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F. M. Hamzeh

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ON THE DESTRUCTION OF MAGNETIC SURFACES IN TOROIDAL SYSTEMS

F. M. Hamzeh

Department of Materials Science and Engineering, Center for the Design of Alloys, and Inorganic Materials Research Division, Lawrence Berkeley Laboratory; University of California Berkeley, California 94720

ABSTRACT

The behavior of the field lines in a torus is analogous to the motion of a non-linear oscillator. If ε, small and positive, is the perturbation parameter, we consider a toroidal system in which terms of higher order than ε are assumed to give nonobservable contributions. For large non-linearity ($x \gg \varepsilon^{1/2}$) we found that two sets of resonances are sufficient to explain the destruction of the magnetic surfaces in the toroidal system. Resonances that transform the unperturbed surfaces into a structure of magnetic islands we call primary resonances, and the secondary resonances transform the bound-state-like contours of a given island into similar structures of secondary magnetic islands. To every primary island we attach two types of stochasticity, external, due to the overlapping of primary resonances and, internal, due to the overlapping of secondary resonances. Depending on the resonance and the system, the destruction of the magnetic surfaces occurs by either or both processes. For small non-linearity ($x \lesssim \varepsilon^{1/2}$) the magnetic contours oscillate in a highly irregular fashion and, therefore, overlap causing orbital instabilities. The orbital instabilities are more pronounced for larger fluxes but do not always destroy the flux surfaces at the sepatrix. Independently of how small is $\varepsilon(>0)$ the flux surfaces are

always destroyed at the separatrix, if not by external, by internal stochasticity. In the immediate neighborhoods of the elliptic singularities, the field lines are orbitally stable, for all non-linearities. These neighborhoods are small and may become observable for small ε.

For the levitron, we calculated the theoretical perturbations for which some of the primary resonances are completely destroyed. These theoretical values are given in Table I and are in good agreement with the numerical results. A typical example of destruction by internal overlapping is shown in Fig. 2; we note the appearance, in Fig. 2 part c, of secondary magnetic islands and contours.

1. Introduction

Rosenbluth, et al.[1] have shown that if resonances overlap, rapid destruction of their island structure occurs. Filonenko, et al.[2] found that in the stellerator, destruction of the magnetic surfaces near the separatrix occurs before the overlapping of primary resonances. It is shown here that in a pertrubed torus the magnetic surfaces at the separatrix are always destroyed, if not by external, by internal overlapping.

We present a general study of formation and destruction of magnetic surfaces in toroidal systems. If r, ϕ and z are the toroidal coordinates, the role of time is played by the z-coordinate while the magnetic islands are observed in the toroidal cross section perpendicular to the z-axis. The magnetic field is taken as the sum of a "stationary" field, (stationary in the sense that it is not a function of z), and a small non-stationary perturbation. The unperturbed system is defined to include all stationary contributions. The behavior of the field lines in the perturbed system is shown to follow non-linear oscillating equations where the rotational transform plays the role of frequency.

In Section 2, the linearized equations in various regions of the primary magnetic island are derived. For various order of magnitude of the non-linearity coefficients various cases were obtained.

In Section 3 we study the structure of the magnetic islands in the limit of large non-linearity. The magnetic islands are shown to contain closed contours centered at each elliptic singularity. Each elliptic singularity, therefore, mark the position of a local magnetic axis for the island. The magnetic contours of adjacent islands of the same

primary resonance are connected through a common contour referred to as a local separatrix. These contours have slow characteristic frequencies and interact with the non-stationary part of the pertrubation. The resonant island contours are referred to as secondary resonances. The behavior of the field lines near the elliptic singularities is treated in Section 3.4.

In Section 4 the behavior of the field line in the limit of small non-linearity is examined.

In Section 5 we evaluate the critical perturbations for the destructions of the magnetic surfaces.

We have published[3] a résumé of this work at its early stage.* All conclusions drawn in that résumé are also drawn here. However, the critical perturbations, carefully determined in the present analysis, are different from the previously reported perturbations and are in better agreement with the numerical results.

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*Erratum for Reference 3.
The following equations are transformed to the correct form by the attached transformations: Eqs. (1.3) and (1.5), ($2\varepsilon \to \varepsilon$). Equation (1.6), ($2\varepsilon^{1/2} \to (2\varepsilon)^{1/2}$). Equation (1.7), ($w \to \omega$, where ω etc.). Equation (1.8), ($\frac{1}{\omega} \to \frac{2}{\omega}$). Equation (2.5), ($\frac{1}{2} \to \frac{1}{\sqrt{2}}$). Equation (2.7), ($\sqrt{2} \to \frac{1}{\sqrt{2}}$; $x_j^{-3/4} \to x_j^{-1/2}$). Equation (2.8), ($8 \to 1$; $\left(\frac{I_j}{x_j}\right)^{1/2} \to I_j^{1/2}$). Table I is not affected except for ε_{jj} = 4.07, 2.72, (3.14), (0.57), (0.11), (0.02), ($\lesssim$0.01) for the resonances from $\nu = 3$ to $\nu = 1/4$, respectively. The conclusions remain unaltered.

2. The Field Line Equations

We consider the magnetic field in the toroidal system of coordinates r, ϕ, z:

$$\vec{B}(r,\phi,z) = \vec{B}^{o}(r,\phi) + \varepsilon\, \vec{B}^{1}(r,\phi,z) \tag{2.1}$$

where $\vec{B}^{o}(r,\phi)$ represent the unperturbed field and is "stationary" in the sense that it is not a function of z, $\varepsilon \geqslant 0$ is a small parameter and $\vec{B}^{1}(r,\phi,z)$ is a non-stationary perturbation field that is periodic with respect to z. The field line equations are:

$$\frac{dr}{B_r} = \frac{r d\phi}{B_\phi} = \frac{dz}{B_z} \tag{2.2}$$

where B_r, B_ϕ and B_z are the field components in the toroidal system expressed as functions of r, ϕ and z.

2.1. The Unperturbed System

We define the unperturbed system to include all stationary terms. In Eq. (2.1) we let $\varepsilon = 0$ and define t "time"* by:

$$dt \equiv \frac{dz}{B_z^{o}} \quad . \tag{2.3}$$

We write the unperturbed field line equations from Eq. (2.2) in the following form:

$$\frac{d\frac{r^2}{2}}{dt} = r\, B_r^{o} \tag{2.4a}$$

$$\frac{d\phi}{dt} = \frac{B_\phi^{o}}{r} \quad . \tag{2.4b}$$

Using the divergence theorem $\vec{\nabla} \cdot \vec{B}^{o} = 0$ and Eqs. (2.4)a,b we derived the first integral:

*"Time" is a convenient coordinate, t, that plays the role of time in the analogy with Hamilton's equations. The systems treated here are time independent. The time dependence is easily introduced by letting ε be a function of time.

$$H\left(\frac{r^2}{2},\phi\right) = \int^{r} B_{\phi}^{o}(r',\phi)\, dr' \tag{2.5}$$

where:

$$\frac{d\frac{r^2}{2}}{dt} = -\frac{\partial H}{\partial \phi} \tag{2.6)a}$$

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial \frac{r^2}{2}} \quad . \tag{2.6)b}$$

The unperturbed magnetic surfaces are represented by surfaces of constant H.* Equations (2.6)a,b are similar to Hamilton's equations where H is the hamiltonian, $\frac{r^2}{2}$ and ϕ are the canonical variables and t plays the role of time.

We normalize the major radius of the torus to one and conveniently let $B_z^o(0,\phi) = 1$. By varying z from zero to 2π the field line does not necessarily return to its original position after having gone around the torus. The problem we are concerned with is a perturbation around a situation in which exact flux surfaces exist. We introduce the action and angle variables (I,θ) corresponding to the canonical variables $(\frac{r^2}{2},\phi)$:

$$I = I(H) = \frac{1}{2\pi}\oint \frac{r^2}{2}\, d\phi \tag{2.7)a}$$

$$\theta(\phi,I) = \frac{\partial S(\phi,I)}{\partial I} \tag{2.7)b}$$

where

$$S(\phi,I) = \int^{\phi} \frac{r^2}{2}\, d\phi \tag{2.7)c}$$

and the frequency:

$$\nu = \nu(I) = \frac{dH}{dI} \quad . \tag{2.8}$$

*The present study does not assume vacuum field.

We have chosen the definition for the flux in Eq. (2.7)a in order to give 2π changes for θ every time ϕ changes by 2π. In the action angle representation the field line equation of the unperturbed system are:

$$\frac{dI}{dt} = 0 \tag{2.9)a}$$

$$\frac{d\theta}{dt} = \nu \quad . \tag{2.9)b}$$

The rotational transform $\frac{R}{2\pi}$, measured in number of field line rotations about the toroidal axis per rotation about the major axis of the torus is defined by:

$$\frac{R}{2\pi} = \frac{\delta\phi}{\delta z} \tag{2.10}$$

The rotational transform is null at the separatrix, in the interval of interest, between the central magnetic axis and the separatrix, we consider $R/2\pi$ to be positive, this can always be achieved by properly orienting the toroidal coordinates with respect to the field lines. (Although we assumed the rotational transform to be positive, for convenience, the problem with negative transform can be treated in a similar fashion.) Let T be the change in the variable t, averaged with respect to ϕ and corresponding to a variation in z equal to 2π. From Eq. (2.3) we get:

$$T = 2\pi(1 - \langle \delta B_z^o \rangle_\phi) \tag{2.11)a}$$

where

$$\langle \delta B_z^o \rangle_\phi = \frac{1}{2\pi} \int_0^{2\pi} d\phi [B_z^o(r,\phi) - B_z^o(0,\phi)] \tag{2.11)b}$$

Near the central magnetic axis $\langle \delta B_z^o \rangle_\phi$ is small and T is approximately equal to 2π.

In accordance with Eq. (2.9)b:

$$\nu = \frac{d\theta}{dt} = \frac{\delta\theta}{\delta\phi} \quad \frac{\delta\phi}{\delta z} \quad \frac{dz}{dt} \quad ,$$

since $\delta\phi = 2\pi$ when $\delta\theta = 2\pi$ changes and $\frac{dz}{dt} = \frac{2\pi}{T}$ we get:

$$\nu = \frac{R}{2\pi}\,\Omega = \frac{R}{2\pi}\,(1 + \langle \delta B_z^o \rangle_\phi) \tag{2.12)a}$$

where higher order terms in $\langle \delta B_z^o \rangle\phi$ are neglected and

$$\Omega = \frac{2\pi}{T} \quad . \tag{2.12)b}$$

The meaning of Ω becomes clear from $dz = \Omega dt$. Equation (2.12)a relate the frequency ν to the rotational transform $R/2\pi$. These two quantities are approximately equal in the immediate neighorhood of the central magnetic axis. In Ref. 3 we approximated the problem by letting Ω equal to 1. In the present paper, we do not make this approximation.

2.2. The Perturbed System

In this section we determine how Eqs. (2.9)a,b are affected by the perturbation. We let $\frac{dr}{dt} = \frac{dr^o}{dt} + \frac{\varepsilon dr^1}{dt}$ and $\frac{d\phi}{dt} = \frac{d\phi^o}{dt} + \frac{\varepsilon d\phi^1}{dt}$ and linearize Eq. (2.2) to obtain:

$$\frac{dr^o}{dt} = B_r^o \tag{2.13)a}$$

$$\frac{d\phi^o}{dt} = \frac{B_\phi^o}{r} \tag{2.13)b}$$

and

$$\frac{dr^1}{dt} = B_r^1 - \frac{Br^o}{B_z^o}\, B_z^1 \tag{2.14)a}$$

$$\frac{d\phi^1}{dt} = \frac{1}{r}\left(B_\phi^1 - \frac{B_\phi^o}{B_z^o} B_z^1\right) . \qquad (2.14)b$$

The perturbations in the action and angle variables due to perturbing $\frac{dr}{dt}$ and $\frac{d\phi}{dt}$ (or equivalently the magnetic field) are given by:

$$\frac{dI}{dt} = \frac{dI}{dH}\left[\left(\frac{\partial H}{\partial r}\right)^o \frac{dr}{dt} + \left(\frac{\partial H}{\partial \phi}\right)^o \frac{d\phi}{dt}\right] \qquad (2.15)a$$

$$\frac{d\theta}{dt} = \left(\frac{\delta\theta}{\delta\phi}\right)^o \frac{d\phi}{dt} \qquad (2.15)b$$

Using Eqs. (2.6)a,b we get:

$$\frac{dI}{dt} = -\frac{\varepsilon r}{\nu}\left[\frac{dr^o}{dt}\frac{d\phi^1}{dt} - \frac{d\phi^o}{dt}\frac{dr^1}{dt}\right] + O(\varepsilon^2) \qquad (2.16)a$$

$$\frac{d\theta}{dt} = \nu\left[1 + \varepsilon\left(\frac{d\phi^1}{dt}\right) / \left(\frac{d\phi^o}{dt}\right)\right] + O(\varepsilon^2) \qquad (2.16)b$$

By substituting Eqs. (2.13)a,b and Eqs. (2.14)a,b into Eqs. (2.16)a,b we get:

$$\frac{dI}{dt} = \frac{\varepsilon}{\nu}\,\gamma\left(\frac{r^2}{2},\phi,z\right) + O(\varepsilon^2) \qquad (2.17)a$$

$$\frac{d\theta}{dt} = \nu\left[1 + \varepsilon\,\pi\left(\frac{r^2}{2},\phi,z\right)\right] + O(\varepsilon^2) \qquad (2.17)b$$

where r, ϕ and z are the coordinates of the unperturbed system and

$$\gamma\left(\frac{r^2}{2},\phi,z\right) = B_\phi^o B_r^1 - B_\phi^1 B_r^o \qquad (2.18)a$$

$$\pi\left(\frac{r^2}{2},\phi,z\right) = \frac{B_\phi^1}{B_\phi^o} - \frac{B_z^1}{B_z^o} \qquad (2.18)b$$

From Eqs. (2.3), (2.5) and (2.7)a,b,c z can be obtained as a function of I, θ and t on one hand and r and ϕ as functions of I and θ on the other. We introduce the functions:

$$\Gamma(I,\theta,t) \equiv \gamma\left[\frac{r^2}{2}(I,\theta)\ ;\ \phi(I,\theta)\ ;\ z(I,\theta,t)\right] \tag{2.19)a}$$

$$\Pi(I,\theta,t) = \pi\left[\frac{r^2}{2}(I,\theta)\ ;\ \phi(I,\theta)\ ;\ z(I,\theta,t)\right] \tag{2.19)b}$$

and write Eqs. (2.17)a,b in the consistent form:

$$\frac{dI}{dt} = \frac{\varepsilon}{\nu}\Gamma(I,\theta,t) + O(\varepsilon^2) = -\varepsilon\frac{\partial H_1}{\partial\theta} + O(\varepsilon^2) \tag{2.20)a}$$

$$\frac{d\theta}{dt} = \nu + \varepsilon\nu\,\Pi(I,\theta,t) + O(\varepsilon^2) = \nu + \varepsilon\frac{\partial H_1}{\partial I} + O(\varepsilon^2)\quad, \tag{2.20)b}$$

where the second equalities are implied by the flux conservation theorem; εH_1 being the perturbation of the hamiltonian. Equations of the form of Eqs. (2.20)a,b were derived for the straight stellerator field[2] and for the levitron.[4]

For an exact flux surface of a rotational transform equal to $R/2\pi$ the field lines are conserved under the transformation: $\phi \to \phi + 2\pi$ $z \to z + \frac{2\pi}{R}\ 2\pi$. This corresponds to a change in θ equal to 2π and a change in t equal to $\frac{2\pi}{\nu}$. We conclude that the functions $\Gamma(I,\theta,t)$ and $\Pi(I,\theta,t)$ are periodic with respect to θ of period 2π and with respect to t of period $\frac{2\pi}{\nu}$.

2.3. The Linearized Equations

Let $\{\nu_i\}$ represent the set of resonant frequencies. From Eq. (2.8) $\nu_i = \nu(I = I_i)$. To $\{\nu_i\}$ corresponds a set of values, $\{I_i\}$, for the action. From Eq. (2.7)a, $\{I_i\}$ are the unperturbed fluxes and are characterized by the transforms $\frac{1}{2\pi}\{R_i\} = \frac{1}{2\pi}\left\{\frac{\nu_i}{\Omega_i}\right\}$.

In the domain of a given resonance ($|\Delta I| \ll I_i$), we let

$$I = I_i + \Delta I \tag{2.21}$$

and Taylor expand $\nu(I)$:

$$\nu(I) = \nu_i + \frac{d\nu_i}{dI}\ \Delta I + \ldots$$

where $\frac{d\nu_i}{dI} = \frac{d\nu}{dI}\ (I = I_i)$.

Let x_i define the non-linearity coefficients:

$$x_i = \frac{I_i}{\nu_i}\ \left|\frac{d\nu_i}{dI}\right| , \tag{2.22}$$

and substitute in the Taylor expansion keeping only the first two terms to get.

$$\left|\frac{\Delta\nu}{\nu_i}\right| = x_i\ \left|\frac{\Delta I}{I_i}\right| \tag{2.23}$$

where $\Delta\nu = \nu - \nu_i$.

Let $\overline{\Delta\nu} = \sqrt{\langle (\Delta\nu)^2 \rangle}$ and $\overline{\Delta I} = \sqrt{\langle (\Delta I)^2 \rangle}$ where the average is taken over a complete period of the slow variable $\Delta\theta = \theta - \nu_i t$ and consider the two following approximations:

(i):

$$\frac{\overline{\Delta\nu}}{\nu_i} >> \varepsilon \quad . \tag{2.24)a}$$

From Eq. (2.23) this condition is equivalent to $\overline{\Delta I} >> \frac{\varepsilon I_i}{x_i}$ and since $\frac{\overline{\Delta I}}{I_i} << 1$ then:

$$\frac{\varepsilon I_i}{x_i} << \overline{\Delta I} << I_i \tag{2.24)b}$$

defines the domain of case (i).

In this domain the second term on the right-hand side of Eq. (2.20)b is much smaller than the first and can be dropped out. Furthermore we linearize Eqs. (2.20)a,b to obtain:

$$\frac{d\Delta I}{dt} = \frac{\varepsilon}{\nu_i} \Gamma(I_i, \Delta\theta + \nu_i t, t) \qquad (2.25)a$$

$$\frac{d\Delta\theta}{dt} = \frac{d\nu_i}{dI} \Delta I \quad . \qquad (2.25)b$$

These are the magnetic island equations.

(ii):

$$\frac{\overline{\Delta\nu}}{\nu} \lesssim \varepsilon \qquad (2.26)a$$

which is equivalent to:

$$\overline{\Delta I} \lesssim \frac{\varepsilon I_i}{x_i} \qquad (2.26)b$$

the linearized equations are:

$$\frac{d\Delta I}{dt} = \frac{\varepsilon}{\nu_i} \Gamma(I_i, \Delta\theta + \nu_i t, t) \qquad (2.27)a$$

$$\frac{d\Delta\theta}{dt} = \frac{d\nu_i}{dI} \Delta I + \varepsilon\nu_i \Pi(I_i, \Delta\theta + \nu_i t, t) \qquad (2.27)b$$

In the action angle plane, the domains of validity of the approximations described above are dependent on the non-linearity coefficients x_i. By using Eqs. (2.24)b and (2.26)b we determine these domains for various orders of magnitude of x_i. The various cases are:

<u>For $x_i \gg \varepsilon^{1/2}$</u>

Case I: Eqs. (2.25)a,b $\quad \varepsilon^{1/2} I_i \simeq \overline{\Delta I} \ll I_i$

Case II': Eqs. (2.27)a,b $\quad 0 < \overline{\Delta I} \ll \varepsilon^{1/2} I_i$

<u>For $x_i \simeq \varepsilon^{1/2}$</u>

Case II: Eqs. (2.27)a,b $\quad 0 < \overline{\Delta I} \ll I_i$

In Sections 3.4 and 4 we show that, for all x_i in the domain $0 < \Delta I \ll \varepsilon^{1/2} I_i$, the Eqs. (2.27)a,b are reduced to a form similar to Eqs. (2.25)a,b.

3. The Structure of Magnetic Islands

For large non-linearity coefficient $x_i \gg \varepsilon^{1/2}$ the magnetic islands are shown to contain closed contours centered at each elliptic singularity. Each elliptic singularity, therefore, mark the position of a local magnetic axis for the island. The magnetic contours of adjacent islands of the same primary resonance are connected through a common contour referred to as local separatrix. These contours have slow characteristic frequencies and are preturbed by the presence of other resonances in the system. Resonant island contours are referred to as secondary resonances. The behavior of the field lines near the elliptic singularities is treated in Section 3.4.

3.1. The Island Contours

In this section we solve Eqs. (2.25)a,b. First we derive an averaging method that determines the contribution to a given resonance due to all the resonant harmonics.

To simplify the problem we separate Γ into symmetric and antisymmetric parts

$$\Gamma(I_i,\theta,t) = \Gamma^S(I_i,\theta,t) + \Gamma^A(I_i,\theta,t) \tag{3.1}$$

where

$$\Gamma^S(I_i,\theta,t) = \frac{1}{2}\,\Gamma(I_i,\theta,t) + \frac{1}{2}\,\Gamma(I_i,-\theta,-t) \tag{3.2a}$$

$$\Gamma^A(I_i,\theta,t) = \frac{1}{2}\,\Gamma(I_i,\theta,t) - \frac{1}{2}\,\Gamma(I_i,-\theta,-t) \quad . \tag{3.2b}$$

We expand Γ^S and Γ^A in Fourier series and rearrange terms to get:

$$\Gamma^S(I_i,\theta,t) = 2\sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty}[\gamma^S_{m,\ell}(I_i)\cos(m\theta + \ell\Omega_i t) + \gamma^S_{m,-\ell}(I_i)\cos(m\theta - \ell\Omega_i t)] \tag{3.3a}$$

$$\Gamma^A(I_i,\theta,t) = 2\sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty}[\gamma^A_{m,\ell}(I_i)\sin(m\theta + \ell\Omega_i t) + \gamma^A_{m,-\ell}(I_i)\sin(m\theta - \ell\Omega_i t)] \tag{3.3)b}$$

where we have made use of the symmetry of the relations Eqs. (3.2)a,b and assumed that only terms that are function of θ and t are present in the perturbation i.e., $\gamma_{m,o} \equiv 0$, and $\gamma_{o,\ell} \equiv 0$. (In fact it was indicated in Section 2.1 that all stationary terms must be included in the unperturbed system representation; the $\gamma_{o,\ell}$ terms should drop out after the averaging indicated below.)

We let $\theta = \Delta\theta + \nu_i t$ and average Eqs. (3.3)a,b over a complete period of t; the only non null contributions come from secular terms where

$$m\nu_i - \ell\Omega_i = 0 \tag{3.4}$$

In other words, $\frac{\nu_i}{\Omega_i}$ is rational. Let ℓ_i, m_i be the lowest integers* to satisfy Eq. (3.4), ℓ_i and m_i depend on ν_i/Ω_i and therefore characterize the resonance. Since $\frac{\nu_i}{\Omega_i} = \frac{R_i}{2\pi} = \frac{\ell_i}{m_i}$ the rotational transform of the resonant surface is rational. Depending on the perturbation, a subgroup of the exact flux surfaces are excited; we call them primary resonances.

To a given primary resonance of $\frac{R_i}{2\pi} = \frac{\ell_i}{m_i}$ there are contributions from all the resonant harmonics characterized by $(m,\ell) = (pm_i, p\ell_i)$ where $p \geqslant 1$. In the vicinity of this resonance by averaging Eq. (2.25)a over fast oscillations we obtain:

$$\frac{d\Delta I}{dt} = \frac{2\varepsilon}{\nu_i}\sum_{p=1}^{\infty}[\gamma^S_{pm_i,-p\ell_i}(I_i)\cos pu - \gamma^A_{pm_i,-p\ell_i}(I_i)\sin pu] \tag{3.5}$$

where $u = -m_i\Delta\theta \equiv -m_i(\theta - \nu_i t) = -m_i\theta + \ell_i\Omega_i t$.

*Another condition on ℓ_i, m_i is to represent a harmonic of the perturbation whose Fourier coefficient has a modulus of the order of magnitude of the maximum Fourier modulus contributing to the same resonance.

In the appendix we evaluated the right-hand side of Eq. (3.5) and got:

$$\frac{d\Delta I}{dt} = \frac{\varepsilon}{\nu_i} f_{\nu_i}(u) \quad , \tag{3.6)a}$$

from Eq. (2.25)b

$$\frac{du}{dt} = \mu \Delta I \tag{3.6)b}$$

where $\mu = - m_i \frac{d\nu_i}{dI}$ and

$$f_{\nu_i}(u) = \frac{1}{T_i} \int_0^{T_i} dt \; \Gamma(I_i, - \frac{u}{m_i} + \nu_i t, t) \quad . \tag{3.7)a}$$

For simplicity, the indice i will be dropped from most equations up to the end of Section 3.2. Equations (3.6)a,b have the first integral:

$$K(\Delta I, u) = \frac{\mu}{2}(\Delta I)^2 + \frac{\varepsilon}{\nu} V(u) \tag{3.8}$$

where

$$V(u) = - \int^u du' \; f_\nu(u') \tag{3.9}$$

Surfaces of constant K represent the island contour equations.

Let us introduce a parameter a_j characterizing the a- contour in the j-island by the initial condition $\Delta I(u = a_j) = 0$. In our notations, by changing j we change from one island of a given resonance to another island of the same resonance where by changing a we change from one contour to another within the j-island. The island contour characterized by a_j is given by:

$$\Delta I = \sigma \sqrt{\frac{2\varepsilon}{\nu}} \left\{ \frac{1}{\mu} [V(a_j) - V(u)] \right\}^{1/2} \tag{3.10)a}$$

for all u's satisfying the condition:

$$[V(a_j) - V(u)]/\mu \geqslant 0 \quad . \tag{3.10)b}$$

$\sigma = \text{sgn}\ \Delta I$ and is equal to ± 1.

By comparing Eqs. (3.5) and (3.6)a one obtains the following expansion for $f_\nu(u)$:

$$f_\nu(u) = 2 \sum_{p=1}^{\infty} \left[\gamma^S_{pm,-p\ell} \cos pu - \gamma^A_{pm,-p\ell} \sin pu \right] \tag{3.7)b}$$

and a similar expansion for $\mu V(u)$. Thus $f_\nu(u)$ and $\mu V(u)$ are periodic functions of u of periods equal to 2π. Assuming that $\mu V(u)$ and its first derivative are smooth functions of u, it follows that their zeros are separated by intervals of π respectively. Therefore, on the closed interval $\frac{u}{m_i} = 0$ to $\frac{u}{m_i} = 2\pi$ the function $\mu V(u)$ has $2m_i$ zeros. Taking $\mu V(u)$ to be finite and continueous function of u, it follows that $\mu V(u)$ has m_i minima and m_i maxima. Let $\{\alpha_j\}_1^{m_i}$ be the set of minima and $\{\beta_j\}_1^{m_i}$ be the set of maxima and arrange the u = 0 axis to obtain

$$0 \leqslant \alpha_j \leqslant a_j \leqslant \beta_j < \alpha_{j+1} \leqslant a_{j+1} \leqslant \beta_{j+1} \ ; \text{ all } j \quad . \tag{3.11}$$

For clarification the reader is advised to examin Fig. 1. In order to obtain the maximum excursion in the action we express, as a first step, the action excursion expression in Eq. (3.10)a in terms of variables of the j-island. Let u_j represent the variable u restricted to vary between β_j and β_{j+1} and let $u_j' = u_j - \alpha_j$. Equation (3.10)a expressed in terms of u_j' is:

$$\Delta I(u_j') = \sigma \sqrt{\frac{2\varepsilon}{\nu}} \left\{ \frac{1}{\mu} \left[V'(a_j') - V'(u_j') \right] \right\}^{1/2} \tag{3.12)a}$$

where

$$V'(u_j') = - \int_{\alpha_j}^{\alpha_j + u_j'} du'\ f_\nu(u') \quad , \tag{3.12)b}$$

and $a_j' = a_j - \alpha_j$.

The maximum excursion in the action for the a_j contour is obtained by maximizing $|\Delta I(u_j', a_j')|$ with respect to u_j' and the maximum excursion in the action is obtained by maximizing the result with respect to a_j'. Let ΔI_M be this maximum and let $\beta_j' = \beta_j - \alpha_j$, then

$$\Delta I_M = |\Delta I(0,\beta_j')| = \sqrt{\frac{2\varepsilon}{\nu}} \left\{\frac{V'(\beta_j')}{\mu}\right\}^{1/2} . \tag{3.13}$$

Since α_j is a minimum of $\frac{V(u)}{\mu}$ one can easily show that $\frac{V'(a_j')}{\mu} \geqslant 0$ and therefore $\frac{V'(\beta_j')}{\mu} \geqslant 0$, necessary condition for Eq. (3.13). Another important property is that the function $\frac{V'(\beta_j')}{\mu}$ does not depend on the j-island of a given resonance and is function of the perturbation at the resonance, say $\sqrt{|V'(\beta_j')|} = F_i(P)$. Where P symbolizes the dependence on the perturbation at the primary resonance.

Let W_i define the resonance width; ($W_i = \mathrm{Max}|\Delta\nu_i|$). From Eq. (2.23):

$$W_i = \left|\frac{d\nu_i}{dI}\right| \Delta I_M \tag{3.14}$$

therefore:

$$W_i = \sqrt{\frac{2\varepsilon x_i}{m_i I_i}} \; F_i(P) \tag{3.15}$$

3.2. The Island Contour Oscillations

Consider the field line cross section that is located at $t = t_1$ in the j-island. At "time" $t = t_1 + \frac{2\pi}{\Omega_i \ell_i}$ it will appear in the j + 1 island. Therefore, a given island is transformed into itself in intervals of $\frac{2\pi}{\nu_i}$. We therefore introduce the "new time" for the island oscillations defined by:

$$\tau = \nu_i t \tag{3.16}$$

Since t is "time" we define sgn δt to be positive. From Eq. (3.6)b sgn $\delta u'_j = \sigma$ sgn μ. Let us call $\sigma' =$ sgn μ, and introduce the variables:

$$w_a = \sigma' u' \tag{3.17)a}$$

$$v_a = \Delta I(u' = \sigma' w_a) \quad , \tag{3.17)b}$$

where j is understood. Equations (3.6)a,b in terms of w_a and v_a become:

$$\frac{dv_a}{d\tau} = \frac{\varepsilon}{\nu^2} f_\nu(\alpha + \sigma' w_a) \tag{3.18)a}$$

$$\frac{d}{d\tau} w_a = \frac{|\mu|}{\nu} v_a \tag{3.18)b}$$

and can be derived from the hamiltonian:

$$K'(v_a, w_a) = \frac{|\mu|}{2\nu} v_a^2 + \frac{\varepsilon\sigma'}{\nu^2} V'(\sigma' w_a) \tag{3.19}$$

It is obvious from Eq. (3.18)b that sgn δw_a is the same as sgn v_a and the island contour oscillations for the a-contour occur for w_a variations between $-a'$ and $+a'$ and, therefore, are of the libration type.

Here we introduce the new action angle variables (η_a, J_a) corresponding to the pair of variables (w_a, v_a):

$$J_a = \frac{1}{4a'} \oint v_a dw_a \tag{3.20)a}$$

$$\eta_a = \frac{\partial S'(J_a, w_a)}{\partial J_a} \tag{3.20)b}$$

where

$$S'(J_a, w_a) = \int^{w_a} v_a dw_a \tag{3.20)c}$$

In this representation the island contour equations are given by:

$$\frac{dJ_a}{d\tau} = 0 \tag{3.21)a}$$

$$\frac{d\eta_a}{d\tau} = \omega_a \tag{3.21)b}$$

where

$$\omega_a = \frac{dK'}{dJ_a} \tag{3.22}$$

J_a is a function of a' and is positive except for $J_a(a' = 0) = 0$. ω_a is also a function of a' and is positive within the island, except for $\omega_a(a' = \beta') = 0$. We therefore refer to the elliptic singularity at $a' = 0$ as the local magnetic axis of the j-island and the island contour at $a' = \beta'$ as the local separatrix. The local separatrix is common for all the islands of a given primary resonance.

In the limit of small oscillation i.e., a' small, the hamiltonian (Eq. (3.19)) reduces to:

$$K'(v_a,w_a) = \frac{|\mu|}{2\nu} v_a^2 + \frac{c}{2\nu} w_a^2 \tag{3.23)a}$$

where

$$c = -\frac{\varepsilon\sigma'}{\nu} \frac{df_\nu}{du} (u = \alpha) \quad . \tag{3.23)b}$$

since α is a minimum of $\sigma' V(u)$ then $\sigma' \frac{df_\nu}{du} (u = \alpha) < 0$, therefore c is positive. In the limit of small oscillations:

$$K' = \frac{c}{2\nu} a'^2 \quad , \tag{3.24}$$

$$J_a = \frac{\pi}{4} a' \sqrt{\frac{c}{|\mu|}} \tag{3.25}$$

and

$$\omega_a = \frac{\sqrt{|\mu|\, c}}{\nu} \cdot \frac{4}{\pi}\, a' \tag{3.26}$$

Before we close this section we introduce the non-linearity coefficients X_a, associated with the slow frequency ω_a of the island contours

$$X_a = \frac{J_a}{\omega_a} \left| \frac{d\omega_a}{dJ} \quad (J = J_a) \right| \tag{3.27}$$

In the small oscillations region $X_a = 1$ (refer to Eqs. (3.25) and (3.26)).

3.3. The Island Contour Perturbation and Secondary Resonances

The oscillations examined in the previous section are subject to a "non-stationary perturbation" related to the presence of other resonances in the system. To simplify our notations we drop the indice a and let dots represent derivative with respect to τ. The island perturbation equations are determined from Eq. (3.18)a,b and Eq. (2.25)a,b, we get:

$$\dot{v} = \frac{\varepsilon}{\nu_i^2} \left[\Gamma\left(I_i, -\frac{\sigma' w}{m_i} + \tau - \frac{\alpha}{m_i} ; \frac{\tau}{\nu_i}\right) - f_{\nu_i}(\sigma' w + \alpha) \right] \tag{3.28a}$$

$$\dot{w} = \frac{|\mu|}{\nu_i}\, v \tag{3.28b}$$

The resulting action and angle perturbations are:

$$\dot{J} = \frac{1}{\omega}\left[\dot{v}\,\frac{\partial K'}{\partial v} + \dot{w}\,\frac{\partial K'}{\partial w}\right] \tag{3.29)a}$$

$$\dot{\eta} = \dot{w}\,\frac{d\eta}{dw} \tag{3.29)b}$$

since $\dot{v}$ and $\dot{w}$ are of the order of ε and $\varepsilon^{1/2}$ respectively, the highest contributions to Eqs. (3.29)a,b corresponds to evaluating the derivatives $\frac{\partial K'}{\partial v}$, $\frac{\partial K'}{\partial w}$ and $\frac{d\eta}{dw}$ on the unperturbed island contour. $\frac{\partial K'}{\partial v}$ and $\frac{\partial K'}{\partial w}$ are, therefore, obtained from Eqs. (3.18)a,b and Hamilton's equations, while:

$$\frac{d\eta}{dw} = \frac{d\eta^\circ/d\tau}{dw^\circ/d\tau} = \frac{\omega\nu_i}{|\mu|v} \quad . \tag{3.30}$$

If we substitute for $\frac{\partial K'}{\partial v}$, $\frac{\partial K'}{\partial w}$ and $\frac{d\eta}{dw}$ in Eqs. (3.29)a,b we get:

$$\dot{J} = \frac{1}{\omega}\sqrt{\frac{2|\mu|\varepsilon^3}{\nu_i^7}}\;\Gamma'(J,\eta,\tau) \tag{3.31)a}$$

$$\dot{\eta} = \omega(1 + O(\varepsilon^{1/2})) \tag{3.31)b}$$

where

$$\Gamma'(J,\eta,\tau) = \sigma\left\{\sigma' V_i'(a') - \sigma' V_i'[\sigma' w(J,\eta)]\right\}^{1/2}\;\Gamma\left[I_i;\tau - \frac{\alpha}{m_i} - \frac{\sigma' w(J,\eta)}{m_i}\;;\;\frac{\tau}{\nu_i}\right] . \tag{3.32}$$

For an obvious reason we have dropped all stationary terms. The function $w(J,\eta)$ is determined from Eqs. (3.20)a,b, and, for most cases, is very difficult to determine explicitely. For libration type motion one important property of w is its periodicity in η.[5] Taking this property into consideration we have shown that the function $\Gamma'(J,\eta,\tau)$ is a periodic function of η and τ.

Equations (3.31)a,b for J and η are similar to Eqs. (2.25)a,b for I and θ. And, similarly, they are valid in the region defined by:

$$\frac{\varepsilon J_K}{X_K} \ll \overline{\Delta J} \ll J_K \quad . \tag{3.33}$$

In order to explain the terms in Eq. (3.33) we need to introduce the concept of secondary resonances. Those are the excited island contours whose slow transforms are rational. Let $\{\omega_K\}$ represent the set of secondary resonances frequencies to which, from Eq. (3.22), correspond $\{J_K\}$ for the action. $\Delta J = J - J_K$ and $\overline{\Delta J} = \sqrt{\langle (\Delta J)^2 \rangle}$ where the average is taking over a complete period of $\Delta\eta = \eta - \omega_K \tau$.

Let p_K and q_K be the prime integers satisfying $\frac{\omega_K}{\Omega'_K} = \frac{p_K}{q_K}$, $(\Omega'_K \simeq \nu_i \Omega_i)$, and let $\xi = -q_K \Delta\eta$. In the domain defined by Eq. (3.33) we average Eqs. (3.31)a,b with respect to τ and get:

$$\Delta\dot{J} = \frac{1}{\omega_K} \sqrt{\frac{2|\mu|\varepsilon^3}{\nu_i^7}} \; f'_{iK}(\xi) \tag{3.34)a}$$

$$\dot{\xi} = M\Delta J \tag{3.34)b}$$

where $f'_{iK}(\xi) = \langle \Gamma'(J_K;\ \omega_K\tau - \frac{\xi}{q_K};\tau) \rangle_\tau$ and $M = -\, q_K \frac{d\omega}{dJ}\,(J = J_K)$.
Equations (3.34)a,b can be derived from the hamiltonian:

$$h(\Delta J,\xi) = \frac{M}{2}(\Delta J)^2 - \frac{1}{\omega_K}\sqrt{\frac{2|\mu|\varepsilon^3}{\nu_i^7}} \int^{\xi} f'_{iK}(\xi')\, d\xi' \tag{3.35}$$

where surfaces of constant h represent the secondary island contours. The maximum excursion in the action, ΔJ_M, is equal to the maximum of $|\Delta J|$ with respect to ξ and h. The secondary resonance width given by:

$$\Delta_{iK} = \left|\frac{d\omega}{dJ}\,(J = J_K)\right| \Delta J_M \quad , \tag{3.36}$$

is equal to:

$$\Delta_{iK} = \left(\frac{2\varepsilon m_i x_i}{\nu_i^6 I_i}\right)^{1/4} \left(\frac{2\varepsilon X_K}{q_K J_K}\right)^{1/2} F'_{iK}(P') \tag{3.37}$$

where $F'_{iK}(P')$ depends on the perturbation at the secondary resonance, symbolized by P'.

3.4. Behavior of the Field Lines Near the Elliptic Singularities

If we apply the method of averaging derived in Section 3.1 to Eqs. (2.27)a,b in the domain of the resonance $\frac{\nu_i}{\Omega_i} = \frac{\ell_i}{m_i}$, we obtain:

$$\frac{d\Delta I}{dt} = \frac{\varepsilon}{\nu_i} f_{\nu_i}(u_j) \tag{3.38)a}$$

$$\frac{du_j}{dt} = \mu\Delta I - \varepsilon\ell_i\Omega_i g_{\nu_i}(u_j) \tag{3.38)b}$$

where $f_{\nu_i}(u_j)$ was defined in Eq. (3.7)a and $g_{\nu_i}(u_j)$ is defined by:

$$g_{\nu_i}(u_j) = \frac{1}{T_i}\int_0^{T_i} dt\Pi(I_i, -\frac{u_j}{m_i} + \nu_i t, t) \tag{3.39}$$

Near the elliptic singularies $u'_j = u_j - \alpha_j \lesssim O(\varepsilon^{1/2})$. We expand $f_{\nu_i}(u_j)$ and $g_{\nu_i}(u_j)$ in term of u'_j:

$$f_{\nu_i}(u_j) = 0 + \frac{u'_j}{1!}\frac{df_{\nu_i}}{du_j}(\alpha_j) + \frac{u'^2_j}{2!}\frac{d^2 f_{\nu_i}}{du_j^2}(\alpha_j) + \ldots \tag{3.40)a}$$

$$g_{\nu_i}(u_j) = g_{\nu_i}(\alpha_j) + 0 + \frac{u_j'^2}{2!} \frac{d^2 g_{\nu_i}}{du_j^2}(\alpha_j) + \ldots \qquad (3.40)b$$

$f_{\nu_i}(\alpha_j)$ is null because, by definition, the $\{\alpha_j\}$ are the minima of $\mu V(u_j)$; and from Eqs. (2.20)a,b:

$$\frac{dg_{\nu_i}}{du_j}(\alpha_j) = \frac{1}{\ell_i \Omega_i \nu_i} \frac{d}{dI} f_{\nu_i}(\alpha_j) = 0 \qquad (3.41)$$

By substituting from Eqs. (3.40)a,b into Eqs. (3.38)a,b we obtain:

$$\frac{d\Delta I'}{dt} = -\varepsilon\mu A u'_j + O(\varepsilon^2) \qquad (3.42)a$$

$$\frac{du'_j}{dt} = \mu \Delta I' + O(\varepsilon^2) \qquad (3.42)b$$

where $\Delta I' = \Delta I - \varepsilon A'$,

$$A' = \frac{\ell_i}{\mu} \Omega_i g_{\nu_i}(\alpha_j) \qquad (3.43)$$

and

$$A = \frac{-1}{\mu\nu_i} \frac{df_{\nu_i}}{du_j}(\alpha_j) \equiv \frac{1}{\mu\nu_i} \frac{d^2V}{du_j^2}(\alpha_j) \geqslant 0 \qquad (3.44)$$

We solved Eqs. (3.42)a,b for the initial value condition $\Delta I'(u'_j = a'_j) = 0$ and got:

$$\frac{\Delta I'^2}{\varepsilon A a_j'^2} + \frac{u_j'^2}{a_j'^2} = 1 \qquad (3.45)$$

which is the equation of an ellipse. However, the contours will not always look like ellipses due to the curvilinear nature of the coordinates $\Delta I'$ and u'_j.

4. Behavior of the Field Lines in the Limit of Small Non-Linearity Coefficients

The field line equations in the limit of small non-linearity coefficients are given by Eqs. (3.38)a,b. However, there are various limits in which Eq. (3.38)a,b can be approximated by Eqs. (3.6)a,b, treated in Section 3. For example, near the elliptic singularity, $a_j' = O(\varepsilon^{1/2})$ and from Eq. (3.45) $\frac{\overline{\Delta I'}}{I} = O(\varepsilon)$. Thus, within the limit of our approximation (i.e., to $O(\varepsilon^2)$), the non-linearity coefficients must satisfy $\varepsilon << x \simeq \varepsilon^{1/2}$, and, therefore, from Eq. (3.40)b, using Eq. (3.41), one can show that

$$\frac{x_i \overline{\Delta I'}}{I_i} \Big/ \varepsilon \left| g_{\nu_i}(u_j) - g_{\nu_i}(\alpha_j) \right| = O(\varepsilon^{-1/2})$$

and, therefore, Eq. (3.38)a,b reduce to Eqs. (3.42)a,b, studied in Section 3.4.

If we let a_j' increase from $a_j' = O(\varepsilon^{1/2})$ to $a_j'^2 = O(\varepsilon^{1/2})$ and substitute from Eqs. (3.40)a,b into Eqs. (3.38)a,b, using Eq. (3.40), we get:

$$\frac{d\Delta I'}{dt} = -\frac{\varepsilon}{\nu_i} f_i' v_j + \frac{\varepsilon}{2\nu_i} f_i'' v_j^2 + O(\varepsilon^2) \qquad (4.1)a$$

$$\frac{dv_j}{dt} = \ell_i \Omega_i \left[\frac{x_i}{I_i} \Delta I' + \frac{\varepsilon}{2} g_i'' v_j^2 + O(\varepsilon^2) \right] \qquad (4.1)b$$

where $\Delta I' = \Delta I - \varepsilon A'$ and A' is given by Eq. (3.43), $v_j = \sigma' u_j'$ and σ' is the signature of μ,

$$f_i' \equiv -\sigma' \frac{df_{\nu_i}}{du_j}(\alpha_j) \geqslant 0 \quad , \qquad (4.2)$$

$$f_i'' \equiv \frac{d^2 f_{\nu_i}}{du_j^2}(\alpha_j) \tag{4.3}$$

and

$$g_i'' \equiv -\sigma' \frac{d^2 g_{\nu_i}}{du_j^2}(\alpha_j) \quad . \tag{4.4}$$

We note here that for perturbations where the dependence of $f_{\nu_i}(u_j)$ on I_i, near α_j, is expressed only as positive powers of I_i, g_i'' is positive or null.

For $a_j'^2 = O(\varepsilon^{1/2})$ the second terms on the right-hand sides of Eqs. (4.1)a,b add but a small perturbation to Eqs. (3.42)a,b, therefore, their solution which satisfies $\Delta I'(u_j' = a_j') = 0$ will have small deviations from the ellipse in Eq. (3.45). These deviations are equally distributed on the inside and the outside of that ellipse in order to satisfy the flux conservation requirement. As a_j' increases, the deviations mentioned above become larger and larger in magnitude (they may also change in shape due to the addition of terms in v_j^3 etc on the right-hand side of Eqs. (4.1)a,b). These deviations from the smooth elliptic shape will cause neighboring contours to overlap above some critical value of a_j', and, therefore, to self destruct. We refer to this type of behavior as orbital instability.

The condition for orbital stability is:

$$\frac{x_i}{I_i}\,\overline{\Delta I'} \gg \varepsilon\left|g_{\nu_i}(u_j) - g_{\nu_i}(\alpha_j)\right| \tag{4.5}$$

and is favored for: (i) large x_i, (ii) small I_i, and (iii) small ε. The second condition, (ii), implies that the magnetic island contours are

orbitally more stable near the central magnetic axis. However, it does not necessarily imply that the magnetic contours are always destroyed at the separatrix. In Section 5 we will show that the destruction of the flux surfaces at the separatrix is due, if not to external, to internal overlapping. This implies that the internal stochasticity differs from the orbital instability, although the two instabilities, in many respects have similar effects.

The high stability regions at $a_j' \simeq O(\varepsilon^{1/2})$ are, sometime, observable for small ε, (depending on the system and the resonance parameters). Their existence in toroidal systems of magnetic fields and, in conservative non-linear oscillating systems is demonstrated in the literature.[4,6] They usually appear as small, but well defined contours in highly unstable backgrounds.

5. Destruction of the Magnetic Surfaces by the Overlapping of Resonances

It is well confirmed that a strong instability with random-like behavior occurs when resonances overlap.[1]

The overlapping of two neighboring resonances will occur if their separation is smaller or equal to the sum of their widths. There is at least one overlapping of resonances below a given frequency ν_i if the sum of the widths of all resonances with frequencies smaller or equal to ν_i is greater or equal to ν_i. We take this as a definition for partial sotchasticity below ν_i, it is equivalent to:

$$\nu_i \leqslant \sum_{\nu_j \leqslant \nu_i} W_j \quad . \tag{5.1}$$

Although the frequencies of all the resonances below ν_i range from zero to ν_i their number is infinite. We let $\{j\}_i^\infty$ represent the set of resonances below ν_i ordered in such a way that $\nu_{j \geqslant i} \leqslant \nu_i$ and $\nu_\infty = 0$. Equation (5.1) is therefore the same as:

$$\nu_i \leqslant \sum_{j=i}^{\infty} W_j \quad , \tag{5.2}$$

where W_j is the width of the j-resonance.

The limit of total stochasticity below ν_i is obtained if all the resonances of frequencies smaller or equal to ν_i satisfies the criterion of partial stochasticity. Equation (5.2) should give an underestimate of the critical perturbation for ν_i, an overestimate is obtained from the limit of total stochasticity below ν_i. The critical limit is somewhere in between. We introduce $C_i \leqslant 1$ to be a positive constant

characteristic of the resonance ν_i such that

$$\nu_i = C_i \sum_{j=i}^{\infty} W_j \tag{5.3}$$

defines the critical perturbation for the i-resonance. (C_i is constant in the sense that it is not explicitely dependent on the resonance parameters; however, C_i varies from one resonance to the other and is dependent on the system.)

The widths of primary resonances are given by Eq. (3.15), if ε_i is the critical perturbation for the i-resonance, from Eq. (5.3) we get:

$$C_i G_i \varepsilon_i^{1/2} = \left(\frac{1}{2x_i}\right)^{1/2} \tag{5.4}$$

where,

$$G_i = \frac{1}{\nu_i} \sum_{j=i}^{\infty} \left(\frac{x_j}{x_i} \frac{1}{I_j} \frac{1}{m_j}\right)^{1/2} F_j(P) \tag{5.5}$$

$F_j(P)$ having the dimension of $(m_j I_j)^{1/2} \nu_j$, we arranged Eq. (5.5) so that G_i is dimensionless. G_i is like a structure factor characterizing the perturbed system below ν_i.

Let us call external overlapping the overlapping of primary resonances and internal overlapping the overlapping of secondary resonances, Eq. (5.4) thus describes the critical perturbation by external overlapping. Let ω_s represent a cutoff frequency which corresponds to the lowest observable island contour. Therefore, the total descruction of the ν_i-resonance by internal overlapping is obtained by the critical perturbation below the ω_s-secondary-resonance. Let ε_s represent the critical perturbation by internal overlapping, from Eqs. (3.37) and (5.3) we get:

$$C_s' G_s' \varepsilon_s^{3/4} = \left(\frac{\nu_i^6 I_i}{2 m_i x_i}\right)^{1/4} \left(\frac{1}{2X_s}\right)^{1/2} \tag{5.6}$$

where C_s'($\leqslant 1$ and positive) is defined, similarly to C_i, for the secondary resonances, and

$$G_s' = \frac{1}{\omega_s} \sum_{K=s}^{\infty} \left(\frac{X_K}{X_s} \frac{1}{J_K} \frac{1}{q_K}\right)^{1/2} F_{iK}'(P') \ . \tag{5.7}$$

Our definition of G_s' for the secondary resonances (Eq. (5.7)) is similar to that of G_i for the primary resonances (Eq. (5.5)). G_s' is also like a structure factor characterizing the secondary system below ω_s.

The secondary resonance of frequency ω_s is in the small oscillations region, treated at the end of Section 3.2. From Eqs. (3.25) and (3.26).

$$X_s = 1 \ . \tag{5.8}$$

From Eqs. (5.8) and (5.6) the limit of internal stochasticity for the ν_i-resonance is obtained:

$$C_s' G_s' \varepsilon_s^{3/4} = \left(\frac{\nu_i^6 I_i}{8 m_i x_i}\right)^{1/4} \tag{5.9}$$

Equation (5.4) for the external stochasticity and Eq. (5.9) for the internal stochasticity give the dependence of the critical perturbations by these two processes on the primary resonance parameters. For a given

resonance, the physical critical perturbation is the smallest of the two above mentioned critical perturbations. From the present analysis it is possible, in an experimental situation, to determine the primary resonance widths explicitly. The values of the critical perturbations, by external overlapping, are, therefore, obtained by directly checking the overlapping of neighboring primary resonances. However, the secondary resonance widths are difficult to determine explicitly from the present analysis.

Secondary resonances are related to the presence of more than one primary resonance in the system. For systems where many primary resonances are possible it is, therefore, logical to assume that the secondary systems are similar to the primary system.[7] This leads us to assume that

$$C_i G_i = e C_s' G_s' \tag{5.10}$$

where e is a constant. If we take the ratio of Eqs. (5.9) and (5.4) and use Eq. (5.10) we get:

$$\varepsilon_s^{3/4} = e \left(\frac{\nu_i^6 I_i x_i}{2m_i} \right)^{1/4} \varepsilon_i^{1/2} \tag{5.11}$$

Equation (5.11) relates the internal critical perturbation to the external critical pertrubation . Thus, from the values of ε_i and the primary resonance parameters, the values of ε_s are determined.

For the stellerator[2], where stochasticity occurred near the separatrix for values smaller than the external stochasticity limit ε_i, Eq. (5.11) shows that this had occurred by internal overlapping.

In general, formula (5.11) asserts that independently of how small is the perturbation, the magnetic surfaces are always destroyed at the separatrix.

For the levitron* we calculated ε_i by directly checking the overlapping of neighboring primary resonances** and determined ε_s from Eq. (5.11), where we have taken $e \simeq 1$ and $\Omega_i \equiv 1$. The results are tabulated in Table I. (For the levitron ε is a tilt angle;[4] in Table I it is given in degrees.) We conclude that for $\nu_i \leqslant 2$ the destruction is caused by internal overlapping. In Table I quantities in parenthesis are the theoretical limits for destruction. ε_c are the numerically measured tilts for which the resonances are completely destroyed.[8] There is a very good agreement between theoretical and numerical values of the critical perturbations. In Fig. 2, we show a typical example of destruction by internal overlapping in the levitron; the secondary magnetic islands and contours appear in part c.

Before we close this section we would like to comment on Eq. (5.10) which says that $C_i G_i$ is proportional to $C_s' G_s'$. It can be argued that the limiting secondary resonance parameters are functions of the primary resonance parameters and the perturbation at the primary resonance. On the other hand, the summation in Eq. (5.5) over the indice

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*The basic features of an average minimum B levitron, are represented by a simplified model which consists of a single filamentary conducting circular loop located at the center of the torus, a straight filamentary conductor along the vertical axis and a uniform vertical magnetic field. The location of the separatrix surface is determined by the ratio of the uniform vertical field to the loop field. The distance from the loop to the separatrix approximates the minor radius of the torus and thus determines the system's aspect ratio.

**The simultaneous destruction of two neighboring resonances of comparable widths occurred when their separation was smaller or equal to the arithmetic average of their widths. This criterion gave the best agreement with the numerical results.

j transforms the dependence of F_j on the perturbation at ν_j into a dependence of G_i on the primary system below ν_i. Similarly G'_s is dependent on the secondary system below ω_s. We have argued before that for systems with many resonances the secondary systems are similar to the primary system. Thus, there is an approximate similarity between the primary system below ν_i and the secondary system below ω_s. This similarity and the dependence of G_i and G'_s on the primary resonance parameters only, suggest that G_i and G'_s can be related. Although this does not prove the proportionality of $C_i G_i$ and $C'_s G'_s$ it does give a strong support in favor of it. The proportionality constant, e, is in general dependent on the system; from Table I, taking e equal to one seems to be a good approximation for the levitron under consideration.

6. Conclusions

It has been established that if resonances overlap, a rapid destruction of their island structure occurs.[1]

1. Thus, if primary resonances overlap, a rapid destruction of their flux surfaces is expected.

2. For large non-linearity ($x \gg \varepsilon^{1/2}$), the field lines are trapped in an effective potential well in the primary resonance domain forming families of closed contours at the elliptic singularities. We refer to the resonant island contours as secondary resonances. Another possible phenomenon of destruction is the overlapping of secondary resonances. Depending on the primary resonance parameters (and the system) destruction may occur by either or both phenomena.

3. The excursion in the action for the primary magnetic island increases as $\varepsilon^{1/2}$ while for the secondary island it increases as $\varepsilon^{3/4}$. Internal overlapping proceeds almost orderly from the local separatrix to the elliptic singularity. Therefore, for islands that are most affected by internal overlapping the observed primary excursion should increase at a lower rate than $\varepsilon^{1/2}$ due to the successive disappearance of outer contours destroyed by secondary resonance overlapping. This is in agreement with the numerical observation by Freis et al.,[4] where the 1 and $\frac{1}{2}$ resonance widths increase as $\varepsilon^{1/2}$ until breadup while the $\frac{3}{2}$, 2, $\frac{5}{2}$ and 3 resonance widths increase as $\varepsilon^{3/2}$.

4. For small non-linearity, except in the immediate neighborhoods of the elliptic singularities, the magnetic contours oscillate in an irregular fashion and overlap causing orbital instabilities. Orbital instabilities are more pronounced for larger fluxes but do not always destroy the flux surfaces at the separatrix.

5. In the immediate neighborhoods of the elliptic singularities the field lines are orbitally stable for all non-linearities. These neighborhoods, therefore, constitute small stable regions which, depending on the resonance parameters and the system, may become observable for small ε. The existence of these high stability regions is demonstrated numerically in the literature where small but well defined contours appear sometimes in highly unstable backgrounds.

6. It is known since Poincaré[9] that a hierarchy of resonances are generated in a non-linear oscillating system. In a system where terms of the first order in ε give the highest order observable contribution, we found that two sets of resonances are sufficient to explain the formation and destruction of the magnetic surfaces. In general, if ε^n, where n is a positive integer, gives the highest order observable contribution then 2n sets of resonances are sufficient.

7. Equation (5.4) for the external stochasticity and Eq. (5.9) for the internal stochasticity give the dependence of the critical perturbations by these two processes on the primary resonance parameters. For a given resonance these relations give the dependence of the critical perturbation on the non-linearity coefficient.

8. From Eq. (5.11), as ν_j approaches zero, the region of internal stochasticity extends over all the x-ε plane thus, independently of how small is $\varepsilon > 0$, the flux surfaces are always destroyed near the separatrix.

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Appendix

The Averaging Method

To evaluate the right-hand side of Eq. (3.5) we consider the Fourier coefficients:

$$\gamma^S_{pm_i,-p\ell_i}(I_i) = \frac{1}{2\pi T_i}\int_0^{T_i} dt \int_0^{2\pi} d\theta \Gamma^S(I_i,\theta,t)\cos(pm_i\theta - p\ell_i\Omega_i t)$$

$$= \frac{1}{2\pi T_i}\int_0^{T_i} dt \int_{\ell_i\Omega_i t}^{\ell_i\Omega_i t - 2\pi m_i}\left(-\frac{du}{m_i}\right)\Gamma^S(I_i,-\frac{u}{m_i}+\nu_i t,t)\cos pu \tag{A.1}$$

where $u = -m_i\Delta\theta = -m_i\theta + \ell_i\Omega_i t$.

Since $\Gamma^S(I_i,\theta,t)$ is periodic in θ and t then:

$$\int_{\ell_i\Omega_i t}^{\ell_i\Omega_i t - 2\pi m_i} du\Gamma^S(I_i,-\frac{u}{m_i}+\nu_i t,t) = \int_0^{-2\pi m_i} du\Gamma^S(I_i,-\frac{u}{m_i}+\nu_i t,t) \; . \tag{A.2}$$

From Eqs. (A-1) and (A-2):

$$\gamma^S_{pm_i,-p\ell_i}(I_i) = \frac{1}{2\pi m_i}\int_{-2\pi m_i}^{0} du\, f^S_{\nu_i}(u)\cos pu \tag{A.3}$$

where

$$f^S_{\nu_i}(u) = \frac{1}{T_i}\int_0^{T_i} dt\Gamma^S(I_i,-\frac{u}{m_i}+\nu_i t,t) \quad . \tag{A.4}$$

Similarly

$$\gamma^A_{pm_i,-p\ell_i}(I_i) = -\frac{1}{2\pi m_i}\int_{-2\pi m_i}^{0} du\, f^A_{\nu_i}(u)\sin pu \tag{A.5}$$

where

$$f^A_{\nu_i}(u) = \frac{1}{T_i}\int_0^{T_i} dt\Gamma^A(I_i,-\frac{u}{m_i}+\nu_i t,t) \quad . \tag{A.6}$$

If we substitute from Eqs. (A.3) and (A.5) into Eq. (3.5) we get:

$$\frac{d\Delta I}{dt} = \frac{2\varepsilon}{\nu_i} \frac{1}{2\pi m_i} \int_{-2\pi m_i}^{0} du' \, f_{\nu_i}^{S}(u') \sum_{p=1}^{\infty} \cos pu \cos pu' + f_{\nu_i}^{A}(u') \sum_{p=1}^{\infty} \sin pu \sin pu' \tag{A.7}$$

In order to evaluate Eq. (A.7) we express $\sum_{p=1}^{\infty} \cos pu \cos pu'$ and $\sum_{p=1}^{\infty} \sin pu \sin pu'$ in terms of delta distributions, consider x in the one dimensional real space, R^1, then:

$$\sum_{k=-\infty}^{+\infty} e^{2\pi ikx} = \sum_{n=-\infty}^{+\infty} \delta(x - n)$$

where n is an integer. If x varies only in an interval of R^1, say $[x_1, x_2]$, $x_1 < x_2$, and if $n_1 = \mathrm{Ent}(x_1) + 1$ and $n_2 = \mathrm{Ent}(x_2)$ then:

$$\sum_{k=-\infty}^{+\infty} e^{2\pi ikx} = \sum_{n=n_1}^{n_2} \delta(x - n)$$

It follows that:

$$\sum_{k=1}^{\infty} \cos 2\pi kx = -\frac{1}{2} + \frac{1}{2} \sum_{n=n_1}^{n_2} \delta(x - n) \quad . \tag{A.8}$$

By using

$$\sum_{p=1}^{\infty} \cos pu \cos pu' = \frac{1}{2} \sum_{p=1}^{\infty} [\cos p(u + u') + \cos p(u - u')]$$

$$\sum_{p=1}^{\infty} \sin pu \sin pu' = \frac{1}{2} \sum_{p=1}^{\infty} [-\cos p(u + u') + \cos p(u - u')]$$

and noting that the interval of variation of u+u' and u-u' is $[0, 2\pi m_i]$ we get:

$$\sum_{p=1}^{\infty} \cos pu \cos pu' = -\frac{1}{2} + \frac{\pi}{2} \sum_{n=0}^{m_i} \delta(u + u' - 2\pi n) + \sum_{n=0}^{+m_i} \delta(u - u' - 2\pi n) \tag{A.9)a}$$

$$\sum_{p=1}^{\infty} \sin pu \sin pu' = \frac{\pi}{2} \sum_{n=0}^{m_i} \delta(u + u' - 2\pi n) + \sum_{n=0}^{+m_i} \delta(u - u' - 2\pi n) \tag{A.9)b}$$

By using $f^S_{\nu_i}(u) = f^S_{\nu_i}(-u)$ and $f^A_{\nu_i}(u) = -f^A_{\nu_i}(-u)$ after substituting Eq. (A-9)a and Eq. (A-9)b into Eq. (A-7) we get:

$$\frac{d\Delta I}{dt} = \frac{2\varepsilon}{\nu_i} \frac{2}{4m_i} \sum_{n=0}^{m_i} f_{\nu_i}(u - 2\pi n) \tag{A.10}$$

where

$$f_{\nu_i}(x) = f^S_{\nu_i}(x) + f^A_{\nu_i}(x) \tag{A.11}$$

From the definition of $f^{S,A}_{\nu_i}(u)$ and the periodicity of $\Gamma(I,\theta,t)$ with respect to θ and t it is straightforward to show that:

$$f_{\nu_i}(u) = f_{\nu_i}(u \pm 2\pi n) \; ; \; n = 0,1,2, \ldots \tag{A.12}$$

By substituting Eq. (A-12) into Eq. (A-10) we get:

$$\frac{d\Delta I}{dt} = \frac{\varepsilon}{\nu_i} f_{\nu_i}(u) \; . \tag{A.13}$$

Equation (A-13) can also be derived directly by averaging Eq. (2.20)a over fast oscillating terms in t. This shows the equivalence of the present analysis and the averaging technique often used in the literature.

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Figure and Table Captions

Fig. 1. Structure of the $\left(\frac{\nu_i}{\Omega_i} = \frac{1}{4}\right)$ - primary resonance in the action angle plane. The various angles shown are defined in the text.

Fig. 2. (a) A closed primary contour of the π resonance at $1+r\cos\phi \simeq 0.51$ and part of the 2π primary resonance at $1+r\cos\phi \simeq 0.61$. (b) The perturbation is doubled and the contour is heavily distorted. (c) The perturbation is doubled again, the contour is completely destroyed and secondary magnetic islands and contours appear. This is a typical example of destruction by internal overlapping. The 2π primary resonance island after reaching its maximum flux in part (c) is seen partially destroyed in part (d). This figure is taken from Ref. 3.

Table I. For the levitron, ε is a tilt angle. The theoretical tilts for which the resonances are completely destroyed are equal to the smallest of ε_i and ε_s. ε_i and ε_s are the limits of external and internal stochasticities, respectively. (*The 3 and 5/2 resonances overlap and are simultaneously destroyed.) ε_c are the numerically measured critical tilts. ν_i/Ω_i are the rotational transforms, x_i the nonlinearity coefficients, and I_i the primary actions, $\Omega_i \simeq 1$. In the levitron $m_i \geqslant 2$, except for the 2π-resonance where $m_i = 1$.

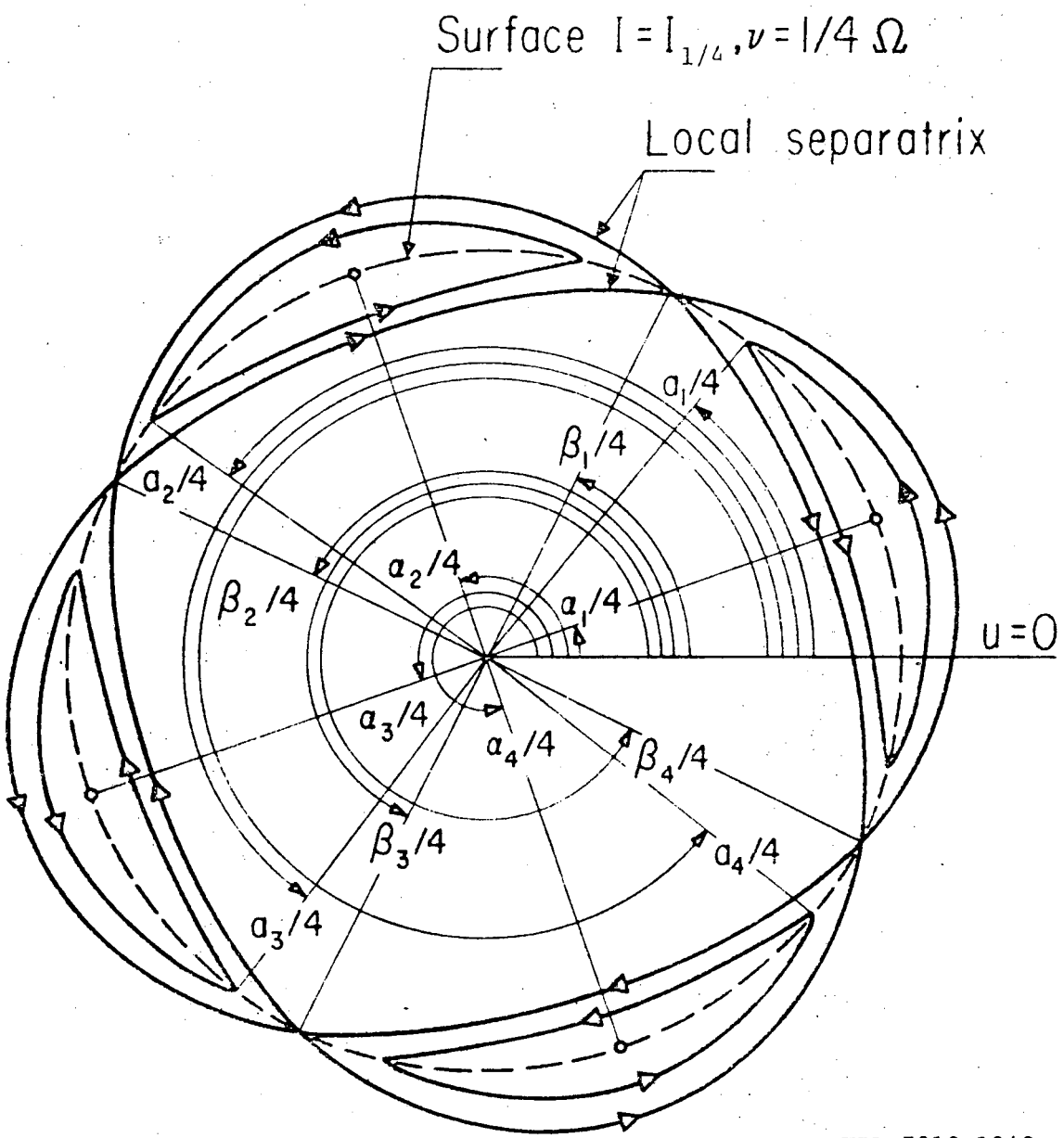

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Fig. 1

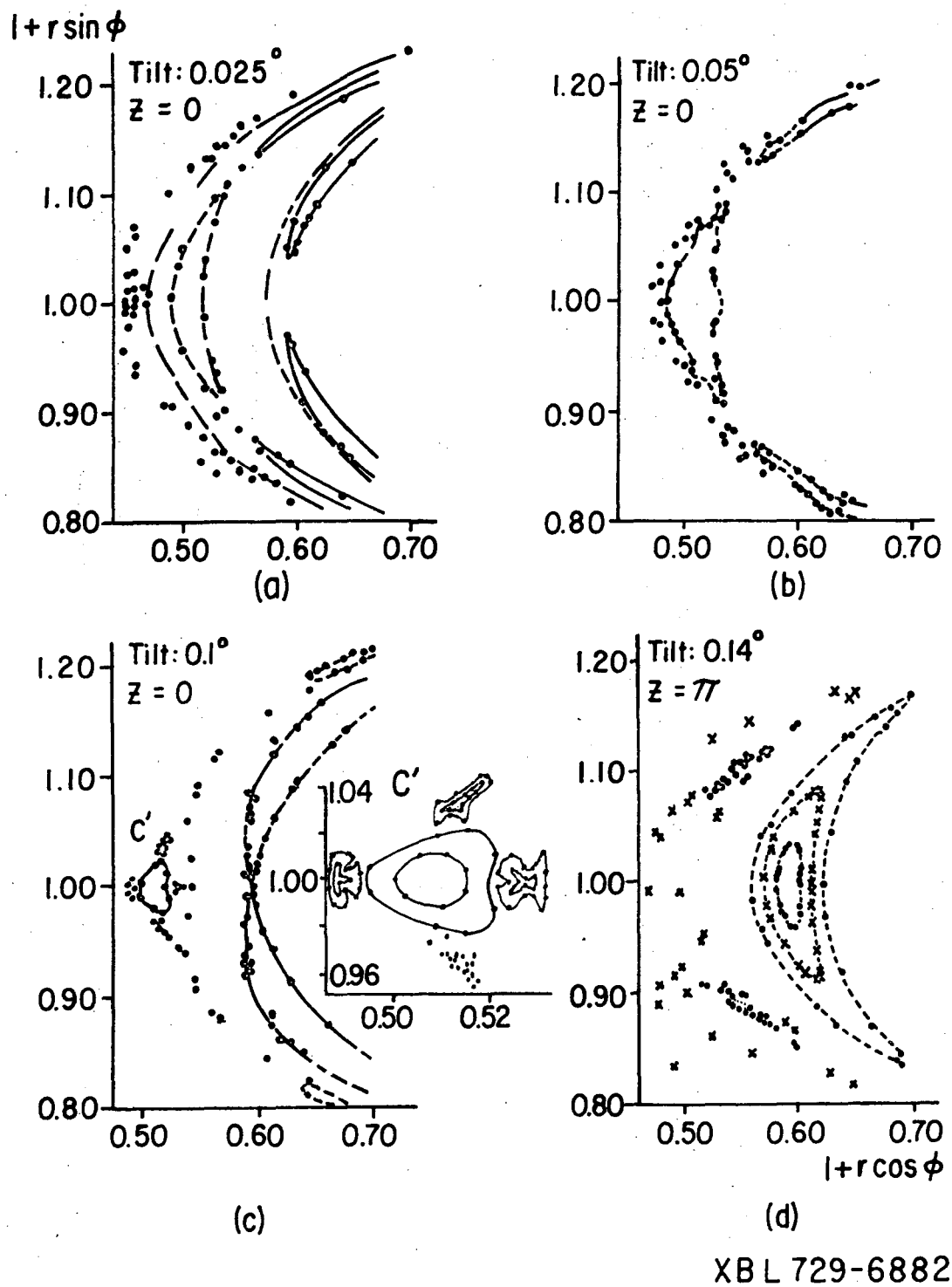

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Fig. 2

$\frac{\nu_i}{\Omega_i} = \frac{\ell_i}{m_i}$	x_i	I_i	ε_i	ε_s	ε_c
6/2	1.256	0.036	(2.75)	3.96	2.50
5/2	1.074	0.040	(2.75)*	2.71	2.50
4/2	1.045	0.050	4.95	(2.74)	>1.00 <3.00
3/2	1.140	0.061	1.68	(0.83)	0.70
1/1	1.300	0.084	0.54	(0.25)	0.30
1/2	2.200	0.120	0.54	(0.067)	0.15
1/4	2.280	0.162	~0.244	(~0.002)	< 0.02

Table I

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