. There are three approximations to the field line equations: (i) $\rightarrow$ type I, (ii) $\rightarrow$ type II, and (iii) $\rightarrow$ type III. For $x \ll \varepsilon^{1/2}$, (iii) alone is in the resonance domain, for $x \sim \varepsilon^{1/2}$ (iii) and (ii) are in the resonance domain, and for $x \gg \varepsilon^{1/2}$ (iii), (ii) and (i) share the resonance domain. Only (i) may possess stable solutions, thus stable magnetic contours are obtained if $x \gg \varepsilon^{1/2}$.

Table II. For the levitron, ε is a tilt angle. The theoretical tilts for which the resonances are completely destroyed are equal to the smallest of ε_i and ε_s. ε_i and ε_s are the limits of external and internal stochasticities, respectively. (*The 3 and 5/2 resonances overlap and are simultaneously destroyed.) ε_c are the numerically measured critical tilts. ν_i/Ω_i are the rotational transforms, x_i the nonlinearity coefficients, and I_i the primary actions.

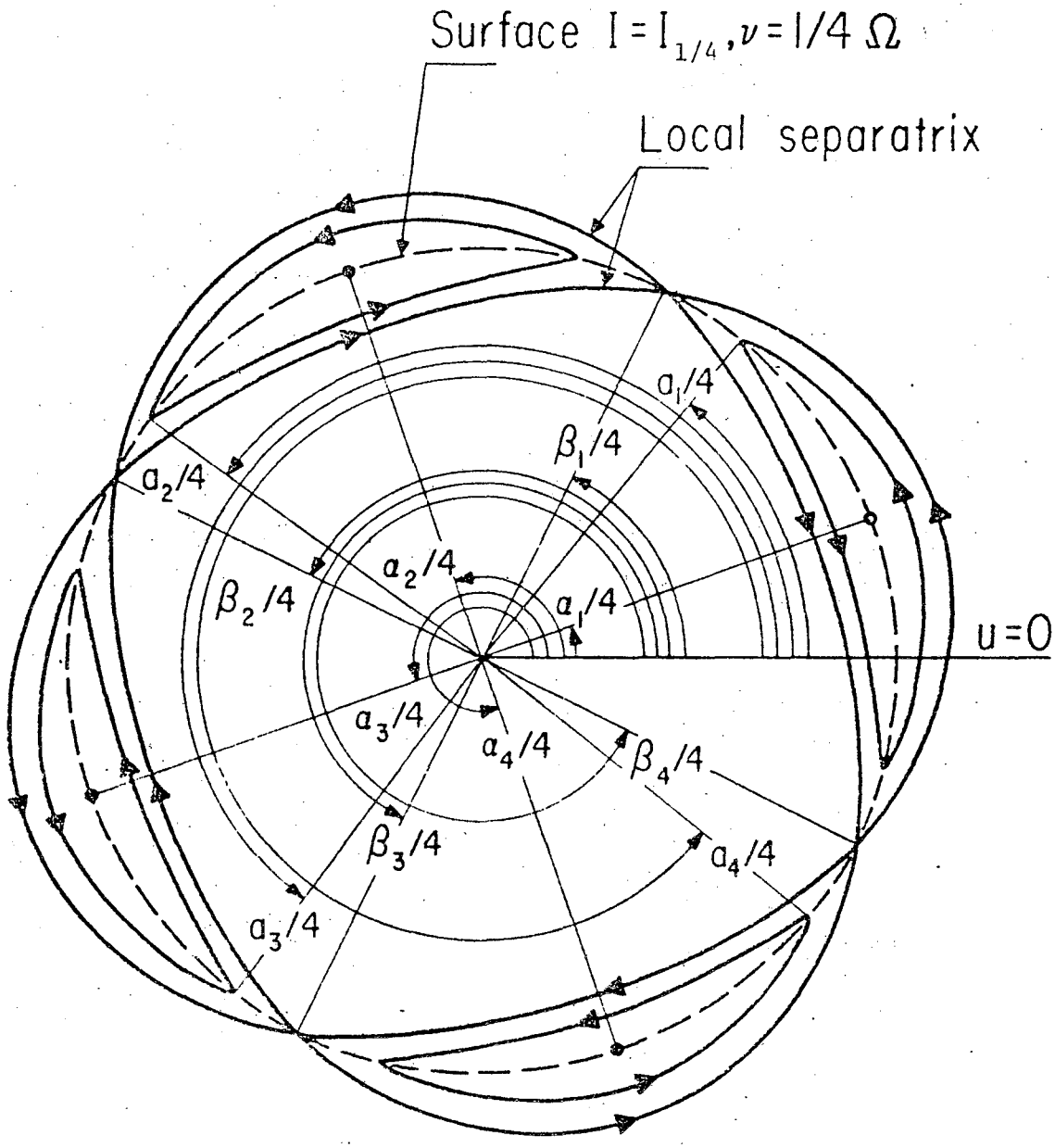

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Fig. 1

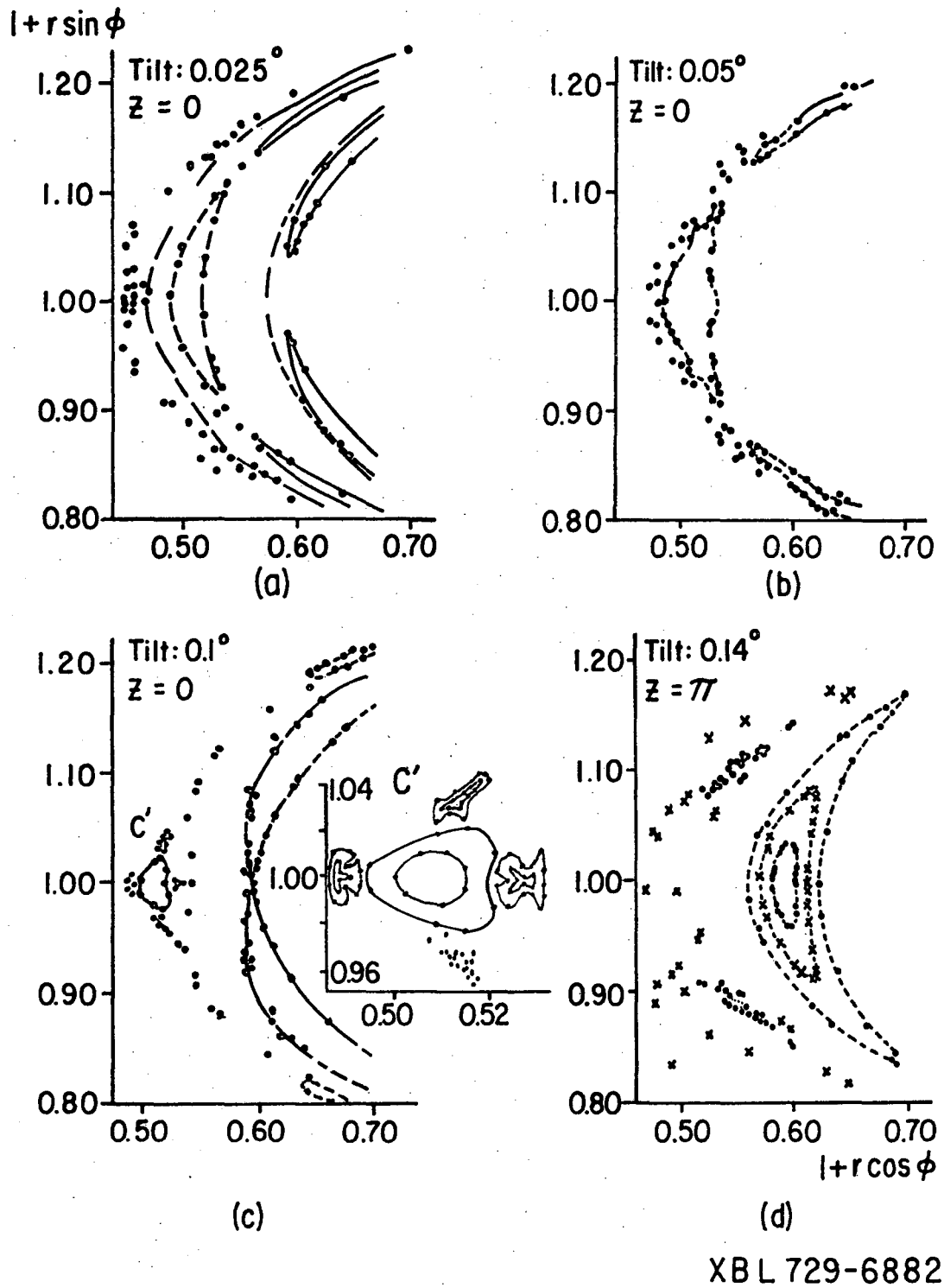

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Fig. 2

x \ $\frac{\overline{\Delta\nu}}{\nu}$	$\ll \varepsilon$	$\simeq \varepsilon$	$\gg \varepsilon$
$\ll \varepsilon^{1/2}$	III		
$\simeq \varepsilon^{1/2}$	III′	II	
$\gg \varepsilon^{1/2}$	III″	II′	I

Table I

XBL 739-1824

ν	x_i	I_i	ε_i	ε_s	ε_c
3	1.256	0.036	(2.75)	5.02	2.50
5/2	1.074	0.040	(2.75)*	2.72	2.50
2	1.045	0.050	4.95	(3.46)	3.00
3/2	1.140	0.061	1.68	(0.82)	0.70
1	1.300	0.084	0.54	(0.27)	0.30
1/2	2.200	0.120	0.54	(0.067)	0.15
1/4	2.280	0.162	~0.244	(~0.002)	< 0.02

Table II

XBL7310-1947

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