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Presented at the IMA Conference on Numerical Methods in Fluid Dynamics, University of Reading, Reading, England, March 29-31, 1982

CALCULATION OF SHOCKS IN OIL RESERVOIR MODELING AND POROUS FLOW

P. Concus

March 1982

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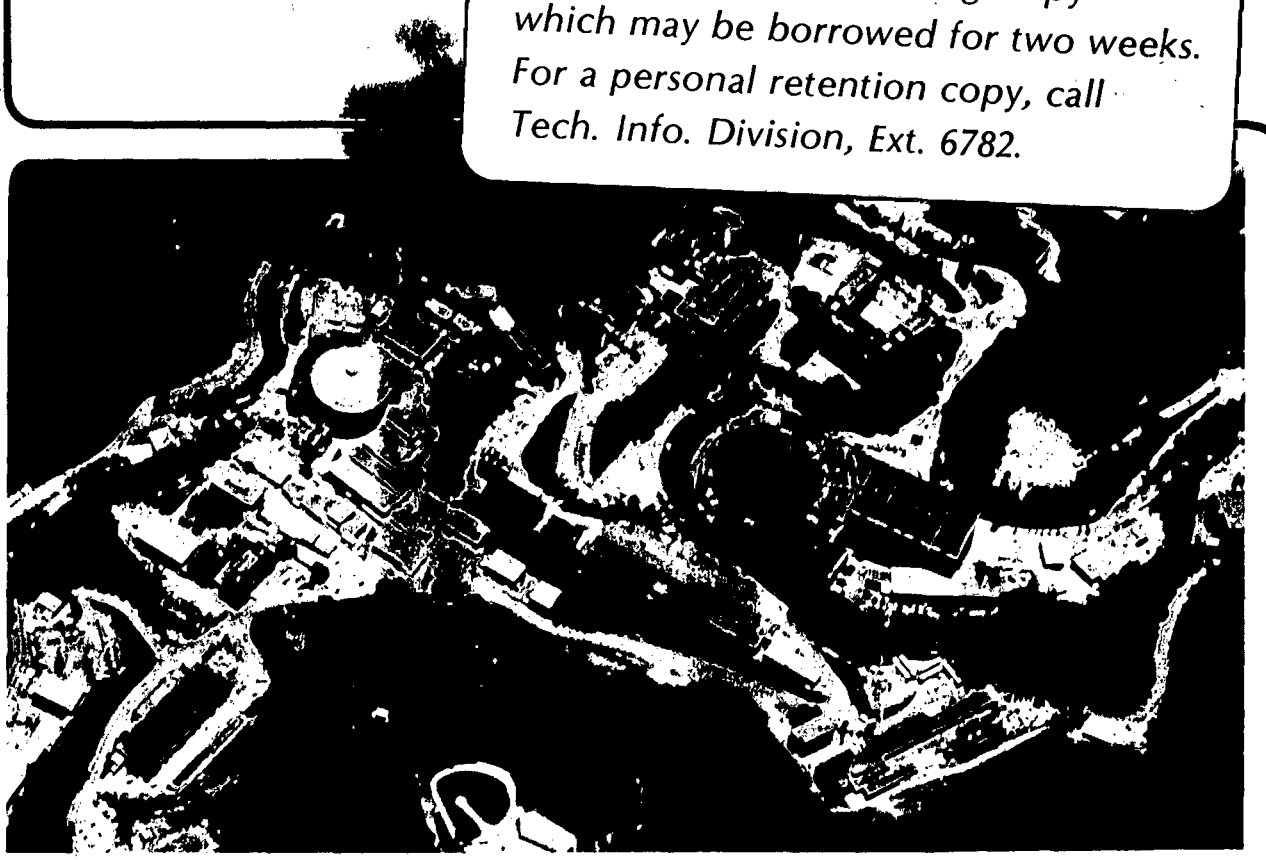

Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

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[1]Supported in part by the Director, Office of Energy Research, Office of Basic Energy Sciences, Engineering, Mathematical, and Geosciences Division of the U.S. Department of Energy under contract DE-AC03-76SF00098.

CALCULATION OF SHOCKS IN OIL RESERVOIR MODELING AND POROUS FLOW

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1. Introduction

In recent years the numerical modeling of fluid displacement through a porous medium has received increased attention, stimulated by the development of enhanced recovery methods for obtaining petroleum from underground reservoirs and the advent of larger, higher-speed computers. For many recovery methods of interest, propagating fronts arise that may be steep or discontinuous. One example is the water flooding of a petroleum reservoir, in which there is forced out residual oil that remains after outflow by decompression has declined.

Some of the work being carried out in the Mathematics Group of the Lawrence Berkeley Laboratory for developing high-resolution numerical methods to solve porous flow problems having propagating discontinuities is discussed here. Such discontinuities usually pose substantial difficulty for conventional discretization methods. Our investigations center on some alternative numerical methods that incorporate analytical information concerning the discontinuities. Such methods have been effective in treating hyperbolic conservation laws arising in gas dynamics and can be adapted in many cases to the equations of porous flow.

One method of interest, the random choice method, can track solution discontinuities sharply and accurately for one space dimension. The first phase of our study adapted this method for solving the Buckley-Leverett equation for immiscible displacement in one space dimension. Extensions to more than one

space dimension for the random choice method were carried out subsequently in our study by means of fractional splitting. Because inaccuracies could be introduced for some problems at discontinuity fronts propagating obliquely to the splitting directions, our efforts are currently being directed at investigating alternatives for multidimensional cases.

A front tracking method for multidimensional problems based on the SLIC scheme developed by Noh & Woodward (1976) was extended to the Burgers equation and porous flow cases. The method assigns to mesh cells a value representing the fraction of the cell lying behind the front. The cell fractions are then appropriately advanced at each time step. Away from the front, random choice method, Godunov's method, or other methods may be used.

Recently a higher order version of Godunov's method, which utilizes piecewise linear rather than piecewise constant segments, as introduced in the MUSCL scheme of van Leer (1979), has been extended to gas dynamics in Eulerian coordinates. Preliminary results for the extension of this work to the porous flow equations for one space dimension have been promising, and an investigation of extension to higher dimensions is being carried out.

Specialized numerical approaches taken by others for porous flow problems with steep fronts can be found, for example, in papers SPE 10499 to SPE 10502 in the Proceedings of the Sixth SPE Reservoir Simulation Symposium (1982)and the references therein, in (K. Miller & R. Miller, 1981), and in (Glimm, Isaacson, Marchesin, & McBryan, 1981).

2. Equations for immiscible displacement

The equations for two-phase immiscible incompressible displacement in a porous medium in the absence of capillary pressure are (Peaceman, 1977)

$$\varphi \frac{\partial s}{\partial t} + \mathbf{q}\cdot\nabla f(s) - \gamma \frac{\partial}{\partial z} g(s) = 0 \tag{2.1}$$

$$\nabla\cdot\mathbf{q} = Q \tag{2.2}$$

$$\mathbf{q} = -\lambda(s)[\nabla p - \gamma\tilde{g}(s)\mathbf{e}_k]\,. \tag{2.3}$$

The porous medium is taken to be homogeneous and isotropic and the interior of the domain free of sources or sinks (i.e., injection or production wells).

In the above equations $s(\mathbf{x},t)$, $0 \le s \le 1$, is the saturation of the wetting fluid (fraction of available pore volume occupied by the fluid). The saturation of the non-wetting fluid is then $1-s$. The independent variables $\mathbf{x}$ and t are the space and time, respectively, and $\mathbf{q}(\mathbf{x},t)$ is the total velocity (sum of the individual velocities of the two fluids). If gravity is present it is assumed to act in the negative z direction, with $\mathbf{e}_k$ the unit vector in the positive z direction. The quantity $p(\mathbf{x},t)$ is the excess over gravitational head of the reduced pressure; here the reduced pressure is the average of the individual phase pressures less the gravitational head. The quantity Q represents the sources and sinks of fluid on the boundary of the domain, and φ is the porosity, which will be assumed constant. The quantity γ, the coefficient of the gravitational term, is the product of the acceleration due to gravity times the density difference between the wetting and nonwetting phases.

Eq. (2.1) is the Buckley-Leverett equation, which for a given $\mathbf{q}$ is hyperbolic. Eq. (2.2) is the incompressibility condition, and (2.3) is Darcy's law. For a given s, (2.2),(2.3) is elliptic.

The functions of saturation $f(s), g(s), \lambda(s)$ and $\tilde{g}(s)$ can be expressed in terms of the empirically determined phase mobilities (ratios of permeability to

viscosity) λ_n and λ_w of the non-wetting and wetting fluids. For immiscible displacement these are

$$f(s) = \lambda_w/\lambda \ , \ g(s) = \lambda_n f$$
$$\tilde{g}(s) = \lambda_n/\lambda \ , \ \lambda(s) = \lambda_n + \lambda_w \ .$$

The quantities f, g, and $\tilde{g}$ are non-negative, and λ is positive.

A distinguishing feature of the immiscible displacement equations is that f and g are non-convex; f typically has one inflection, as depicted for a model case in Fig. 1, and g has two, as depicted in Fig. 2. Thus weak solutions may have combinations of propagating shock and expansion waves in contact.

Attempts to solve (2.1),(2.2),(2.3), subject to appropriate boundary conditions, by standard discretization methods such as finite difference or finite element methods can give rise to substantial difficulty. Inaccuracies may arise near a moving front, or an incorrect weak solution may be obtained. To circumvent these difficulties, the first phase of our study initiated an attempt to adapt the random choice method to solving problems of fluid displacement in porous media.

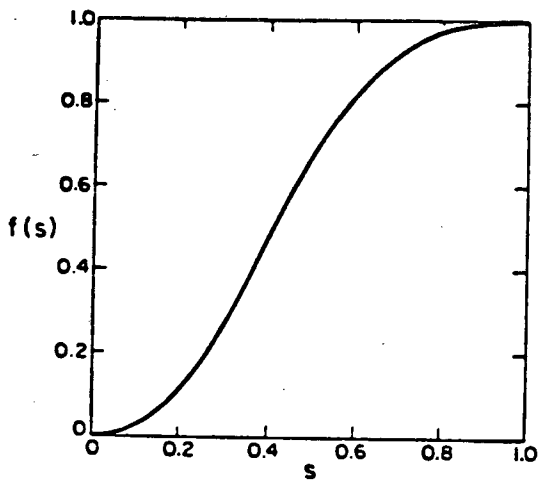

Fig. 1

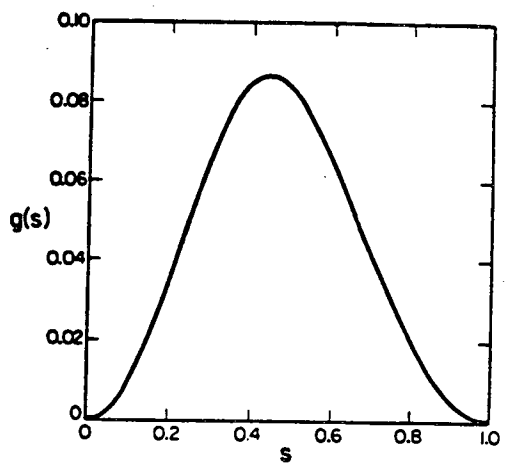

Fig. 2

3. Random choice method

The random choice method, which was formulated originally for solving the equations of gas dynamics, is a numerical method that incorporates the accurate propagation of solution discontinuities. It is based on a mathematical construction of Glimm (1965) that was developed into a practical and efficient computational algorithm by Chorin (1976,1977). It was first adapted to porous flow problems in (Concus & Proskurowski, 1979), and subsequently in (Glimm, Marchesin, and McBryan, 1979).

For a single nonlinear conservation law,

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial z}\psi(s) = 0 \tag{3.1}$$

to which (2.1) reduces in one space dimension, the random choice method advances a solution in time as follows. The solution $s(z,t_j)$ at time t_j is represented by a piecewise-constant function on a spatial grid of spacing Δz, where the function is equal to $s_i^j = s(z_i,t_j)$ in the interval $z_i - \frac{1}{2}\Delta z < z \le z_i + \frac{1}{2}\Delta z$. Then the solution of (3.1) is constructed analytically by the method of characteristics for this piecewise-constant initial data by solving a sequence of Riemann problems (3.1) with initial data

$$s(z,t_j) = \begin{cases} s_i^j & , \quad z \le z_i + \frac{1}{2}\Delta z \\ s_{i+1}^j & , \quad z > z_i + \frac{1}{2}\Delta z \ . \end{cases} \tag{3.2}$$

So long as the time increments Δt satisfy the Courant-Friedrichs-Lewy condition $(\Delta t/\Delta z)\cdot\max|\psi'(s)| < \frac{1}{2}$ (< 1 for forms of the method using half time-steps on staggered grids), the waves propagating from the individual mesh-point discontinuities will not interact during a given time step. This permits the solution of (3.1) to be obtained during the step by joining together the separate Riemann problem solutions. The above procedure is common with other methods, such as

Godunov's method. The distinguishing feature of the random choice method is that it obtains the new piecewise-constant representation of the solution at the new time by sampling the exact solution at a point within each spatial interval. In this way moving discontinuities remain perfectly sharp, since no intermediate values are introduced by the method, at the price of introducing a small amount of statistical uncertainty. The method is essentially first order and is observed to give good results for one-dimensional problems.

4. Riemann problems

The practicality of the random choice method depends upon being able to solve the Riemann problems efficiently. For the immiscible displacement problem the function $\psi(s)$, which is a linear combination of $f(s)$ and $g(s)$, has either one or two inflections, depending upon the relative magnitudes of q and γ. If in (2.1) the gravity term $\gamma g(s)$ is small compared with the transport term $qf(s)$ then there is only one inflection, for which the Riemann problem solution is given in (Concus & Proskurowski, 1979). For the case of two inflections the solution is given in (Albright, Anderson, & Concus, 1980), (Anderson & Concus, 1980), and for a special case in (Proskurowski, 1980).

A typical example for a case in which two inflections occur is depicted in Fig. 3, which is the case given in Fig. 3 of (Anderson & Concus, 1980). The Riemann problem solution is obtained by applying the following general conditions, which must hold along any curve of discontinuity of $s(z,t)$. Let $s_- = \lim_{z \to z_-} s(z,t)$ and $s_+ = \lim_{z \to z_+} s(z,t)$ be the limiting values from the left and right at a discontinuity. Then there must hold (Lax, 1973),(Oleinik, 1963),

(i) Rankine-Hugoniot jump condition: The curve of discontinuity is a straight line with slope

$$\frac{dz}{dt} = \frac{\psi(s_+) - \psi(s_-)}{s_+ - s_-}$$

(ii) Generalized entropy condition:

$$\frac{\psi(s_+) - \psi(s)}{s_+ - s} \leq \frac{\psi(s_+) - \psi(s_-)}{s_+ - s_-}$$

for any s between s_+ and s_-.

For the case $s_i^j = 0$ and $s_{i+1}^j = 1$ one obtains the solution of (3.1),(3.2) depicted in Fig. 4. Fig. 3 depicts the corresponding concave hull of $\psi(s)$, whose points of tangency with $\psi(s)$ determine the shock propagation speeds. The two shocks shown in Fig. 4 propagate to the left and right, respectively, from the initial discontinuity. The characteristics from the left of the discontinuity intersect the leftward travelling shock, and those from the right intersect the rightward travelling shock. Between the two shocks is an expansion wave with its fan of characteristics emanating from the initial discontinuity.

The solution depicted in Fig. 4 is then sampled at a value of z at the later time to obtain the new value to be used for the interval in the piecewise constant representation of s. The sampling details and further discussion of the Riemann problem for these equations are given in (Albright, Anderson, & Concus, 1980) and (Anderson & Concus, 1980).

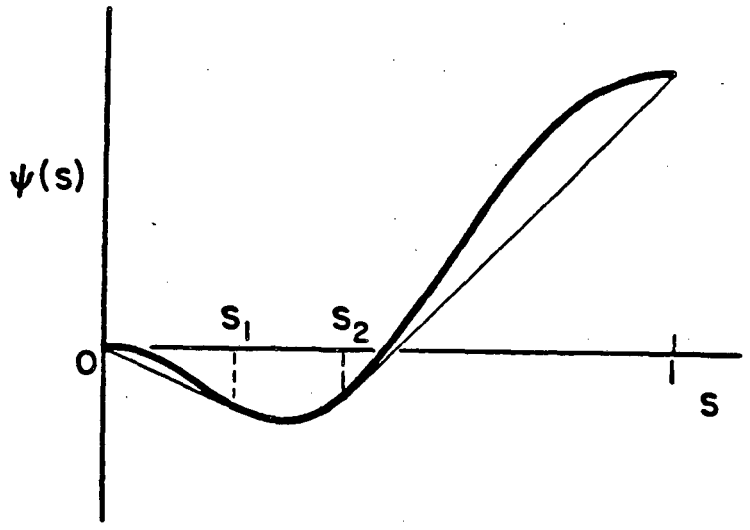

Fig. 3

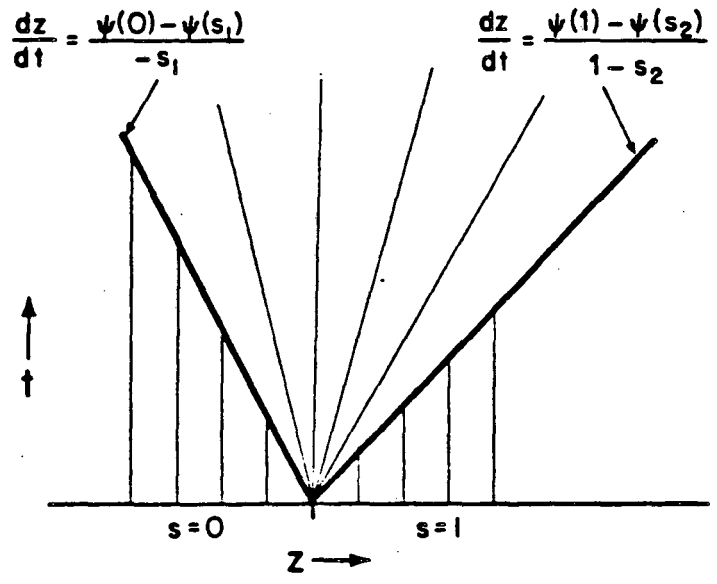

Fig. 4

5. Fractional splitting for multidimensional problems

For a one-dimensional problem, (2.1),(2.2),(2.3) reduces to the single conservation law of the form (3.1), since q is constant in the interior for this case. For a multidimensional problem the standard technique for solving (2.1),(2.2),(2.3) is to solve successively (2.2),(2.3) for p (and $\mathbf{q}$) taking s to be fixed at its approximate solution for the current time, and then to advance (2.1) one time step considering $\mathbf{q}$ fixed, to obtain an approximate solution for s at the new time.

Advancing (2.1) is carried out using the random choice method in (Albright & Concus, 1980), (Albright, Anderson, & Concus, 1980), and (Anderson & Concus, 1980) by means of fractional splitting. Specifically, one solves successively the one-dimensional problems

$$\varphi \frac{\partial s}{\partial t} + u^{(n)} f(s) = 0$$

$$\varphi \frac{\partial s}{\partial t} + w^{(n)} \frac{\partial}{\partial z} f(s) - \gamma \frac{\partial}{\partial z} g(s) = 0 ,$$

where $\mathbf{q} = (u, w)$.

Although this technique is efficient and gives acceptable results for many problems of interest, it can be inaccurate for cases in which a shock front is advancing obliquely to the splitting directions (see, for example, [Crandall & Majda, 1980]).

6. Front tracking method

As an alternative to operator splitting, a front tracking method was developed in (Lotstedt, 1981) to follow shock discontinuities in Burgers' equation and the equation for two-phase porous flow. This method is based on the method

of Noh & Woodward (1976) and Chorin (1980).

A function F_{ij} is defined whose value at the mesh cell (i,j) is the fraction of the cell that is behind the front. Thus most cells have the value one behind the front, or zero ahead of the front, with $0 < F_{ij} < 1$ along the band of cells enclosing the front. For the cells (i,j) in which $0 < F_{ij} < 1$ a line segment is drawn approximating the position of the front, based on not only F_{ij} but the fractions in the neighboring cells. At each time step the cell fractions are advanced, first in the x-direction and then in the y-direction.

In (Lotstedt, 1981) the details of the front tracking scheme are given. The line segments approximating the front within each cell are permitted to have oblique slope and need not be parallel to the mesh lines. The method is tested on the inviscid Burgers equation and porous flow equations in two space dimensions. The numerical solution is calculated for discontinuous initial data and is found to agree very well with known solutions. For the case of a physically unstable interface, which can occur for porous displacement when a more mobile fluid displaces a less mobile one, the method is able to resolve and follow the fingering of the surface as it develops.

Fig. 5 depicts successive positions of the discontinuity front as calculated by the method. This example, which is presented in (Lotstedt, 1981), is for two-phase miscible displacement with a source at (0,0) and a sink at (1,1). For this miscible displacement case, (2.1),(2.2),(2.3) hold with $f(s) = s$ and $\lambda(s) = (s+\tau(1-s))^4$, $\tau > 0$. Initially the square is occupied entirely by the fluid to be displaced, and at the boundary the normal derivative of s is zero. A uniform spatial grid with spacing 1/40 was used for the calculation. For this problem the front is physically unstable, a property that Fig. 5 depicts as being captured by the method. Other, more stabilized, front-tracking methods, such as the one in (Glimm, Isaacson, Marchesin, & McBryan, 1981), are reported to yield for this

same problem a solution that does not have observable fingering.

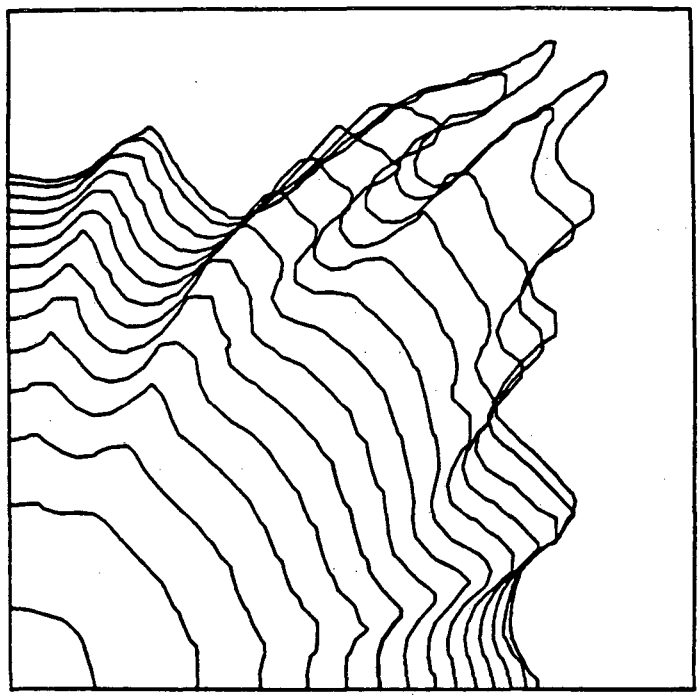

Fig. 5

7. Higher order Godunov method

Another alternative to the random choice method with splitting for multidimensional problems included in our study is that of a higher order Godunov method. Although first order Godunov methods generally are too diffusive to follow steep fronts sharply, the higher order methods introduced recently by van Leer (1979) for gas dynamics in Lagrangian coordinates appear to circumvent a large portion of this shortcoming. Godunov's method (first order) was introduced in (Godunov, 1959). It and the random choice method are similar in that they both require the solution of Riemann problems corresponding to piecewise constant initial data. Whereas the random choice method obtains the new approximation at the new time by sampling, Godunov's method obtains it by a conservative differencing scheme. Shocks are not kept sharp by it, as intermediate values are introduced. However, when Godunov's method is combined with operator splitting to solve multidimensional problems, shocks travelling obliquely to the mesh do not give rise to the same degree of difficulty as might be encountered in the random choice method. The second-order Godunov methods hold promise of giving good results for solving multidimensional porous flow problems.

For solving (3.1) with $\gamma = 0$, the condition that $\psi'(s) > 0$ simplifies Godunov's method to first order upwind differencing

$$s_{j+\frac{1}{2}}^{n+\frac{1}{2}} = s_j^n$$

$$s_j^{n+1} = s_j^n + \frac{\Delta t}{\Delta z}\left[\psi(s_{j-\frac{1}{2}}^{n+\frac{1}{2}}) - \psi(s_{j+\frac{1}{2}}^{n+\frac{1}{2}})\right].$$

The second order (in t and z) scheme obtains $s_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ by solving an initial value problem with piecewise linear rather than piecewise constant data. This gives for (3.1) with $\psi'(s) > 0$

$$s_{j+1/2}^{n+1/2} = s_j^n + \delta_j \left[\frac{1}{2} - \frac{\Delta t}{2\Delta z} \psi'(s_j^n) \right],$$

where δ_j is a centered difference approximation to $\Delta z \left.\frac{\partial s}{\partial z}\right|_j$, subject to certain constraints.

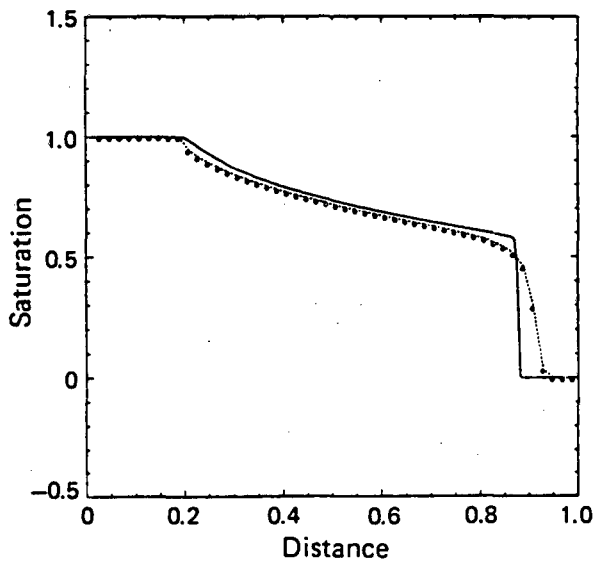

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Fig. 6

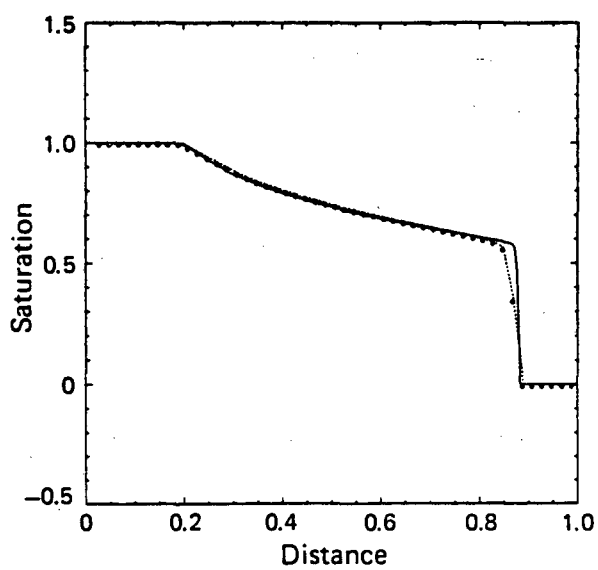

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Fig. 7

The extensions to van Leer's approach developed in (Woodward & Colella, 1980) and (Colella, 1982) are being adapted to the porous flow equations with promising results (Bell, Colella, Concus, & Glaz, 1982). A comparison between the first order and second order Godunov methods for a one dimensional case is depicted in Figs. 6 and 7. For these problems $s(z,t)=1$ for $0 \le z \le 0.2$ and $s(z,0)=0$ for $0.2 < z \le 1$. The dashed curves represent the solution of (3.1) with $\psi(s)=f(s)=s^2/(s^2+0.5(1-s)^2)$ and $\Delta z = 0.02$. The first order method is depicted in Fig. 6 and the second order one in Fig. 7. The plotting routine indicates the data points by placing circles below them, more or less tangent to the interpolating curve. The solid lines in the figures represent the solution for $\Delta z = 0.0025$ using the second order method, which is essentially the exact solution for this case (Data points are not indicated). The time step Δt was taken to be 0.1 Δz for all cases, which corresponds to a CFL number of about 0.2. Precise computer running time comparisons for optimally designed programs are not yet available, but for the experimental programs the computer running time for the second-order method was substantially less than twice that for the corresponding first-order method. The improvement in the solution using the second-order over the first-order method is easily seen for this problem.

Acknowledgments

This work was supported in part by the Director, Office of Energy Research, Office of Basic Energy Sciences, Engineering, Mathematical, and Geosciences Division of the U.S. Department of Energy under contract DE-AC03-76SF00098.

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This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

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