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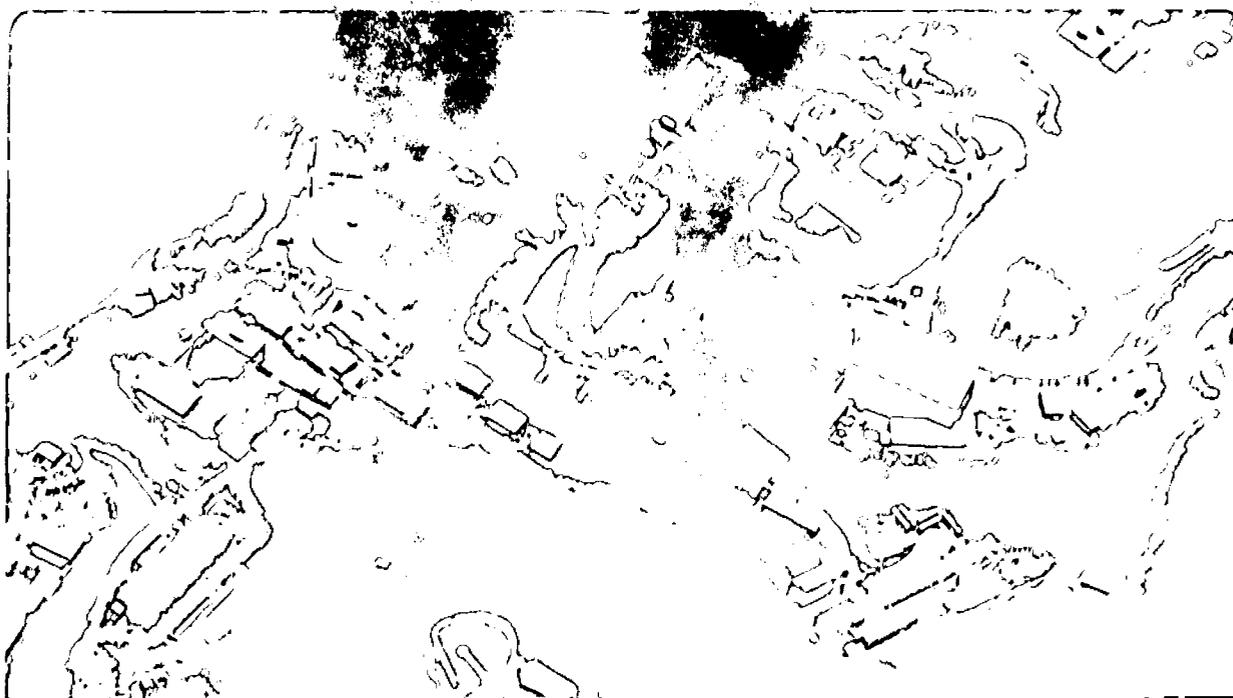
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ENTROPY OF SELF-AVOIDING SURFACES ON THE LATTICE

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Abstract

We compute the entropy of latticized, self-avoiding planar surfaces in three dimensions, by computer simulation of surface fluctuations. Our data is consistent with the formula  $N_S \sim S^\gamma e^{\epsilon S}$ , where  $N_S$  is the number of surfaces of area  $S$ , and  $\gamma = - .5 \pm .05$ ,  $\epsilon = .531 \pm .003$ . We also determine the phase structure of a field theory based on interacting surfaces, which reduces to  $Z_2$  lattice gauge theory in the "non-interacting" limit.

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It is widely believed that non-abelian gauge theories are in some way related to a quantum theory of surfaces [1]. The connection certainly seems to hold in strong-coupling lattice gauge theory, where any expectation value can be expressed in terms of a sum over latticized surfaces known as polymers [2]. This obvious relationship between gauge theories and surfaces motivates a study of the entropy of random, non-overlapping surfaces on the lattice. In this letter we will compute numerically the entropy of self-avoiding (i.e. non-overlapping and non-intersecting) surfaces, having the topology of a sphere, on a 3-dimensional lattice. These planar surfaces are the simplest type of polymer contributing to the functional integral of lattice gauge theory. It has been argued by Greensite and Bardakci [3] that the entropy of this type of surface, in  $D$  dimensions, controls whether or not QCD confines in  $D$  dimensions at weak couplings. Our calculation is for  $D = 3$  dimensions, using, with some modifications, the technique of polymer simulations discussed in ref. [4]. We have previously used this technique to do numerical simulations of compact QED with external sources [4].

We will also determine the phase structure of a field theory of interacting surfaces, whose "non-interacting" limit is  $Z_2$  lattice gauge-theory. The interaction is taken to be proportional to the length of intersections. We find that while a repulsive interaction seems to have little effect on the phase structure, an attractive interaction can induce a new "anti-frozen" phase in the theory.

Our starting point is the observation, due to Osterwalder and

Seiler [5], that the number  $\bar{N}_S(\Lambda)$  of connected lattice surfaces (polymers) of area  $S$  that include a certain set of links  $\Lambda$  (such as a Wilson loop), is bounded asymptotically by

$$\bar{N}_S(\Lambda) \leq a(\Lambda) e^{bS}, \quad (1)$$

where  $a(\Lambda)$  is a constant depending on  $\Lambda$ , and  $b$  is a constant depending only on the number of spatial dimensions  $D$ . This bound is crucial in proving confinement at strong-couplings in lattice gauge theory. If we drop the requirement that the polymers include the set of links  $\Lambda$ , the bound is presumably

$$\bar{N}_S \leq a' e^{bS} \cdot n_{\text{sites}}, \quad (2)$$

where the factor  $n_{\text{sites}}$  (= no. of lattice sites) appears because each connected polymer can be translated to  $n_{\text{sites}}$  different lattice positions. Note that a polymer is forbidden to overlap itself, i.e. pass through any plaquette more than once. Latticized surfaces without this restriction are known to grow super-exponentially [6], perhaps like  $S!$  [7].

Now consider the set of self-avoiding (i.e. non-overlapping, non-intersecting) surfaces with the topology of a sphere. Since this set of surfaces is a small subset of all possible closed polymers, it must also obey a bound of the form (2). It is therefore reasonable to assume an asymptotic behavior in the  $S \rightarrow \infty$  limit

$$N_S \sim a'' S^\gamma e^{\epsilon S} \cdot n_{\text{sites}} \quad (3)$$

where  $N_S$  is the number of self-avoiding surfaces of area  $S$  with spherical (i.e. closed and "planar") topology. Our objective is to compute the constants  $\gamma$  and  $\epsilon$ , and the strategy is to set up a statistical system associated with a fluctuating, self-avoiding surface. Let us therefore define the partition function

$$\begin{aligned} Z &= \sum_Q e^{-\beta S(Q)} \\ &= \sum_S N_S e^{-\beta S} \end{aligned} \quad (4)$$

where  $\{Q\}$  is the set of all "spherical" self-avoiding surfaces on the lattice, and  $S(Q)$  is the surface area of surface  $Q$ . Consider the expectation value of the average surface area

$$\begin{aligned} \langle S \rangle &= \frac{1}{Z} \sum_Q S(Q) e^{-\beta S(Q)} \\ &= \frac{\sum_S S N_S e^{-\beta S}}{\sum_S N_S e^{-\beta S}} \end{aligned} \quad (5)$$

For  $\beta \gg \epsilon$ ,  $\langle S \rangle \ll 1$ . As  $\beta \rightarrow \epsilon$  from above, large surfaces contribute to (5), and it makes sense to replace the sums over  $S$  by integrals. Then, from (3) and (4) we have

$$\langle S \rangle = \frac{\gamma + 1}{\beta - \epsilon} \quad (6)$$

At  $\beta_C = \epsilon$  there is a phase transition:  $\langle S \rangle \rightarrow \infty$  on an infinite lattice, while on a finite lattice  $\langle S \rangle$  will approach some substantial fraction of the total number of plaquettes at  $\beta_C = \epsilon$ . In order to compute  $\gamma$  and  $\epsilon$ , we have only to compute  $\langle S \rangle$  near the transition by Monte-Carlo

techniques, and fit the data to (6) (the best procedure is actually a straight-line fit to  $\langle S \rangle^{-1}$ ).

To set up the Monte-Carlo procedure, let us associate, with every plaquette  $p$  on the lattice, a number  $r_p = 0$  or  $1$ . Plaquettes on the surface  $Q$  have  $r_p = 1$ , all others have  $r_p = 0$ . The lattice is initialized with a single cube of 6 plaquettes excited (i.e.  $r_p = 1$ ), all other plaquettes have  $r_p = 0$ . The basic operation for modifying surfaces, the "change-a-cube" or "CC" operation, is to reset  $r_p$  on all 6 plaquettes on a give cube  $K$  by

$$r_p \rightarrow (r_p + 1) \text{ mod } 2, \quad p \in K. \quad (7)$$

This operation will take a closed surface  $Q_1$  into a closed surface  $Q_2$  by stretching or shrinking the surface over one cube. However, we must ensure that if  $Q_1$  is connected, self-avoiding, and has the topology of  $S^2$ , then  $Q_2$  also has these properties.

In the first place, we require that the CC operation locally modify an existing surface  $Q_1$ , and not just excite a disconnected cube. A necessary requirement is that at least one of the plaquettes in cube  $K$  belong to  $Q$ ; i.e. not all  $r_p = 0$  on  $K$ . Also, we don't want to completely destroy the initial surface (all  $r_p = 1$ ). So we require:

- R1) The CC operation is not applied to cube  $K$   
if all plaquettes in  $K$  have the same value  
of  $r_p$ .

With this restriction, the CC operation will modify surfaces by

stretching or shrinking them by one cube at a time, as in Fig. 1(a).

The next requirement is that the surface is self-avoiding, which is accomplished by the following restriction:

- R2) For each link  $\ell$  on the surface  $Q$ , exactly two of the four plaquettes on the lattice which contain  $\ell$  have  $r_p = 1$ .

This restriction eliminates all intersecting surfaces, since links along the lines of intersection will not satisfy the condition R2.

A CC operation is only acceptable if the resulting surface  $Q_2$  satisfies R2.

Finally, the CC operation must not change the connectedness of the surface, i.e. change a connected surface into two disconnected surfaces, or change the topology from a sphere to a torus. This is accomplished by one last restriction [see Fig. 1(b)]:

- R3) The CC operation must not be applied to cube  $K$  if two plaquettes on opposite faces of  $K$  have  $r_p = 0(1)$  while all other plaquettes have  $r_p = 1(0)$ .

To see why R3 is effective, consider the way in which a connected surface  $Q$  can be changed into two disconnected surfaces  $Q_a, Q_b$  by the CC operation acting on a single cube  $K$ . This is only possible if the two disconnected surfaces  $Q_a, Q_b$  are joined by the cube  $K$ , as shown in Fig. 1(b) and 1(c). But the operation shown in Fig. 1(b) is forbidden by R3, while the operation in Fig. 1(c) is suppressed by R2. So it is impossible to change a connected surface into two

disconnected surfaces. Similarly, if  $Q_a$ ,  $Q_b$  are connected by paths which do not run through  $K$ , the operations of Fig. 1(b) and 1(c) would create or destroy a handle. All such modifications which would change the connectedness of a surface are therefore suppressed by  $R_2$  and  $R_3$ .

The Monte-Carlo procedure is now as follows: The lattice is initialized by exciting a single cube of plaquettes. An iteration consists of sweeping through all cubes on the lattice, which in 3 dimensions are in 1-1 correspondence with the sites on the dual lattice. At each dual site, the corresponding cube is examined to see if a CC operation is allowed, according to restrictions  $R_1$ ,  $R_2$ ,  $R_3$ . If a CC operation is allowed, the standard Metropolis algorithm [with Action =  $\beta S(Q)$ ] is applied to determine whether to operate on the cube. Averaging the total surface area, we arrive at a numerical determination of  $\langle S \rangle$  for any given  $\beta$ .

In Fig. 2 we have plotted  $\langle S \rangle^{-1}$  vs.  $\beta$ , determined on a  $10 \times 10 \times 10$  lattice. Fitting this data to the inverse of eq. (6), we arrive at the values

$$\begin{aligned} \epsilon &= .531 \pm .003 \quad , \\ \gamma &= -.50 \pm .05 \quad , \end{aligned} \tag{8}$$

which determine the asymptotic behavior of the entropy of self-avoiding surfaces.

Let us now consider theories in which surfaces not only fluctuate, but can be created and destroyed, i.e. "field theories" of surfaces.

If we remove restrictions R1, R2, R3, then it is not hard to see, from standard duality arguments [8], that the monte-carlo procedure outlined above is just a numerical simulation of  $Z_2$  lattice gauge theory in the polymer formulation (see ref. [4] for monte-carlo studies of U(1) gauge theory by polymer simulation). If restriction R2 only is imposed, then the procedure simulates a field theory of self-avoiding (but not necessarily planar) surfaces.

A theory of self-avoiding surfaces can of course be viewed as a theory in which intersecting surfaces have extremely strong repulsive interactions along the length of their intersections. This suggests considering a generalized theory which interpolates between  $Z_2$  lattice gauge theory ( a "free" theory of surfaces), and the field theory of self-avoiding surfaces,

$$Z = \sum_Q e^{-[\beta_A S(Q) + \beta_I L(Q)]} \quad (9)$$

where the sum runs over all polymers generated by the unrestricted CC operation, i.e. all the polymers of  $Z_2$  lattice gauge theory.

$S(Q)$  is the surface area (no. of plaquettes) of polymer  $Q$ , while  $L(Q)$  is the length (no. of links) of intersections. Note that a link belongs to a surface intersection if all four plaquettes containing the link have  $r_p = 1$ .

One question of interest, in a theory of interacting surfaces, is whether the mutual repulsion among closed surfaces would tend to suppress large polymers, and in that way perhaps eliminate a transition between the large- and small- $\beta_A$  coupling regimes. A further question

is whether new phases are produced if the surfaces are highly attractive along their intersections.

To study the phase structure of the generalized surface theory in (9), we compute by the Monte-Carlo procedure the average surface area per plaquette

$$S_p = \langle S \rangle / N_p = \frac{1}{N_p} \frac{\partial}{\partial \beta_A} \ln Z, \quad (10)$$

which is analogous to the plaquette energy in lattice gauge theory. A discontinuity in  $S_p$  as a function of  $\beta_A$  signals a first-order phase transition, while a sudden, steep (but continuous) change is usually associated with a second-order transition.

In Fig. 3 we show the phase structure of the 3-dimensional interacting surface theory in the  $\beta_A - \beta_I$  plane, as determined by the Monte-Carlo method. It seems that the self-avoiding property ( $\beta_I \rightarrow \infty$ ) does not make much difference even to the value of  $\beta_A^{\text{critical}}$ ; so presumably large non-intersecting surfaces can still form and give rise to long-range correlations at the second-order transition point. At  $\beta_I \ll 0$  (surfaces attract) we do find a new "antifrozen" phase of some kind. The frozen ( $\beta_A \gg 0$ ), antifrozen, and disordered phases meet at a bicritical point [9], which is the juncture of two lines of second-order, and one line of first-order, phase transitions.

#### Acknowledgement

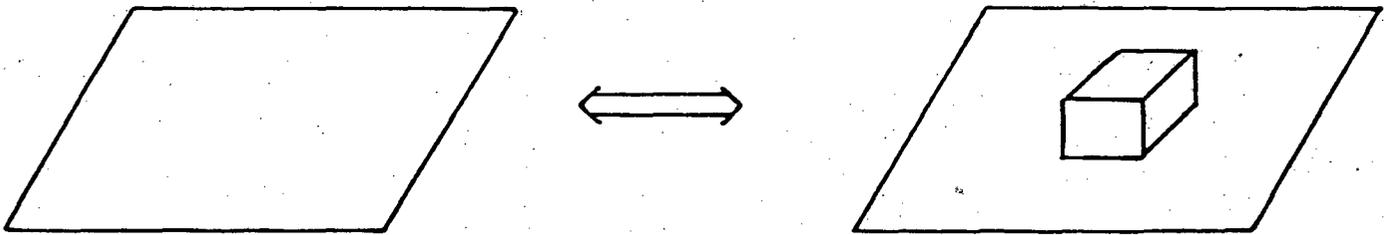
We would like to thank Veit Elser for helpful discussions.

References

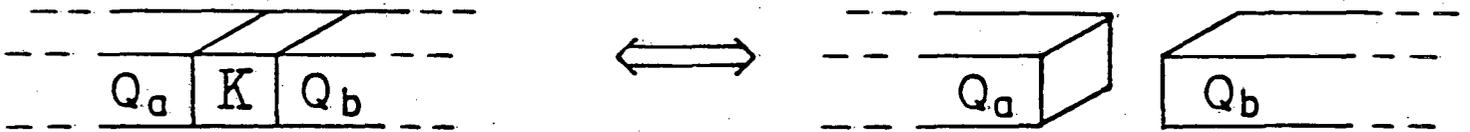
- [1] K. Wilson, Phys. Rev. D10 (1974) 2445; A.A. Migdal, Nucl Phys. B189 (1981) 253; D. Foerster, Nucl. Phys. B170 [FS1] (1980) 107; K. Bardakci and J. Greensite, Berkeley preprint UCB-PTH-82/12.
- [2] G. Münster, Nucl. Phys. B180 [FS2] (1981) 23.
- [3] K. Bardakci and J. Greensite, Berkeley preprint UCB-PTH-82/9.
- [4] T. Sterling and J. Greensite, Lawrence Berkeley Laboratory preprint LBL-14769.
- [5] K. Osterwalder and E. Seiler, Ann. Phys. 110 (1978) 440.
- [6] D. Weingarten, Phys. Lett. 90B (1980) 280.
- [7] T. Eguchi and H. Kawai, Phys. Lett. 114B (1982) 247.
- [8] R. Balian, J.M. Drouffe, and C. Itzykson, Phys. Rev. D11 (1975) 2098.
- [9] M. Fisher and D. Nelson, Phys. Rev. Lett. 32 (1974) 1350.

Figure Captions

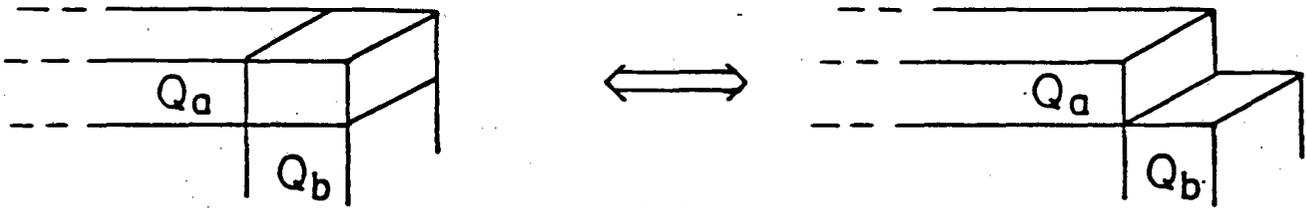
- Fig. 1 Allowed and forbidden CC operations: a) an allowed operation; b) operation suppressed by R3; c) operation suppressed by R2.
- Fig. 2 Plot of  $\langle S \rangle^{-1}$  vs.  $\beta$  near  $\beta_{\text{critical}} = .53$ . Data is taken on a  $10 \times 10 \times 10$  lattice.
- Fig. 3 Phase diagram in the  $\beta_A - \beta_I$  plane of the interacting surface theory. Solid dots are 2nd order transition points, open circles are 1st order transition points; the bi-critical point is denoted by an open triangle.



(a)



(b)



(c)

FIG. 1

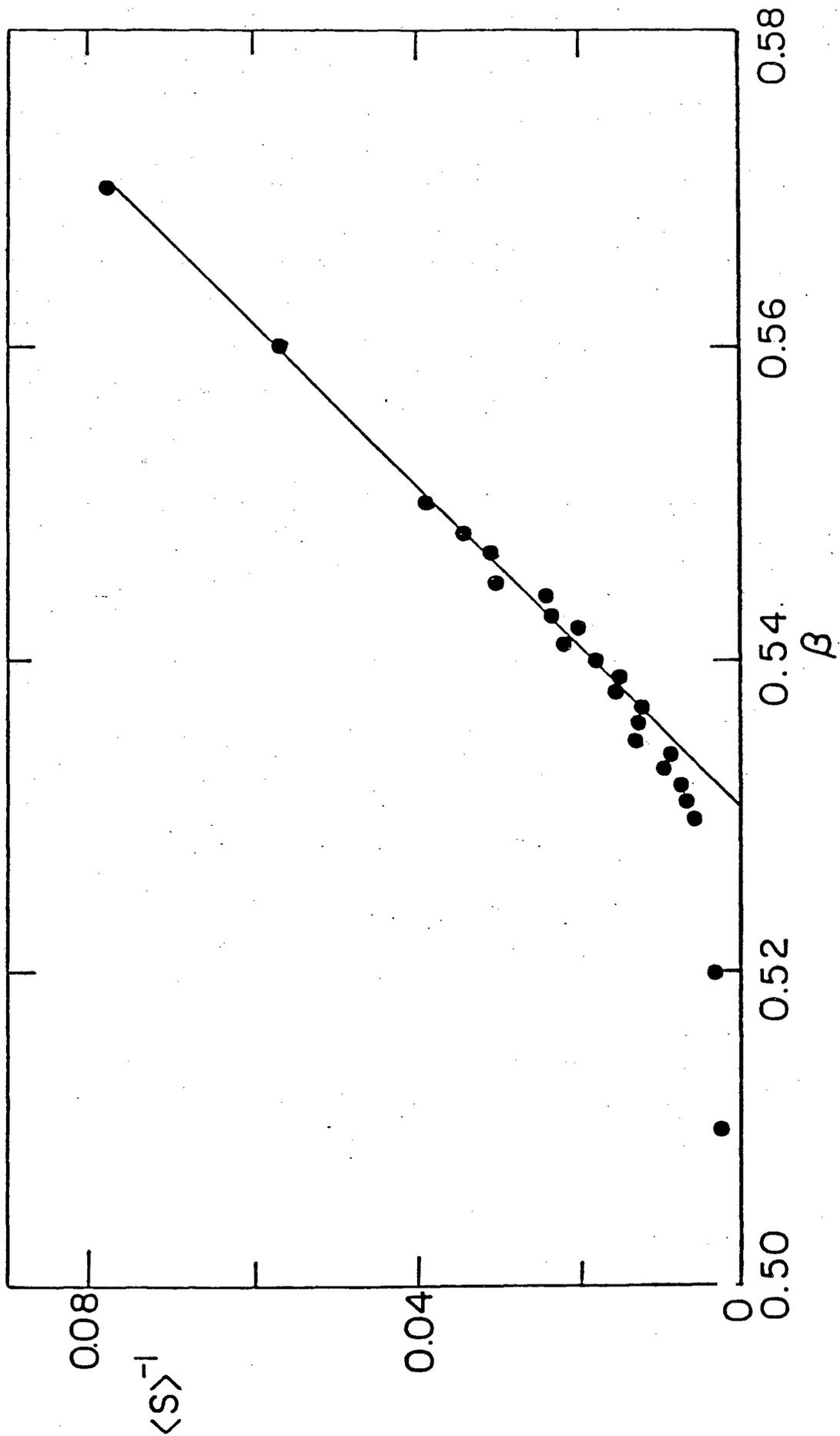


FIG. 2

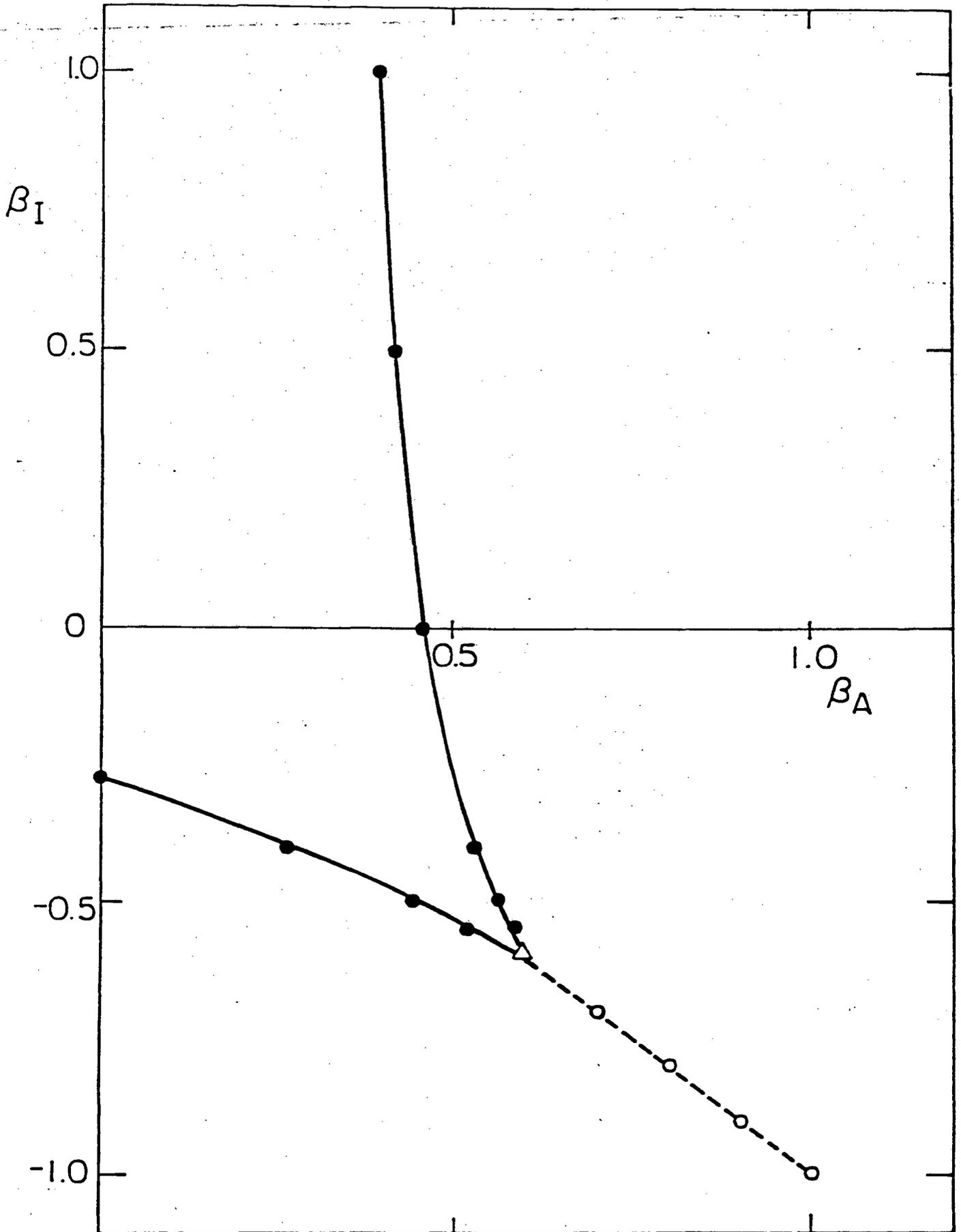


FIG. 3

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