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Quantum Theory of Nonequilibrium Processes. I *

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ABSTRACT

Green's function techniques for studying nonequilibrium quantum processes are discussed. Perturbation expansions and Green's function equations of motion are developed for noncorrelated and correlated initial states of a system. A transition, from the Kadanoff-Baym Green's function equations of motion to the Boltzmann equation, and specifications of the respective limit, are examined in detail.

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1. Introduction

Nonequilibrium Green's function techniques, initiated by Schwinger [1] and Kadanoff and Baym [2], have received in the past much attention, in particular in connection with plasma, laser, and chemical reactions problems. This study has been motivated by an application of the techniques to high-energy nuclear collisions. While some of the results obtained here have been derived before, a coherent and systematic presentation of the subject has been lacking up to this time. This paper is intended to be a self-contained introduction to the nonequilibrium Green's function techniques. Several new formal derivations and results are presented for the first time.

The nonequilibrium Green's function methods allow one to study a time evolution of a many-particle quantum system, and a particular numerical example will be presented in a following paper of the series. We solve there equations of motion with self-energies for 1-particle Green's functions in an idealized nuclear system, and compare the results with a classical Markovian dynamics from the Boltzmann equation. Knowing the 1-particle Green's functions one may evaluate 1-particle quantities in a given system. The many-particle information about the system is cast into self-energies in the Green's function equations of motion. Guided by the perturbation expansion for the Green's functions, one may attempt approximations to the self-energies.

In Sect. 2 of the present paper we introduce a generalized Green's function for a nonstationary quantum state of a system. In the case of a noncorrelated initial state of a system, the Green's function possesses a perturbation expansion analogous to a ground state perturbation expansion of a chronological Green's function. The introduced Green's function coincides, in particular ranges of variation of its arguments, with conventional 1-particle Green's functions. In Appendices A, B, and C, related to Sect. 2, we discuss

respectively the Wick decomposition, a variational derivation of the perturbation expansion, and the modified rules of the perturbation theory. In Sect. 3 the equations of motion for Green's functions, self-energies, lowest-order approximations to the self-energy, and conservation laws are discussed. In Appx. D it is shown that the self-energy for the generalized Green's function may be introduced, and its properties may be studied, without a direct reference to the perturbation expansion. In Appx. E we study, basing on the equations of motion and the self-energy perturbation expansion, the properties of Green's functions in a state of thermodynamic equilibrium. In Appx. F, a T-matrix approximation to the self-energy is discussed.

From the equations of motion for the Green's functions, the so-called Kadanoff-Baym equations, the Boltzmann equation may be obtained, at an assumption of slow time and space variations in a system. The transition to the Boltzmann equation is presented in Sect. 4. For the Boltzmann equation to be of use in describing the system's dynamics, the dynamics given by the Boltzmann equation must be insensitive to uncertainties in particle energies and momenta. In Appx. G the conditions for the transition to the Boltzmann equation are analyzed for a system in thermodynamic equilibrium in the Boltzmann statistics limit, with self-energies in the Born approximation.

In Sect. 5 of the paper, we present the Green's function techniques for a correlated initial state of a system, prepared through the imaginary-time evolution. In Appx. H, we discuss a perturbation expansion and equations of motion for Green's functions for a general correlated initial state.

2. Green's functions and the perturbation expansion

We shall consider a nonrelativistic system of fermions or bosons with a hamiltonian

$$\hat{H} = \int d\underset{\sim}{x}\ \hat{\Psi}^{\dagger}(\underset{\sim}{x}) \left(-\frac{\nabla^2}{2m}\right) \hat{\Psi}(\underset{\sim}{x}) + \frac{1}{2} \int d\underset{\sim}{x} \int d\underset{\sim}{y}\ \hat{\Psi}^{\dagger}(\underset{\sim}{x})\ \hat{\Psi}^{\dagger}(\underset{\sim}{y})\ V\ (\underset{\sim}{x} - \underset{\sim}{y})\ \hat{\Psi}(\underset{\sim}{y})\ \hat{\Psi}(\underset{\sim}{x}) \quad . \tag{2.1}$$

The field operators satisfy the commutation relations

$$\hat{\Psi}(\underset{\sim}{x})\ \hat{\Psi}^{\dagger}(\underset{\sim}{y}) \pm \hat{\Psi}^{\dagger}(\underset{\sim}{y})\ \hat{\Psi}(\underset{\sim}{x}) = \delta(\underset{\sim}{x} - \underset{\sim}{y}) \quad , \tag{2.2a}$$

$$\hat{\Psi}(\underset{\sim}{x})\ \hat{\Psi}(\underset{\sim}{y}) \pm \hat{\Psi}(\underset{\sim}{y})\ \hat{\Psi}(\underset{\sim}{x}) = 0 \quad . \tag{2.2b}$$

The upper signs refer to fermions, while the lower to bosons. Spin and isospin of particles will be ignored. The reader familiar with the ground-state Green's function methods, e.g. [3], will be able to introduce particle spin and isospin indices at any stage of the consideration.

We are interested in an evolution of a system, which is specified at an initial time t_0 with a density operator $\hat{\rho}$. A quantity in which the evolution may be studied is a 1-particle Green's function

$$\mp i\ G^{<}(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) = \langle\hat{\Psi}^{\dagger}_H(\underset{\sim}{x}_2,t_2)\hat{\Psi}_H(\underset{\sim}{x}_1,t_1)\rangle \quad . \tag{2.3}$$

The symbol $\langle\cdot\rangle$ denotes an expectation value with respect to the initial state, $\mathrm{Tr}(\hat{\rho}\cdot)/\mathrm{Tr}(\hat{\rho})$, and the field operators in (2.3) are in the Heisenberg picture. For $t_2 = t_1$, the r.h.s. of Eq. (2.3) is the 1-particle density matrix. For example, the spatial density of particles is

$$n(\underset{\sim}{x}_1,t_1) = \langle\hat{n}_H(\underset{\sim}{x}_1,t_1)\rangle = \mp iG^{<}(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_1,t_1) \quad . \tag{2.4}$$

The 1-particle density matrix, Fourier-transformed in relative variables, constitutes the so-called Wigner function

$$f(\underset{\sim}{p};\underset{\sim}{R},T) = \int d\underset{\sim}{r}\ e^{-i\underset{\sim}{p}\underset{\sim}{r}}\ \langle\hat{\Psi}^{\dagger}_H(\underset{\sim}{R} - \underset{\sim}{r}/2,T)\hat{\Psi}_H(\underset{\sim}{R} + \underset{\sim}{r}/2,T)\rangle \quad . \tag{2.5}$$

The Wigner function is an expectation value of the operator that corresponds, according to the Weyl's [4] postulate of quantum mechanics, to a classical momentum and space particle density. Let us expand the initial density operator in a basis of momentum and energy eigenstates $\{\Psi_\lambda\}$

$$\hat{\rho} = \sum_{\lambda\lambda'} \rho_{\lambda\lambda'} |\Psi_\lambda\rangle\langle\Psi_{\lambda'}| \quad . \tag{2.6}$$

Upon introduction of (2.6) into (2.5) and insertion of the unity-operator expansion in between the field operators in (2.5), it can be shown that

$$f(\underset{\sim}{p};\underset{\sim}{R},T) = \frac{1}{Tr(\hat{\rho})} \sum_{\lambda\lambda'\eta} \rho_{\lambda\lambda'} (2\pi)^3 \delta(\underset{\sim}{p} - ((\underset{\sim}{P}_\lambda + \underset{\sim}{P}_{\lambda'})/2 - \underset{\sim}{P}_\eta))$$

$$\times \langle\Psi_{\lambda'}|\hat{\Psi}^\dagger_H(\underset{\sim}{R},T)|\Psi_\eta\rangle\langle\Psi_\eta|\hat{\Psi}_H(\underset{\sim}{R},T)|\Psi_\lambda\rangle \quad . \tag{2.7}$$

The Green's function (2.3), Fourier-transformed in all its relative (microscopic) variables $\underset{\sim}{r} = \underset{\sim}{x}_1 - \underset{\sim}{x}_2$, $t = t_1 - t_2$, at fixed macroscopic variables $\underset{\sim}{R} = (\underset{\sim}{x}_1 + \underset{\sim}{x}_2)/2$, $T = (t_1 + t_2)/2$, may be considered a generalization of the Wigner function to a distribution not only in momentum and space but also in energy

$$\begin{aligned}
\mp iG^<(\underset{\sim}{p},\omega;\underset{\sim}{R},T) &= \int d\underset{\sim}{r} \int dt\, e^{-i\underset{\sim}{p}\underset{\sim}{r}+i\omega t} (\mp i)G^<(\underset{\sim}{r},t;\underset{\sim}{R},T) \\
&= \frac{1}{Tr(\hat{\rho})} \sum_{\lambda\lambda'\eta} \rho_{\lambda\lambda'} (2\pi)^3 \delta(\underset{\sim}{p} - ((\underset{\sim}{P}_\lambda + \underset{\sim}{P}_{\lambda'})/2 - \underset{\sim}{P}_\eta)) \\
&\times 2\pi\delta(\omega - ((E_\lambda + E_{\lambda'})/2 - E_\eta)) \\
&\times \langle\Psi_{\lambda'}|\hat{\Psi}^\dagger_H(\underset{\sim}{R},T)|\Psi_\eta\rangle\langle\Psi_\eta|\hat{\Psi}_H(\underset{\sim}{R},T)|\Psi_\lambda\rangle \quad .
\end{aligned} \tag{2.8}$$

Here

$$G^<(\underset{\sim}{r},t;\underset{\sim}{R},T) \equiv G^<(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) \quad ,$$

and such a simplified notation will be frequently employed in the future. We

have

$$f(\underset{\sim}{p};\underset{\sim}{R},T) = \int \frac{d\omega}{2\pi} (\mp i)\, G^{<}(\underset{\sim}{p},\omega;\underset{\sim}{R},T) \quad . \tag{2.9}$$

The Green's function

$$iG^{>}(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) = \langle \hat{\Psi}_H(\underset{\sim}{x}_1,t_1)\hat{\Psi}^{\dagger}_H(\underset{\sim}{x}_2,t_2)\rangle \tag{2.10}$$

may be seen as corresponding to a density for an addition of a particle to a system (a density of holes)

$$\begin{aligned} iG^{>}(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = \frac{1}{Tr(\hat{\rho})} \sum_{\lambda\lambda 1} & \rho_{\lambda\lambda'}\, (2\pi)^3\, \delta(\underset{\sim}{p} - (\underset{\sim}{P}_1 - (\underset{\sim}{P}_\lambda + \underset{\sim}{P}_{\lambda'})/2)) \\ & \times 2\pi\, \delta(\omega - (E_1 - (E_\lambda + E_{\lambda'})/2)) \\ & \times \langle \Psi_{\lambda'}|\hat{\Psi}_H(\underset{\sim}{R},T)|\Psi_1\rangle\langle\Psi_1|\hat{\Psi}^{\dagger}_H(\underset{\sim}{R},T)|\Psi_\lambda\rangle \quad . \end{aligned} \tag{2.11}$$

From the commutation relations it follows that

$$i(G^{>} - G^{<})(\underset{\sim}{x}_1,t,\underset{\sim}{x}_2,t) = \delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) \quad , \tag{2.12}$$

and consequently

$$\int \frac{d\omega}{2\pi}\, iG^{>}(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = 1 \mp f(\underset{\sim}{p};\underset{\sim}{R},T) \quad . \tag{2.13}$$

The consideration of the Wigner function and the Fourier-transformed Green's functions (2.8) and (2.11) as densities must be done with care because the functions are generally not positive definite. They are, however, always real.

When working with a ground state of a system, one usually deals with a chronological Green's function

$$iG^{c}(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) = \langle T^{c}[\hat{\Psi}_H(\underset{\sim}{x}_1,t_1)\hat{\Psi}^{\dagger}_H(\underset{\sim}{x}_2,t_2)]\rangle \quad , \tag{2.14}$$

where T^c is the chronological-ordering operator. There holds

$$\begin{aligned} G^{c}(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) = {} & \theta(t_1 - t_2)\, G^{>}(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) \\ & + \theta(t_2 - t_1)\, G^{<}(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) \end{aligned} \tag{2.15}$$

with

$$\theta(t) = \begin{cases} 1 \quad , & \text{for } t > 0 \quad , \\ 0 \quad , & \text{for } t < 0 \quad , \end{cases}$$

and the 1-particle density matrix may be obtained from (2.14) in the limit $t_2 = t_1^+ = t_1 + \varepsilon$. The hermitian conjugate of (2.14) yields the antichronological Green's function

$$iG^a(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) = \langle T^a[\hat{\Psi}_H(\underset{\sim}{x}_1,t_1)\hat{\Psi}_H^\dagger(\underset{\sim}{x}_2,t_2)]\rangle \quad , \tag{2.16}$$

$$G^a(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) = \theta(t_1 - t_2)G^<(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) + \theta(t_2 - t_1)G^>(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) \tag{2.17}$$

The spectrally decomposed chronological and antichronological Green's functions possess propagator forms.

We shall now consider an expectation value of an operator with one time argument $\langle\hat{O}_H(t)\rangle$. As a consequence of that consideration we shall introduce, for a nonstationary state of a system, a Green's function possessing a perturbation expansion analogous to the ground-state chronological Green's function perturbation-expansion. In particular ranges of variation of its arguments, the introduced Green's function will coincide with the Green's functions (2.3),(2.10),(2.14), and (2.16).

We have

$$\hat{O}_H(t) = \hat{U}(t_o,t)\ \hat{O}_I(t)\ \hat{U}(t,t_o) \quad , \tag{2.18}$$

where $\hat{O}_I$ is in the interaction picture and $\hat{U}$ is the interaction picture evolution operator. For $t > t_o$

$$\hat{U}(t,t_o) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} T^c\left[\int_{t_o}^{t} dt_1 \ldots \int_{t_o}^{t} dt_n\ \hat{H}_I^1(t_1) \ldots \hat{H}_I^1(t_n)\right]$$

$$= T^c \left[\exp\left(-i \int_{t_0}^{t} dt' \, \hat{H}_I^1 (t')\right)\right] \quad , \tag{2.19}$$

with $\hat{H}_I(t)$ the interaction hamiltonian in the interaction picture. For $t > t_0$

$$\hat{U}(t_0,t) = T^a \left[\exp\left(-i \int_{t}^{t_0} dt' \, \hat{H}_I^1 (t')\right)\right] \quad . \tag{2.20}$$

Let us see how one obtains the conventional Feynmann diagrams for a ground state of a system. The Heisenberg and the interaction pictures coincide in that case at time $t = 0$. The operator $\hat{H}_I(t)$ gets a factor $\exp(-\varepsilon|t|)$, which switches the interaction on and off at $t = \pm\infty$. The noninteracting ground state $|\Phi\rangle$ is assigned to the system at $t = -\infty$, and the interacting ground state is obtained on the basis of the Gell-Mann and Low theorem: $|\Psi\rangle = U(0,-\infty)|\Phi\rangle$. For the expectation value of an operator we have

$$\begin{aligned}
\langle\Psi|\hat{O}_H(t)|\Psi\rangle &= \langle\Phi|\hat{U}(-\infty,0)\hat{O}_H(t)\hat{U}(0,-\infty)|\Phi\rangle \\
&= \langle\Phi|\hat{U}(-\infty,0)\hat{U}(0,t)\hat{O}_I(t)\hat{U}(t,0)\hat{U}(0,-\infty)|\Phi\rangle \\
&= \langle\Phi|\hat{U}(-\infty,t)\hat{O}_I(t)\hat{U}(t,-\infty)|\Phi\rangle \\
&= \langle\Phi|\hat{U}(-\infty,+\infty)\hat{U}(+\infty,t)\hat{O}_I(t)\hat{U}(t,-\infty)|\Phi\rangle \quad ,
\end{aligned} \tag{2.21}$$

where in the last equalities we exploit a group property of the $\hat{U}$ operators. For a nondegenerate state $|\Psi\rangle$, the state $U(+\infty,-\infty)|\Phi\rangle$ is up to an (infinite) phase factor equal to $|\Phi\rangle$, and into the last expression of (2.21) one can insert a projection operator on $|\Phi\rangle$

$$\begin{aligned}
&\langle\Psi|\hat{O}_H(t)|\Psi\rangle \\
&= \langle\Phi|\hat{U}(-\infty,+\infty)|\Phi\rangle\langle\Phi|\hat{U}(+\infty,t)\hat{O}_I(t)\hat{U}(t,-\infty)|\Phi\rangle
\end{aligned}$$

$$= \frac{\langle\Phi|\hat{U}(+\infty,-\infty,t)\hat{O}_I(t)\hat{U}(t,-\infty)|\Phi\rangle}{\langle\Phi|\hat{U}(+\infty,-\infty)|\Phi\rangle}$$

$$= \frac{\langle\Phi|T^c[\exp(-i\int_{-\infty}^{\infty}dt'\hat{H}_I^1(t'))\hat{O}_I(t)]|\Phi\rangle}{\langle\Phi|T^c[\exp(-i\int_{-\infty}^{\infty}dt'\hat{H}_I^1(t'))]|\Phi\rangle} \tag{2.22}$$

On obtaining the second equality we exploit the fact that

$$1 = \langle\Phi|\Phi\rangle = \langle\Phi|\hat{U}(-\infty,+\infty)\hat{U}(+\infty,-\infty)|\Phi\rangle$$
$$= \langle\Phi|\hat{U}(-\infty,+\infty)|\Phi\rangle\langle\Phi|\hat{U}(+\infty,-\infty)|\Phi\rangle \quad , \tag{2.23}$$

and we obtain the last equality in (2.22) by introducing the expansion (2.19). For the chronological Green's function we have in analogy to (2.22)

$$iG^c(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) = \langle\Phi|T^c[\hat{\Psi}_H(\underset{\sim}{x}_1,t_1)\hat{\Psi}_H^\dagger(\underset{\sim}{x}_2,t_2)]|\Phi\rangle$$

$$= \frac{\langle\Phi|T^c[\exp(-i\int_{-\infty}^{\infty}dt'\hat{H}_I^1(t'))\hat{\Psi}_I(x_1,t_1)\hat{\Psi}_I^\dagger(x_2,t_2)]|\Phi\rangle}{\langle\Phi|T^c[\exp(-i\int_{-\infty}^{\infty}dt'\hat{H}_I^1(t'))]|\Phi\rangle} \quad . \tag{2.24}$$

Upon application of the Wick decomposition to (2.24) (also to (2.22)), one obtains the usual Feynman rules: the denominator cancels the disconnected diagrams.

The above scheme, however, cannot be aplied to the nonstationary state expectation values. The basic reason for that is the fact that, in general, within the evolution, no state of a system in future may be identified with any of states in the past.

Let us return to the expectation value of an operator with respect to a state specificed at t_o,

$$\langle\hat{O}_H(t)\rangle = \langle\hat{U}(t_o,t)\hat{O}_I(t)\hat{U}(t,t_o)\rangle$$

$$= \langle T^a[\exp(-i \int_t^{t_o} dt' \hat{H}_I^1(t'))]\hat{O}_I(t) T^c[\exp(-i \int_{t_o}^{t} dt' \hat{H}_I^1(t'))]\rangle \quad . \tag{2.25}$$

The perturbative evaluation of (2.25) may be put in a form analogous to the usual Feynman diagrams, when one joins the exponential functions from the left and right of the $\hat{O}$-operator, and one introduces a time-ordering operator T that recognizes whether the field operators belong to the chronological or antichronological parts of the product. Accordingly we introduce a contour running along the time axis, Fig. 1, and a T operator ordering along the contour. (In conection with future applications, the contour may be imagined as lying in the complex time plane.) We assign the time arguments of the field operators to the contour. The T operator, reduced to the part of the contour running forward or backward in time, will become the chronological or antichronological ordering operator, respectively. The parts of the contour will be named the chronological and antichronological branches, respectively. The T operator will order all operators, from the antichronological branch, to the left of operators from the chronological branch. We can rewrite Eq. (2.25) in the form

$$\langle \hat{O}_H(t)\rangle = \langle T[\exp(-i \oint_{t_o}^{t_o} dt' \hat{H}_I^1(t'))\hat{O}_I(t)]\rangle \quad , \tag{2.26}$$

where $\oint_{t_o}^{t_o}$ stands for the integral along the contour, further denoted by $\oint$. By inserting extra $\hat{U}$ operators into (2.25), one may elongate the contour, so that it would run beyond the time t (one may also deform the contour).

We define a Green's function on the contour, i.e., with the time arguments from the contour,

$$iG(\underset{\sim}{x}_1, t_1, \underset{\sim}{x}_2, t_2) = \langle T[\hat{\Psi}_H(\underset{\sim}{x}_1, t_1)\hat{\Psi}_H^\dagger(\underset{\sim}{x}_2, t_2)]\rangle \quad , \tag{2.27}$$

and we have

$$iG(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) = \langle T[\exp(-i \oint dt' \hat{H}_I^1(t'))\hat{\Psi}_I(\underset{\sim}{x}_1,t_1)\hat{\Psi}_I^\dagger(\underset{\sim}{x}_2,t_2)]\rangle \ , \quad (2.28)$$

with the contour running above the largest argument of the Green's function. With Eq. (2.28), the Green's function will possess a perturbation expansion analogous to the ground-state expansion. The Green's function (2.27) equals

$$G(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2)$$

$$= \theta(t_1,t_2)G^>(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) + \theta(t_2,t_1)G^<(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) \ , \quad (2.29)$$

with the function $\theta(t_1,t_2)$ defined on a contour:

$$\theta(t_1,t_2) = \begin{cases} 1, \text{ if } t_1 \text{ is later on a contour than } t_2 \ , \\ 0, \text{ if earlier} \ . \end{cases}$$

On restricting the variation of the arguments of the introduced generalized Green's function to the separate branches of the contour, one gets the conventional Green's functions: chronological, antichronological, and the functions with a fixed order of $\hat{\Psi}$ and $\hat{\Psi}^\dagger$, $G^<$ and $G^>$.

About the initial state specified at t_0, we assume that its density operator commutes with the particle-number operator. Furthermore, we assume that the initial state admits the Wick decomposition (is noncorrelated). The density operators of such states are generally of the form $\hat{\rho} = \exp(\hat{\mathcal{A}})$, with $\hat{\mathcal{A}}$ being a 1-particle operator. The vacuum and for fermions the Hartree-Fock states, arising from applications to the vacuum of sets of 1-particle creation operators, correspond to the limiting cases of such density operators. The Wick decomposition is discussed in Appx. A.

The Feynman rules, which result from an application of the Wick decomposition to (2.28), are similar to the conventional ground-state Feynman rules. The difference is such that all the time integrations do not run from $-\infty$ to $+\infty$, but along the contour. The top of the contour must be above or

equal to the largest time argument of the evaluated Green's function. More precisely, within a chosen part of a diagram, the internal time integrations must run to the largest external time in that part of a diagram. The disconnected diagrams vanish, because there the integration contours may be reduced to t_0. (1 + disconnected diagrams) is an expansion of

$$1 = \langle 1 \rangle = \langle T\,[\exp(-i \int dt' \hat{H}_I^1(t'))] \rangle \quad .$$

When the kinetic energy operator is taken as a 1-particle hamiltonian defining the interaction picture, then the Feynman rules for evaluating $iG(\underset{\sim}{x}_1, t_1, \underset{\sim}{x}_2, t_2)$ are the following:

1. Draw all topologically distinct connected and directed diagrams. Particle lines run continuously; one sequence of lines runs from $(\underset{\sim}{x}_2, t_2)$ to $(\underset{\sim}{x}_1, t_1)$.
2. A particle line running from $(\underset{\sim}{x}', t')$ to $(\underset{\sim}{x}, t)$ represents a noninteracting Green's function

$$iG^0(\underset{\sim}{x}, t, \underset{\sim}{x}', t') = \langle T\,[\hat{\Psi}_I(\underset{\sim}{x}, t)\hat{\Psi}_I^\dagger(\underset{\sim}{x}', t')] \rangle \tag{2.30}$$

3. To an interaction line there corresponds a factor $-iV(\underset{\sim}{x} - \underset{\sim}{x}')\,\delta(t,t')$. The function $\delta(t,t')$ is defined on a contour: it is equal to $\delta(t - t')$ on the chronological branch and to $-\delta(t - t')$ on the antichronological branch.
4. To a single particle line that forms a closed loop or is linked by the same interaction line, there corresponds a function $iG^{0<}$.
5. For fermions attribute to the diagram a factor $(-1)^F$, where F is the number of particle loops.
6. Integrate all internal vertices over a whole space and in time over a directed contour from t_0 to t_0.

One of the possible variational derivations of Feynman diagrams is discussed in Appx. B.

When evaluating a particular type of a Green's function $iG^{\gtrless}$, iG^{a}, or iG^{c}, it may be convenient to divide the contour into the two branches in the perturbation theory rules. The resulting rules are presented in Appx. C.

3. Green's function equations of motion

3.1 Equations of motion and the self-energies

Using the perturbation expansion, one can define the self-energy Σ as an irreducible part of the Green's function. In Appx. D we introduce the self-energy in a formal manner and analyze its properties without direct reference to the diagrams. The self-energy may also in principle be introduced variationally. The Green's function satisfies equations of motion with self-energy

$$\left(i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m}\right) G(1,1') = \delta(1,1') + \int d2\, \Sigma(1,2)G(2,1') \quad , \tag{3.1}$$

$$\left(-i \frac{\partial}{\partial t_{1'}} + \frac{\nabla_{1'}^2}{2m}\right) G(1,1') = \delta(1,1') + \int d2\, G(1,2)\Sigma(2,1') \quad , \tag{3.2}$$

which correspond to the Dyson equations

$$G(1,1') = G^0(1,1') + \int d2 \int d3\, G^0(1,2)\Sigma(2,3)G(3,1') \quad , \tag{3.3}$$

$$G(1,1') = G^0(1,1') + \int d2 \int d3\, G(1,2)\Sigma(2,3)G^0(3,1') \quad , \tag{3.4}$$

We use here the notations $1 \equiv (\underset{\sim}{x}_1, t_1)$, $\int d1 \equiv \int dt_1 \int d\underset{\sim}{x}_1$, $\delta(1,1') = \delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_{1'})\delta(t_1, t_{1'})$. The self-energy has a form analogous to (2.29)

$$\Sigma(1,2) = \Sigma^\delta(1,2) + \theta(t_1,t_2)\Sigma^>(1,2) + \theta(t_2,t_1)\Sigma^<(1,2) \quad , \tag{3.5}$$

with Σ^δ being a singular part of Σ on the contour.

On fixing the time arguments of the Green's function in Eqs. (3.1) and (3.2) at opposite sides of the contour, one finds, with (2.29) and (3.5), the equations

$$\left(i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m}\right) G^{\gtrless}(1,1')$$

$$= \int d\underset{\sim}{x}_2 \, \Sigma_{HF}(\underset{\sim}{x}_1, \underset{\sim}{x}_2 ; t_1) \, G^{\gtrless}(\underset{\sim}{x}_2, t_1, 1')$$

$$+ \int_{t_0}^{t_1} d2 \, (\Sigma^{>}(1,2) - \Sigma^{<}(1,2)) G^{\gtrless}(2,1')$$

$$- \int_{t_0}^{t_{1'}} d2 \, \Sigma^{\gtrless}(1,2)(G^{>}(2,1') - G^{<}(2,1')) \quad , \tag{3.6}$$

$$\left(-i \frac{\partial}{\partial t_{1'}} + \frac{\nabla_{1'}^2}{2m}\right) G^{\gtrless}(1,1')$$

$$= \int d\underset{\sim}{x}_2 \, G^{\gtrless}(1, \underset{\sim}{x}_2, t_{1'}) \Sigma_{HF}(\underset{\sim}{x}_2, \underset{\sim}{x}_1, t_{1'})$$

$$+ \int_{t_0}^{t_1} d2 \, (G^{>}(1,2) - G^{<}(1,2)) \Sigma^{\gtrless}(2,1')$$

$$- \int_{t_0}^{t_{1'}} d2 \, G^{\gtrless}(1,2)(\Sigma^{>}(2,1') - \Sigma^{<}(2,1')) \quad . \tag{3.7}$$

The function Σ_{HF}, exhibited here, corresponds to the singular part of the self-energy, which may in principle be found diagramatically. The time integrations in (3.7) run along the time axis and the limits are explicitly indicated. Equations (3.7) are actually the hermitian conjugates of Eqs. (3.6). Equations (3.6) and (3.7) are known as the Kadanoff-Baym equations.

With the use of the advanced and retarded functions (see Appx. D)

$$F^{\pm}(1,2) = F^{\delta}(1,2) \pm \theta(\pm(t_1 - t_2))(F^{>}(1.2) - F^{<}(1,2)) \quad , \tag{3.8}$$

equations (3.6) and (3.7) may be written as

$$\left(i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m}\right) G^{\gtrless}(1,1')$$

$$= \int_{t_0}^{\infty} d2\, \Sigma^{+}(1,2)\, G^{\gtrless}(2,1') + \int_{t_0}^{\infty} d2\, \Sigma^{\gtrless}(1,2) G^{-}(2,1') \quad , \tag{3.9}$$

$$\left(- i \frac{\partial}{\partial t_{1'}} + \frac{\nabla_{1'}^2}{2m}\right) G^{\gtrless}(1,1')$$

$$= \int_{t_0}^{\infty} d2\, G^{\gtrless}(1,2)\, \Sigma^{-}(2,1') + \int_{t_0}^{\infty} d2\, G^{+}(1,2) \Sigma^{\gtrless}(2,1') \quad . \tag{3.10}$$

On subtracting Eq. (3.9) for $G^{<}$ from Eq. (3.9) for $G^{>}$, one finds the equation satisfied by the retarded and advanced functions.

$$\left(i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m}\right) G^{\pm}(1,1') - \int_{t_0}^{\infty} d2\, \Sigma^{\pm}(1,2) G^{\pm}(2,1') = \delta(1 - 1') \tag{3.11}$$

Analogously, from Eqs. (3.10) one finds

$$\left(-i \frac{\partial}{\partial t_{1'}} + \frac{\nabla_{1'}^2}{2m}\right) G^{\pm}(1,1') - \int_{t_0}^{\infty} d2\, G^{\pm}(1,2) \Sigma^{\pm}(2,1') = \delta(1 - 1') \tag{3.12}$$

With Eqs. (3.11) and (3.12), a general solution of Eqs. (3.9) and (3.10) may be written as

$$G^{\gtrless}(1,1') = \int_{t_0}^{\infty} d2 \int_{t_0}^{\infty} d3\, G^{+}(1,2)\, \Sigma^{\gtrless}(2,3)\, G^{-}(3,1')$$

$$+ \int d\underset{\sim}{x}_2 \int d\underset{\sim}{x}_3\, G^{+}(1,\underset{\sim}{x}_2,t_0)\, G^{\gtrless}(\underset{\sim}{x}_2,t_0,\underset{\sim}{x}_3,t_0)\, G^{-}(\underset{\sim}{x}_3,t_0,1') \quad , \tag{3.13}$$

where the second term accounts for the initial conditions. Equation (3.13) may be considered a generalized fluctuation-dissipation theorem, as will be seen below.

Let us consider the field-operator equation of motion

$$i \frac{\partial}{\partial t_1} \hat{\Psi}_H(1) = [\hat{\Psi}_H(1), \hat{H}] \quad . \tag{3.14}$$

On evaluating the commutator one finds

$$\left(i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} \right) \hat{\Psi}_H(1) = \int d\underset{\sim}{x}_2 \; V(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) \hat{\Psi}_H^\dagger(\underset{\sim}{x}_2, t_1) \hat{\Psi}_H(\underset{\sim}{x}_2, t_1) \hat{\Psi}_H(1)$$

$$\equiv \hat{j}_H(1) \quad . \tag{3.15}$$

On taking the side-by-side time-ordered product of Eq. (3.15) with its hermitian conjugate, making use of Eqs. (3.1),(3.2), and the definition of the Green's function, one is able to show that

$$\langle T[\hat{j}_H(1) \; \hat{j}_H^\dagger(1')] \rangle + i \; \delta(t_1, t_{1'}) \langle [\hat{j}_H(1), \hat{\Psi}_H^\dagger(1')]_\pm \rangle$$

$$= i \Sigma(1,1') + i \int d2 \int d3 \; \Sigma(1,2) \; G(2,3) \; \Sigma(3,1') \quad , \tag{3.16}$$

where $[\cdot\,,\cdot]_\pm$ stands for the anticommutator in the fermion case and the commutator in the boson case. From Eq. (3.16) we can identify the divergent part of the self-energy

$$\Sigma^\delta(1,1') = \delta(t_1, t_{1'}) \langle [\hat{j}_H(1), \hat{\Psi}_H^\dagger(\underset{\sim}{x}_{1'}, t_1)]_\pm \rangle$$

$$= \delta(t_1, t_{1'}) \langle [\hat{\Psi}_H(1), \hat{j}_H^\dagger(\underset{\sim}{x}_{1'}, t_1)]_\pm \rangle \quad , \tag{3.17}$$

where the last equality follows from the differentiation of the equal-time field-operator commutation relation. On inserting the value of the current $\hat{j}_H$ into (3.17), we find

$$\Sigma^{\delta}(1,1') = \delta(t_1,t_{1'}) \Big[\delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_{1'}) \int d\underset{\sim}{x}_2\, V(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)$$

$$\times \langle\hat{\psi}^{\dagger}_H(\underset{\sim}{x}_2,t_1)\hat{\psi}_H(\underset{\sim}{x}_2,t_1)\rangle \mp V(\underset{\sim}{x}_1 - \underset{\sim}{x}_{1'})$$

$$\times \langle\hat{\psi}^{\dagger}_H(\underset{\sim}{x}_{1'},t_1)\,\hat{\psi}_H(\underset{\sim}{x}_1,t_1)\rangle\Big]$$

$$\equiv \delta(t_1,t_{1'})\,\Sigma_{HF}(\underset{\sim}{x}_1,\underset{\sim}{x}_{1'};t_1) \quad . \tag{3.18}$$

The self-energy (3.18) is the Hartree-Fock self-energy, with the first term being a direct (Hartree) term and the second the exchange term. For the self-energies $\Sigma^{\gtrless}$, it follows from (3.16) that these are the irreducible parts of the current-product expectation values

$$\mp i\,\Sigma^{<}(1,1') = \langle\hat{j}^{\dagger}_H(1')\;\hat{j}_H(1)\rangle_{irred} \quad , \tag{3.19}$$

$$i\,\Sigma^{>}(1,1') = \langle\hat{j}_H(1)\;\hat{j}^{\dagger}_H(1')\rangle_{irred} \quad . \tag{3.20}$$

Irreducibility means here that one excludes from the expectation values those diagrams that can be cut in between the end-points in such a way that the cut passes only through a single particle line. If we define the expectation values in the presence of an external current J coupled to the field operators

$$\langle\hat{O}_H\rangle_J = \frac{\langle T[\hat{O}_H\hat{S}_H]\rangle}{\langle T[\hat{S}_H]\rangle} \quad , \tag{3.21}$$

with

$$\hat{S}_H = \exp\,(-i \int d1(\hat{\psi}^{\dagger}_H(2)J(2) + J^{*}(2)\hat{\psi}_H(2)) \quad , \tag{3.22}$$

and J being a Grassman current in the fermion case, then

$$G^{+}(1,1') = \left(\frac{\delta}{\delta J(1')}\,\langle\hat{\psi}_H(1)\rangle_J\right)_{J=0} \quad , \tag{3.23}$$

and

$$G^{-}(1,1') = \left(\frac{\delta}{\delta J^{*}(1)}\,\langle\hat{\psi}^{\dagger}_H(1')\rangle_J\right)_{J=0} \quad . \tag{3.24}$$

The variational derivatives in (3.23) and (3.24) are carried out as if the current were the same or both branches of the contour. With Eqs. (3.23),(3.24), and (3.19), we can rewrite Eq. (3.13) for $G^<$ (similarly for $G^>$) into the form

$$\langle\hat{\psi}_H^\dagger(1')\hat{\psi}_H(1)\rangle$$

$$= \int_{t_0}^{\infty} d2 \int_{t_0}^{\infty} d3 \left.\frac{\delta\langle\hat{\psi}_H^\dagger(1')\rangle_J}{\delta J^*(2)}\right|_{J=0} \langle\hat{j}_H^\dagger(2)\hat{j}_H(3)\rangle_{irred} \left.\frac{\delta\langle\hat{\psi}_H(1)\rangle_J}{\delta J(3)}\right|_{J=0}$$

$$+ \int d\underset{\sim}{x}_2 \int d\underset{\sim}{x}_3 \left.\frac{\delta\langle\hat{\psi}_H^\dagger(1')\rangle_J}{\delta J^*(\underset{\sim}{x}_2,t_0)}\right|_{J=0} \langle\hat{\psi}^\dagger(\underset{\sim}{x}_2,t_0)\hat{\psi}(\underset{\sim}{x}_3,t_0)\rangle \left.\frac{\delta\langle\hat{\psi}_H(1)\rangle_J}{\delta J(\underset{\sim}{x}_3,t_0)}\right|_{J=0} . \quad (3.25)$$

With $\hat{j}_H$ and $\hat{j}_H^\dagger$ being the sources of the fields $\hat{\psi}_H$ and $\hat{\psi}_H^\dagger$, the expressions (3.19) and (3.20), similar to the definitions of the functions $G^{\gtrless}$, sugggest that $\mp i\Sigma^<$ and $i\Sigma^>$ correspond respectively to the particle production (scattering-in) and absorption (scattering-out or hole production) rates. The condition of irreducibility may be considered as a removal of the effect of the repeated interactions in the medium. We have in fact, respectively, for t_1 on the contour earlier and later than t_1',

$$\int d2 \int d3\ \Sigma(1,2)G(2,3)\,\Sigma(3,1')$$

$$= \int_{t_0}^{\infty} d2 \int_{t_0}^{\infty} d3\ \Sigma^{\gtrless}(1,2)\ G^-(2,3)\,\Sigma^-(3,1')$$

$$+ \int_{t_0}^{\infty} d2 \int_{t_0}^{\infty} d3\ \Sigma^+(1,2)\ G^{\gtrless}(2,3)\,\Sigma^-(3,1')$$

$$+ \int_{t_0}^{\infty} d2 \int_{t_0}^{\infty} d3\ \Sigma^+(1,2)\ G^+(2,3)\,\Sigma^{\gtrless}(3,1') \quad , \qquad (3.26)$$

cf. Eq. (3.16). On taking the expectation value of Eq. (3.15) in the presence of the external current, and making a variation with respect to the current, one finds

$$\left(i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} \right) G^+(1,1') - \frac{\delta \langle \hat{j}_H(1) \rangle_J}{\delta J(1')} \bigg|_{J=0} = \delta(1,1') \quad , \tag{3.27}$$

and from comparison with Eq. (3.11)

$$\frac{\delta \langle \hat{j}_H(1) \rangle_J}{\delta J(1')} \bigg|_{J=0} = \int_{t_0}^{\infty} d2\, \Sigma^+(1,2)\, G^+(2,1') \tag{3.28}$$

$$= \int_{t_0}^{\infty} d2\, \Sigma^+(1,2) \frac{\delta \langle \hat{\Psi}_H(2) \rangle_J}{\delta J(1')} \bigg|_{J=0} \quad .$$

From the above follows

$$\Sigma^+(1,2) = \frac{\delta \langle \hat{j}_H(1) \rangle_J}{\delta \langle \hat{\Psi}_H(2) \rangle_J} \bigg|_{J=0} \quad . \tag{3.29}$$

Similarly one has

$$\int_{t_0}^{\infty} d2\, G^-(1,2)\, \Sigma^-(2,1') = \frac{\delta \langle \hat{j}_H^\dagger(1') \rangle_J}{\delta J^*(1)} \bigg|_{J=0} \quad , \tag{3.30}$$

and

$$\Sigma^-(1,2) = \frac{\delta \langle \hat{j}_H^\dagger(2) \rangle_J}{\delta \langle \hat{\Psi}_H^\dagger(1) \rangle_J} \bigg|_{J=0} \quad . \tag{3.31}$$

The functions with which we are dealing in this section obtain simple forms in a uniform system in equilibrium. Results, following from the Green's function equations of motion, for a system achieving a uniform equilibrium are discussed in Appx. E. Before reading the Appendix, we advise the reader to get acquainted with Appx. C and the next subsection.

The Green's function equations of motion ,Eqs. (3.6) and (3.7), were first obtained by Kadanoff and Baym [2], by means of an analytic continuation of equations satisfied by temperature Green's functions in the imaginary time corresponding to the temperature. A contour method has been applied by Schwinger [1] to study the equations of motion of a quantum oscillator in an exernal field. Schwinger employed a matrix notation for functions and their multiplications on the contour. The Kadanoff-Baym equations have been independently derived with a contour method by Keldysh [5] and by Fujita [6,7]. Other papers concerning Green's function equations of motion under different hamiltonians, transition from the equations to kinetic equations (Sect. 4 of the present paper), application of Green's function methods to various problems, are Refs. [8-17].

3.2 Perturbative evaluation of self-energies

The two lowest order diagrams for the self-energy are presented in Fig. 2. The perturbation theory rules give for these diagrams

$$\Sigma^1(1,1') = \delta(t_1,t_{1'})[\delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_{1'}) \int d\underset{\sim}{x}_2\, V(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)$$
$$\times\ (\mp i)\ G^{0<}(\underset{\sim}{x}_2,t_1,\underset{\sim}{x}_2,t_1) + V(\underset{\sim}{x}_1 - \underset{\sim}{x}_{1'})$$
$$\times\ iG^{0<}(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_{1'},t_1)] \quad . \tag{3.32}$$

The approximation to the self-energy, relying on the noninteracting Green's functions, may seem reasonable only for times close to the initial time t_0. On replacing the functions G^0 by G in (3.32), one sums a whole class of diagrams, and one obtains the expression for the self-consistent Hartee-Fock energy, Eq. (3.18).

Next-order self-energy diagrams, Fig. 3, are named the Born diagrams, direct and exchange. The direct diagram gives the following contributions to the self-energies:

$$\Sigma^{\gtrless}_{Bd}(1,1') = \pm \int d\underset{\sim}{x}_2 \int d\underset{\sim}{x}_{2'}\, V(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)\, V(\underset{\sim}{x}_{2'} - \underset{\sim}{x}_{1'})\, G^{\gtrless}(1,1')$$

$$\times\, G^{\gtrless}(\underset{\sim}{x}_2,t_1,\underset{\sim}{x}_{2'},t_{1'})\, G^{\lessgtr}(\underset{\sim}{x}_{2'},t_{1'},\underset{\sim}{x}_2,t_1) \quad , \tag{3.33}$$

while the exchange diagram contributions are

$$\Sigma^{\gtrless}_{Be}(1,1') = - \int d\underset{\sim}{x}_2 \int d\underset{\sim}{x}_{2'}\, V(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)\, V(\underset{\sim}{x}_{2'} - \underset{\sim}{x}_{1'})\, G^{\gtrless}(1,\underset{\sim}{x}_{2'},t_{1'})$$

$$\times\, G^{\gtrless}(\underset{\sim}{x}_2,t_1,1')\, G^{\lessgtr}(\underset{\sim}{x}_{2'},t_{1'},\underset{\sim}{x}_2,t_1) \quad , \tag{3.34}$$

where we already use the functions G instead of G^0. The self-energies (3.33) and (3.34) correspond to the lowest order scattering with particles of the medium.

A self-energy approximation, in which diagrams of all orders in a 2-body scattering with particles of the medium are summed, is called the T-matrix approximation. This approximation is presented in Appx. F.

The RPA approximation, which will not be discussed in detail, consists in the summation of bubble diagrams in the interaction. Besides the particle Green's function equations, one deals with equations of motion of a polarization insertion that describes phonons (density fluctuations). The physical picture is such that the particles induce an emission and absorption of phonons.

3.3 Conservation laws

In many physical processes, an essential role is played by conservation laws. When approximating the Green's function equations of motion, on choosing diagrams for the self-energy, one may obtain equations that violate conservation laws. The problem of conservation laws, in a system of particles with a potential interaction, has been considered by Baym and Kadanoff [18]. We shall summarize here the results of these authors, by presenting the approximations to the equations of motion that yield the conservation laws for particle number, momentum, and energy.

From Eq. (3.15) there follows an equation for the Green's function on a contour

$$\left(i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m}\right) G(1,1')$$

$$= \delta(1,1') \mp i \int d\underset{\sim}{x}_2 \; V(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) G_2(1,\underset{\sim}{x}_2,t_1:1',\underset{\sim}{x}_2,t_1^+) \quad , \tag{3.35}$$

where the 2-particle Green's function is

$$i^2 G_2(1,2:1',2') = \langle T[\hat{\Psi}_H(1)\,\hat{\Psi}_H(2)\,\hat{\Psi}_H^\dagger(2')\,\hat{\Psi}_H^\dagger(1')]\rangle \quad , \tag{3.36}$$

and t_1^+ denotes a time infinitesimally later on a contour than t_1. The hermitian conjugate of Eq. (3.15) yields another equation

$$\left(-i \frac{\partial}{\partial t_{1'}} + \frac{\nabla_1^2}{2m}\right) G(1,1')$$

$$= \delta(1,1') \mp i \int d\underset{\sim}{x}_2 \; G_2(1,\underset{\sim}{x}_2,t_{1'};1',\underset{\sim}{x}_2,t_{1'}^+) \; V(\underset{\sim}{x}_2 - \underset{\sim}{x}_{1'}) \quad . \tag{3.37}$$

From the definition of the function G_2 it follows that

$$G_2(1,2:1^+,2^+) = G_2(2,1;2^+,1^+) \quad . \tag{3.38}$$

If the approximate Green's function obeys both an equation of the form (3.35) and an equation of the form (3.37), and the approximate function G_2 satisfies the condition (3.38), then the conservation laws are satisfied.

For the density of particles (2.4), one finds from (3.35) and (3.37)

$$\frac{\partial}{\partial t_1} \langle \hat{n}_H(1)\rangle + \underset{\sim}{\nabla}_1 \cdot \langle \hat{\underset{\sim}{j}}_H(1)\rangle = 0 \quad , \tag{3.39}$$

where the particle flux is

$$\langle \hat{\underset{\sim}{j}}_H(1)\rangle = -\frac{i}{2m} (\underset{\sim}{\nabla}_1 - \underset{\sim}{\nabla}_{1'})(\mp i)\; G^<(1,1')\Big|_{1'=1} \quad . \tag{3.40}$$

Momentum and energy conservation laws cannot strictly be written in a local form. For the total momentum

$$\langle \hat{\underset{\sim}{P}}_H(t_1)\rangle = \int d\underset{\sim}{x}_1 \left(-\frac{i}{2}\right)(\underset{\sim}{\nabla}_1 - \underset{\sim}{\nabla}_{1'})(\mp i)\, G^<(1,1')\Big|_{1'=1} , \qquad (3.41)$$

one finds from (3.35),(3.37), and (3.38), $d\langle \hat{\underset{\sim}{P}}_H(t)\rangle/dt = 0$. By using the Green's function equations of motion (3.35) and (3.37), the expectation value of the hamiltonian (2.1) may be expressed in terms of the 1-particle Green's function

$$\langle \hat{H}_H(t_1)\rangle$$

$$= \frac{1}{4}\int d\underset{\sim}{x}_1 \left[\left(i\frac{\partial}{\partial t_1} - i\frac{\partial}{\partial t_{1'}}\right) - \frac{1}{2m}(\nabla_1^2 + \nabla_{1'}^2)\right](\mp i)\, G^<(1,1')\Big|_{1'=1} . \qquad (3.42)$$

For the hamiltonian expectation value (3.42), one finds from (3.35),(3.37), and (3.38), $d\langle \hat{H}_H(t)\rangle/dt = 0$.

What conclusions, concerning self-energy, can one draw from Eqs. (3.35),(3.37), and (3.38)? One may confront the equations with self-energy (3.1) and (3.2) with Eqs. (3.35) and (3.37). In order that the conservation laws be satisfied, Σ must be such that a term ΣG can be written as $\mp iVG_2$ and $G\Sigma$ as $\mp iG_2V$, with the same G_2 in both cases. G_2 must satisfy the condition (3.38). Typical approximations to the self-energy, like Hartree, Hartree-Fock, Born, T-matrix, and RPA, yield the conservation laws [18,2]. However, when one includes an arbitrary diagram in the self-energy, then the conditions (3.35),(3.37), and (3.38), generate a number of other graphs of the same order and similar topological structure, which must be simultaneously included to comply with the conservation laws. It may be worth mentioning that the conservation laws enforce the use of the full Green's functions G in the construction of self-energy. (One takes into account skeleton diagrams, irreducible with respect to the self-energy, in the construction.)

4. Boltzmann equation

4.1 Boltzmann equation

Under proper conditions, the Boltzmann equation for the Wigner function can be derived from the Kadanoff-Baym equations. We shall assume that the temporal and spatial changes in a system are small, and the evolution does not differ much from a free evolution of a uniform system.

On subtracting Eq. (3.7) from Eq. (3.6) for $G^<$, with $t_1 = t_{1'} = T$, and on taking a Fourier transform in spatial microscopic variables, we find an equation

$$
\begin{aligned}
& i\left(\frac{\partial}{\partial T} + \frac{\underset{\sim}{p}}{m}\cdot\nabla_{\underset{\sim}{R}}\right) G^<(\underset{\sim}{p};\underset{\sim}{R},T) \\
&= \int d\underset{\sim}{r}\, e^{-i\underset{\sim}{p}\underset{\sim}{r}} \int d\underset{\sim}{r}'\, \Sigma_{HF}(\underset{\sim}{r} - \underset{\sim}{r}':\underset{\sim}{R} + \underset{\sim}{r}'/2,T) G^<(\underset{\sim}{r}':\underset{\sim}{R} - (\underset{\sim}{r} - \underset{\sim}{r}')/2,T) \\
&- \int d\underset{\sim}{r}\, e^{-i\underset{\sim}{p}\underset{\sim}{r}} \int d\underset{\sim}{r}'\, \Sigma_{HF}(\underset{\sim}{r} - \underset{\sim}{r}';\underset{\sim}{R} - \underset{\sim}{r}'/2,T) G^<(\underset{\sim}{r}';\underset{\sim}{R} + (\underset{\sim}{r} - \underset{\sim}{r}')/2,T) \\
&+ \int d\underset{\sim}{r}\, e^{-i\underset{\sim}{p}\underset{\sim}{r}} \int_{-\infty}^{0} dt' \int d\underset{\sim}{r}'\, \Sigma^>(\underset{\sim}{r} - \underset{\sim}{r}',-t';\underset{\sim}{R} + \underset{\sim}{r}'/2,\ T + t'/2) \\
&\times G^<(\underset{\sim}{r}',t':\underset{\sim}{R} - (\underset{\sim}{r} - \underset{\sim}{r}')/2,\ T + t'/2) \\
&+ \int d\underset{\sim}{r}\, e^{-i\underset{\sim}{p}\underset{\sim}{r}} \int_{0}^{+\infty} dt' \int d\underset{\sim}{r}'\, \Sigma^>(\underset{\sim}{r} - \underset{\sim}{r}',-t';\underset{\sim}{R} - \underset{\sim}{r}'/2,\ T - t'/2) \\
&\times G^<(\underset{\sim}{r}',t':\underset{\sim}{R} + (\underset{\sim}{r} - \underset{\sim}{r}')/2,\ T - t'/2) \\
&- (\ \ldots\ {}^< \ldots G^> \ldots\ ) \quad , \qquad (4.1)
\end{aligned}
$$

where we have introduced relative variables in the functions and we have adopted $t_0 \to -\infty$. The omission of the fourth microscopic variable in the l.h.s. Green's function is an abbreviation for

$$G^<(\underset{\sim}{p};\underset{\sim}{R},T) \equiv \int \frac{d\omega}{2\pi} G^<(\underset{\sim}{p},\omega:\underset{\sim}{R},T) = G^<(\underset{\sim}{p},t = 0:\underset{\sim}{R},T) \quad .$$

We have not written explicitly the last two terms at the r.h.s. of (4.1), which enter the r.h.s. with a "-" sign, have the same structure as the third and fourth r.h.s. terms, but contain $\Sigma^<$ and $G^>$ instead of $\Sigma^>$ and $G^<$. In a freely evolving uniform system the functions have no dependence on macroscopic variables, and the Green's functions

$$G^{\gtrless}(\underset{\sim}{p},\omega) = 2\pi\delta(\omega - \omega_p^0)\, G^{\gtrless}(\underset{\sim}{p}) \text{ with } \omega_p^0 = p^2/2m \quad .$$

We shall evaluate the r.h.s. of Eq. (4.1), which makes the evolution different from a free one, on ignoring the dependence of the functions on macroscopic variables, with the Green's functions as for a freely evolving uniform system $G^{\gtrless}(\underset{\sim}{p},\omega) = 2\pi\,\delta(\omega - \omega_p^0)G^{\gtrless}(\underset{\sim}{p};\underset{\sim}{R},T)$. In that case the terms with Σ_{HF} cancel out. The third and fourth terms may be combined, similarly the fifth and the sixth, and the r.h.s. of Eq. (4.1) takes a form

$$\int d\underset{\sim}{r}\, e^{-i\underset{\sim}{p}\underset{\sim}{r}} \int_{-\infty} dt' \int d\underset{\sim}{r}'\, \Sigma^>(\underset{\sim}{r} - \underset{\sim}{r}',-t';\underset{\sim}{R},T) \int \frac{d\underset{\sim}{p}'}{(2\pi)^3}\, G^<(\underset{\sim}{p}';\underset{\sim}{R},T)$$

$$\times\, e^{i\underset{\sim}{p}'\underset{\sim}{r}' - i\omega_{p'}^0 t'} - \int d\underset{\sim}{r}\, e^{-i\underset{\sim}{p}\underset{\sim}{r}} \int_{-\infty}^{\infty} dt' \int d\underset{\sim}{r}'\, \Sigma^<(\underset{\sim}{r} - \underset{\sim}{r}',-t';\underset{\sim}{R},T)$$

$$\times \int \frac{d\underset{\sim}{p}'}{(2\pi)^3}\, G^>(\underset{\sim}{p}';\underset{\sim}{R},T)\, e^{i\underset{\sim}{p}'\underset{\sim}{r}' - i\omega_{p'}^0 t'} \quad . \tag{4.2}$$

Upon completion of the integrations over microscopic coordinates and times in (4.2), we obtain an equation

$$i\left(\frac{\partial}{\partial T} + \frac{\underset{\sim}{p}}{m}\underset{\sim}{\nabla}_{\underset{\sim}{R}}\right) G^<(\underset{\sim}{p};\underset{\sim}{R},T)$$

$$= \Sigma^>(\underset{\sim}{p},\omega_p^0;\underset{\sim}{R},T)G^<(\underset{\sim}{p};\underset{\sim}{R},T) - \Sigma^<(\underset{\sim}{p},\omega_p^0;\underset{\sim}{R},T)G^>(\underset{\sim}{p};\underset{\sim}{R},T) \quad . \tag{4.3}$$

With (2.9) and (2.13), Eq. (4.3) becomes

$$\left(\frac{\partial}{\partial T} + \frac{\underset{\sim}{p}}{m}\cdot\underset{\sim}{\nabla}_{\underset{\sim}{R}}\right) f\,(\underset{\sim}{p};\underset{\sim}{R},T)$$

$$= (\mp i)\Sigma^{<}(\underset{\sim}{p},\omega_p^o;\underset{\sim}{R},T)(1 \mp f(\underset{\sim}{p};\underset{\sim}{R},T)) - i\Sigma^{>}(\underset{\sim}{p},\omega_p^o;\underset{\sim}{R},T)\, f(\underset{\sim}{p};\underset{\sim}{R},T) \quad , \qquad (4.4)$$

which is just the Boltzmann equation.

In the Fourier-transformed variables the sum of Born diagrams for self-energy, Eqs. (3.33) and (3.34), gives

$$\Sigma_B^{\gtrless}\,(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = \pm \int \frac{d\underset{\sim}{p}_1 d\omega_1}{(2\pi)^4} \int\frac{d\underset{\sim}{p}' d\omega'}{(2\pi)^4} \int\frac{d\underset{\sim}{p}_1' d\omega_1'}{(2\pi)^4}$$

$$\times\ (2\pi)^4\ \delta(\underset{\sim}{p} + \underset{\sim}{p}_1 - \underset{\sim}{p}' - \underset{\sim}{p}_1')\ \delta(\omega + \omega_1 - \omega' - \omega_1')$$

$$\times\ [(V(\underset{\sim}{p} - \underset{\sim}{p}'))^2 \mp V(\underset{\sim}{p} - \underset{\sim}{p}')V(\underset{\sim}{p} - \underset{\sim}{p}_1')]$$

$$\times\ G^{\lessgtr}\,(\underset{\sim}{p}_1,\omega_1;\underset{\sim}{R},T)\ G^{\gtrless}\,(\underset{\sim}{p}',\omega';\underset{\sim}{R},T)\ G^{\gtrless}\,(\underset{\sim}{p}_1',\omega_1';\underset{\sim}{R},T) \quad . \qquad (4.5)$$

Upon introduction of the Wigner functions we find

$$\mp i\Sigma_B^{<}(\underset{\sim}{p},\omega_p^o;\underset{\sim}{R},T) = \int\frac{dp_1}{(2\pi)^3} \int\frac{dp'}{(2\pi)^3} \int\frac{dp_1}{(2\pi)^3}\,(2\pi)^4\ \delta(\underset{\sim}{p} + \underset{\sim}{p}_1 - \underset{\sim}{p}' - \underset{\sim}{p}_1')$$

$$\times\ \delta(\omega_p^o + \omega_{p_1}^o - \omega_{p'}^o - \omega_{p_1'}^o)\ \frac{1}{2}\ (V(\underset{\sim}{p} - \underset{\sim}{p}') \mp V(\underset{\sim}{p} - \underset{\sim}{p}_1'))^2$$

$$\times\ (1 \mp f(\underset{\sim}{p}_1;\underset{\sim}{R},T))f(\underset{\sim}{p}';\underset{\sim}{R},T)f(\underset{\sim}{p}_1';\underset{\sim}{R},T) \quad , \qquad (4.6)$$

and

$$i\Sigma_B^{>}(\underset{\sim}{p},\omega_p^o;\underset{\sim}{R},T) = \int\frac{dp_1}{(2\pi)^3} \int\frac{dp'}{(2\pi)^3} \int\frac{dp_1}{(2\pi)^3}\,(2\pi)^4\ \delta(\underset{\sim}{p} + \underset{\sim}{p}_1 - \underset{\sim}{p}' - \underset{\sim}{p}_1')$$

$$\times\ \delta(\omega_p^o + \omega_{p_1}^o - \omega_{p'}^o - \omega_{p_1'}^o)\ \frac{1}{2}\ (V(\underset{\sim}{p} - \underset{\sim}{p}') \mp V(\underset{\sim}{p} - \underset{\sim}{p}_1'))^2$$

$$\times\ f(\underset{\sim}{p}_1;\underset{\sim}{R},T)(1 \mp f(\underset{\sim}{p}';\underset{\sim}{R},T))(1 \mp f(\underset{\sim}{p}_1';\underset{\sim}{R},T)) \quad . \qquad (4.7)$$

A convective derivative of the distribution function constitutes the l.h.s. of Eq. (4.4), and the r.h.s. of the equation accounts for changes in the distribution caused by interactions. With $i\Sigma^{>}$ a scattering-out rate and f a density of initial states, the second r.h.s. term in (4.4) accounts for scattering-out from $\underset{\sim}{p}$; with $\mp i\Sigma^{<}$ a scattering-in rate and $(1 \mp f)$ a density of final states, the first term accounts for scattering-in. Equation (4.7) for $i\Sigma^{>}$ sums over 2-body scattering processes, a symmetrized cross section is in a Born approximation, the δ functions correspond to momentum and energy conservation in collisions. With respective densities of states, integrations run over particles with which scattering occurs and over final states. $i\Sigma^{>}$ is a collision frequency of a particle with momentum $\underset{\sim}{p}$. Equation (4.6) describes inverse processes with respect to those in (4.7), in which particles of the medium scatter and one of them emerges with a momentum $\underset{\sim}{p}$. See also the form of self-energy in the T-matrix approximation Eq. (F.25).

In the equilibrium the r.h.s. of the Boltzmann equation must vanish, and a detailed balance equation is satisfied

$$\mp i\Sigma^{<}(\underset{\sim}{p},\omega_p^0)(1 \mp f(\underset{\sim}{p})) = i\Sigma^{>}(\underset{\sim}{p},\omega_p^0)f(\underset{\sim}{p}) \quad . \tag{4.8}$$

If we disturb the equilibrium by adding or removing particles of momentum $\underset{\sim}{p}$, we have for the associated disturbance $\delta f(\underset{\sim}{p};\underset{\sim}{R},T)$ an equation

$$\left(\frac{\partial}{\partial T} + \frac{\underset{\sim}{p}}{m}\cdot\underset{\sim}{\nabla}_{\underset{\sim}{R}}\right) \delta f(\underset{\sim}{p};\underset{\sim}{R},T) = -\Gamma(\underset{\sim}{p},\omega_p^0)\ \delta f(\underset{\sim}{p};\underset{\sim}{R},T) \tag{4.9}$$

with $\Gamma = (i\Sigma^{>} \pm (\mp i)\Sigma^{<}) = i(\Sigma^{>} - \Sigma^{<})$. For a disturbance independent of $\underset{\sim}{R}$, from (4.9) follows

$$\delta f(\underset{\sim}{p};T) = \delta f(\underset{\sim}{p};T = 0)\ e^{-\Gamma(\underset{\sim}{p},\omega_p^0)T} \quad , \tag{4.10}$$

and this equation Fourier-transformed in macroscopic times reads

$$\delta f(\underset{\sim}{p};\Omega) = \frac{\delta f(\underset{\sim}{p};T = 0)}{\Gamma(\underset{\sim}{p},\omega_p^o)-i\Omega} \quad . \tag{4.11}$$

The function Γ sets the rate at which equilibrium is reached.

When a local equilibrium is established in a system, i.e. Eq. (4.8) is approximately satisfied at every $(\underset{\sim}{R},T)$ (cf. r.h.s. of Eq. (4.4)), then the rate of changes of a local distribution f may be arbitrarily small. The time-rate will depend on the scale of spatial inhomogeneities L, $\Omega \sim v/L$, where v is a characteristic particle velocity. A system in a local equilibrium can be described with a set of hydrodynamic equations, local conservation laws of particle number, momentum, and energy.

Before we proceed further let us note the following. On using the function Γ, a formal solution to the Boltzmann equation (4.4) in a homogeneous system may be written as

$$f(\underset{\sim}{p};T) = f(\underset{\sim}{p};T = 0) \exp\left(-\int_0^T dT' \, \Gamma(\underset{\sim}{p},\omega_p^o;T')\right)$$

$$+ \int_0^T dT' \, (\mp i)\Sigma^<(\underset{\sim}{p},\omega_p^o;T') \exp\left(-\int_{T'}^T dT'' \, \Gamma(\underset{\sim}{p},\omega_p^o;T'')\right). \tag{4.12}$$

In the low-density limit $i\Sigma^> \gg \mp i\Sigma^<$ and $\Gamma \approx i\Sigma^>$. At high occupations of states, the quantities $i\Sigma^>$ and $i\Sigma^<$ may be comparable. For bosons, when $i\Sigma^<$ is larger than $i\Sigma^>$, the function Γ is negative (from Eq. (4.8) it follows that this may not occur at equilibrium). From Eq. (4.12) it follows that in case of a negative Γ function, the occupation of a state increases exponentially; we may call this a laser effect.

An assumption leading to the Boltzmann equation was the slow variation of the functions with the macroscopic variables. If it were possible to obtain

the Boltzmann equation, variation in macroscopic variables away from equilibrium would be set by magnitude of the function Γ. When we pass from Eq. (4.1) to (4.2), the variations in macroscopic variables must be compared with variations in microscopic variables; the latter are determined by energies and momenta in the system.

Upon sketching the transition to the Boltzmann equation, we shall now rederive the equation in a more formal manner, so that the approximations involved will become explicit.

4.2 Derivation of the Boltzmann equation reassessed

We start with the Kadanoff-Baym equations in forms (3.9) and (3.10), which can be written as

$$\int_{t_0}^{\infty} d2\ (G^{-1})^{+}(1,2)G^{\gtrless}(2,1') = \int_{t_0}^{\infty} d2\ \Sigma^{\gtrless}(1,2)G^{-}(2,1') \quad , \tag{4.13}$$

$$\int_{t_0}^{\infty} d2\ G^{\gtrless}(1,2)(G^{-1})^{-}(2,1') = \int_{t_0}^{\infty} d2\ G^{+}(1,2)\,\Sigma^{\gtrless}(2,1') \quad . \tag{4.14}$$

We take a limit $t_0 \to -\infty$, and in a moment it will become clear how this limit should be understood. Each side in the above equations is of the form

$$\int d^4x_2 f(x_1,x_2)u(x_2,x_{1'}) \quad , \tag{4.15}$$

and upon introducing $x = x_1 - x_{1'}$, $X = (x_1 + x_{1'})/2$, $x' = x_2 - x_{1'}$, we may rewrite (4.15) into

$$\int d^4x' f(x - x';\ X + x'/2)u(x':X + (x' - x)/2) \quad . \tag{4.16}$$

We shall Fourier transform the sides of Eqs. (4.13) and (4.14), and we shall average the results over a certain range of frequencies; i.e., we evaluate expressions of the form

$$\int \frac{d\omega}{2\pi} F(\omega) \quad \int dt \int d\underset{\sim}{r}\ e^{i\omega t}\ e^{-i\underset{\sim}{p}\underset{\sim}{r}} \quad \int dt' \int d\underset{\sim}{r}'$$

$$\times\ f(\underset{\sim}{r} - \underset{\sim}{r}',\ t - t';\underset{\sim}{R} + \underset{\sim}{r}'/2,\ T + t'/2)$$

$$\times\ u(\underset{\sim}{r}',t';\underset{\sim}{R} + (\underset{\sim}{r} - \underset{\sim}{r}')/2,\ T + (t - t')/2) \quad , \tag{4.17}$$

where F is a function used for the averaging. We take the width of the function F small in comparison with characteristic energies in the system and sufficiently large that

$$\int \frac{d\omega}{2\pi} F(\omega) e^{i\omega t}$$

is sharply peaked around $t = 0$ in comparison with the variation of the functions f and u in macroscopic times. (In connection with the spatial variables, an extra averaging over momenta in (4.17) may be indispensable in a low-temperature system.) We shall assume that the properties of the functions f and u are such that the dominant contribution to the integral (4.17) comes from small values of $\underset{\sim}{r}$, $(\underset{\sim}{r} - \underset{\sim}{r}')$, and t', small in comparison with the variation of the functions f and u with the macroscopic variables. We provide a certain analysis of the function properties in Appx. G. Under the above assumptions we may expand the functions

$$f(x - x';X + x'/2) \approx (1 + \frac{1}{2} x' \frac{\partial}{\partial X})\ f(x - x';X),$$

$$u(x';X + (x - x')/2) \approx \left(1 + \frac{1}{2} (x - x') \frac{\partial}{\partial X}\right) u(x';X) \quad .$$

Upon expansion of the functions f and u in (4.17) and introduction of the Fourier transforms of the functions, we find for (4.17)

$$\int d^4x\ e^{ipx} \int d^4x' \int \frac{d^4p_1}{(2\pi)^4} e^{-ip_1(x-x')} \int \frac{d^4p_2}{(2\pi)^4} e^{-ip_2x'}$$

$$\times \left[f(p_1;X)\ u(p_2;X) + \frac{1}{2} x' \frac{\partial f}{\partial X} (p_1;X)\ u(p_2;X)\right.$$

$$\left. + \frac{1}{2} (x - x')\ f(p_1;X) \frac{\partial u}{\partial X} (p_2;X)\right] \quad , \tag{4.18}$$

where for a while we omit the averaging function over frequencies. Use of an identity

$$x\, e^{-ipx} = i \frac{\partial}{\partial p} e^{-ipx} \quad ,$$

integration by parts, and integrations over x and x' lead to the expression

$$f(p;X)\, u(p;X) + \frac{i}{2} \left(\frac{\partial f}{\partial p}(p;X) \frac{\partial u}{\partial X}(p;X) - \frac{\partial f}{\partial X}(p;X) \frac{\partial u}{\partial p}(p;X) \right) . \quad (4.19)$$

In connection with (4.19) we define a generalized Poisson bracket

$$[f,u] = \frac{\partial f}{\partial p}\frac{\partial u}{\partial X} - \frac{\partial f}{\partial X}\frac{\partial u}{\partial p} \quad . \quad (4.20)$$

Upon application of the above procedure to each side of Eqs. (4.13) and (4.14) and subtraction of the equations from one another, one finds a so-called generalized Boltzmann equation (see the properties of the functions, Appx. D)

$$[\mathrm{Re}(G^{-1})^{+}, iG^{\gtrless}] - [i\Sigma^{\gtrless}, \mathrm{Re}G^{+}] = G^{<}\Sigma^{>} - G^{>}\Sigma^{<} \quad . \quad (4.21)$$

For the sake of clarity we have omitted the variables (p;X). Implicitly, to each side of (4.21) an averaging is applied over a range of frequencies large in comparison with the rate of change of the functions. However, the range of averaging in frequencies must be small in comparison with characteristic energies in the system, so the equation can be closed. A connection between (4.21) and (4.3) may be seen when one notices that

$$\mathrm{Re}(G^{-1})^{+}\,(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = \omega - p^2 2m - \mathrm{Re}\Sigma^{+}\,(\underset{\sim}{p},\omega;\underset{\sim}{R},T) \quad , \quad (4.22)$$

and applies (4.20) to the first term at the r.h.s. of (4.21).

In the above derivation, we have obtained the generalized Boltzmann equation by retaining the lowest terms in a certain expansion. The role of a small parameter in the expansion is played by (characteristic time of

variations)$^{-1}$/e, where e corresponds to characteristic energies. When a system is away from equilibrium, or when we ask about deviations from equilibrium, the rate of variations from the Boltzmann equation is of the order of Γ. (The magnitude of Γ may even be considered a lower bound for the rate of temporal changes, because nonhomogeneity of the system or the presence of an external potential in the self-energy may enforce higher rates.) A different situation occurs when we consider the evolution of a local equilibrium, but we postpone the discussion of that case until the end of the subsection[1]. According to the above, on obtaining the generalized Boltzmann equation, terms of second order in Γ/e are omitted. Consequently one can disregard such terms in Eq. (4.21), which otherwise consists of the first-order terms. Of second order is the second term at the l.h.s. of (4.21). Further, the Green's functions appearing in Eq. (4.21) and used in construction of self-energies can be taken in the zeroth order. When establishing a zeroth order, one should cautiously deal with ReΣ^+: there occur situations when Re$\Sigma^+ \gg \Gamma$, e.g. for long-range interactions (problem considered in the T-matrix approximation at the end of Appx. F), further when ReΣ^+ contains an external potential or for fermions close to zero temperature. For a strong short-range potential, when special effects due to Pauli principle are absent, we may expect Re$\Sigma^+ \sim \Gamma$. In the latter case, the zeroth-order equation for the Green's function is (from (4.13) and (4.19))

$$(\omega - p^2/2m)\, G^{\gtrless}(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = 0 \quad , \tag{4.23}$$

which together with (2.13) and (2.9) gives

$$iG^{>}(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = 2\pi\, \delta(\omega - \omega_{\underset{\sim}{p}})(1 \mp f(\underset{\sim}{p};\underset{\sim}{R},T)) \quad , \tag{4.24a}$$

$$\mp iG^{<}(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = 2\pi\, \delta(\omega - \omega_{\underset{\sim}{p}}) f(\underset{\sim}{p};\underset{\sim}{R},T) \quad , \tag{4.24b}$$

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[1]The author is grateful to Prof. G. Baym for pointing out the two cases.

with $\omega_{\underset{\sim}{p}} = p^2/2m$. Upon insertion of the functions (4.24) into Eq. (4.21) and integration over ω, one finds Eq. (4.4).

If $\mathrm{Re}\Sigma^+ \gg \Gamma$, it may be necessary to retain $\mathrm{Re}\Sigma^+$ in the zeroth-order equation for Green's functions

$$(\omega - p^2/2m - \mathrm{Re}\Sigma^+(\underset{\sim}{p},\omega;\underset{\sim}{R},T))\, G^{\gtrless}(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = 0 \quad . \tag{4.25}$$

If one can ignore the dependence of $\mathrm{Re}\Sigma^+$ on ω, $\mathrm{Re}\Sigma^+(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = \mathrm{Re}\Sigma^+(\underset{\sim}{p};\underset{\sim}{R},T)$, then from (4.25) follow the forms of Green's functions (4.24) with $\omega_{\underset{\sim}{p}} = p^2/2m + \mathrm{Re}\Sigma^+(\underset{\sim}{p};\underset{\sim}{R},T)$. Upon insertion of the functions into (4.21), one finds

$$\left(\frac{\partial}{\partial T} + \frac{\underset{\sim}{p}}{m}\,\frac{\partial}{\partial \underset{\sim}{R}} + \frac{\partial \mathrm{Re}\Sigma^+(\underset{\sim}{p};\underset{\sim}{R},T)}{\partial \underset{\sim}{p}}\cdot\frac{\partial}{\partial \underset{\sim}{R}} - \frac{\partial \mathrm{Re}\Sigma^+(\underset{\sim}{p};\underset{\sim}{R},T)}{\partial \underset{\sim}{R}}\cdot\frac{\partial}{\partial \underset{\sim}{p}}\right) f(\underset{\sim}{p};\underset{\sim}{R},T)$$

$$= (\mp i)\,\Sigma^<(\underset{\sim}{p},\omega_{\underset{\sim}{p}};\underset{\sim}{R},T)(1 \mp f(\underset{\sim}{p};\underset{\sim}{R},T)) - i\Sigma^>(\underset{\sim}{p},\omega_{\underset{\sim}{p}};\underset{\sim}{R},T) f(\underset{\sim}{p};\underset{\sim}{R},T) \quad . \tag{4.26}$$

An equation of this form, written for fermions close to zero temperature, for momenta close to the Fermi surface, is known as a Landau-Silin equation. A Vlason equation emerges from (4.26), when one neglects the r.h.s., and for the self-energy takes the Hartree term, independent of $\underset{\sim}{p}$.

In the case when it is not possible to ignore the dependence of $\mathrm{Re}\Sigma^+$ on ω, but Γ is small around $\mathrm{Re}(G^{-1})^+ = 0$, we may introduce an occupation number corresponding to the zero of $\mathrm{Re}(G^{-1})^+$, and obtain a kinetic equation for the number. We parametrize the behavior of $G^{\gtrless}$ in ω, around a solution $\omega_{\underset{\sim}{p}}$ of $\mathrm{Re}(G^{-1})^+ = 0$, with

$$\mp iG^<(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = 2\pi\ \delta(\mathrm{Re}(G^{-1})^+(\underset{\sim}{p},\omega;\underset{\sim}{R},T))\, n(\underset{\sim}{p};\underset{\sim}{R},T)$$

$$= Z(\underset{\sim}{p};\underset{\sim}{R},T)\ 2\pi\ \delta(\omega - \omega_{\underset{\sim}{p}})\, n(\underset{\sim}{p};\underset{\sim}{R},T) \quad , \tag{4.27}$$

and

$$iG^>(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = Z(\underset{\sim}{p};\underset{\sim}{R},T)\ 2\pi\ \delta(\omega - \omega_{\underset{\sim}{p}})(1 \mp n(\underset{\sim}{p};\underset{\sim}{R},T)) \quad , \tag{4.28}$$

where

$$Z^{-1}(\underset{\sim}{p};\underset{\sim}{R},T) = 1 - \frac{\partial \mathrm{Re}\Sigma^{+}(\underset{\sim}{p},\omega;\underset{\sim}{R},T)}{\partial\omega}\Bigg|_{\omega=\omega_{\underset{\sim}{p}}} \tag{4.29}$$

We rely in this parametrization on the equality $G^{>} - G^{<} = G^{+} - G^{-}$ (see Appx. D) and the forms of $G^{\pm}$ following from (3.11) and (4.19) in the zeroth order with respect to Γ/e

$$G^{\pm}(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = \frac{1}{\omega - p^2/2m - \mathrm{Re}\Sigma^{+}(\underset{\sim}{p},\omega;\underset{\sim}{R},T) \pm i\varepsilon} \quad . \tag{4.30}$$

We insert the function $G^{<}$ into (4.21), in the form given by the first of equalities in (4.27), so the δ-function can be removed from under the Poisson bracket. In subsequent steps we exploit the fact that with

$$e(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = p^2/2m + \mathrm{Re}\Sigma^{+}(\underset{\sim}{p},\omega;\underset{\sim}{R},T) \quad ,$$

and $x = \underset{\sim}{p},\underset{\sim}{R},T$, we have

$$\frac{\partial e(p,\omega;R,T)}{\partial x}\Bigg|_{\omega=\omega_{\underset{\sim}{p}}} = \frac{\partial e(\underset{\sim}{p},\omega_{\underset{\sim}{e}};\underset{\sim}{R},T)}{\partial x} - \frac{\partial\omega_{\underset{\sim}{p}}}{\partial x}\,\frac{\partial e(\underset{\sim}{p},\omega;\underset{\sim}{R},T)}{\partial\omega}\Bigg|_{\omega=\omega_{\underset{\sim}{p}}}$$

$$= Z^{-1}\,\frac{\partial\omega_{\underset{\sim}{p}}}{\partial x} \quad .$$

Upon integration over frequencies we find the equation

$$\left(\frac{\partial}{\partial T} + \frac{\underset{\sim}{p}}{m}\cdot\frac{\partial}{\partial\underset{\sim}{R}} + \frac{\partial\mathrm{Re}\Sigma^{+}(\underset{\sim}{p},\omega_{\underset{\sim}{p}};\underset{\sim}{R},T)}{\partial\underset{\sim}{p}}\cdot\frac{\partial}{\partial\underset{\sim}{R}} - \frac{\partial\mathrm{Re}\Sigma^{+}(\underset{\sim}{p},\omega_{\underset{\sim}{p}};\underset{\sim}{R},T)}{\partial\underset{\sim}{R}}\cdot\frac{\partial}{\partial\underset{\sim}{p}}\right)n(\underset{\sim}{p};\underset{\sim}{R},T)$$

$$= Z(\underset{\sim}{p};\underset{\sim}{R},T)(\mp i)\,\Sigma^{<}(\underset{\sim}{p},\omega_{\underset{\sim}{p}};\underset{\sim}{R},T)(1 \mp n(\underset{\sim}{p};\underset{\sim}{R},T))$$

$$- Z(\underset{\sim}{p};\underset{\sim}{R},T)\,i\Sigma^{>}(\underset{\sim}{p},\omega_{\underset{\sim}{p}};\underset{\sim}{R},T)n(\underset{\sim}{p};\underset{\sim}{R},T)) \tag{4.31}$$

Let us now turn to the evolution of local equilibrium. The rates of temporal variations of the Green's functions will be set by a scale of nonhomogeneities in a system and may be arbitrarily small. The Fourier-transformed Eqs. (4.13),(4.14),(3.11), expanded according to Eq.

(4.19), provide us (Eq. (4.21)) with the conditions for a local equilibrium and the form of the functions (see Appx. E). Apart from the limitations on the rate of temporal variations inherent to the ordinary Boltzmann equation (next subsection and Appx. G), it follows that the rate must be much smaller than Γ. (Note that since Γ now does not fix the rates of macroscopic variations, there is no need for Γ being small.) Depending on the smoothness of the expected equilibrium functions and the rates of macroscopic variations, the averaging in (4.17) may be lifted. As far as the evolution is concerned, the following occurs. The forms of the local equilibrium functions depend on a few parameters, which can be determined from the local particle, momentum, and energy densities. Although the evolution can be studied by using the kinetic equations, it is more convenient to use the local conservation laws, the hydrodynamic equations. These equations may be deduced from Eqs. (3.35) and (3.37), and in the momentum and energy cases the derivation involves an expansion of G_2 over the interaction range (cf. Refs. [18,19]). The G_2 in the equations (in the pressure) can then be perturbatively expressed in terms of the 1-particle equilibrium Green's functions. Apart from the case of the ordinary Boltzman equation, the derivation of the conservation laws from a kinetic equation may be quite involved, especialy in case of a full generalized Boltzmann equation (4.21). One has to trace down the correspondence between Eqs. (4.13),(4.14), and (3.35),(3.37), in the procedure leading to the kinetic equation. The effort is not necessarily rewarding, because in general the part of the pressure explicitly depending on the interaction cannot be directly expressed in terms of the actual G and Σ.

4.3 Conditions leading to the Boltzmann equation

From the derivation of the Boltzmann equation and analysis of the rejected terms, it follows that use of the Boltzmann equation in a homogeneous medium is conditioned by a weak sensitivity of the Boltzmann equation dynamics

to uncertainties in energies of the order of Γ. In a nonhomogeneous medium, with Γ setting also the rates of spatial variations, use of the Boltzmann equation would be additionally conditioned by weak sensitivity of the dynamics to uncertainties in momenta of the order of $\Gamma m/p$.

The Boltzmann equation limit can be directly seen when considering a disturbance of an equilibrium, by an addition or removal of particles of a momentum $\underset{\sim}{p}$. For a uniform disturbance, it follows from the Kadanoff-Baym equations that

$$\mp i\delta G^{<}(\underset{\sim}{p};t,t') = G^{+}(\underset{\sim}{p};t-0)\,\delta f(\underset{\sim}{p};T=0)\,G^{-}(\underset{\sim}{p};0-t') \quad . \tag{4.32}$$

Equation (4.32) Fourier-transformed in microscopic and macroscopic times reads (see (E.4))

$$\begin{aligned}
&\mp i\delta G^{<}(\underset{\sim}{p},\omega;\Omega)\\
&= \delta f(\underset{\sim}{p};T=0)\,G^{+}(\underset{\sim}{p},\omega+\Omega/2)\,G^{-}(\underset{\sim}{p},\omega-\Omega/2)\\
&= \delta f(\underset{\sim}{p};T=0)(G^{-}(\underset{\sim}{p},\omega-\Omega/2)-G^{+}(\underset{\sim}{p},\omega+\Omega/2))\\
&\quad\times\frac{1}{(G^{+})^{-1}(\underset{\sim}{p},\omega+\Omega/2)-(G^{-})^{-1}(\underset{\sim}{p},\omega+\Omega/2)}\\
&= \delta f(\underset{\sim}{p};T=0)\,\Big(\frac{1}{\omega-\Omega/2-p^2/2m-\mathrm{Re}\Sigma^{+}(\underset{\sim}{p},\omega-\Omega/2)-i\Gamma(\underset{\sim}{p},\omega-\Omega/2)/2}\\
&\quad-\frac{1}{\omega+\Omega/2-p^2/2m-\mathrm{Re}\Sigma^{+}(\underset{\sim}{p},\omega+\Omega/2)+i\Gamma(\underset{\sim}{p},\omega+\Omega/2)/2}\Big)\\
&\quad\times\frac{1}{\Omega-\mathrm{Re}\Sigma^{+}(\underset{\sim}{p},\omega+\Omega/2)+\mathrm{Re}\Sigma^{+}(\underset{\sim}{p},\omega-\Omega/2)+i(\Gamma(\underset{\sim}{p},\omega+\Omega/2)+\Gamma(\underset{\sim}{p},\omega-\Omega/2))/2} \quad .
\end{aligned} \tag{4.33}$$

If details in the ω-dependence of $\delta G^{<}$ within the range of Γ are not important, and the self-energies vary weakly within that range, we can approximate the r.h.s. of (4.33) with

$$Z(\underset{\sim}{p})2\pi\delta(\omega-\omega_{\underset{\sim}{p}})\,\frac{f(\underset{\sim}{p},T=0)}{\Gamma(\underset{\sim}{p},\omega_{\underset{\sim}{p}})-iZ^{-1}(\underset{\sim}{p})\Omega} \quad , \tag{4.34}$$

$$-i\Omega f(\underset{\sim}{p};\Omega) = \int\frac{d\omega}{2\pi}\left(\frac{i}{\omega+\Omega/2+i\epsilon} - \frac{i}{\omega-\Omega/2-i\epsilon}\right)$$

$$\times \int\frac{d\underset{\sim}{p}_1}{(2\pi)^3}\int\frac{d\underset{\sim}{p}'}{(2\pi)^3}\int\frac{d\underset{\sim}{p}'_1}{(2\pi)^3}(2\pi)^3\,\delta(\underset{\sim}{p}+\underset{\sim}{p}_1-\underset{\sim}{p}'-\underset{\sim}{p}'_1)$$

$$\times\,[(V(\underset{\sim}{p}-\underset{\sim}{p}'))^2 \mp V(\underset{\sim}{p}-\underset{\sim}{p}')V(\underset{\sim}{p}-\underset{\sim}{p}'_1)]\times F(\underset{\sim}{p},\underset{\sim}{p}_1,\underset{\sim}{p}',\underset{\sim}{p}'_1,\omega:\Omega)\quad, \tag{4.35}$$

with the auxiliary function

$$F(\underset{\sim}{p},\underset{\sim}{p}_1,\underset{\sim}{p}',\underset{\sim}{p}'_1,t;T)$$

$$= G^{>}(\underset{\sim}{p},t;T)G^{>}(\underset{\sim}{p}_1,t;T)G^{<}(\underset{\sim}{p}'_1,-t;T)G^{<}(\underset{\sim}{p}'_1,-t;T)$$

$$- G^{<}(\underset{\sim}{p},t;T)G^{<}(\underset{\sim}{p}_1,t;T)G^{>}(\underset{\sim}{p}'_1,-t;T)G^{>}(\underset{\sim}{p}'_1,-t;T) \tag{4.36}$$

Proceeding toward the Boltzmann equation we approximate the function F with

$$F(\underset{\sim}{p},\underset{\sim}{p}_1,\underset{\sim}{p}',\underset{\sim}{p}'_1,\omega;T) = 2\pi\delta(\omega-\Delta e)F(\underset{\sim}{p},\underset{\sim}{p}_1,\underset{\sim}{p}',\underset{\sim}{p}'_1;T)\quad, \tag{4.37}$$

where

$$F(\underset{\sim}{p},\underset{\sim}{p}_1,\underset{\sim}{p}',\underset{\sim}{p}'_1;T)$$

$$= (1\mp f(\underset{\sim}{p};T))(1\mp f(\underset{\sim}{p}_1;T))\,f(\underset{\sim}{p}';T)f(\underset{\sim}{p}'_1;T)$$

$$- f(\underset{\sim}{p};T)f(\underset{\sim}{p}_1;T)(1\mp f(\underset{\sim}{p}';T))(1\mp f(\underset{\sim}{p}'_1;T)\quad, \tag{4.38}$$

and

$$\Delta e = \omega^0_p + \omega^0_{p_1} - \omega^0_{p'} - \omega^0_{p'_1}\quad. \tag{4.39}$$

We shall look under what circumstances we can ignore the $\Omega/2$ terms in the denominators at the r.h.s. of (4.35), and replace the respective expresion in the bracket with $2\pi\delta(\omega)$. (The r.h.s. of (4.35) corresponds then to the r.h.s. of Eq. (4.21); the first-order terms in the Ωs from denominators correspond to the terms at the l.h.s. of (4.21).) With $\Omega\sim\Gamma$, the accuracy of the approximation will reveal the accuracy of the Boltzmann equation (the frequency structure of the function F, Eq. (4.37), will be valid with an accuracy of the order Γ). With (4.37) we may rewrite the r.h.s. of (4.35) into

in the most involved case of $\mathrm{Re}\Sigma^{+}$. Equation (4.34) occurs to represent a solution to the kinetic equation (4.31) (with an adequate boundary condition). For the simpler cases of $\mathrm{Re}\Sigma^{+}$, proper approximations to (4.33) represent solutions to Eqs. (4.26) or (4.4).

If Γ is comparable with particle energies in a given system, the dynamics may not be described with the Boltzmann equation. This is to be attributed to the time-energy uncertainty principle because of the appearance of $\hbar$ in the comparison of particle energies with the time of variations in a system. A situation like that occurs in high-energy nuclear collisions which are the topic of a following paper of the series. Let us mention that in the low-density limit $\Gamma \approx i\Sigma^{>}$ and $\hbar\Gamma^{-1}$ constitutes a mean time between succesive particle-particle collisions. In the Born or T-matrix approximations to the self-energies $\hbar\Gamma^{-1}$ is of the order of $1/(n\sigma v)$, where n is a density of particles, σ - an average total particle-particle cross-section, and v - an average particle velocity.

In judging the applicability of the Boltzmann equation to a given system, one has not only to take into account the magnitude of particle energies. The possibility of describing a system with the Boltzmann equation can be further limited by the properties of an interaction. In Appx. G we examine values of microscopic variables that give dominant contributions to integrals (4.17) of Green's functions with self-energies. The values of variables correspond to energy and momentum dependence of self-energies. The analysis of Appx. G is performed for a system in thermodynamic equilibrium with self-energies in the Born approximation. Below, we analyze the equation of motion for the Wigner function, Fourier-transformed in macroscopic time.

Equation (4.1), with the self-energies in the Born approximation, Fourier-transformed in the macroscopic time, yields the following equation for the momentum distribution in a homogeneous system:

$$\int \frac{d\underset{\sim}{p}_1}{(2\pi)^3} \int \frac{d\underset{\sim}{p}'}{(2\pi)^3} \int \frac{d\underset{\sim}{p}_1'}{(2\pi)^3} \left(\frac{i}{\Delta e + \Omega/2 + i\varepsilon} - \frac{i}{\Delta e - \Omega/2 - i\varepsilon} \right)$$

$$\times\ (2\pi)^3\ \delta(\underset{\sim}{p} + \underset{\sim}{p}_1 - \underset{\sim}{p}' - \underset{\sim}{p}_1')((V(\underset{\sim}{p} - \underset{\sim}{p}')^2 \mp V(\underset{\sim}{p} - \underset{\sim}{p}')V(\underset{\sim}{p} - \underset{\sim}{p}_1'))$$

$$\times\ F(\underset{\sim}{p}, \underset{\sim}{p}_1, \underset{\sim}{p}', \underset{\sim}{p}_1'; \Omega) \quad . \tag{4.40}$$

The dependence of the remaining part of (4.40) on Δe will be decisive in replacing the dfference in the bracket by $2\pi\delta(\Delta e)$. The dependence will be set by properties of the particle distribution and properties of the potential. From (4.40), it follows that for the Boltzmann equation we must have $p\Delta p/m \gg \Gamma$, with p a characteristic particle momentum and Δp a scale of variation of the particle distribution in momentum. The same condition must also be satisfied when Δp is taken as the scale of variation of the potential with the momentum transfer. For a potential with a spatial range η, the condition reduces to $p/\eta m \gg \Gamma$. The condition states that the interaction time, evaluated as the time of flight through the potential range, must be much smaller than the time between the collisions. The condition related to the potential is classical, as opposed to the condition related to the particle distribution.

In Appx. F we present an analysis similar to the above, for self-energies in the T-matrix approximation. Conditions for the Boltzmann equation, related to the interaction, read $p\Delta p/m \gg \Gamma$ and $\Delta\omega \gg \Gamma$, with Δp and $\Delta\omega$ scales of variation of the scattering matrix with momentum transfer and energy, respectively.

5. Dynamics for a correlated initial state.

The perturbation expansion and Green's function equations of motion, outlined in Sects. 2 and 3, apply to initial states that admit a Wick decomposition. This is a considerable limitation when the evolution starts at a finite time. If one wanted e.g to include a strong repulsion in the nuclear interaction at small distances, then the thus-far obtained results would be of a little use. In Appendix H we present a perturbation expansion and Green's function equations of motion for a completely general initial state. Below we shall discuss a practical method of switching on the correlations in the initial state.

A correlated initial state may be prepared from a noncorrelated state through an imaginary-time evolution. The technique is applicable when the initial state can be defined as a lowest eigenvalue state of certain operator $\hat{\mathcal{H}}$ (also when the initial state is defined with the equilibrium density operator). In contrast to the perturbation expansion of Appx. H, the resulting perturbation expansion will not contain correlation matrices $\tilde{G}^{o}_{k}$. It should be noted that, when the noncorrelated state is a nondegenerate lowest eigenvalue state of an operator $\hat{\mathcal{H}}^{o}$, and when the imaginary-time evolution lasts infinitely long, then the Goldstone expansion may be obtained for the correlated state. Implementation of the imaginary-time evolution into the nonequilibrium Green's function method has been suggested in Ref. [20]; see also Ref. [21].

Let us take a certain state $|\Phi\rangle$ and expand it in a basis of eigenstates $\{\Psi_n\}$ of an operator $\hat{\mathcal{H}}$

$$|\Phi\rangle = \sum_n a_n |\Psi_n\rangle \quad . \tag{5.1}$$

Then

$$\lim_{\tau\to\infty} \frac{e^{-\tau\hat{\mathcal{H}}}|\Phi\rangle}{e^{-\tau\hat{\mathcal{H}}}|\Phi\rangle} = \lim_{\tau\to\infty} \frac{\sum_n a_n e^{-\tau\varepsilon_n}|\Psi_n\rangle}{(\sum_n |a_n|^2 e^{-2\tau\varepsilon_n})^{1/2}} = |\Phi_0\rangle \quad , \tag{5.2}$$

where $\{\varepsilon_n\}$ are the eigenvalues of $\hat{\mathcal{H}}$, and $|\Psi_0\rangle$ is the normalized projection of $|\Phi\rangle$ onto the lowest eigenvalue subspace onto which the projection does not vanish. According to (5.2), a projecting out of the lowest eigenvalue state corresponds to an imaginary-time evolution from $i\tau = i\infty$ to $i\tau = 0$. On assigning the resulting state to a system at a time t_0 we put $i\tau \equiv t - t_0$. When evaluating an expectation value at a time t_0

$$\langle\Phi_0|\hat{O}|\Psi_0\rangle = \lim_{\tau\to\infty} \frac{\langle\Phi|e^{-\tau\hat{\mathcal{H}}}\hat{O}\,e^{-\tau\hat{\mathcal{H}}}|\Phi\rangle}{\langle\Phi|e^{-\tau\hat{\mathcal{H}}}\,e^{-\tau\hat{\mathcal{H}}}|\Phi\rangle} \quad , \tag{5.3}$$

we deal with an evolution running along a contour from Fig. 4. When evaluating expectation values of Heisenberg picture operators at $t_1 > t_0$, we deal with the evolution contour extended along the real time axis, Fig. 5. If the pure state expectation values at the r.h.s. of (5.3) are replaced by those with respect to a general density operator, then the imaginary-time evolution occurs to project out, similarly as for a pure state, a part of the density operator within a subspace of a lowest ε_n. An initial state of a real evolution, specified with an equilibrium density operator of a temperature $T = (k_B\beta)^{-1}$, corresponds to an imaginary evolution that starts from a unity operator $\hat{1}$

$$\langle\hat{O}\rangle = \frac{\mathrm{Tr}(e^{-\beta\hat{\mathcal{H}}}\,\hat{O})}{\mathrm{Tr}(e^{-\beta\hat{\mathcal{H}}})} \quad . \tag{5.4}$$

Because of a cyclic property of the trace, it is not important how the imaginary evolution interval is positioned with respect to the real time axis;

contours corresponding to (5.4) and to an expectation value of a Heisenberg picture operator are presented in Fig. 6. In the zero-temperature limit $\beta \to \infty$, one obtains, starting from the operator $\hat{1}$, a projection operator onto the $\hat{\mathcal{H}}$ lowest eigenvalue subspace. This procedure is more slowly convergent than starting with an imaginary evolution from an adequately chosen state, e.g. a lowest eigenvalue state of a 1-particle operator $\hat{\mathcal{H}}^0$. In the numerical calculation of the next paper of the series we start from such a state, and the imaginary evolution lasts a finite time. The contours from Fig. 7 correspond to the finite time imaginary evolution.

We shall now find a Green's function perturbation expansion. We begin with a consideration of the expectation value of an operator with one time argument. The operator expectation value, with respect to a state obtained at t_0 through an imaginary evolution, will be denoted by $\langle\langle \hat{O}_H(t) \rangle\rangle$. We have

$$\langle\langle \hat{O}_H(t) \rangle\rangle = \langle\langle \hat{U}_S(t_0,t)\ \hat{O}\ \hat{U}_S(t,t_0) \rangle\rangle \quad , \tag{5.5}$$

where the Schrödinger picture evolution operator

$$\hat{U}_S(t,t') = e^{-i\hat{H}(t-t')} \quad . \tag{5.6}$$

From the method of preparation of the state at t_0, it follows that

$$\langle\langle \hat{O}_H(t) \rangle\rangle$$

$$= \frac{\langle \hat{U}_S(t_0-i\tau_0,t_0)\hat{U}_S(t_0,t)\ \hat{O}\ \hat{U}_S(t,t_0)\hat{U}_S(t_0,t_0+i\tau_0) \rangle}{\langle \hat{U}_S(t_0-i\tau_0,t_0+i\tau_0) \rangle} \tag{5.7}$$

(one might eventually introduce at once the limit $\tau_0 \to \infty$). The evolution operator for imaginary times is

$$\hat{U}_S(t,t') = e^{-i\hat{\mathcal{H}}(t-t')} \quad . \tag{5.8}$$

The expectation values $\langle \cdot \rangle$ in (5.7) are taken with respect to a state from which the imaginary evolution starts. With a notation

$$\hat{H}(t) = \begin{cases} \hat{\mathcal{H}} & \text{for imaginary times} \quad , \\ H & \text{for real times} \quad , \end{cases} \tag{5.9}$$

the evolution operator on a contour from Fig. 7b is

$$\hat{U}_S(t,t') = T[\exp(-i \int_{t'}^{t} dt_1 \hat{H}(t_1))] \quad , \tag{5.10}$$

with the integral running along the contour interval from t' to t. The expectation value (5.7) may be written as

$$<<\hat{O}_H(t)>> = \frac{<T[\hat{U}_S(t_o-i\tau_o,t_o+i\tau_o)\ \hat{O}_S(t)]>}{<\hat{U}_S(t_o-i\tau_o,t_o+i\tau_o)>} \tag{5.11}$$

We write the index S at the $\hat{O}$ operator to stress that this is a Schrödinger picture operator--most often independent of time. The time argument of the operator determines the position in the operator product, where the operator is to be placed by the time-ordering operator on the contour. We may generalize the expectation value of an $\hat{O}$ operator at a time t, $<<\hat{O}(t)>>$, to imaginary times, by defining the expectation value with a r.h.s. of (5.11).

We shall assume a partition of $\hat{H}(t)$

$$\hat{H}(t) = \hat{H}^0(t) + \hat{H}^1(t) \quad , \tag{5.12}$$

with $\hat{H}^0(t)$ a 1-particle operator. The free evolution operator is

$$\hat{U}^0(t,t') = T[\exp(-i \int_{t'}^{t} dt_1\ \hat{H}^0(t_1))] \quad . \tag{5.13}$$

There holds

$$\begin{aligned} \hat{U}_S(t,t') &= T[\hat{U}^0(t,t') \exp\ (-i \int_{t'}^{t} dt_1\ \hat{H}^1_S(t_1))] \\ &= \hat{U}^0(t,t') + (-i) \int_{t'}^{t} dt_1 \hat{U}^0(t,t_1)\ \hat{H}^1(t_1)\ \hat{U}^0(t_1,t') \\ &\quad + (-i)^2 \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \hat{U}^0(t,t_1)\hat{H}^1(t_1)\hat{U}^0(t_1,t_2)\hat{H}^1(t_2)\hat{U}^0(t_2,t') \\ &\quad + \ldots \quad . \end{aligned} \tag{5.14}$$

Equation (5.14) may be verified by ascertaining that the r.h.s. of (5.14) satisfies the same differential equation on a contour as (5.10), with the same boundary condition at $t = t'$. Accordingly we have, for an expectation value defined with the r.h.s. of (5.11),

$$\langle\langle\hat{O}(t)\rangle\rangle$$

$$= \frac{\langle T[\hat{U}^0(t_0-i\tau_0,t_0+i\tau_0)\ \exp\ (-i \oint dt_1\hat{H}^1_S(t_1))\hat{O}_S(t))]\rangle}{\langle T[\hat{U}^0(t_0-i\tau_0,t_0+i\tau_0)\ \exp\ (-i \oint dt_1\hat{H}^1_S(t_1))]\rangle} \quad . \tag{5.15}$$

We shall assume that the operator $\hat{\mathcal{H}}$ and a density operator of the initial state of the imaginary evolution both commute with a particle-number operator. Further, we shall assume that the initial state of the imaginary evolution admits the Wick decomposition. I.e., we shall assume that noninteracting many-particle Green's functions (defined by expectation values of the type $\langle\langle\cdot\rangle\rangle$ with $\hat{H}^1(t) \equiv 0$) factorize into 1-particle Green's functions

$$iG^0(1,1') = \langle\langle T[\hat{\psi}(1)\hat{\psi}^\dagger(1')]\rangle\rangle^0$$

$$= \frac{\langle T[\hat{U}^0(t_0-i\tau_0,t_0+i\tau_0)\ \hat{\psi}_S(1)\hat{\psi}^\dagger_S(1')]\rangle}{\langle\hat{U}^0(t_0-i\tau_0,t_0+i\tau_0)\rangle} \quad . \tag{5.16}$$

The problem of the Wick decomposition within the imaginary-time evolution is discussed in Appx. A.

If both $\hat{\mathcal{H}}^1$ and $\hat{H}^1$ are the same 2-body potential interaction, and the difference between the evolution generators in real and imaginary times lies only in the 1-particle parts of the generators, then the Feynman rules for evaluating the Green's function

$$iG(1,1') = \langle\langle T[\hat{\psi}(1)\hat{\psi}^\dagger(1')]\rangle\rangle$$

$$= \frac{\langle T[\hat{U}_S(t_0-i\tau_0,t_0+i\tau_0)\ \hat{\psi}_S(1)\hat{\psi}^\dagger_S(1')]\rangle}{\langle\hat{U}_S(t_0-i\tau_0,t_0+i\tau_0)\rangle} \tag{5.17}$$

are such as in Sec. 2. The time integrals are carried along the contour from Fig. 7b. The contour must extend above the largest of the real times of the evaluated Green's function. A reduction of a contour may occur only for real times, and a minimal contour is the one from Fig. 7a. The disconnected diagrams do not now vanish, but are cancelled by the denominator in (5.17). With

$$\hat{H}^{o}(t) = \int d\underset{\sim}{x}_1 \int d\underset{\sim}{x}_{1'} \hat{\Psi}^{\dagger}(\underset{\sim}{x}_1)\, h(\underset{\sim}{x}_1,\underset{\sim}{x}_{1'};t)\, \hat{\Psi}(\underset{\sim}{x}_{1'}) \quad , \tag{5.18}$$

and the function h for real times

$$h(\underset{\sim}{x}_1,\underset{\sim}{x}_{1'};t) = -\frac{\nabla_1^2}{2m}\,\delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_{1'}) \quad , \tag{5.19}$$

the Green's function equations of motion on the contour have a form

$$i\frac{\partial}{\partial t_1} G(1,1') - \int d\underset{\sim}{x}_2\, h(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t_1)\, G(\underset{\sim}{x}_2,t_1,1')$$

$$= \delta(1,1') + \oint d2\, \Sigma(1,2)G(2,1') \quad , \tag{5.20a}$$

$$-i\frac{\partial}{\partial t_{1'}} G(1,1') - \int d\underset{\sim}{x}_2\, G(1,\underset{\sim}{x}_2,t_{1'})\, h(\underset{\sim}{x}_2,\underset{\sim}{x}_{1'};t_{1'})$$

$$= \delta(1,1') + \oint d2\, G(1,2)\, \Sigma(2,1') \quad . \tag{5.20b}$$

The function $\delta(1,1') = \delta(t_1,t_{1'})\delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_{1'})$, and the function $\delta(t_1,t_{1'})$ is defined on a contour in the complex time plane

$$\oint dt_{1'}\,\delta(t_1,t_{1'})F(t_{1'}) = F(t_1) \quad . \tag{5.21}$$

For real times, the Kadanoff-Baym equations have similar forms to (3.6) and (3.7), but they contain extra contributions from imaginary parts of a contour. An equation for $G^{<}(1,1')$, where t_1 is real, has e.g. the form

$$\left(i\frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m}\right) G^{<}(1,1') = \int d\underset{\sim}{x}_2\, \Sigma_{HF}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t_1)\, G^{<}(\underset{\sim}{x}_2,t_1,1')$$

$$+ \int_{t_0+i\tau_0}^{t_1} d2\, \Sigma^{>}(1,2)G^{<}(2,1') + \int_{t_1}^{t_{1'}} d2\, \Sigma^{<}(1,2)G^{<}(2,1')$$

$$+ \int_{t_{1'}}^{t_o - i\tilde{\tau}_o} d2\, \Sigma^{<}(1,2) G^{>}(2,1') \quad . \tag{5.22}$$

In Ref. [2] the Kadanoff-Baym equations have been derived for an initial state of a real evolution specified with an equilibrium density operator (Fig. 6). A nontrivial evolution for the real times has been achieved by disturbing the system with an external potential.

6. Final remark

The investigating the Green's function methods in the paper has lead to several new results. We have clarified the transition from the Green's function equations to the kinetic equations, and the analysis of the thermodynamic equlibrium on the basis of the Green's function equations of motion. We developed the Green's function methods on a contour in the complex time plane, and the perturbation expansion and Green's function equations for a general initial state. The results should be of use in problems of nuclear physics, as well as in other branches of physics.

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Appendix A. Wick decomposition

The consideration will be confined to initial states specified by density operators commuting with a particle-number operator. For such states, an expectation value of a field-operator product vanishes, if the product contains a different number of the $\hat{\psi}$ operators from a number of $\hat{\psi}^{\dagger}$.

A state admitting the Wick decomposition is a state for which expectation values of products of the interaction-picture field-operators (many-particle Green's functions) factorize into expectation values of pairs of operators (1-particle Green's functions)

$$\langle \hat{A}\hat{B} \ldots \hat{Y}\hat{Z} \rangle = \hat{A}^{\cdot}\hat{B}^{\cdot} \ldots \hat{Y}^{\cdot\cdot}\hat{Z}^{\cdot\cdot} + \hat{A}^{\cdot}\hat{B}^{\cdot\cdot} \ldots \hat{Y}^{\cdot}\hat{Z}^{\cdot\cdot} + \ldots$$

$$= \text{sum over all possible contracted products,} \tag{A.1}$$

where the contraction

$$\hat{A}^{\cdot}\hat{B}^{\cdot} = \langle \hat{A}\hat{B} \rangle \quad . \tag{A.2}$$

For the fermion operators, when one rearranges the order of the operators in a given term of (A.1), with intention to bring a contracted pair of operators next to one another, the term is to be multiplied by the sign of the performed permutation. In Eq. (A.2) the operators are in the same order as they appear in (A.1). When deriving Feynman diagrams, one applies the Wick decomposition to operators which are in a specific time-order at the l.h.s. of (A.1).

We shall show that states specified by density operators of the form

$$\hat{\rho} = \exp(\hat{\mathcal{A}}) \quad , \tag{A.3}$$

with $\hat{\mathcal{A}}$ being a 1-particle operator, admit the Wick decomposition. The l.h.s. of Eq. (A.1) will satisfy the same differential evolution equations in every field-operator argument, as the r.h.s. Therefore the consideration of the factorisation (A.1) may be limited to a one set of times, e.g. when all the interaction-picture operators are taken at the initial time t_0.

We have

$$\hat{\mathcal{A}} = \sum_j \mathcal{A}_j \hat{a}_j^\dagger \hat{a}_j \quad , \tag{A.4}$$

with $\mathcal{A}_j$ real and $\hat{a}_j^\dagger$ being creation operators of the 1-particle basis, that diagonalizes $\hat{\mathcal{A}}$. Let us expand $\hat{\rho}$ in a basis of the Hartree-fock states $|\{n_j\}\rangle$ - the states that arise from applications to the vacuum of sets of 1-particle creation operators

$$|\{n_j\}\rangle = \left(\prod_j \frac{1}{\sqrt{n_j!}} (\hat{a}_j^\dagger)^{n_j}\right)|0\rangle \quad .$$

We have

$$\langle\{m_j\}|\hat{\rho}|\{n_j\}\rangle = \delta_{\{m_j\},\{n_j\}} \exp\left(\sum_j \mathcal{A}_j n_j\right) \quad , \tag{A.5}$$

and

$$\hat{\rho} = \sum_{\{n_j\}} \exp\left(\sum_j \mathcal{A}_j n_j\right)|\{n_j\}\rangle\langle\{n_j\}| \quad . \tag{A.6}$$

In the boson case, in order that the state gives a finite particle-number expectation value, we must have $\mathcal{A}_j < 0$. Let us note that an unoccupied 1-particle state j_0 corresponds to $\mathcal{A}_{j_0} \to -\infty$. The vacuum corresponds to all $\mathcal{A}_j \to -\infty$. For fermions, a projection operator onto the Hartree-Fock state mat be obtained from $\hat{\rho}/\mathrm{Tr}(\hat{\rho})$ in the limit of $\mathcal{A}_j \to -\infty$ for the unoccupied states j, and $\mathcal{A}_j \to +\infty$ for the occupied states. For fermions

$$\hat{\rho}/\mathrm{Tr}(\hat{\rho}) = \prod_j [\langle\hat{a}_j\hat{a}_j^\dagger\rangle\hat{a}_j\hat{a}_j^\dagger + \langle\hat{a}_j^\dagger\hat{a}_j\rangle\hat{a}_j^\dagger\hat{a}_j] \quad ,$$

with $\langle\hat{a}_j^\dagger\hat{a}_j\rangle = (e^{-\mathcal{A}_j} + 1)^{-1}$.

We shall prove that for a density operator (A.3) and a set of anihilation and creation operators $\hat{\alpha}_a, \hat{\alpha}_b, \ldots, \hat{\alpha}_y, \hat{\alpha}_z$, there holds

$$\frac{\mathrm{Tr}(\hat{\rho}\hat{\alpha}_a\hat{\alpha}_b \ldots \hat{\alpha}_y\hat{\alpha}_z)}{\mathrm{Tr}(\hat{\rho})} = \frac{\mathrm{Tr}(\hat{\rho}\hat{\alpha}_a\hat{\alpha}_b)}{\mathrm{Tr}(\hat{\rho})} \frac{\mathrm{Tr}(\hat{\rho}\hat{\alpha}_c \ldots \hat{\alpha}_y\hat{\alpha}_z)}{\mathrm{Tr}(\hat{\rho})}$$

$$\mp \frac{\mathrm{Tr}(\hat{\rho}\hat{\alpha}_a\hat{\alpha}_c)}{\mathrm{Tr}(\hat{\rho})} \frac{\mathrm{Tr}(\hat{\rho}\hat{\alpha}_b\hat{\alpha}_d \ldots \hat{\alpha}_y\hat{\alpha}_z)}{\mathrm{Tr}(\hat{\rho})} + \ldots + \frac{\mathrm{Tr}(\hat{\rho}\hat{\alpha}_a\hat{\alpha}_z)}{\mathrm{Tr}(\hat{\rho})} \frac{\mathrm{Tr}(\hat{\rho}\hat{\alpha}_b\hat{\alpha}_c \ldots \hat{\alpha}_y)}{\mathrm{Tr}(\hat{\rho})} \quad . \tag{A.7}$$

The subsequent applications of (A.7) lead to Eq. (A.1) for the annihilation and creation operators. The latter implies Eq. (A.1) for the field operators, since these are linear combinations of annihilation and creation operators. We have

$$\hat{\alpha}_a\hat{\rho} = \zeta_a\hat{\rho}\hat{\alpha}_a \quad , \tag{A.8}$$

where $\zeta_a = e^{\mathcal{A}_a}$, if $\hat{\alpha}_a = \hat{a}_a$, and $\zeta_a = e^{-\mathcal{A}_a}$ if $\hat{\alpha}_a = \hat{a}_a$.

Let us take

$$\mathrm{Tr}(\hat{\rho}\hat{\alpha}_a\hat{\alpha}_b \ldots \hat{\alpha}_y\hat{\alpha}_z) \quad ,$$

and for fermions let us anticommute, and for bosons commute, the operator $\hat{\alpha}_a$ to the right. Upon application of (A.8) we find

$$\frac{\mathrm{Tr}(\hat{\rho}\hat{\alpha}_a\hat{\alpha}_b \ldots \hat{\alpha}_y\hat{\alpha}_z)}{\mathrm{Tr}(\hat{\rho})} = \frac{[\hat{\alpha}_a,\hat{\alpha}_b]_\pm}{1 \pm \zeta_a} \frac{\mathrm{Tr}(\hat{\rho}\hat{\alpha}_c \ldots \hat{\alpha}_y\hat{\alpha}_z)}{\mathrm{Tr}(\hat{\rho})}$$

$$\mp \frac{[\hat{\alpha}_a,\hat{\alpha}_c]_\pm}{1 \pm \zeta_a} \frac{\mathrm{Tr}(\hat{\rho}\hat{\alpha}_b\hat{\alpha}_d \ldots \hat{\alpha}_y\hat{\alpha}_z)}{\mathrm{Tr}(\hat{\rho})} + \ldots + \frac{[\hat{\alpha}_a,\hat{\alpha}_z]_\pm}{1 \pm \zeta_a} \frac{\mathrm{Tr}(\hat{\rho}\hat{\alpha}_b \ldots \hat{\alpha}_y)}{\mathrm{Tr}(\hat{\rho})} \quad , \tag{A.9}$$

where $[\cdot,\cdot]_+$ denotes an anticommutator, and $[\cdot,\cdot]_-$ a commutator. For two operators (A.9) takes form

$$\frac{\mathrm{Tr}(\hat{\rho}\hat{\alpha}_a\hat{\alpha}_b)}{\mathrm{Tr}(\hat{\rho})} = \frac{[\hat{\alpha}_a,\hat{\alpha}_b]_\pm}{1 \pm \zeta_a} \quad . \tag{A.10}$$

A combination of (A.9) and (A.10) yields (A.7).

Upon arriving at (A.1) or (A.7), one may take limits of $\mathcal{A}_j \to -\infty$, and in the fermion case $\mathcal{A}_j \to +\infty$, proving thereby the Wick decomposition for the limiting forms of $\hat{\rho}/\mathrm{Tr}(\hat{\rho})$. It should be pointed out, however, that for the vacuum state, or the fermion Hartree-Fock states, a more direct proof of the Wick decomposition may be obtained through the introduction of an operator

normal-product and an application of the Wick's theorem [3].

Now we shall show that all states which admit the Wick decomposition are described by the density operators of the form (A.3), involving eventually the limiting forms of $\hat{\rho}/\mathrm{Tr}(\hat{\rho})$ with $\mathcal{A}_j \to \pm\infty$. Let us take a state that admits the Wick decomposition and is described by a density operator $\hat{\rho}'$. We want to show that there exists a density operator of the form (A.3) such that

$$\hat{\rho}'/\mathrm{Tr}(\hat{\rho}') = \hat{\rho}/\mathrm{Tr}(\hat{\rho}) \quad . \tag{A.11}$$

When taking expectation values of arbitrary field-operator products from the sides of Eq. (A.11), we have by assumption the Wick decompositions for the both sides. In order to prove (A.11) it is sufficient to show that there exists an operator $\hat{\rho}$, of the form (A.3), such that the expectation values of pairs of the operators agree with those from $\hat{\rho}'$. The 1-particle density matrix $\langle\hat{\Psi}^\dagger(\underset{\sim}{x}')\hat{\Psi}(\underset{\sim}{x})\rangle = \mathrm{Tr}(\hat{\rho}'\hat{\Psi}^\dagger(\underset{\sim}{x}')\hat{\Psi}(\underset{\sim}{x}))/\mathrm{Tr}(\hat{\rho}')$ is hermitian and may be diagonalized. Generally

$$\langle\hat{\Psi}^\dagger(\underset{\sim}{x}')\hat{\Psi}(\underset{\sim}{x})\rangle = \sum_{j'j} \langle\hat{a}^\dagger_{j'}\hat{a}_j\rangle\varphi^*_{j'}(\underset{\sim}{x}')\varphi_j(x) \quad , \tag{A.12}$$

where $\hat{a}_j$ are annihilation operators of a basis of states with wavefunctions $\varphi_j(\underset{\sim}{x})$. For a basis that diagonalizes (A.12), $\langle\hat{a}^\dagger_{j'}\hat{a}_j\rangle = \langle\hat{a}^\dagger_j\hat{a}_j\rangle\delta_{j'j}$. For fermions $\langle\hat{a}^\dagger_j\hat{a}_j\rangle \leq 1$, which holds for any state of the system and any 1-particle state, as can be seen by expanding the density operator in a Hartree-Fock basis. We construct the operator $\hat{\mathcal{A}}$, Eqs. (A.3) and (A.4), using the operators of the basis that diagonalizes (A.12), with

$$\mathcal{A}_j = -\log\left(\frac{1}{\langle\hat{a}^\dagger_j\hat{a}_j\rangle} \mp 1\right) \quad . \tag{A.13}$$

The case of $\langle\hat{a}^\dagger_j\hat{a}_j\rangle = 0$ is to be understood as $\mathcal{A}_j \to -\infty$, and $\langle\hat{a}^\dagger_j\hat{a}_j\rangle = 1$ for fermions as $\mathcal{A}_j \to +\infty$. With Eq. (A.13), the expectation values of the pairs of the field operators, obtained from $\hat{\rho}$, agree with those from $\hat{\rho}'$, and this

completes the proof.

Let us now discuss a Wick decomposition in connection with an imaginary-time evolution (Sect. 5). In that case a Wick decomposition will stand for a factorisation of a noninteracting expectation value of a field-operator product:

$$\langle\langle \hat{A}\hat{B} \ldots \hat{Y}\hat{Z} \rangle\rangle^{0} = \hat{A}^{\cdot}\hat{B}^{\cdot} \ldots \hat{Y}^{\cdot\cdot}\hat{Z}^{\cdot\cdot} + \hat{A}^{\cdot}\hat{B}^{\cdot\cdot} \ldots \hat{Y}^{\cdot}\hat{Z}^{\cdot\cdot} + \ldots$$

$$= \text{sum over all possible contracted products} \tag{A.14}$$

with the contraction

$$\hat{A}^{\cdot}\hat{B}^{\cdot} = \langle\langle \hat{A}\hat{B} \rangle\rangle^{0} \quad . \tag{A.15}$$

The operators at the l.h.s. of (A.14) are to be ordered according to the time-arguments from a contour in the complex time plane. The contour may be arbitrarily deformed along the real axis (but not along the imaginary axis), running back and forth, as is in fact in general occuring with an evolution contour appropriate for the l.h.s. of Eq. (A.1). When all operators have real time-arguments, then Eqs. (A.14) and (A.15) reduce to Eqs. (A.1) and (A.2). Expectation values in (A.1) and (A.2) are to be taken with respect to the density operator

$$\hat{\rho} = e^{-\tau\hat{\mathcal{H}}^{0}} \hat{\tilde{\rho}}\, e^{-\tau\hat{\mathcal{H}}^{0}} \quad , \tag{A.16}$$

where $\hat{\mathcal{H}}^{0}$ is the generator and τ the duration of the imaginary evolution, and $\hat{\tilde{\rho}}$ specifies the initial state of the imaginary evolution. The real-evolution part of an operator $\hat{U}^{0}$ from (A.14) and (A.15), remaining upon extraction of the imaginary part into (A.16), corresponds to the interaction picture of the operators in (A.1) and (A.2).

The l.h.s. of Eq. (A.14) will satisfy the same differential evolution equations on a contour, in every field-operator argument as the r.h.s. A consideration of the factorisation may therefore be limited to one set of times, and we shall take for convenience all operators at an initial time of

the real evolution t_0. From the previous discussion it follows that the density matrix (A.16) must be of the form (A.3) with (A.4).
Equation (A.16) and the Baker-Campbell-Hausdorff (BCH) formula [22], imply then that $\hat{\tilde{\rho}}$ must also be of the form (A.3). With (A.16) and (A.3) for $\hat{\rho}$, the BCH formula will express the density operator $\hat{\tilde{\rho}}$ as an exponential of an infinite series of commutators of $\hat{\mathcal{A}}$ and $\hat{\mathcal{H}}^0$. With $\hat{\mathcal{A}}$ and $\hat{\mathcal{H}}^0$ being 1-particle operators, the series will consist of 1-particle operators. Let

$$\hat{\tilde{\rho}} = \exp(\hat{\mathcal{C}}) \quad , \quad \hat{\rho} = \exp(\hat{\mathcal{A}}) \quad ,$$

with

$$\hat{\mathcal{C}} = \sum_l \mathcal{C}_l \hat{c}_l^\dagger \hat{c}_l \quad ,$$

and in a basis that diagonalizes $\hat{\mathcal{H}}^0$

$$\hat{\mathcal{A}} = \sum_{mn} \mathcal{A}_{mn} \hat{b}_m^\dagger \hat{b}_n \quad , \quad \hat{\mathcal{H}}^0 = \sum_m h_m \hat{b}_m^\dagger \hat{b}_m \quad ,$$

and

$$\hat{c}_l^\dagger = \sum_m \alpha_{lm} \hat{b}_m^\dagger \quad .$$

Then, on studying matrix elements of $\hat{\rho}$ between 1-particle states, one finds that the matrix $\mathcal{A}$ equals

$$\mathcal{A} = \log(e^{-\tau(h_m + h_n)} \sum_l \alpha_{lm} \alpha_{ln}^* e^{\mathcal{C}_l}) \quad .$$

The logarithm of a matrix in the bracket is well-defined, because the matrix is hermitian and positive definite.

Appendix B. Variational derivation of the perturbation expansion

We introduce a Green's function in the presence of an external potential

$$iG(1,1')_U = \frac{\langle T[\hat{\Psi}_H(1)\hat{\Psi}_H^\dagger(1')\hat{S}_H]\rangle}{\langle T[\hat{S}_H]\rangle} \quad , \tag{B.1}$$

where

$$\hat{S}_H = \exp(-i\int d1''\ U(1'')\hat{\Psi}_H(1'')\hat{\Psi}_H^\dagger(1'')) \quad . \tag{B.2}$$

Here $1\equiv(\underset{\sim}{x}_1,t_1)$, $\int d1 \equiv \int dt_1 \int d\underset{\sim}{x}_1$. When a given expression will be varied with respect to the potential U, we shall assume that U is different on each branch of the contour. In the final results the potential U is to be put equal to 0.

It can be shown (see Eqs. (3.14), (3.15), (3.35), and (3.36)) that the Green's function (B.1) satisfies the equation

$$(i\frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U(1))G(1,1')_U = \delta(1,1') \mp i\int d2\ V(1,2)G_2(1,2;1',2^+)_U \quad , \tag{B.3}$$

with the 2-particle Green's function

$$i^2G_2(1,2;1',2')_U = \frac{\langle T[\hat{\Psi}_H(1)\hat{\Psi}_H(2)\hat{\Psi}_H^\dagger(2')\hat{\Psi}_H^\dagger(1')\hat{S}_H]\rangle}{\langle T[\hat{S}_H]\rangle} \quad . \tag{B.4}$$

Here $\delta(1,1') \equiv \delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_{1'})\delta(t_1,t_{1'})$, $V(1,1') = V(\underset{\sim}{x}_1 - \underset{\sim}{x}_{1'})$ $\times\ \delta(t_1,t_{1'})$.

Next we introduce a noninteracting Green's function

$$iG^0(1,1')_U = \frac{\langle T[\hat{\Psi}_I(1)\hat{\Psi}_I^\dagger(1')\hat{S}_I]\rangle}{\langle T[\hat{S}_I]\rangle} \quad , \tag{B.5}$$

where

$$\hat{S}_I = \exp(-i\int d1''\ U(1'')\hat{\Psi}_I(1'')\hat{\Psi}_I^\dagger(1'')) \quad . \tag{B.6}$$

The noninteracting Green's function satisfies the equation

$$(i\frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U(1))G^0(1,1')_U = \delta(1,1') \quad . \tag{B.7}$$

Equations (B.3), (B.7), and the identity

$$\frac{\delta}{\delta U(2)} G(1,1')_U = (G_2(1,2:1',2^+)_U - G(1,1')_U G(2,2^+)_U) \quad , \tag{B.8}$$

imply an equation for the Green's function

$$G(1,1')_U = G^0(1,1')_U + i\int d1'' \int d2\, G^0(1,1'')_U V(1'',2)$$
$$\times(\frac{\delta}{\delta U(2)} G(1'',1')_U \mp G(1'',1')_U G(2,2^+)_U) \quad . \tag{B.9}$$

From (B.9) a perturbation expansion for G may be obtained. The first-order terms arise from replacement of G on r.h.s. with G^0. The second-order terms emerge from insertion of the first-order terms into the r.h.s., and the procedure may be continued. In the first step a knowledge of $\delta G^0/\delta U$ is necessary. Commonly one would conclude, from an equation following from (B.7)

$$(i\frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U(1)) \frac{\delta}{\delta U(2)} G^0(1,1')_U = \delta(1,2)G^0(2,1')_U \quad : \tag{B.10}$$

and Eq. (B.7), that

$$\frac{\delta}{\delta U(2)} G^0(1,1')_U = G^0(1,2)_U G^0(2,1')_U \quad . \tag{B.11}$$

The subsequent insertions into Eq. (B.9) and use of Eq. (B.11) yield the perturbation expansion known from Sect. 2.

Since we never referred to the properties of an initial state, the above would indicate that the perturbation expansion relies always on the noninteracting 1-particle Green's functions G^0, independent of the properties. In fact an eventual error arises when one concludes Eq. (B.11)

from Eq. (B.10), and one ignores the possibility of adding a solution of the homogenous differential equation. The presence of a homogenous equation solution is set by the initial conditions at t_0. From the definition (B.5) it follows that

$$\frac{\delta}{\delta U(2)} G^0(1,1')_U = \mp(G_2^0(1,2:1',2^+)_U - G^0(1,1')_U G^0(2,2^+)_U) \quad , \qquad (B.12)$$

and the higher variational derivatives of G^0 introduce the higher noninteracting many-particle Green's functions into the expansion. The adoption of (B.11), instead of the more general (B.12), corresponds to the assumption of the factorisation of all initial many-particle Green's functions. (The reading of Appendix H may clarify this aspect of the problem.)

Appendix C. Perturbation theory rules upon separation of the contour into branches

The rules serve for the evaluation of a specific Green's function type: $iG^{<}$, $iG^{>}$, iG^{c}, or iG^{a}. (The rules may also be employed in evaluation of a specific type of self-energy or other functions.)

1. Draw a line dividing the plane into two parts that will correspond to the two time-branches - chronological and antichronological. Place the points, corresponding to the function arguments, at one branch or the opposite branches, according to the type of the evaluated function. Draw all topologically-distinct connected and directed diagrams. The diagrams, which are differently cut by the division line, are distinct. The division line may not pass through the potential.
2. A particle line represents $iG^{o<}$, $iG^{o>}$, iG^{oc}, or iG^{oa}, depending on the line start and end positions.
3. To an interaction line there corresponds a factor $-iV(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) \times \delta(t_1 - t_2)$.
4. To an interaction line at the antichronological branch there corresponds a factor (-1).
5. To a single particle-line, that forms a closed loop or is linked by the same interaction line, there corresponds a function $iG^{o<}$.
6. For fermions attribute to a diagram a factor $(-1)^F$, where F is the number of particle loops.
7. Integrate all internal vertices over a whole space, and in time from t_o to the maximum argument of the evaluated function.
8. To every antichronological-side time-integration there corresponds a factor (-1).

Jointly the rules 4 and 8 give a factor (-1) for every antichronological-branch interaction which gets both vertices integrated. The rule 7 necessitates a

complement. Let the division line into branches be a dashed line and let us introduce a vertical time-axis, see Fig. 8. Then the diagrams from Fig. 8 give expressions that differ only with a sign. Generally it is sufficient to carry the internal time-integrations in a given part of a diagram up to the maximum external time, because jointly the integrations above that time cancel out.

Appendix D. Functions on a contour

Let us take a contour with a top at a time t_{max}, see Fig. 9. The Green's function on a contour possesses the following symmetry property: when $t_1 > t_2$ then $G(\underset{\sim}{x}_1,t_1,\underset{\sim}{x}_2,t_2) = G(\underset{\sim}{x}_1,t_1^T,\underset{\sim}{x}_2,t_2)$, where t_1^T has the same time-axis value as t_1, but lies at the opposite side of the contour $((t_1^T)^T \equiv t_1)$. The same occurs for the second argument of the Green's function, when $t_2 > t_1$. We shall use below the following notation: $1 \equiv (\underset{\sim}{x}_1,t_1)$, $1^T \equiv (\underset{\sim}{x}_1,t_1^T)$, $\int d1 \equiv \int dt_1 \int d\underset{\sim}{x}_1$, $\delta(1,1') \equiv \delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_{1'})\delta(t_1,t_{1'})$. If the functions F and C, that have no singularities for equal time-variables on the contour, possess the above symmetry, then the function

$$E(1,1') = \int d2\ F(1,2)C(2,1') \tag{D.1}$$

also possesses the symmetry. Only the time-integration is relevant here. Let e.g. $t_1 > t_{1'}$, then for $t_2 > t_1$ $F(1,2)C(2,1') = F(1,2^T)C(2^T,1')$, and the sides of this equality enter the integral with opposite signs. The integration above t_1 cancels out. With F having no singularity, only the values of F for $t_2 < t_1$ enter the integral, and from the symmetry of F follows the symmetry of E in t_1. The value of E depends only on values of F and C for $t < t_1$. Let us consider a possibility of singularities in F or C for equal time-variables on the contour. In order that E possesses the symmetry, the effect of a singularity must not depend on the side of the contour at which the singularity is placed. Generally such singularities are of the form

$$\left(\frac{\partial}{\partial t_1}\right)^n \delta(t_1,t_2) \quad . \tag{D.2}$$

If both F and C possess singularities of this type, then the singularity of E is also the type (D.2). From the symmetry property of the functions, it follows that the considered functions are of the form

$$F(1,2) = F^{\delta}(1,2) + \theta(t_1,t_2)F^{>}(1,2) + \theta(t_2,t_1)F^{<}(1,2) \quad , \tag{D.3}$$

with F^{δ} the singular part of the function (of the type (D.2)), and with the functions $F^{\gtrless}$ defined on the time-axis.

The space of the functions having the symmetry property, and supplemented with the form (D.2) of singularities, is closed with respect to the operation defined by (D.1). We assume that G has an inverse in that space

$$\int d2\ G^{-1}(1,2)G(2,1') = \int d2\ G(1,2)G^{-1}(2,1') = \delta(1,1') \quad . \tag{D.4}$$

From (D.4) and the symmetry of the functions, it follows, that G^{-1} does not depend on the choice of contour (i.e. t_{max}, Fig. 9) and that G^{-1} with time-arguments <t depends only on G with time-arguments <t.

The inverse of G^0, Eq. (2.30), is

$$G^{0-1}(1,2) = (i\frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m})\delta(1,2) \quad , \tag{D.5}$$

which follows from the equations of motion of the interaction-picture field-operators, and Eqs. (2.29) and (2.12). The (proper) self-energy will be defined by

$$\Sigma(1,2) = G^{0-1}(1,2) - G^{-1}(1,2) \quad . \tag{D.6}$$

From Eq. (2.27), it follows that under complex conjugation we have $[iG(1,2)]^{*}=iG(2^T,1^T)$, and further

$$[iG(1,2)]^{\dagger} = iG(1^T,2^T) \quad . \tag{D.7}$$

For the functions $iG^{\gtrless}$, Eqs. (2.3) and (2.10), we have $[iG^{\gtrless}(1,2)]^{\dagger}=iG^{\gtrless}(1,2)$, i.e. the functions $iG^{\gtrless}$ are hermitian. Let us study the properties of G^{-1} under conjugation. Upon taking hermitian conjugates of the sides of Eq. (D.4), we find, with (D.7),

$$\int d2\ (-)G(1^T,2^T)[G^{-1}(2,1')]^{\dagger} = \delta(1',1) \quad .$$

We change the sequence of integration of the sides of the contour, i.e. in the functions we change the argument 2 into 2^T, and simultaneously we change the overall sign,

$$\oint d2\ G(1^T,2)[G^{-1}(2^T,1')]^\dagger = \delta(1',1) \quad .$$

Changing 1 and 1' into 1^T and $1'^T$ and using

$$\delta(1,1') = \delta(1',1) = -\delta(1^T,1'^T) \quad , \tag{D.8}$$

we find

$$\oint d2\ G(1,2)[-G^{-1}(2^T,1'^T)]^\dagger = \delta(1,1') \quad ,$$

which implies

$$[iG^{-1}(1,2)]^\dagger = iG^{-1}(1^T,2^T) \quad , \tag{D.9}$$

comp. (D.7).

We shall consider functions of the form (D.3), for which

$$[iF(1,2)]^\dagger = iF(1^T,2^T) \quad . \tag{D.10}$$

To this class belong the functions G^o, G, G^{o-1}, G^{-1}, Σ, and other functions with which we shall deal in this series. From Eqs. (D.10), (D.3), (D.2), (D.8), and the equality $\theta(t_2,t_1) = \theta(t_1^T,t_2^T)$, it follows that

$$[F^\delta(1,2)]^\dagger = F^\delta(1,2) \quad , \quad [iF^{\gtrless}(1,2)]^\dagger = iF^{\gtrless}(1,2) \quad . \tag{D.11}$$

We shall define, on the time-axis, the retarded and advanced functions

$$F^+(1,2) = F^\delta(1,2) + (F^>(1,2) - F^<(1,2))\theta(t_1-t_2) \quad , \tag{D.12a}$$

$$F^-(1,2) = F^\delta(1,2) - (F^>(1,2) - F^<(1,2))\theta(t_2-t_1) \quad , \tag{D.12b}$$

where the singular part is taken such as on the chronological branch. We have the relations

$$F^+(1,2) - F^-(1,2) = F^>(1,2) - F^<(1,2) \quad , \tag{D.13}$$

and

$$[F^+(1,2)]^\dagger = F^-(1,2) \quad . \tag{D.14}$$

We may define the hermitian functions

$$\mathrm{Re}F^{+}(1,2) = \frac{1}{2}(F^{+}(1,2) + F^{-}(1,2))$$

$$= F^{\delta}(1,2) + \frac{1}{2}\epsilon(t_1-t_2)(F^{>}(1,2) - F^{<}(1,2)) \quad , \tag{D.15}$$

$$\mathrm{Im}F^{+}(1,2) = \frac{1}{2i}(F^{+}(1,2) - F^{-}(1,2)) = \frac{1}{2i}(F^{>}(1,2) - F^{<}(1,2)) \quad , \tag{D.16}$$

where $\epsilon(t_1-t_2) = \theta(t_1-t_2)-\theta(t_2-t_1)$. We have

$$\mathrm{Re}F^{+}(1,2) = F^{\delta}(1,2) + i\epsilon(t_1-t_2)\mathrm{Im}F^{+}(1,2) \quad . \tag{D.17}$$

We shall show that a Fourier-transform in relative variables, of a hermitian function, is real (see Eqs. (2.5), (2.8), (2.11)). We use a 4-dimensional notation

$$H(p;X) = \int d^4x\ e^{ipx}H(X+x/2,X-x/2) \quad . \tag{D.18}$$

We have

$$H^{*}(p;X) = \int d^4x\ e^{-ipx}H^{*}(X+x/2,X-x/2) = \int d^4x\ e^{-ipx}H^{\dagger}(X-x/2,X+x/2)$$

$$= \int d^4x\ e^{-ipx}H(X-x/2,X+x/2) = H(p;X) \quad . \tag{D.19}$$

For the further purposes of the paper, we shall define the hermitian functions

$$A(1,2) = -2\mathrm{Im}G^{+}(1,2) = i(G^{>}(1,2) - G^{<}(1,2)) \quad , \tag{D.20}$$

$$\Gamma(1,2) = -2\mathrm{Im}\Sigma^{+}(1,2) = i(\Sigma^{>}(1,2) - \Sigma^{<}(1,2)) \quad . \tag{D.21}$$

From (2.12) follows

$$\int \frac{d\omega}{2\pi} A(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = 1 \quad . \tag{D.22}$$

According to (D.15), upon identification of the singular part of the self-energy (Sect. 3),

$$\mathrm{Re}\Sigma^{+}(\underset{\sim}{p},\omega;\underset{\sim}{R},T) = \Sigma_{HF}(\underset{\sim}{p};\underset{\sim}{R},T) + \int \frac{d\omega'}{2\pi} \frac{\Gamma(\underset{\sim}{p},\omega';\underset{\sim}{R},T)}{\omega - \omega'} \quad . \tag{D.23}$$

Appendix E. Thermodynamic equilibrium

We shall discuss a system that has achieved a uniform equilibrium. The functions G and Σ will depend only on differences of the arguments, and we may introduce Fourier transforms

$$F(\underset{\sim}{x}-\underset{\sim}{x}',t-t') = \int \frac{d\underset{\sim}{p}\, d\omega}{(2\pi)^4}\, e^{i\underset{\sim}{p}(\underset{\sim}{x}-\underset{\sim}{x}')} e^{-i\omega(t-t')} F(\underset{\sim}{p},\omega) \quad . \tag{E.1}$$

We write the Fourier-transformed functions $G^{\gtrless}$ in the following way

$$\mp iG^{<}(\underset{\sim}{p},\omega) = f(\underset{\sim}{p},\omega)A(\underset{\sim}{p},\omega) \quad , \tag{E.2a}$$

$$iG^{>}(\underset{\sim}{p},\omega) = (1 \mp f(\underset{\sim}{p},\omega))A(\underset{\sim}{p},\omega) \quad , \tag{E.2b}$$

with $A = i(G^{>} - G^{<})$ (Eq. (D.20)), and we define with Eqs. (E.2) the function f. For a freely evolving system

$$A^{0}(\underset{\sim}{p},\omega) = 2\pi\delta(\omega - p^2/2m) \quad . \tag{E.3}$$

From the Fourier-transformed Eq. (3.11) ($t_0 \to -\infty$) we obtain

$$G^{\pm}(\underset{\sim}{p},\omega) = \frac{1}{\omega - p^2/2m - \Sigma^{\pm}(\underset{\sim}{p},\omega)} \quad , \tag{E.4}$$

and we find that A has a Lorentzian shape

$$A(\underset{\sim}{p},\omega) = \frac{\Gamma(\underset{\sim}{p},\omega)}{(\omega - p^2/2m - \mathrm{Re}\Sigma^{+}(\underset{\sim}{p},\omega))^2 + (\Gamma(\underset{\sim}{p},\omega)/2)^2} \tag{E.5}$$

(see the end of Appx. D). We have [2]

$$\int \frac{d\omega}{2\pi} A(\underset{\sim}{p},\omega) = 1 \quad , \tag{E.6}$$

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[2] In connection with convergence problems encountered in a nuclear application of the Green's function method in Ref. [23], we would like to mention that there exists a normalization condition for the function Γ. The condition can be deduced from Eqs. (3.19) and (3.20), and the form of the current. The condition, relating the width-function Γ to the bare interaction and the particle-hole fluctuation-function, has a form

and

$$\mathrm{Re}\Sigma^{+}(\underset{\sim}{p},\omega) = \Sigma_{HF}(\underset{\sim}{p}) + \mathcal{P}\int\frac{d\omega'}{2\pi}\,\frac{\Gamma(\underset{\sim}{p},\omega')}{\omega - \omega'} \quad . \tag{E.7}$$

The function $A(\underset{\sim}{p},\omega)$, the so-called spectral function, may be interpreted as a function weighting the frequencies ω for a given momentum $\underset{\sim}{p}$. The total weight of A is equal to 1. With $\mp iG^{<}(\underset{\sim}{p},\omega)$ and $iG^{>}(\underset{\sim}{p},\omega)$ respectively distributions of particles and holes in momenta and energies, the function $f(\underset{\sim}{p},\omega)$ obtains an interpretation of the occupation of states $(\underset{\sim}{p},\omega)$.

A sum of the Born diagrams for the self-energy, Eqs. (3.33) and (3.34), gives in a stationary uniform system

$$\mp i\Sigma_B^{<}(\underset{\sim}{p},\omega) = \int\frac{d\underset{\sim}{p}_1 d\omega_1}{(2\pi)^4}\int\frac{d\underset{\sim}{p}' d\omega'}{(2\pi)^4}\int\frac{d\underset{\sim}{p}_1' d\omega_1'}{(2\pi)^4}\,(2\pi)^4\delta(\underset{\sim}{p} + \underset{\sim}{p}_1 - \underset{\sim}{p}' - \underset{\sim}{p}_1')$$

$$\times\ \delta(\omega + \omega_1 - \omega' - \omega_1')\ \tfrac{1}{2}(V(\underset{\sim}{p}-\underset{\sim}{p}') \mp V(\underset{\sim}{p}-\underset{\sim}{p}_1'))^2 A(\underset{\sim}{p}_1,\omega_1)A(\underset{\sim}{p}',\omega')$$

$$\times\ A(\underset{\sim}{p}_1',\omega_1')(1 \mp f(\underset{\sim}{p}_1,\omega_1))f(\underset{\sim}{p}',\omega')f(\underset{\sim}{p}_1',\omega_1') \quad , \tag{E.8a}$$

$$i\Sigma_B^{>}(\underset{\sim}{p},\omega) = \ldots\ f(\underset{\sim}{p}_1,\omega_1)(1 \mp f(\underset{\sim}{p}',\omega'))(1 \mp f(\underset{\sim}{p}_1',\omega_1')) \quad . \tag{E.8b}$$

In (E.8b) we do not write explicity that part of the r.h.s. which is the same as in Eq. (E.8a). The expression (E.8a) accounts for scattering of particles $(\underset{\sim}{p}',\omega')$ and $(\underset{\sim}{p}'_1,\omega'_1)$ into states $(\underset{\sim}{p},\omega)$ and $(\underset{\sim}{p}_1,\omega_1)$, with a

---

$$\int\frac{d\omega}{2\pi}\,\Gamma(\underset{\sim}{p},\omega) = P_d \mp P_e(\underset{\sim}{p}) \quad .$$

We shall quote here only the direct term, which may be written as

$$P_d = \int\frac{d\underset{\sim}{p}'}{(2\pi)^3}\,(V(\underset{\sim}{p}'))^2 D(\underset{\sim}{p}') = \int\frac{d\underset{\sim}{p}'}{(2\pi)^3}\int\frac{d\omega'}{2\pi}\,(V(\underset{\sim}{p}'))^2 D(\underset{\sim}{p}',\omega') \quad ,$$

where $D(1,2) = \langle\tilde{\hat{n}}_H(1)\tilde{\hat{n}}_H(2)\rangle$, and $\tilde{\hat{n}}(1) = \hat{n}(1) - \langle\hat{n}(1)\rangle$. (Comp. Eq. (2.11) of Ref. [24].)

Born-approximation scattering cross-section. The function $\mp i\Sigma^{<}_{B}(\underset{\sim}{p},\omega)$ is seen as a scattering-in rate into $(\underset{\sim}{p},\omega)$ due to these processes. Equation (E.8b) accounts for the inverse processes, and describes the scattering-out rate from $(\underset{\sim}{p},\omega)$. See also the forms of self-energies in the T-matrix approximation, Eq. (F.22).

On Fourier-transforming Eqs. (3.9) and (3.10), and subtracting the resulting equations from one another, one finds the detailed balance equation

$$\Sigma^{<}(\underset{\sim}{p},\omega)G^{>}(\underset{\sim}{p},\omega) = \Sigma^{>}(\underset{\sim}{p},\omega)G^{<}(\underset{\sim}{p},\omega) \quad . \tag{E.9}$$

Equation (E.9) expresses the equilibrium between the scattering-in of particles into $(\underset{\sim}{p},\omega)$ and the scattering-out from $(\underset{\sim}{p},\omega)$. The detailed balance equation will allow us to find a form of the function $f(\underset{\sim}{p},\omega)$. We shall present below an argumentation which generalizes an argumentation given sometimes in considerations of the Boltzmann equation.

The self-energy Σ may be expressed solely in terms of the Green's functions G, on using skeleton (irreducible) diagrams, see Ref. [25]. Examples of such diagrams are the Hartree-Fock and Born diagrams. Diagrams for $-i\Sigma^{\gtrless}$, obtained according to Appx. C in the space-time representation, possess the following, important for the subsequent discussion, properties in the Fourier-transformed variables (the top of the contour is pulled to $+\infty$ before introducing the transforms): energy and momentum are conserved in the vertices, an interaction line corresponds to $-iV(\underset{\sim}{p})$, at an antichronological branch we have for a particle line $iG^{a}(\underset{\sim}{p},\omega)$, for every interaction a factor (-1), for every internal interaction-vertex a factor (-1) (jointly a factor (-1) for every internal interaction), at a chronological branch a particle line corresponds to $iG^{c}(\underset{\sim}{p},\omega)$, particle lines crossing the division line correspond to $iG^{\gtrless}(\underset{\sim}{p},\omega)$, all independent momenta and energies are to be integrated over. From the relation $[iG^{c}(1,1')]^{\dagger}=iG^{a}(1,1')$, follows $[iG^{c}(\underset{\sim}{p},\omega)]^{*}=iG^{a}(\underset{\sim}{p},\omega)$, according to Appx. D.

Let us take a certain diagram that gives a contribution to $-i\Sigma^<$. From the fact that $-i\Sigma^<$ is real (Appx. D), it follows that some other diagram gives to $-i\Sigma^<$ a conjugate contribution, which differs from the previous in a replacement of all iG^a with iG^c, iG^c with iG^a, and $-iV$ with iV, without affecting $iG^<$ and $iG^>$. Let us analyze a conversion of $-i\Sigma^<$ into $-i\Sigma^>$. In a given contribution the conversion consists in replacement of all iG^a with iG^c and iG^c with iG^a, in change of signs in potentials (in principle excluding the extreme potentials, but there are exactly two of those in every diagram of $-i\Sigma^{\gtrless}$), and in replacement of all $iG^<$ with $iG^>$, and $iG^>$ with $iG^<$.

In every term of $-i\Sigma^<$, the number of $iG^<$ is larger by one than the number of $iG^>$, and an opposite holds in $-i\Sigma^>$. The total momentum and energy carried through the division line are equal to $(\underset{\sim}{p},\omega)$ in every term of $-i\Sigma^{\gtrless}(\underset{\sim}{p},\omega)$. Let us assume that we insert the expressions for the self-energies into the microscopic-balance equation. We shall have terms originating from various diagrams, differing in structure. However, it follows from the above discussion, that for each l.h.s. term we shall have a respective r.h.s. term, that will differ only in the replacement of $G^{\gtrless}$ with $G^{\lessgtr}$. It seems natural that the equality should hold separately in every corresponding pair, and that a certain relation between $G^>$ and $G^<$ should be responsible for the equality. Upon separating-out integrations and factors containing G^a, G^c, and V, the postulated equalities have the forms

$$G^>(\underset{\sim}{p},\omega)G^>(\underset{\sim}{p}_1,\omega_1)\ \ldots\ G^>(\underset{\sim}{p}_n,\omega_n)G^<(\underset{\sim}{p}',\omega')G^<(\underset{\sim}{p}_1',\omega_1')\ \ldots\ G^<(\underset{\sim}{p}_n',\omega_n')$$

$$= G^<(\underset{\sim}{p},\omega)G^<(\underset{\sim}{p}_1,\omega_1)\ \ldots\ G^<(\underset{\sim}{p}_n,\omega_n)G^>(\underset{\sim}{p}',\omega')G^>(\underset{\sim}{p}_1',\omega_1')\ \ldots\ G^>(\underset{\sim}{p}_n',\omega_n')\ , \tag{E.10}$$

while

$$\underset{\sim}{p} = \underset{\sim}{p}' + \underset{\sim}{p}_1' + \ldots + \underset{\sim}{p}_n' - (\underset{\sim}{p}_1 + \ldots + \underset{\sim}{p}_n) \quad ,$$

$$\omega = \omega' + \omega_1' + \ldots + \omega_n' - (\omega_1 + \ldots + \omega_n) \quad ,$$

i.e.

$$\underset{\sim}{p}' + \underset{\sim}{p}_1' + \ldots + \underset{\sim}{p}_n' = \underset{\sim}{p} + \underset{\sim}{p}_1 + \ldots + \underset{\sim}{p}_n \quad ,$$

$$\omega' + \omega_1' + \ldots + \omega_n' = \omega + \omega_1 + \ldots + \omega_n \quad . \tag{E.11}$$

From (E.10) it follows that $G^>$ and $G^<$ should be related by a factor, and with the conservation laws (E.11) the most general form of the factor is

$$e^{\beta(\omega - \underset{\sim}{v}\underset{\sim}{p} - \mu)} \quad ,$$

where we readily use a conventional notation. From

$$G^>(\underset{\sim}{p},\omega) = e^{\beta(\omega - \underset{\sim}{v}\underset{\sim}{p} - \mu)} G^<(\underset{\sim}{p},\omega) \tag{E.12}$$

we find for the occupation

$$f(\underset{\sim}{p},\omega) = \frac{1}{\exp(\beta(\omega - \underset{\sim}{v}\underset{\sim}{p} - \mu)) \pm 1} \quad . \tag{E.13}$$

Results of the present Appendix refer to a system whose noncorrelated initial state has been specified at $t_0 \to -\infty$. For an equilibrium system, specified with a density operator $\hat{o} = \exp(-\beta(\hat{H} - \underset{\sim}{v}\hat{\underset{\sim}{p}} - \mu\hat{N}))$, the form (E.13) of the f function follows directly from the Green's function definitions. The remaining results of the Appendix may be obtained for such a system with an evolution-contour method in the complex time plane (Sect. 5).

At the end, we would like to mention that a uniform equilibrium may not exist within the constraints put on the system. One encounters such a situation when one finds singularities in the retarded or advanced functions in regions where the functions should be analytic according to their definitions.

Appendix F. T-matrix approximation

We shall formulate the T-matrix approximation to the self-energy starting from Eqs. (3.35), (3.36). On comparing (3.35) with (3.1), we find a relation

$$-i\Sigma(1,1') = \mp\int d2 \int d1''(-i)V(1,2)i^2G_2(1,2;1'',2^+)(-i)G^{-1}(1'',1') \quad , \quad (F.1)$$

where $V(1,2) = V(\underset{\sim}{x}_1-\underset{\sim}{x}_2)\delta(t_1,t_2)$.

The rules for evaluating i^2G_2 are essentially the same as the rules for iG. One needs only to make a sign change of the diagram in cases when the lines running between the end-points of a diagram cross. In the T-matrix approximation we sum for i^2G_2 the ladder diagrams which correspond to repeated interactions between particles, Fig. 10. We define the T-matrix with the diagrams presented in Fig. 11

$$-i\langle 1,2|T|1',2'\rangle = -iV(1,2)\delta(1,1')\delta(2,2') + \int d1'' \int d2''(-i)V(1,2)$$
$$\times\ iG(1,1'')iG(2,2'')(-i)\langle 1'',2''|T|1',2'\rangle = -iV(1,2)\delta(1,1')\delta(2,2')$$
$$+ \int d1'' \int d2''(-i)\langle 1,2|T|1'',2''\rangle iG(1'',1')iG(2'',2')(-i)V(1',2') \quad , \quad (F.2)$$

where we use the notation as in [2]. Then

$$i^2G_2(1,2;1',2') = iG(1,1')iG(2,2') \mp iG(1,2')iG(2,1')$$
$$+ \int d1'' \int d2'' \int d1''' \int d2'''(iG(1,1'')iG(2,2'') \mp iG(1,2'')iG(2,1''))$$
$$\times\ (-i)\langle 1'',2''|T|1''',2'''\rangle iG(1''',1')iG(2''',2') \quad , \quad (F.3)$$

and from (F.1) we find

$$\Sigma(1,1') = \int d2 \int d2' iG(2',2)(\langle 1,2|T|2',1'\rangle \mp \langle 1,2\ T\ 1',2'\rangle) \quad , \quad (F.4)$$

comp. (3.18).

The time arguments of the scattering matrix pairwise coincide for a potential interaction

$$\langle 1,2|T|1',2'\rangle = \delta(t_1,t_2)\delta(t_{1'},t_{2'})\langle \underset{\sim}{x}_1,\underset{\sim}{x}_2|T|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}\rangle \quad . \quad (F.5)$$

With the notation

$$\langle \underset{\sim}{x}_1,\underset{\sim}{x}_2|\mathcal{G}(t_1,t_{1'})|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}\rangle = iG(1,1')G(\underset{\sim}{x}_2,t_1,\underset{\sim}{x}_{2'},t_{1'}) \quad , \quad (F.6)$$

we have from (F.2)

$$<\underset{\sim}{x}_1,\underset{\sim}{x}_2|T(t_1,t_{1'})|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}> = V(\underset{\sim}{x}_1-\underset{\sim}{x}_2)[\delta(t_1,t_{1'})\delta(\underset{\sim}{x}_1-\underset{\sim}{x}_{1'})\delta(\underset{\sim}{x}_2-\underset{\sim}{x}_{2'})$$

$$+ \int d1'' \int d\underset{\sim}{x}_{2''} <\underset{\sim}{x}_1,\underset{\sim}{x}_2|\mathcal{G}(t_1,t_{1''})|\underset{\sim}{x}_{1''},\underset{\sim}{x}_{2''}><\underset{\sim}{x}_{1''},\underset{\sim}{x}_{2''}|T(t_{1''},t_{1'})|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}>]$$

$$= [\delta(t_1,t_{1'})\delta(\underset{\sim}{x}_1-\underset{\sim}{x}_{1'})\delta(\underset{\sim}{x}_2-\underset{\sim}{x}_{2'}) + \int d1'' \int d\underset{\sim}{x}_{2''}<\underset{\sim}{x}_1,\underset{\sim}{x}_2|T(t_1,t_{1''})|\underset{\sim}{x}_{1''},\underset{\sim}{x}_{2''}>$$

$$\times <\underset{\sim}{x}_{1''},\underset{\sim}{x}_{2''}|\mathcal{G}(t_{1''},t_{1'})|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}>]V(\underset{\sim}{x}_{1'}-\underset{\sim}{x}_{2'}) \quad . \tag{F.7}$$

With the definition

$$[<\underset{\sim}{x}_1,\underset{\sim}{x}_2|F(t_1,t_{1'})|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}>] = [<\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}|F(t_{1'},t_1)|\underset{\sim}{x}_1,\underset{\sim}{x}_2>]^* \quad , \tag{F.8}$$

we have from (F.6) and (D.7)

$$[i<\underset{\sim}{x}_1,\underset{\sim}{x}_2|\mathcal{G}(t_1,t_{1'})|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}>]^\dagger = i<\underset{\sim}{x}_1,\underset{\sim}{x}_2|\mathcal{G}(t_1^T,t_{1'}^T)|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}> \quad . \tag{F.9}$$

and from (F.7)

$$[i<\underset{\sim}{x}_1,\underset{\sim}{x}_2|T(t_1,t_{1'})|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}>]^\dagger = i<\underset{\sim}{x}_1,\underset{\sim}{x}_2|T(t_1^T,t_{1'}^T)|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}> \quad . \tag{F.10}$$

From (F.4) and (F.5) follows

$$\Sigma(1,1') = \int d\underset{\sim}{x}_2 \int d\underset{\sim}{x}_{2'} iG(\underset{\sim}{x}_{2'},t_{1'},\underset{\sim}{x}_2,t_1)$$

$$\times (<\underset{\sim}{x}_1,\underset{\sim}{x}_2|T(t_1,t_{1'})|\underset{\sim}{x}_{2'},\underset{\sim}{x}_{1'}> \mp <\underset{\sim}{x}_1,\underset{\sim}{x}_2|T(t_1,t_{1'})|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}>) \quad , \tag{F.11}$$

and further

$$\Sigma^{\gtrless}(1,1') = \int d\underset{\sim}{x}_2 \int d\underset{\sim}{x}_{2'} iG^{\lessgtr}(\underset{\sim}{x}_{2'},t_{1'},\underset{\sim}{x}_2,t_1)$$

$$\times (<\underset{\sim}{x}_1,\underset{\sim}{x}_2|T^{\gtrless}(t_1,t_{1'})|\underset{\sim}{x}_{2'},\underset{\sim}{x}_{1'}> \mp <\underset{\sim}{x}_1,\underset{\sim}{x}_2|T^{\gtrless}(t_1,t_{1'})|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}>) \quad . \tag{F.12}$$

Proceeding similarly as in the case of the 1-particle Green's function equations of motion (Subsect. 3.1), one finds from (F.7) the equations

$$<\underset{\sim}{x}_1,\underset{\sim}{x}_2|T^{\gtrless}(t_1,t_{1'})|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}> = V(\underset{\sim}{x}_1-\underset{\sim}{x}_2)\int_{t_0}^{\infty} d1'' \int d\underset{\sim}{x}_{2''}$$

$$\times [<\underset{\sim}{x}_1,\underset{\sim}{x}_2|\mathcal{G}^+(t_1,t_{1''})|\underset{\sim}{x}_{1''},\underset{\sim}{x}_{2''}><\underset{\sim}{x}_{1''},\underset{\sim}{x}_{2''}|T^{\gtrless}(t_{1''},t_{1'})|\underset{\sim}{x}_{1'},\underset{\sim}{x}_{2'}>$$

$$+ \langle \underset{\sim}{x}_1, \underset{\sim}{x}_2 | \mathcal{G}^{\gtrless}(t_1, t_{1''}) | \underset{\sim}{x}_{1''}, \underset{\sim}{x}_{2''} \rangle \langle \underset{\sim}{x}_{1''}, \underset{\sim}{x}_{2''} | T^-(t_{1''}, t_{1'}) | \underset{\sim}{x}_{1'}, \underset{\sim}{x}_{2'} \rangle] \quad , \qquad (F.13)$$

schematically $T^{\gtrless} = V\mathcal{G}^+T^{\gtrless} + V\mathcal{G}^{\gtrless}T^-$, and

$$\langle \underset{\sim}{x}_1, \underset{\sim}{x}_2 | T^{\pm}(t_1, t_{1'}) | \underset{\sim}{x}_{1'}, \underset{\sim}{x}_{2'} \rangle$$

$$= V(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)[\delta(t_1 - t_{1'})\delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_{1'})\delta(\underset{\sim}{x}_2 - \underset{\sim}{x}_{2'}) + \int_{t_0}^{\infty} dl'' \int d\underset{\sim}{x}_{2''}$$

$$\times \langle \underset{\sim}{x}_1, \underset{\sim}{x}_2 | \mathcal{G}^{\pm}(t_1, t_{1''}) | \underset{\sim}{x}_{1''}, \underset{\sim}{x}_{2''} \rangle \langle \underset{\sim}{x}_{1''}, \underset{\sim}{x}_{2''} | T^{\pm}(t_{1''}, t_{1'}) | \underset{\sim}{x}_{1'}, \underset{\sim}{x}_{2'} \rangle] \quad , \qquad (F.14)$$

schematically $T^{\pm} = V + V\mathcal{G}^{\pm}T^{\pm}$. From (F.13) and (F.14) it follows that $T^{\gtrless}$ may be written in the form of a generalized optical theorem $T^{\gtrless} = T^+\mathcal{G}^{\gtrless}T^-$, i.e.

$$\langle \underset{\sim}{x}_1, \underset{\sim}{x}_2 | T^{\gtrless}(t_1, t_{1'}) | \underset{\sim}{x}_{1'}, \underset{\sim}{x}_{2'} \rangle = \int_{t_0}^{\infty} dl'' \int d\underset{\sim}{x}_{2''} \int_{t_0}^{\infty} dl''' \int d\underset{\sim}{x}_{2'''}$$

$$\times \langle \underset{\sim}{x}_1, \underset{\sim}{x}_2 | T^+(t_1, t_{1''}) | \underset{\sim}{x}_{1''}, \underset{\sim}{x}_{2''} \rangle \langle \underset{\sim}{x}_{1''}, \underset{\sim}{x}_{2''} | \mathcal{G}^{\gtrless}(t_{1''}, t_{1'''}) | \underset{\sim}{x}_{1'''}, \underset{\sim}{x}_{2'''} \rangle$$

$$\times \langle \underset{\sim}{x}_{1'''}, x_{2'''} | T^-(t_{1'''}, t_{1'}) | \underset{\sim}{x}_{1'}, \underset{\sim}{x}_{2'} \rangle \quad . \qquad (F.15)$$

If we inserted (F.15) into (F.12), then with (F.6) we would obtain an expression for $\Sigma^{\gtrless}$ similar to the sum of Eqs. (3.33) and (3.34) with the matrices $T^{\pm}$ instead of the respective potentials V.

In below we shall use the Fourier-transformed 2-particle functions, which we introduce in the following way

$$\langle \underset{\sim}{g} | F(\underset{\sim}{P}, \omega : \underset{\sim}{R}, T) | \underset{\sim}{g}' \rangle = \int d(t - t') e^{i\omega(t - t')}$$

$$\times \int d[\tfrac{1}{2}((\underset{\sim}{x}_1 + \underset{\sim}{x}_2) - (\underset{\sim}{x}_{1'} + \underset{\sim}{x}_{2'}))] e^{-i\underset{\sim}{P}\cdot\frac{1}{2}((\underset{\sim}{x}_1 + \underset{\sim}{x}_2) - (\underset{\sim}{x}_{1'} + \underset{\sim}{x}_{2'}))}$$

$$\times \int d(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) e^{-i\underset{\sim}{g}(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)} \int d(\underset{\sim}{x}_{1'} - \underset{\sim}{x}_{2'}) e^{i\underset{\sim}{g}'(\underset{\sim}{x}_{1'} - \underset{\sim}{x}_{2'})}$$

$$\times \langle \underset{\sim}{x}_1, \underset{\sim}{x}_2 | F(t, t') | \underset{\sim}{x}_{1'}, \underset{\sim}{x}_{2'} \rangle \quad , \qquad (F.16)$$

where $\underset{\sim}{R} = (\underset{\sim}{x}_1 + \underset{\sim}{x}_2 + \underset{\sim}{x}_{1'} + \underset{\sim}{x}_{2'})/4$, and $T = (t + t')/2$. From (F.10)

it follows for the Fourier-transformed T-matrices that

$$[i\langle \underset{\sim}{q}|T^{\gtrless}(\underset{\sim}{P},\omega;\underset{\sim}{R},T)|\underset{\sim}{q}'\rangle]^* = i\langle \underset{\sim}{q}'|T^{\gtrless}(\underset{\sim}{P},\omega;\underset{\sim}{R},T)|\underset{\sim}{q}\rangle \quad , \tag{F.17}$$

and also

$$[\langle \underset{\sim}{q}|T^{+}(\underset{\sim}{P},\omega;\underset{\sim}{R},T)|\underset{\sim}{q}'\rangle]^* = \langle \underset{\sim}{q}'|T^{-}(\underset{\sim}{P},\omega;\underset{\sim}{R},T)|\underset{\sim}{q}\rangle \quad . \tag{F.18}$$

Let us see what form the T-matrix approximation results take in a uniform equilibrium. For the self-energies, Eq. (F.12), we find

$$\Sigma^{\gtrless}(\underset{\sim}{p},\omega) = \mp\int\frac{d\underset{\sim}{p}_1 d\omega_1}{(2\pi)^4}\, G^{\lessgtr}(\underset{\sim}{p}_1,\omega_1)\, i(\langle\frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2}|T^{\gtrless}(\underset{\sim}{p}+\underset{\sim}{p}_1,\omega+\omega_1)|\frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2}\rangle$$

$$\mp\langle\frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2}|T^{\gtrless}(\underset{\sim}{p}+\underset{\sim}{p}_1,\omega+\omega_1)|\frac{\underset{\sim}{p}_1-\underset{\sim}{p}}{2}\rangle) \quad . \tag{F.19}$$

From (F.17) and the symmetry under the interchange of particles, it follows that the symmetrized matrices $iT^{\gtrless}$ in (F.16) are real. Equation (F.19) may be understood in the following way. In the 2-particle Green's function equation, the function T plays a somewhat similar role to Σ in the 1-particle Green's function equation. We may expect that $iT^{\gtrless}$ in (F.19) constitute scattering-out and -in rates into noncorrelated 2-particle states. Then the integral and a Green's function in (F.19) are the summation over initial or final states of a remaining particle. The generalized optical theorem, Eq. (F.15), takes in the momentum-energy representation a form

$$\langle \underset{\sim}{p}|T^{\gtrless}(\underset{\sim}{P},\omega)|\underset{\sim}{p}'\rangle = \int\frac{d\underset{\sim}{p}_1}{(2\pi)^3}\int\frac{d\underset{\sim}{p}_1'}{(2\pi)^3}\,\langle \underset{\sim}{p}|T^{+}(\underset{\sim}{P},\omega)|\underset{\sim}{p}_1\rangle$$

$$\times\,\langle \underset{\sim}{p}_1|\mathcal{G}^{\gtrless}(\underset{\sim}{P},\omega)|\underset{\sim}{p}_1'\rangle\langle \underset{\sim}{p}_1'|T^{-}(\underset{\sim}{P},\omega)|\underset{\sim}{p}'\rangle \quad . \tag{F.20}$$

For the symmetrized matrices T occuring in (F.19), one finds from the optical theorem, with (F.6),

$$\langle\frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2}|T^{\gtrless}(\underset{\sim}{p}+\underset{\sim}{p}_1,\omega+\omega_1)|\frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2}\rangle \mp \langle\frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2}|T^{\gtrless}(\underset{\sim}{p}+\underset{\sim}{p}_1,\omega+\omega_1)|\frac{\underset{\sim}{p}_1-\underset{\sim}{p}}{2}\rangle$$

$$= i\int\frac{d\underset{\sim}{p}'d\omega'}{(2\pi)^4}\int\frac{d\underset{\sim}{p}_1'd\omega_1'}{(2\pi)^4}(2\pi)^4\delta(\underset{\sim}{p}+\underset{\sim}{p}_1-\underset{\sim}{p}'-\underset{\sim}{p}_1')\delta(\omega+\omega_1-\omega'-\omega_1')$$

$$\times\frac{1}{2}\left|\langle\frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2}|T^{+}(\underset{\sim}{p}+\underset{\sim}{p}_1,\omega+\omega_1)|\frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2}\rangle\right.$$

$$\left.\mp\langle\frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2}|T^{+}(\underset{\sim}{p}+\underset{\sim}{p}_1,\omega+\omega_1)|\frac{\underset{\sim}{p}_1-\underset{\sim}{p}}{2}\rangle\right|^2 G^{\gtrless}(\underset{\sim}{p}',\omega')G^{\gtrless}(\underset{\sim}{p}_1',\omega_1') \quad , \tag{F.21}$$

which confirms the conjecture concerning $iT^{\gtrless}$. From (F.21) and (F.19), with (E.2),

$$\mp i\Sigma^{<}(\underset{\sim}{p},\omega) = \int\frac{d\underset{\sim}{p}_1d\omega_1}{(2\pi)^4}\int\frac{d\underset{\sim}{p}'d\omega'}{(2\pi)^4}\int\frac{d\underset{\sim}{p}_1'd\omega_1'}{(2\pi)^4}(2\pi)^4\delta(\underset{\sim}{p}+\underset{\sim}{p}_1-\underset{\sim}{p}'-\underset{\sim}{p}_1')$$

$$\times\ \delta(\omega+\omega_1-\omega'-\omega_1')\ \frac{1}{2}\left|\langle\frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2}|T^{+}(\underset{\sim}{p}+\underset{\sim}{p}_1,\omega+\omega_1)|\frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2}\rangle\right.$$

$$\left.\langle\frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2}|T^{+}(\underset{\sim}{p}+\underset{\sim}{p}_1,\omega+\omega_1)|\frac{\underset{\sim}{p}_1-\underset{\sim}{p}}{2}\rangle\right|^2 A(\underset{\sim}{p}_1,\omega_1)A(\underset{\sim}{p}',\omega')A(\underset{\sim}{p}_1',\omega_1')$$

$$\times\ (1\mp f(\underset{\sim}{p}_1,\omega_1))f(\underset{\sim}{p}',\omega')f(\underset{\sim}{p}_1',\omega_1') \quad , \tag{F.22a}$$

$$i\Sigma^{>}(\underset{\sim}{p},\omega) = \ldots\ f(\underset{\sim}{p}_1,\omega_1)(1\mp f(\underset{\sim}{p}',\omega'))(1\mp f(\underset{\sim}{p}_1',\omega_1')) \quad , \tag{F.22b}$$

comp. (E.8).

The $T^{\pm}$ matrices satisfy in an equilibrium system the equations, from (F.14),

$$\langle\underset{\sim}{p}|T^{\pm}(\underset{\sim}{P},\omega)|\underset{\sim}{p}'\rangle = V(\underset{\sim}{p}-\underset{\sim}{p}') + \int\frac{d\underset{\sim}{p}_1}{(2\pi)^3}\int\frac{d\underset{\sim}{p}_1'}{(2\pi)^3}V(\underset{\sim}{p}-\underset{\sim}{p}_1)$$

$$\times\ \langle\underset{\sim}{p}_1|\mathcal{G}^{\pm}(\underset{\sim}{P},\omega)|\underset{\sim}{p}_1'\rangle\langle\underset{\sim}{p}_1'|T^{\pm}(\underset{\sim}{P},\omega)|\underset{\sim}{p}'\rangle \quad . \tag{F.23}$$

Upon omitting the self-energies in the 1-particle Green's functions in $\mathcal{G}$, Eqs. (F.23) become

$$\langle \underset{\sim}{p} | T^{\pm}(\underset{\sim}{P},\omega) | \underset{\sim}{p}' \rangle = V(\underset{\sim}{p}-\underset{\sim}{p}') + \int \frac{d\underset{\sim}{p}_1}{(2\pi)^3} V(\underset{\sim}{p}-\underset{\sim}{p}_1)$$

$$\times \frac{1 \mp f(\underset{\sim}{P}/2+\underset{\sim}{p}_1) \mp f(\underset{\sim}{P}/2-\underset{\sim}{p}_1)}{\omega - P^2/4m - p_1^2/m \pm i\varepsilon} \langle \underset{\sim}{p}_1 | T^{\pm}(\underset{\sim}{P},\omega) | \underset{\sim}{p}' \rangle \quad . \tag{F.24}$$

Let us now discuss the T-matrix approximation in connection with the Boltzmann equation. In the T-matrix approximation, the scattering-in rate in the Boltzmann equation would have a form

$$\mp i\Sigma^{<}(\underset{\sim}{p},\omega_p^o:\underset{\sim}{R},T) = \int \frac{d\underset{\sim}{p}_1}{(2\pi)^3} \int \frac{d\underset{\sim}{p}'}{(2\pi)^3} \int \frac{d\underset{\sim}{p}_1'}{(2\pi)^3} (2\pi)^4 \delta(\underset{\sim}{p} + \underset{\sim}{p}_1 - \underset{\sim}{p}' - \underset{\sim}{p}_1')$$

$$\times \delta(\omega_p^o + \omega_{p_1}^o - \omega_{p'}^o - \omega_{p_1'}^o) \frac{1}{2} \left| \langle \frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2} | T^+(\underset{\sim}{p}+\underset{\sim}{p}_1,\omega_p^o+\omega_{p_1}^o;\underset{\sim}{R},T) | \frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2} \rangle \right.$$

$$\left. \mp \langle \frac{\underset{\sim}{p}-\underset{\sim}{p}_1}{2} | T^+(\underset{\sim}{p}+\underset{\sim}{p}_1,\omega_p^o+\omega_{p_1}^o;\underset{\sim}{R},T) | \frac{\underset{\sim}{p}_1-\underset{\sim}{p}}{2} \rangle \right|^2 (1 \mp f(\underset{\sim}{p}_1:\underset{\sim}{R},T)) f(\underset{\sim}{p}':\underset{\sim}{R},T) f(\underset{\sim}{p}_1':\underset{\sim}{R},T) \quad , \tag{F.25}$$

and the scattering-out rate an analogous form to (F.25). In the Boltzmann equation limit, we would demand that the scattering matrix T^+ satisfies Eq. (F.24), with all functions in the equation referring to an $(\underset{\sim}{R},T)$ location in macroscopic variables. Let us see what must be the properties of the T-matrix, in order that the Kadanoff-Baym equations can be approximated with the Boltzmann equation. We take for simplicity a homogenous system, and proceed in an analogous manner to Subsect. 4.3. Eq. (4.1), and Eqs. (F.12), (F.15) ($t_o \to -\infty$), yield an equation for the distribution function

$$-i\Omega f(\underset{\sim}{p};\Omega) = \int \frac{d\omega}{2\pi} \left(\frac{i}{\omega + \Omega/2 + i\varepsilon} - \frac{i}{\omega - \Omega/2 - i\varepsilon} \right)$$

$$\times \int \frac{d\Omega'}{2\pi} \int \frac{d\Omega''}{2\pi} \int \frac{d\Omega'''}{2\pi} \int \frac{d\Omega^{IV}}{2\pi} 2\pi\delta(\Omega - \Omega' - \Omega'' - \Omega''' - \Omega^{IV})$$

$$\times \int \frac{d\underset{\sim}{p}_1}{(2\pi)^3} \int \frac{d\underset{\sim}{p}'}{(2\pi)^3} \int \frac{d\underset{\sim}{p}_1'}{(2\pi)^3} (2\pi)^3 \delta(\underset{\sim}{p} + \underset{\sim}{p}_1 - \underset{\sim}{p}' - \underset{\sim}{p}_1')$$

$$\times \int \frac{d\omega'}{2\pi} \left\langle \frac{\underset{\sim}{p} - \underset{\sim}{p}_1}{2} \middle| T^+(\underset{\sim}{p}+\underset{\sim}{p}_1, \omega'+(\Omega''+\Omega^{IV})/2 : \Omega''') \frac{\underset{\sim}{p}' - \underset{\sim}{p}_1'}{2} \right\rangle$$

$$\times \left\langle \frac{\underset{\sim}{p}' - \underset{\sim}{p}_1'}{2} \middle| T^-(\underset{\sim}{p}+\underset{\sim}{p}_1, \omega'-(\Omega''+\Omega''')/2 : \Omega^{IV}) \middle| \left(\left| \frac{\underset{\sim}{p} - \underset{\sim}{p}_1}{2} \right\rangle \mp \left| \frac{\underset{\sim}{p}_1 - \underset{\sim}{p}}{2} \right\rangle \right) \right.$$

$$\times \left(F^>(\underset{\sim}{p}, \underset{\sim}{p}_1, \omega+\omega' : \Omega') F^<(\underset{\sim}{p}', \underset{\sim}{p}_1', \omega'+(\Omega^{IV}-\Omega''')/2 : \Omega'') \right.$$

$$\left. - F^<(\underset{\sim}{p}, \underset{\sim}{p}_1, \omega+\omega' : \Omega') F^>(\underset{\sim}{p}', \underset{\sim}{p}_1', \omega'+(\Omega^{IV}-\Omega''')/2 : \Omega'') \right) \quad , \qquad (F.26)$$

with the auxiliary functions

$$F^{\gtrless}(\underset{\sim}{p}, \underset{\sim}{p}_1, t : T) = -G^{\gtrless}(\underset{\sim}{p}, t; T) G^{\gtrless}(\underset{\sim}{p}_1, t; T) \quad ,$$

for which we shall use

$$F^{\gtrless}(\underset{\sim}{p}, \underset{\sim}{p}_1, \omega : T) = F^{\gtrless}(\underset{\sim}{p}, \underset{\sim}{p}_1 : T) 2\pi\delta(\omega - \omega_p^o - \omega_{p_1}^o) \quad ,$$

with

$$F^>(\underset{\sim}{p}, \underset{\sim}{p}_1 : T) = (1 \mp f(\underset{\sim}{p} : T))(1 \mp f(\underset{\sim}{p}_1 : T)) \quad ,$$

$$F^<(\underset{\sim}{p}, \underset{\sim}{p}_1 : T) = f(\underset{\sim}{p} : T) f(\underset{\sim}{p}_1 : T) \quad .$$

The Boltzmann equation follows from (F.26), when one ignores all the macroscopic-frequency contributions to the microscopic frequencies. The similar occurs for Eq. (F.24) and Eq. (F.14) written in the Fourier-transformed variables. The approximations are possible when $p\Delta p/m \gg \Gamma$ holds both for Δp being a scale of variations of particle distribution in momentum, and for Δp being a scale of variation of T^+ in a momentum transfer. Also $\Delta\omega \gg \Gamma$ must hold, where $\Delta\omega$ – a scale of variation of a T^+ matrix in frequency, following from Eq. (F.24). These are the conditions for the Boltzmann equation, within the T-matrix approximation.

On deriving the kinetic equations, we have considered the cases $\mathrm{Re}\Sigma^+ \sim \mathrm{Im}\Sigma^+$ and $\mathrm{Re}\Sigma^+ \gg \mathrm{Im}\Sigma^+$. We shall now examine $\mathrm{Re}\Sigma^+$ and $\mathrm{Im}\Sigma^+$ in the T-matrix approximation, in the low-density limit. From Eq. (F.16) written

schematically as $\Sigma^{\gtrless} = \mp iG^{\lessgtr} T^{\gtrless}$, follows $\mathrm{Re}\Sigma^{+} = \mp iG^{<}\mathrm{Re}T^{+} \mp i\mathrm{Re}G^{+}T^{<}$, and $\mathrm{Im}\Sigma^{+} = iG^{<}\mathrm{Im}T^{+} \pm i\mathrm{Im}G^{+}T^{<}$. In the low-density limit $G^{<} \sim n$, $T^{<} \sim n^{2}$, and the T^{+}-matrix becomes a free-scattering matrix. We shall discard the exchange term of the scattering matrix. On evaluating Σ, we take a forward element of the T-matrix, which we shall denote in a simplified way as T(0). Taking the scattering matrix for a certain characteristic relative momentum p, we get the estimates in the low-density limit $\mathrm{Re}\Sigma^{+} \approx n\mathrm{Re}T^{+}(0)$ and $\mathrm{Im}\Sigma^{+} \approx n\mathrm{Im}T^{+}(0)$. Parametrizing the scattering matrix with a gaussian in the momentum transfer q

$$T^{+}(\underset{\sim}{q}) = T^{+}(0)e^{-\frac{1}{4}\eta^{2}q^{2}} ,$$

we find from the optical theorem a condition for $|\mathrm{Re}T^{+}(0)| \gg |\mathrm{Im}T^{+}(0)|$ in the form

$$1 \gg \frac{m}{8\pi}\frac{1}{\eta^{2}p}\left|T^{+}(0)\right|(1 - e^{-2\eta^{2}p^{2}}) \quad . \tag{F.27}$$

The parameter η has a meaning of an interaction range, and $|T^{+}(0)|$ of a full interaction strength. For a weak long-range interaction satisfying (F.27), $\mathrm{Re}\Sigma^{+} \gg \mathrm{Im}\Sigma^{+}$.

Appendix G. Thermodynamic equilibrium analysis of conditions for the Boltzmann equation

The conditions for passing from the Kadanoff-Baym equations to the Boltzmann equation can be analyzed in some detail in a state of equilibrium, in the Boltzmann-statistics limit. As in Subsect. 4.1 we shall assume well-defined free energies for particles. We shall examine the values of the microscopic variables entering the integrals of the self-energies $\Sigma^{\gtrless}$ with the Green's functions in Eq. (4.1) (see also (4.17)). The values of the microscopic variables, as compared with the scales of macroscopic variations in a system, determine the order of magnitude of the terms neglected in the Boltzmann equation.

The equilibrium distribution function is of the form $f(\underset{\sim}{p}) = \exp(-\beta(p^2/2m - \mu))$, where μ is the chemical potential, and $\beta = T^{-1}$, with T the temperature ($k_B = 1$). The conditions for the Boltzmann equation which we shall find will be valid for distributions whose behaviour with momentum does not depart much from that of the equilibrium distribution. We shall use the direct Born approximation to the self-energies, in which approximation it is possible to obtain analytic expression for the self-energies. We have

$$\mp iG^{<}(\underset{\sim}{p},t) = e^{-\beta(p^2/2m - \mu) - i(p^2/2m)t} \quad , \tag{G.1}$$

$$iG^{>}(\underset{\sim}{p},t) = e^{-i(p^2/2m)t} \quad . \tag{G.2}$$

For a gaussian potential $V(\underset{\sim}{r})=V_0\exp(-r^2/\eta^2)$, one finds in the direct Born approximation

$$\mp i\Sigma^{<}_{Bd}(\underset{\sim}{p},t) = V_0^2\left[\frac{m^2\eta^4}{4(\beta m^2 + \beta^2 + t^2)}\right]^{3/2}$$

$$\times \exp\left[2\beta\mu - \frac{p^2}{2m}\,\frac{2\beta t^2 + \beta^2 m\eta^2 + it(\beta m\eta^2 - \beta^2 + t^2)}{\beta m\eta^2 + \beta^2 + t^2}\right] \quad , \tag{G.3}$$

$$i\Sigma^{>}_{Bd}(\underset{\sim}{p},t) = V_o^2\left[\frac{m^2\eta^4}{4(\beta m\eta^2 + t^2 + 2it\beta)}\right]^{3/2}$$

$$\times \exp\left[\beta\mu - \frac{p^2}{2m}\,\frac{\beta t^2(\beta m\eta^2 + t^2) + it(2\beta^2t^2 + (\beta m\eta^2 + t^2)^2}{(\beta m\eta^2 + t^2)^2 + 4\beta^2t^2}\right] \quad . \qquad (G.4)$$

Let us at first assume that the integrations over space coordinates are already completed in the integrals of the self-energies with Green's functions in (4.1), and only the integrations over the microscopic times t' remain. Our task will be the determination of the values of t' that enter the t' integrations of $\Sigma^{\gtrless}(\underset{\sim}{p},-t')G^{\lessgtr}(\underset{\sim}{p},t')$. The values of t' entering the integrations correspond to the frequency dependence of the self-energies, close to the energy shell, and simultaneously the values of t' define the time in which the energy conservation is being realized in particle interactions.

We have

$$\Sigma^{>}_{Bd}(\underset{\sim}{p},-t')G^{<}(\underset{\sim}{p},t')$$

$$\alpha\ \frac{1}{(\beta m\eta^2 + t'^2 - 2it'\beta)^{3/2}}\exp\left[-\frac{\beta p^2}{2m}\,\frac{t'^2(\beta m\eta^2 + t'^2) + 2i\beta t'^3}{(\beta m\eta^2 + t'^2)^2 + 4\beta^2t'^2}\right] , \qquad (G.5)$$

$$\Sigma^{<}_{Bd}(\underset{\sim}{p},-t')G^{>}(\underset{\sim}{p},t')$$

$$\alpha\ \frac{1}{(\beta m\eta^2 + \beta^2 + t'^2)^{3/2}}\exp\left[-\frac{\beta p^2}{2m}\,\frac{2t'^2 + \beta m\eta^2 + 2it'\beta}{\beta m\eta^2 + \beta^2 + t'^2}\right] \quad . \qquad (G.6)$$

For momenta $p \lesssim (m/\beta)^{1/2}$, the main contribution to the t' integrals, both of (G.5) and (G.6), will come from the times $|t'| \lesssim (\beta m\eta^2 + \beta^2)^{1/2}$. The quantity $\eta(\beta m)^{1/2}$ corresponds to an average interaction time defined as a time of flight through an interaction range. The mean momentum in the system equals approximately $(3/2)(m/\beta)^{1/2}$. (The mean kinetic energy equals $3/2\beta$.) In the high-temperature limit defined with $\beta m\eta^2 \gg \beta^2$, for momenta $p \gtrsim (m/\beta)^{1/2}$ (in case of (G.6) $p \ll \eta m/\beta$), the main contribution to the t' integrals will come from the times $|t'| \lesssim \eta m/p$. In the opposite limit of

temperatures, the analysis of the t' integrations of (G.5) and (G.6) is hindered for large momenta by the oscilatory factors in (G.5) and (G.6). Upon putting $\beta m\eta^2 = 0$, the author has performed an analysis of the integrals in the complex time plane. One can estimate that for momenta $2/\eta \gg p \gg (m/\beta)^{1/2}$ the main contribution to the t' integral of (G.5) will come from the times $|t'| \lesssim 2m/p^2$, and for momenta $p \gg 2\eta$ from the times $|t'| \lesssim \eta m/p$. In the case of (G.6), for momenta $p \gg (m/\beta)^{1/2}$, one finds that the times $|t'| \lesssim \beta$ will always contribute to the integral. The analysis of the integrals of (G.5) and (G.6) may be summarized with a statement that the condition for the Boltzmann equation is a slow variation of the functions in macroscopic times, as compared with $\eta(\beta m)^{1/2}$ and β (more specifically as compared with $(\beta m\eta^2+\beta^2)^{1/2}$).

We may next study the values of microscopic spatial coordinates that enter the integrals of self-energies with Green's functions. It is necessary to find the values of $\underset{\sim}{r} - \underset{\sim}{r}'$ and $\underset{\sim}{r}'$ that enter the integrals

$$\int d\underset{\sim}{r}\, e^{-ip\underset{\sim}{r}} \int d\underset{\sim}{r}'\, \Sigma^{>,<}_{Bd}(\underset{\sim}{r}-\underset{\sim}{r}',-t')G^{>,<}(\underset{\sim}{r}',t') \quad , \tag{G.7}$$

for the times t', which we have determined earlier. The variables $\underset{\sim}{r} - \underset{\sim}{r}'$ entering the integral correspond to the momentum dependence of the self-energies, and the variables define the range in which the momentum conservation is being realized in particle interactions. The variables $\underset{\sim}{r}'$ entering the integrals correspond to the dependence of distribution functions on momentum, and indirectly also correspond to the dependence of self-energies on frequency. (To the variables $\underset{\sim}{r}'$ a meaning can be attributed, of a range in which a particle feels interaction.) The analysis of (G.7) is simplified by the fact that Green's functions and self-energies, possesing gaussian forms in momentum, (G.1)-(G.4), posses also gaussian forms in spatial coordinates

$$iG^{<}(\underset{\sim}{r},t) = [\frac{m}{2\pi(\beta + it)}]^{3/2}\exp[\beta\mu - \frac{mr^2}{2(\beta + it)}] \quad , \tag{G.8}$$

$$iG^{>}(\underset{\sim}{r},t) = [\frac{m}{2\pi it}]^{3/2}\exp[- \frac{mr^2}{2it}] \quad , \tag{G.9}$$

$$\mp i\Sigma^{<}_{Bd}(\underset{\sim}{r},t) = V_o^2[\frac{m^3\eta^4}{8\pi(2\beta t^2 + \beta^2 m\eta^2 + it(\beta m\eta^2 - \beta^2 + t^2))}]^{3/2}$$

$$\times \exp[2\beta\mu - \frac{mr^2}{2}\frac{\beta m\eta^2 + \beta^2 + t^2}{2\beta t^2 + \beta^2 m\eta^2 + it(\beta m\eta^2 - \beta^2 + t^2)}] \quad , \tag{G.10}$$

$$i\Sigma^{>}_{Bd}(\underset{\sim}{r},t) = V_o^2[\frac{m^3\eta^4}{8\pi(-\beta t^2 + it(\beta m\eta^2 + t^2))}]^{3/2}$$

$$\times \exp[\beta\mu - \frac{mr^2}{2}\frac{\beta m\eta^2 + t^2 + 2it\beta}{-\beta t^2 + it(\beta m\eta^2 + t^2)}] \quad . \tag{G.11}$$

In the limit $\beta m\eta^2 \gg \beta^2$, one finds that for momenta $p \gtrsim (m/\beta)^{1/2}$ the main contribution to the integrals (G.7) comes from variables $\lesssim \eta$, while for small momenta $(m/\beta)^{1/2} \gtrsim p \gtrsim \eta^{-1}$ from variables $\lesssim \eta p(\beta/m)^{1/2}$. In the limit $\beta^2 \gg \beta m\eta^2$, the main contribution to the integrals, for momenta $p \lesssim (m/\beta)^{1/2}$, will come from variables $\lesssim (\beta/m)^{1/2}$. For large momenta $p \gg (m/\beta)^{1/2}$, the main contribution to the integral (G.7) of functions $\Sigma^{<}$ and $G^{>}$ will come from the variables $\lesssim p\beta/m$. In case of the integral of $\Sigma^{>}$ and $G^{<}$, the main contribution for momenta $p \gg (m/\beta)^{1/2}$ will come from values of the variables $|\underset{\sim}{r}'| \lesssim p\beta/m$ and $|\underset{\sim}{r} - \underset{\sim}{r}'| \lesssim p^{-1}$. From the above analysis it follows that condition for the Boltzmann equation is the slow variation of the functions in macroscopic spatial coordinates, as compared with η and $p\beta m$, where p corresponds to particle momenta in consideration (for $p \lesssim (m/\beta)^{1/2}$ a quantity $(\beta/m)^{1/2}$ should be taken for comparison).

The conditions, of small $\eta(\beta m)^{1/2}$ and η in comparison with macroscopic variations in a system, are classical, because these conditions do not involve $\hbar$. The conditions will determine the possibility of describing the dynamics

of a system with the Boltzmann equation in the limit $\beta m\eta^2 >> \beta^2$, which is the limit of small interaction time and range inverses in comparison with variation of a particle distribution in momentum. The conditions, of small β and $p\beta/m$ in comparison with macroscopic variations in a system, are purely quantum. The quantities β and $p\beta/m$ are related solely to the particle distribution.

The conditions of small β and $p\beta/m$ would have emerged independently of the approximation used for the self-energy. The value β of a macroscopic time must appear in thermodynamic equilibrium for an arbitrary momentum, irrespective of the particle statistics, irrespective whether a problem is nonrelativistic or relativistic, whether particle production and anihilation is taken into consideration. This is a consequence of the relations between self-energies and Green's functions in a state of thermodynamic equilibrium (see Appendix E: also a third paper of the series). Due to these relations the expressions corresponding to scattering-in and -out from a given momentum (such as (G.5) and (G.6)) are shifted in the complex time plane by $i\beta$. The first moments of the expressions would differ by $i\beta$. The statement concerning the value of a macroscopic time is subject to the fact that one of the processes can be ignored: e.g. scattering-in for large momenta, for fermions close to zero-temperature - the scattering-out below the Fermi surface, and scattering-in above the surface.

Appendix H. Perturbation expansion and Green's function equations of motion for a general initial state

Perturbation theory rules for a general initial state have been outlined, whitout a full derivation, by Fujita in Refs. [14,15]. In the opinion of the present author, the rules given by Fujita are errorous.

About the initial state we shall only assume that its density operator commutes with a particle number operator. For the Green's function on a contour we have an equality (2.28). We define a time-ordered contraction of two operators $\hat{A}$ and $\hat{B}$ by

$$\hat{A}^{\cdot}\hat{B}^{\cdot} = T(\hat{A}\,\hat{B}) - N(\hat{A}\,\hat{B}) \quad , \tag{H.1}$$

where a normal operator-product $N(\cdot)$ is defined with respect to the vacuum. Upon writing an exponential in (2.28) in a form of a series, we apply the Wick theorem to every term of the series.

We have

$$\begin{aligned} T(\hat{A}\hat{B}\hat{C}\ldots\hat{X}\hat{Y}\hat{Z}) = {} & N(\hat{A}\hat{B}\hat{C}\ldots\hat{X}\hat{Y}\hat{Z}) + N(\hat{A}^{\cdot}\hat{B}^{\cdot}\hat{C}\ldots\hat{X}\hat{Y}\hat{Z}) + N(\hat{A}^{\cdot}\hat{B}\hat{C}^{\cdot}\ldots\hat{X}\hat{Y}\hat{Z}) \\ & + \ldots + N(\hat{A}^{\cdot}\hat{B}^{\cdot}\hat{C}^{\cdot\cdot}\ldots\hat{X}^{\cdot\cdot}\hat{Y}\hat{Z}) + \ldots \\ & + N(\hat{A}^{\cdot}\hat{B}^{\cdot\cdot}\hat{C}^{\cdot\cdot\cdot}\ldots\hat{X}^{\cdot}\hat{Y}^{\cdot\cdot}\hat{Z}^{\cdot\cdot\cdot}) + \ldots \quad . \end{aligned} \tag{H.2}$$

If we were taking a vacuum expectation value of the time-ordered product, then only a sum over all combinations of contractions, of the fully contracted products, would remain at the r.h.s. of (H.2). If we take an expectation value with respect to a certain initial state, then the expectation values of noncontracted operators in the normal products may be expressed through many-particle Green's functions

$$i^k G_k^{0<}(1,\ldots,k;1',\ldots,k')$$

$$= (\mp 1)^k \langle \hat{\Psi}_I^\dagger(k') \ldots \hat{\Psi}_I^\dagger(1')\hat{\Psi}_I(1) \ldots \hat{\Psi}_I(k)\rangle \quad . \tag{H.3}$$

For a k-particle Green's function we adopt a decomposition into products of 1-particle Green's functions and correlation matrices

$$G^{0<} = S(\Pi G^{0<} + \tilde{G}_2^0 \Pi G^{0<} + \ldots + \tilde{G}_k^0) \quad . \tag{H.4}$$

S is an operator symmetrizing Green's function arguments according to particle statistics, and Eq. (H.4) defines a k-particle correlation matrix $\tilde{G}_k^0$. For a 2-particle Green's function, Eq. (H.4) has e.g. a form

$$G_2^{0<}(1,2:1',2') = G^{0<}(1,1')G^{0<}(2,2') \mp G^{0<}(1,2')G^{0<}(1',2) + \tilde{G}_2^0(1,2;1',2') \quad . \tag{H.5}$$

Upon applying the Wick's theorem to every term of the series from (2.28), we express expectation values of noncontracted operators through functions $i^k G_k^{0<}$, and to the functions we apply (H.4). In the resulting decomposition, to every term in which a pair of operators $\hat{\Psi}_I$ and $\hat{\Psi}_I^\dagger$ is contracted according to (H.1), there corresponds a term, in which a function $iG^{0<}$ with this pair of operators replaces the contraction. If we sum the corresponding terms with one another, we get a function iG^0, Eq. (2.30), for the pair of operators. Upon consequent application of the procedure to all terms of the series, the only 1-particle functions remaining in the decomposition are the functions iG^0. We may say that we dress the vacuum functions with a medium. The Feynman rules that follow are such as in Sect. 2

- an additional rule reads:

Each k vertices ($k \geqslant 2$), into which previously particle lines were running in, may be connected with k vertices, from which particle lines were running out, through a k-particle correlation matrix $i^k\tilde{G}^o_k$. For fermions, upon assigning specific correlation-matrix arguments to interaction vertices in a diagram, assign to the diagram a factor $(-1)^F$, where F - a number of particle loops in the diagram. The number of loops evaluate by joining the correlation-matrix arguments with functions iG^o, i.e. on having $i^k\tilde{G}_k(1,2,\ldots,k;1',2',\ldots,k')$ join 1' with 1, 2' with 2, ..., k' with k.

The connected 1-order diagrams for $iG(1,1')$ are now of the form presented in Fig. 12. The function $i^2\tilde{G}^o_2$ is denoted by a bubble with directed lines. Let us mention, that correlation matrices may not be directly connected with one another. From a way in which correlation matrices were defined, a value of a matrix does not depend on an assignment of the time-arguments to the branches of a contour. Due to this, when determining a maximum time for internal time-integrations in a subdiagram, there is no need to take into account arguments of correlations matrices connected to a subdiagram. Further if certain subdiagram is connected solely to correlation matrices, as in Fig. 12e or 12f, then the whole diagram vanishes, bacause internal time-integrations in a subdiagram may be reduced to t_o.[3] If a diagram may be cut between the end-points in such a way that the cut passes only through correlation matrices, then the diagram does not depend upon

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[3] Let us mention, that for that reason, on reffering to "1-particle correlation-matrices" $iG^{o<}$, a cancellation occurs for the f^4 terms in the Boltzmann collision integral. Analogous cancellations occur in collision integrals for emission and absorption of bosons - third paper of this series.

assignment of the end-points to the branches of a contour.

In the above expansion we need to know the correlation matrices for all times larger than t_0. There holds

$$\hat{\Psi}_I(\underset{\sim}{x},t) = \int d\underset{\sim}{y}\, \hat{\Psi}(\underset{\sim}{y})(\hat{\Psi}_I(\underset{\sim}{x},t)\hat{\Psi}^\dagger(\underset{\sim}{y}) \pm \hat{\Psi}^\dagger(\underset{\sim}{y})\hat{\Psi}_I(\underset{\sim}{x},t)) \quad , \tag{H.6}$$

which follows from the fact that the r.h.s. satisfies the same differential equation as the l.h.s., with the same boundary condition at $t = t_0$. Equation (H.6) may be written as

$$\hat{\Psi}_I(\underset{\sim}{x},t) = \int d\underset{\sim}{y}\, \hat{\Psi}(\underset{\sim}{y})(iG^{0>}(\underset{\sim}{x},t,\underset{\sim}{y},t_0) - iG^{0<}(\underset{\sim}{x},t,\underset{\sim}{y},t_0)) \quad . \tag{H.7}$$

The hermitian conjugation of (H.7) yields

$$\hat{\Psi}_I^\dagger(\underset{\sim}{x},t) = \int d\underset{\sim}{y}\, \hat{\Psi}^\dagger(\underset{\sim}{y})(iG^{0>}(\underset{\sim}{y},t_0,\underset{\sim}{x},t) - iG^{0<}(\underset{\sim}{y},t_0,\underset{\sim}{x},t)) \quad . \tag{H.8}$$

From (H.5), (H.7) and (H.8), there follows

$$\begin{aligned} i^2\tilde{G}_2^0(1,2;1',2') = {} & \int d\underset{\sim}{x} \int d\underset{\sim}{y} \int d\underset{\sim}{x}' \int d\underset{\sim}{y}'\, i(G^{0>} - G^{0<})(1,\underset{\sim}{x},t_0) \\ & \times i(G^{0>} - G^{0<})(2,\underset{\sim}{y},t_0)\; i^2\tilde{G}_2^0(\underset{\sim}{x},t_0,\underset{\sim}{y},t_0;\underset{\sim}{x}',t_0,\underset{\sim}{y}',t_0) \\ & \times i(G^{0>} - G^{0<})(\underset{\sim}{x}',t_0,1')\; i(G^{0>} - G^{0<})(\underset{\sim}{y}',t_0,2') \quad , \end{aligned} \tag{H.9}$$

and analogous identities hold for higher correlation matrices. The correlation matrices in the existing rules may be replaced by the correlation matrices at t_0 and the $iG^{0\lessgtr}$ lines running to the correlation-matrix arguments at t_0. A rule referring to correlation matrices may be now modified in the following way:

Draw two horizontal marginal lines $t_0^<$ and $t_0^>$, which correspond to the two ends of a time contour. At this lines mark the correlation matrices occuring in a diagram. Each matrix occurs simultaneously at $t_0^<$ and at $t_0^>$. To particle lines running to $t_0^<$ and from $t_0^>$ there correspond functions $iG^{0<}$, and to lines running in the opposite direction

functions $iG^{o>}$. To a line running downward, $iG^{o<}$, there corresponds an extra factor (-1).

A diagram from Fig. 12d decomposes now into 2^4 diagrams, some of which are presented in Fig. 13. Changing the order of summation of diagrams we may dress the lines iG^o running to and from $t_o^{\gtrless}$. We have a possibility of introducing a self-energy.

A self-energy will be defined diagrammatically as an irreducible part of the Green's function. We single out a self energy Σ_C, which begins with a correlation matrix and ends with a potential. The function Σ^C begins with a potential and ends with a correlation, while Σ begins and ends with a potential. There exists no self-energy that would begin and end with a correlation, because respective diagrams vanish. A Dyson equation for the Green's function is of the form

$$G = G^o + G^o\Sigma^C G + G^o\Sigma G + G^o\Sigma_C G \quad , \tag{H.10}$$

and we have on a contour

$$\begin{aligned}\Sigma_C(1,2) &= \Sigma_C(1,\underset{\sim}{x}_2)(\delta(t_o^>,t_2) - \delta(t_o^<,t_2)) \\ &= -\Sigma_C(1,\underset{\sim}{x}_2)\delta(t_o - t_2) \quad ,\end{aligned} \tag{H.11}$$

and

$$\Sigma^C(1,2) = (\delta(t_1,t_o^<) - \delta(t_1,t_o^>))\Sigma^C(\underset{\sim}{x}_1,2)$$
$$= \delta(t_1 - t_o)\Sigma^C(\underset{\sim}{x}_1,2) \quad . \tag{H.12}$$

Upon applying G^{o-1} to both sides of (H.10) and exploiting $G^{o-1}G^{o\gtrless} = 0$, we get

$$G^{o-1}G = 1 + \Sigma G + \Sigma_C G \quad . \tag{H.13}$$

On restricting, at the l.h.s. of (H.13), the variation of the function arguments to opposite branches of a contour, we obtain the following generalized Kadanoff-Baym equations

$$(i\frac{\partial}{\partial t_1} + \nabla_1^2)G^{\gtrless}(1,1') = \int d\underset{\sim}{x}_2\, \Sigma_{HF}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t_1)G^{\gtrless}(\underset{\sim}{x}_2,t_1,1')$$
$$+ \int_{t_o}^{t_1} d2(\Sigma^> - \Sigma^<)(1,2)G^{\gtrless}(2,1') - \int_{t_o}^{t_{1'}} d2(\Sigma^{\gtrless} + \Sigma_C)(1,2)(G^> - G^<)(2,1') \quad . \tag{H.14}$$

Let us mention, that from the Green's functions in the last terms of Eqs. (H.10), (H.13), and (H.14), one should in principle exclude the parts of the functions that end with a correlation. However respective contributions from the functions at $t_o^<$ and $t_o^>$ cancel out. A procedure analogous to the above leads to a second pair of equations

$$(-i\frac{\partial}{\partial t_{1'}} + \frac{\nabla_{1'}^2}{2m})G^{\gtrless}(1,1') = \int d\underset{\sim}{x}_2 G^{\gtrless}(1,\underset{\sim}{x}_2,t_{1'})\Sigma_{HF}(\underset{\sim}{x}_2,\underset{\sim}{x}_{1'};t_{1'})$$
$$+ \int_{t_o}^{t_1} d2(G^> - G^<)(1,2)(\Sigma^{\gtrless} + \Sigma^C)(2,1') - \int_{t_o}^{t_{1'}} d2G^{\gtrless}(1,2)(\Sigma^> - \Sigma^<)(2,1') \quad . \tag{H.15}$$

In the Kadanoff-Baym equations (3.6) and (3.7), for $t_1 = t_{1'} = t_o$, the Hartree-Fock energy yields a sole contribution to the Green's function

evolution. The correlations (scattering) built up only with time. In Eqs. (H.14) and (H.15), at $t_1 = t_{1'} = t_o$, the extra r.h.s. contributions come respectively from Σ_C and Σ^C. At an initial moment, from diagrams,

$$\Sigma_C(\underset{\sim}{x}_1,t_o,\underset{\sim}{x}_{1'}) = \mp iV(\underset{\sim}{x}_1-\underset{\sim}{x}_2)\tilde{G}_2^o(\underset{\sim}{x}_1,t_o,\underset{\sim}{x}_2,t_o;\underset{\sim}{x}_{1'},t_o,\underset{\sim}{x}_2,t_o) \quad , \qquad \text{(H.16)}$$

which inserted into (H.14) yields a result that agrees with (3.35), as it should.

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Figure Captions

Fig. 1. Contour along the time axis for an evaluation of the operator expectation value.

Fig. 2. Lowest-order diagrams for the self-energy.

Fig. 3. Born diagrams for the self-energy.

Fig. 4. Contour in the complex time plane corresponding to the evaluation of an operator expectation value with respect to a state of a lowest ε_n.

Fig. 5. Contour corresponding to the evaluation of an expectation value of a Heisenberg picture operator at $t_1 > t_0$.

Fig. 6. Contours corresponding to the evaluation of expectation values with respect to the equilibrium density operator.

Fig. 7. Contours corresponding to the evaluation of the operator expectation values, for a finite time of the imaginary evolution.

Fig. 8. Diagrams that yield expressions which cancel with one another.

Fig. 9. Contour along the time axis.

Fig. 10. Ladder diagrams for the 2-particle Green's function.

Fig. 11. Diagrams defining the T-matrix.

Fig. 12. First-order diagrams for a 1-particle Green's function.

Fig. 13. Some of the diagrams corresponding to the diagram from Fig. 12d, upon modification of the additional perturbation theory rule.

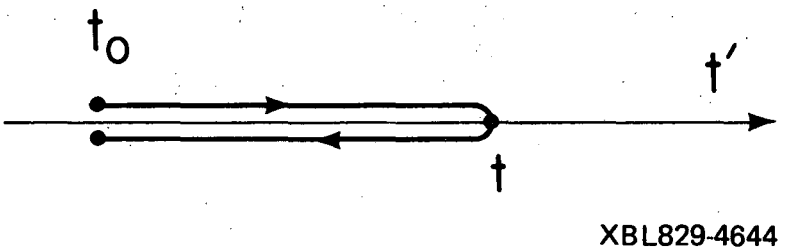

Fig. 1

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Fig. 2

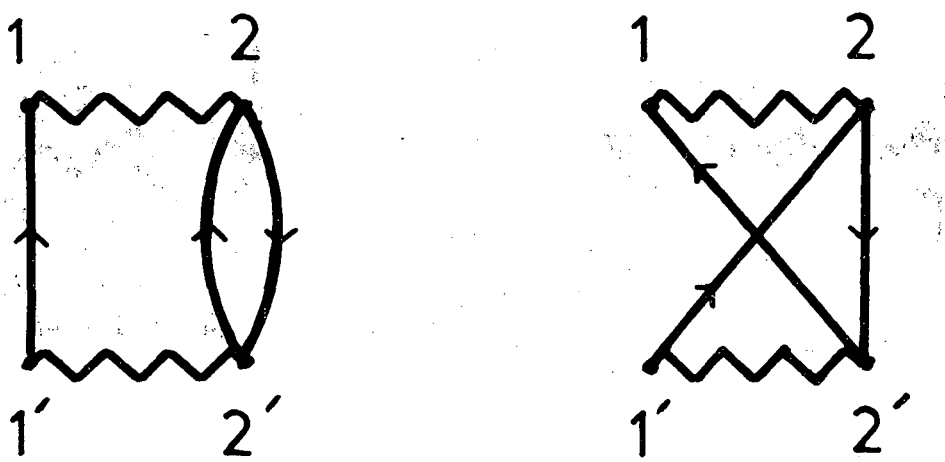

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Fig. 3

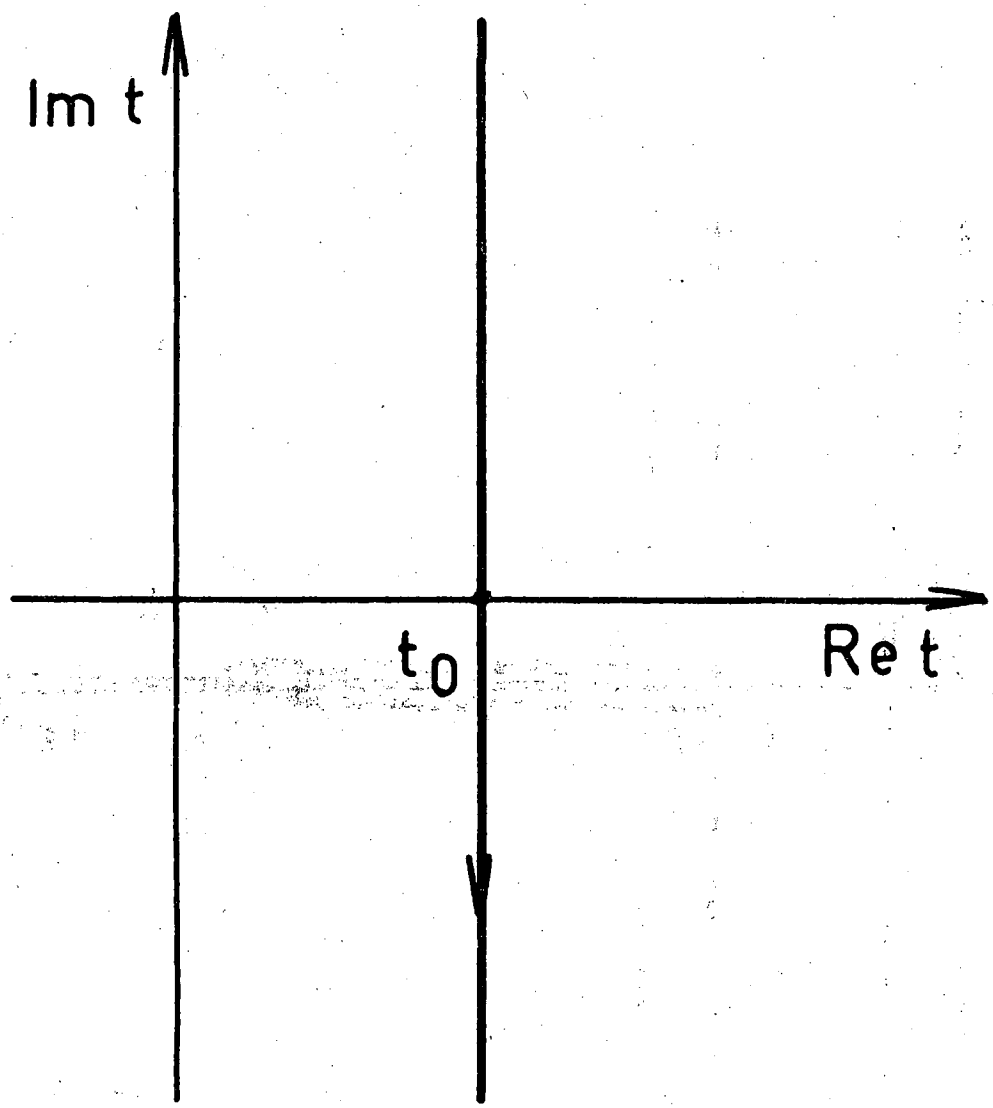

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Fig. 4

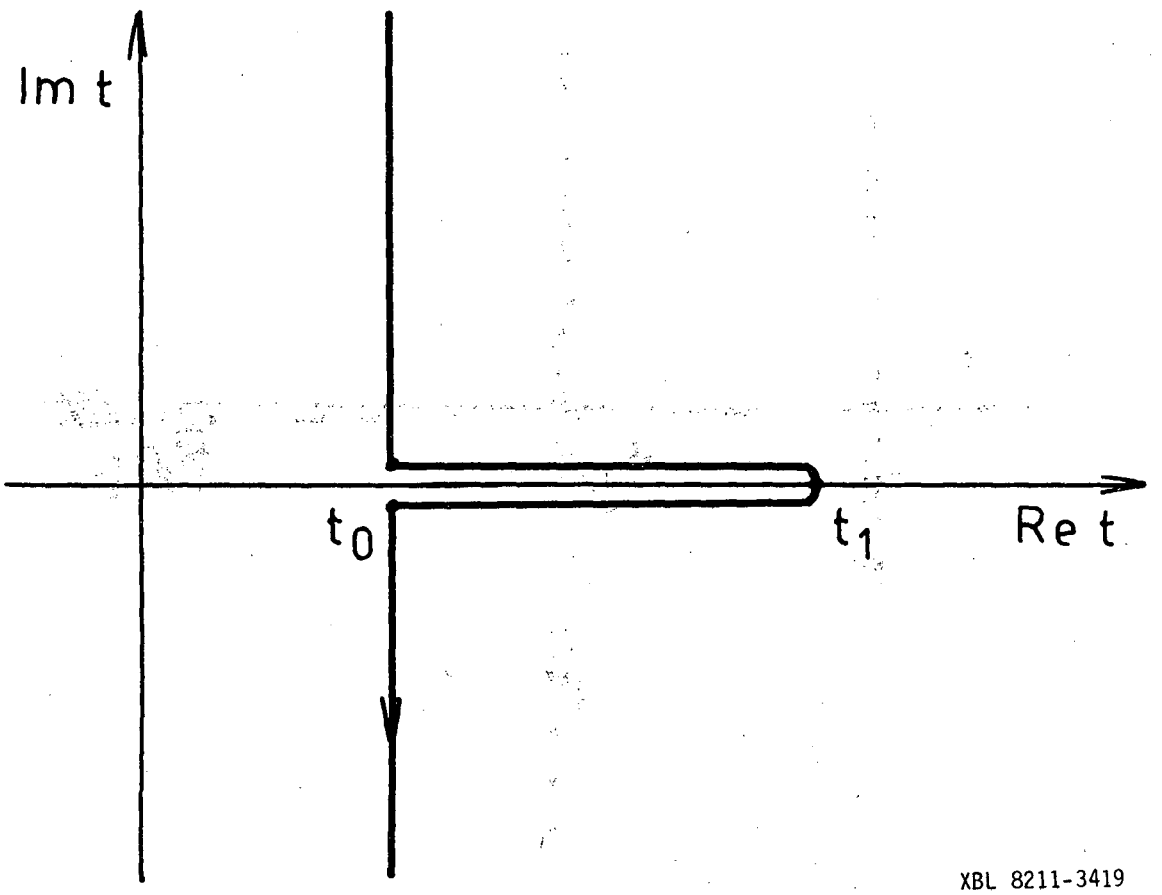
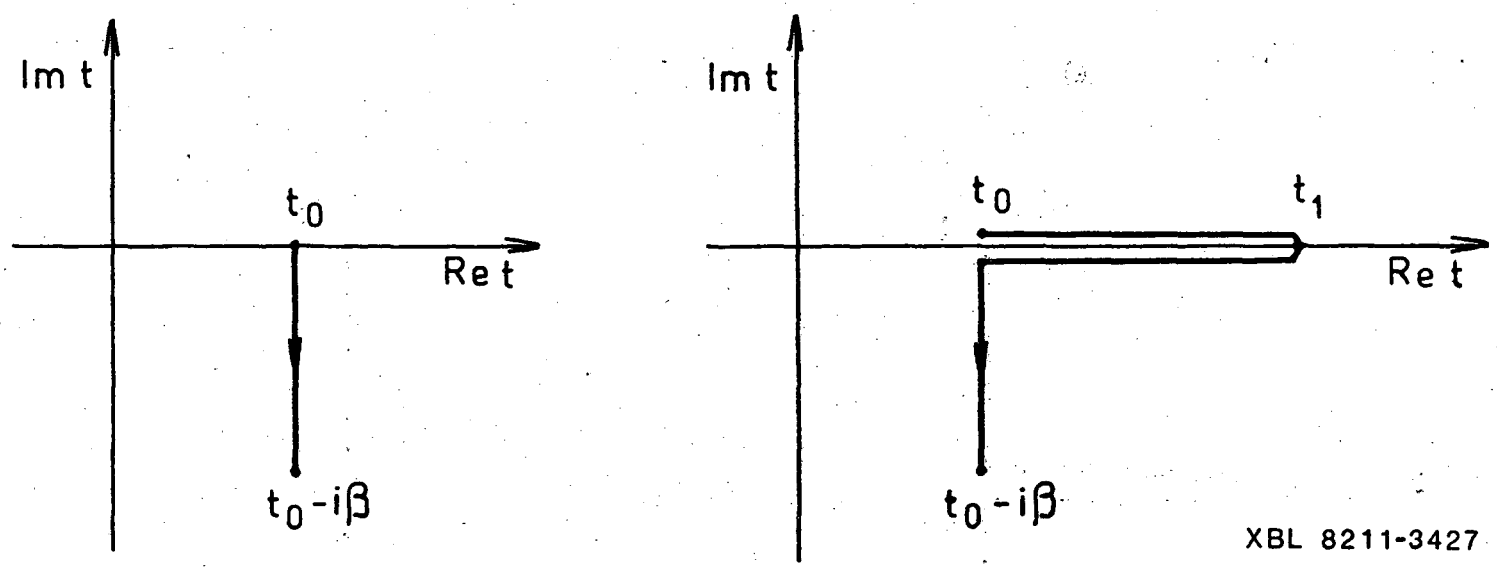


Fig. 5



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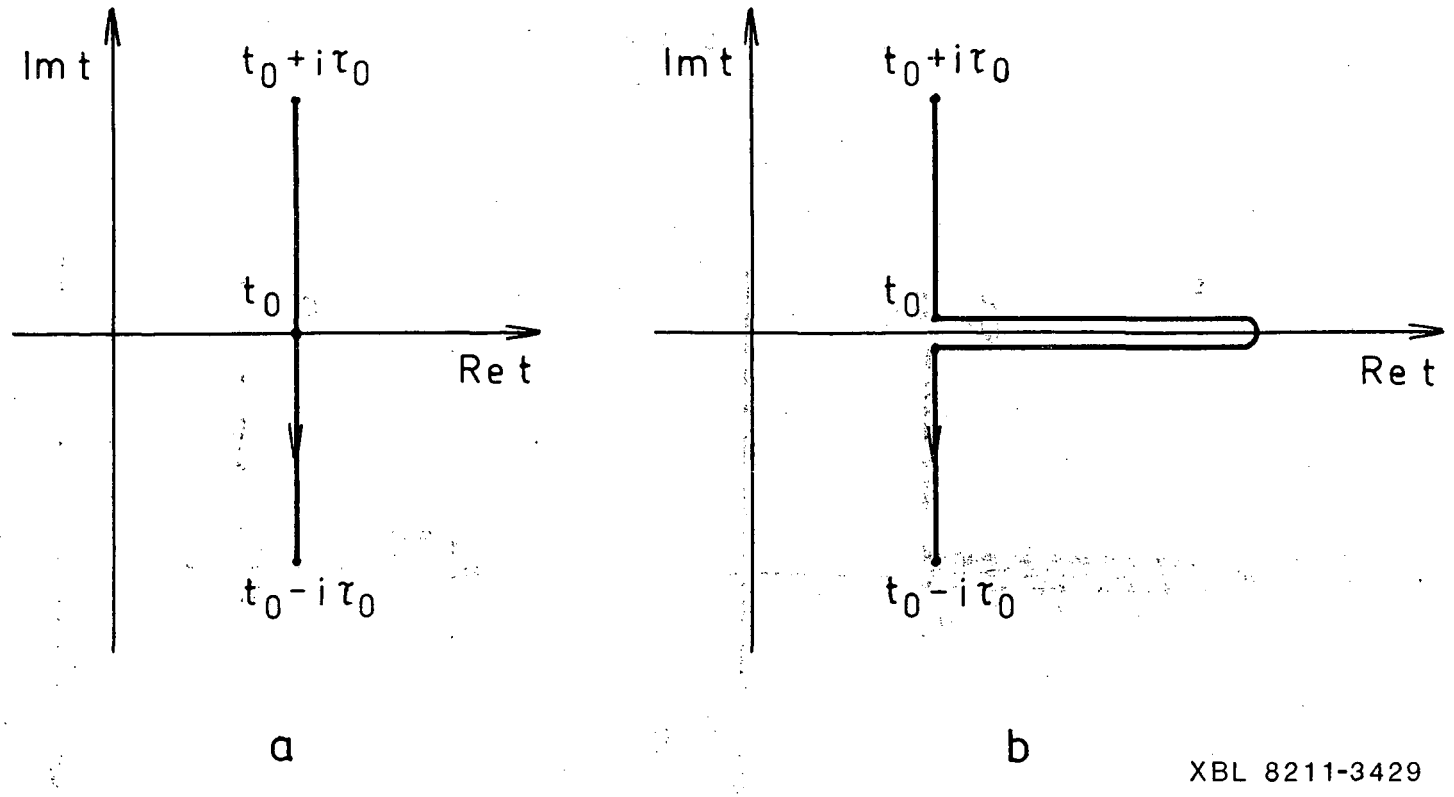
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$t_0 + i\tau_0$
t_0
Re t
$t_0 - i\tau_0$
a
Im t
$t_0 + i\tau_0$
t_0
Re t
$t_0 - i\tau_0$
b
XBL 8211-3429

Fig. 7

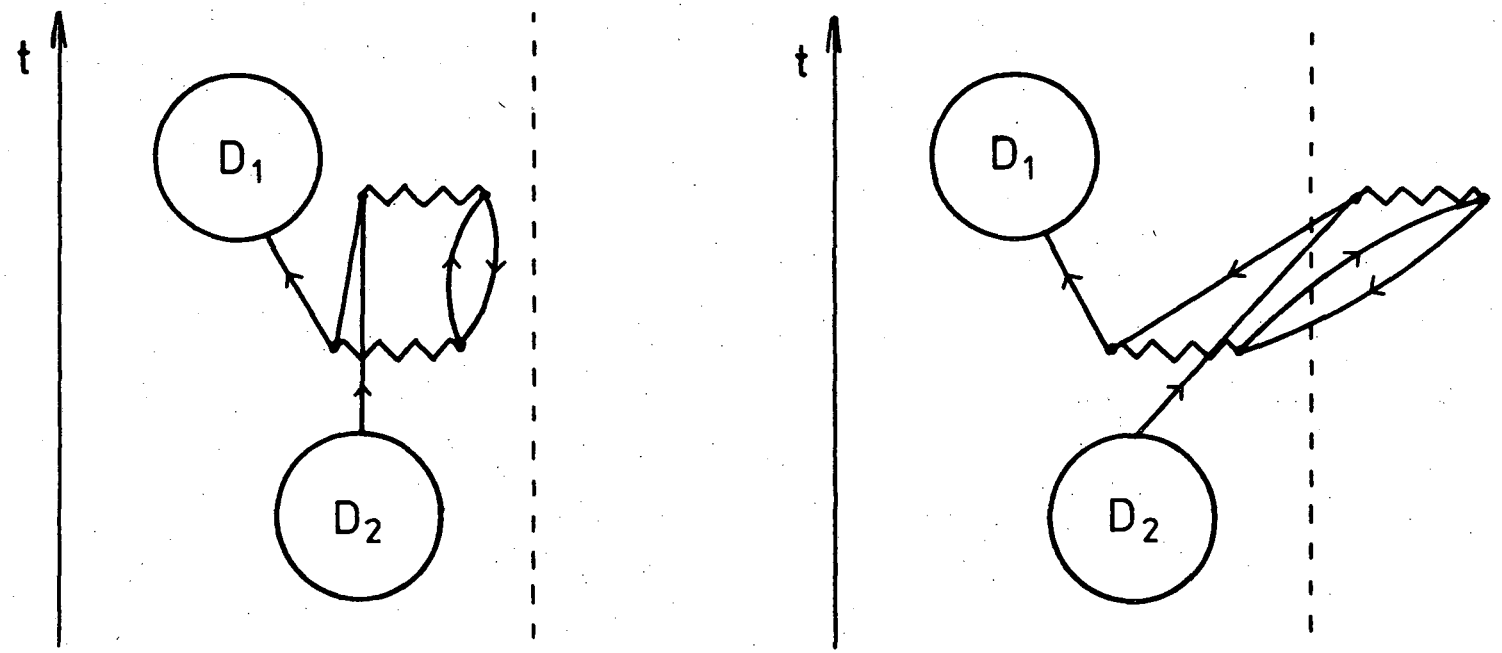

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Fig. 8

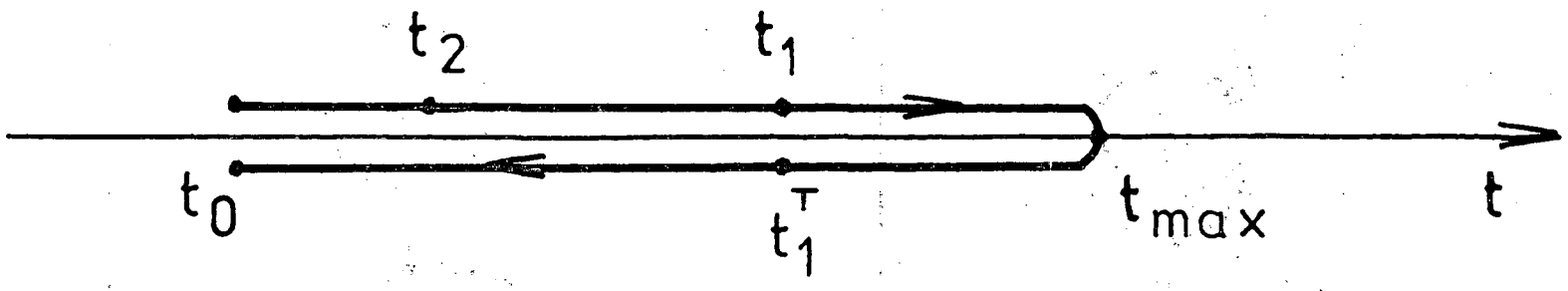

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Fig. 9

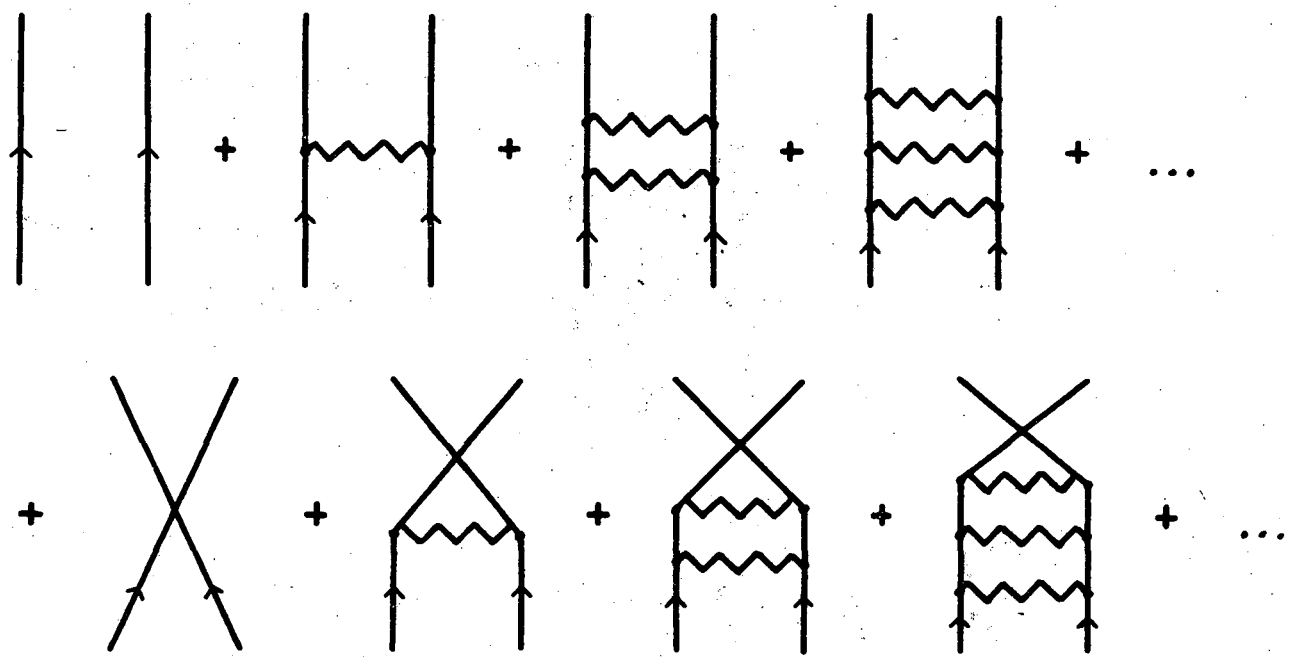

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Fig. 10

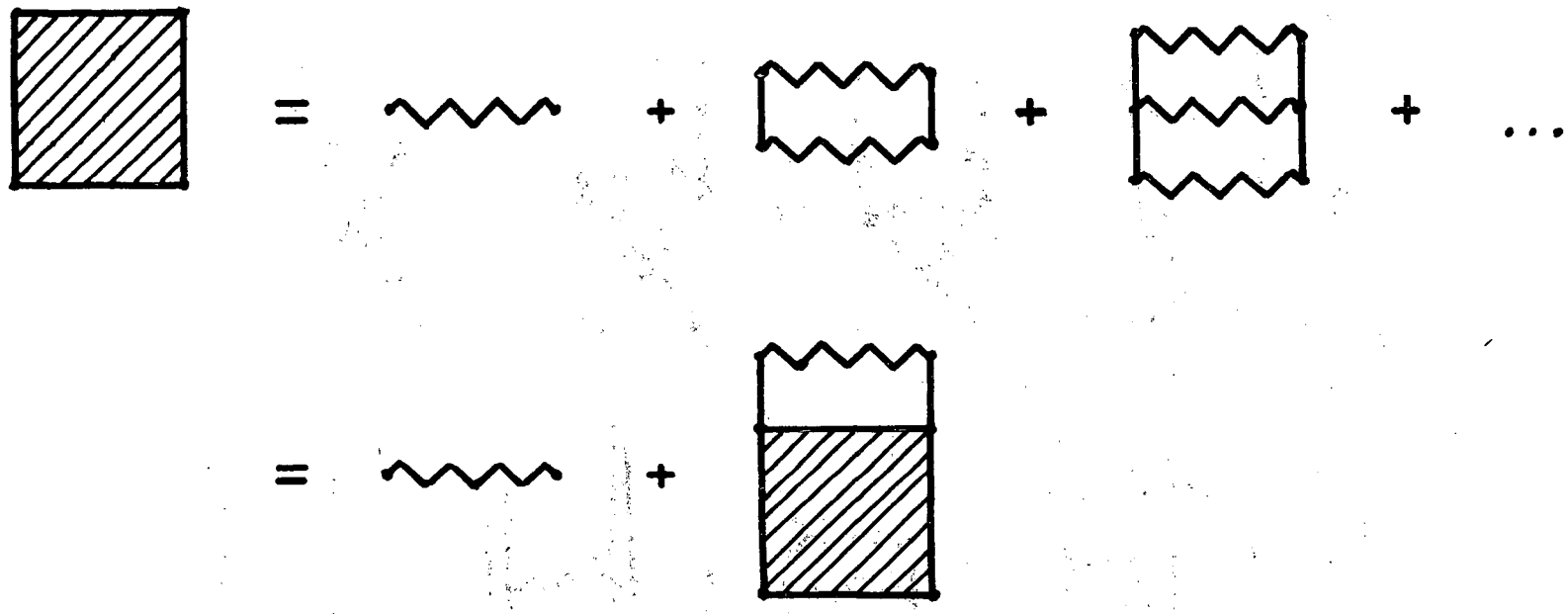

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Fig. 11

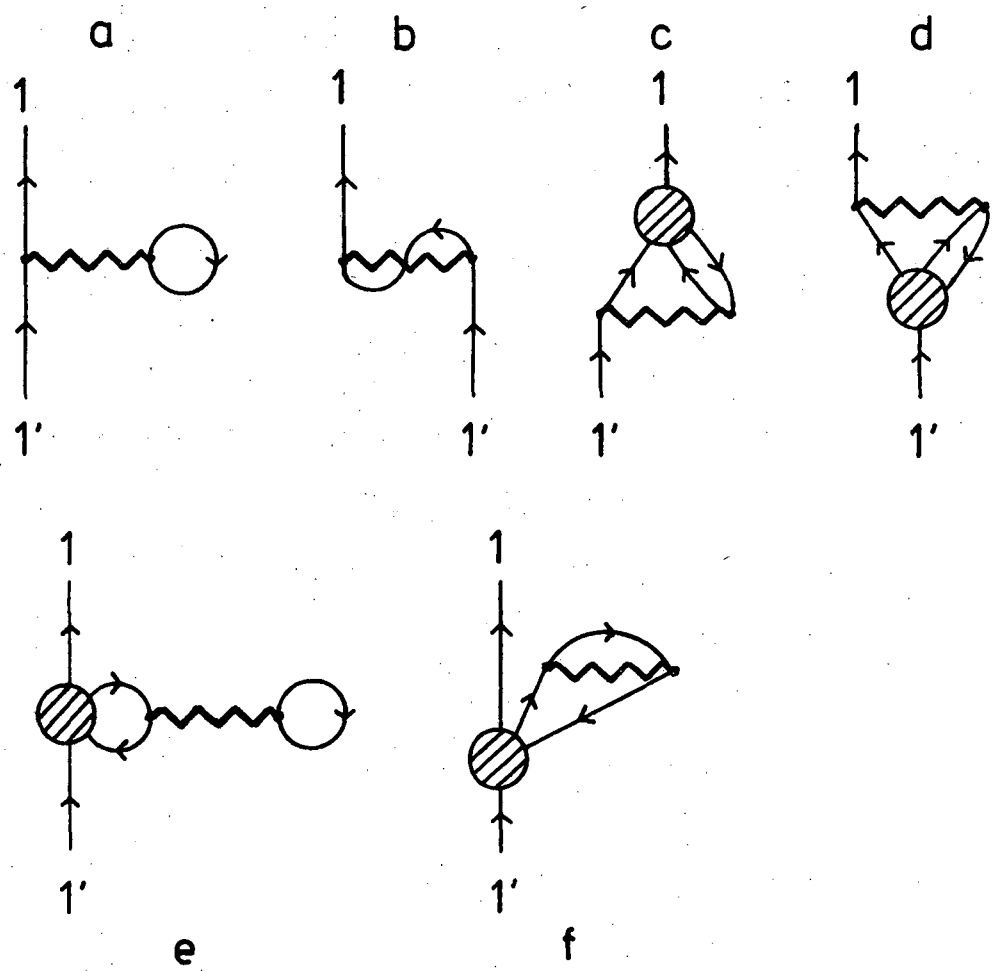

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Fig. 12

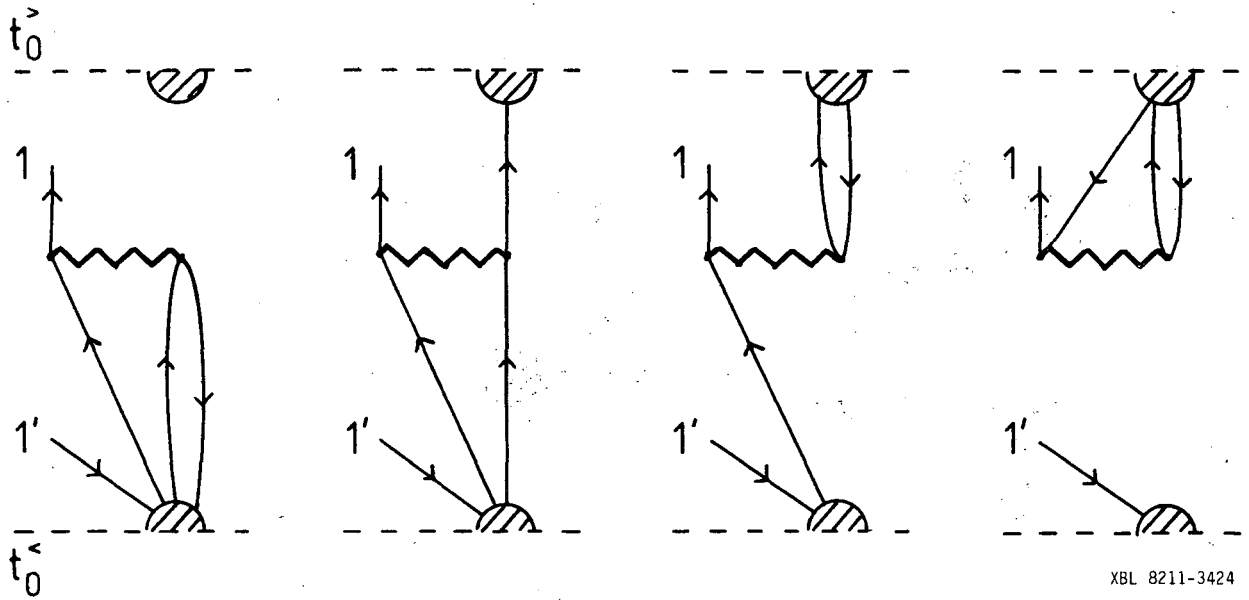

Fig. 13

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