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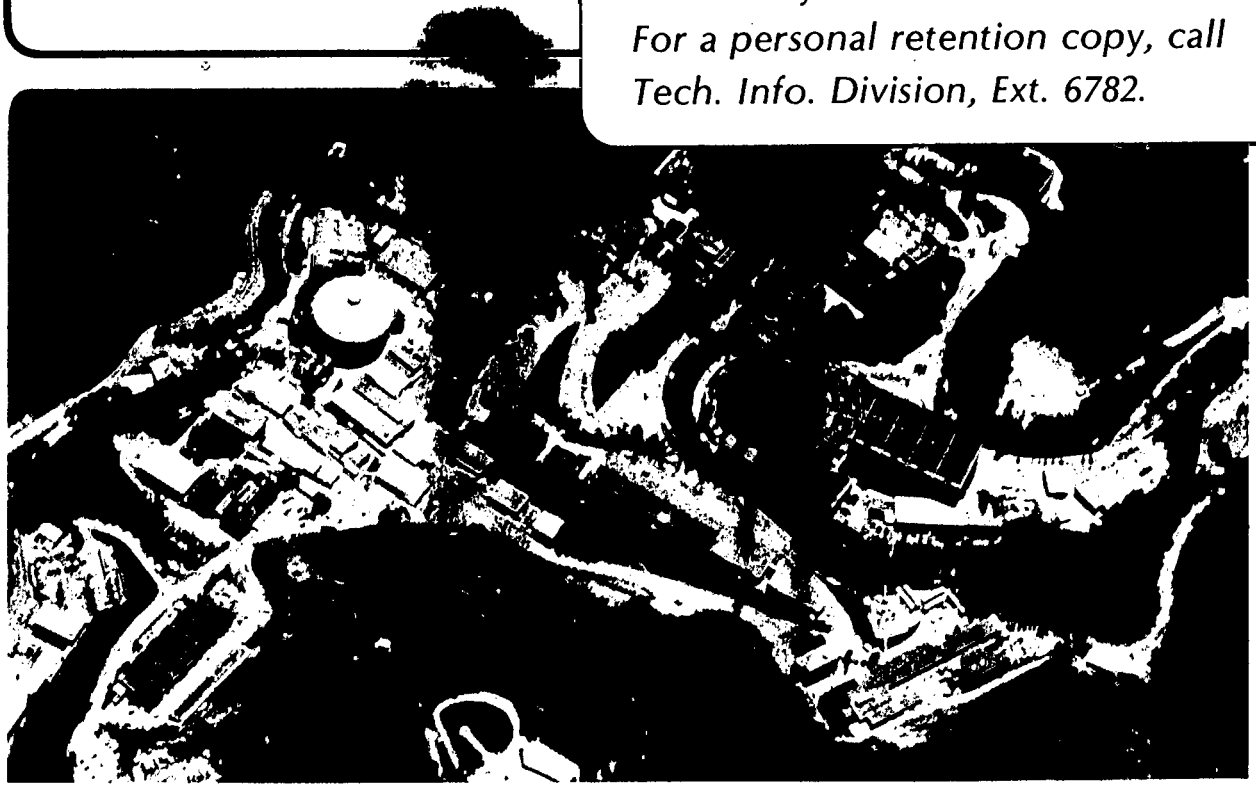

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THE LIMITED ANGLE PROBLEM IN TOMOGRAPHY AND SOME RELATED MATHEMATICAL PROBLEMS[1]

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THE LIMITED ANGLE PROBLEM IN TOMOGRAPHY AND SOME RELATED MATHEMATICAL PROBLEMS

1. INTRODUCTION

Consider the problem of recovering a compactly supported function $f(\underline{x})$, $\underline{x} \in R^n$, from a collection of its "one dimensional projections" given by

$$Pf(w,t) = \int_{\langle x,w\rangle = t} f$$

$$w \in S^{n-1} \quad .$$

Here S^{n-1} denotes the unit sphere in R^n , and $\langle x,w\rangle$ denotes the usual inner product in R^n .

The most common applications call for $n = 2$ (X-ray tomography) or $n = 3$ (Nuclear Magnetic Resonance Imaging) and therefore we will occasionally specialize our considerations to these cases.

One of the main concerns in this lecture is the effect that a limitation of the range over which the directions w can be chosen has on the quality of the reconstruction.

If one had no restriction on the choice of directions w , more precisely if one knew $Pf(w,t)$ for all $w \in S^{n-1}$ one could use the formulas of Radon [1]. In the two dimensional case we have

$$f(x,y) = \frac{1}{2\pi^2} \int_0^{\pi} d\theta \text{ p.v.} \int_{-\infty}^{\infty} \frac{1}{t - s} \frac{\partial}{\partial s} Pf(\theta,s) \quad .$$

with $t = x \cos\theta + y \sin\theta$.

The formula depends on the dimension n but some features of it are independent of n : the projections are <u>first</u> "filtered" by the application of some operator -- differential if n is odd, integral if n is even -- <u>then</u> for each $w \in S^{n-1}$ the result is "backprojected" to get a function of $\underline{x}$ -- i.e. the function is made constant along the hyperplanes $\langle \underline{x}, \underline{w} \rangle = t$, <u>finally</u> these backprojections are added up with uniform weight over S^{n-1} .

Thus Radon's formula has the form of a sum of filtered backprojection. The history behind the formula is interesting. Although Radon, in 1917, certainly proved it first for general n , the Dutch physicist H. Lorentz knew of this formula for $n = 3$ around the end of the century [2]. A paper in 1906 gives credit to Lorentz [3]. Later on Uhlenbeck [4], Cramer and Wold [5], Bracewell [6], Cormack [7] -- and probably many others had to rediscover this result on their own. An independent attack, not related to Radon's formula was conceived and carried out by Hounsfield [7].

2. THE LIMITED ANGLE PROBLEM

A problem of practical interest is that of recovering f from its projections $Pf(w,t)$ in a given limited range. For convenience the following discussion deals with R^2 and we use the notation

$$Pf(\theta,t)$$

for the projections.

Although one can prove that knowing $Pf(\theta,t)$ for $-\alpha < \theta < \alpha$, with $\alpha > 0$ arbitrary, determines f uniquely it is clear that the "quality" of the reconstruction should depend on α . We have the case of an ill-posed inversion problem and the degree of ill-conditioning will depend seriously on the range of views $-\alpha < \theta < \alpha$.

Assume that all the projections $Pf(\theta,t)$ - $\frac{\pi}{2} < -\alpha \leq \theta \leq \alpha < \frac{\pi}{2}$ are known. The problem

is to recover f . It is convenient to visualize the data as the result of applying a linear operator A to the unknown function f . For mathematical expediency we take $f \in L^2(D)$ which gives $Pf(\theta,t) \in L^2(I,w)$, for each θ . Here D denotes the unit disk in R^2 , $L^2(D)$ the space of square integrable functions on D , and $L^2(I,w)$ the space of functions on the unit interval I , square integrable with respect to some appropriate weight w .

In this fashion the operator A maps $L^2(D)$ into $\oplus L^2(I,w)$, with the direct sum extending over all $-\alpha \leq \theta \leq \alpha$. If b -- the data -- denotes the set of all available projections we have

$$Af = b \quad .$$

It is now convenient to bring in the singular value -- singular vector decomposition of A .

Take ψ_i to be the orthonormal eigenvectors of AA^* , with positive real eigenvalues μ_i^2 , $\mu_i > 0$. Then set $\varphi_i = A^*\psi_i/\mu_i$ and get for the smallest f which minimizes the error $Af - b$ the expression

$$\tilde{f} = \Sigma \frac{(b,\psi_i)}{\mu_i} \phi_i$$

A crucial role is played by the spectral decomposition of AA^* . This is an operator from $\oplus L^2(I,w)$ to itself. This space can be expressed as a discrete direct sum

$$\oplus L^2(I,w) = \sum_{n=0}^{\infty} V_n$$

where the subspace V_n is made up of functions of the factorized form

$$k(\theta)\ C_n^1(\cos t)\sqrt{1 - t^2},$$

with $k(\theta) \in L^2(-\alpha,\alpha)$ and $C_n^1(\cos t)$ a Gegenbauer polynomial.

One can see that AA^* leaves each one of the spaces V_n invariant, and that its action on V_n is given by

$$((AA^*)_n\ k)(\theta) = \int_{-\alpha}^{\alpha} \frac{\mathbb{C}_n^1(\cos(\theta - \zeta))}{\mathbb{C}_n^1(1)}\, k(\zeta)\, d\zeta$$

If $k_{i,n}(\theta)$ denote the eigenfunctions of $(AA^*)_n$ then the eigenfunctions of AA^* are given by

$$k_{i,n}(\theta)\ \mathbb{C}_n^1(\cos t)\sqrt{1 - t^2}$$

For all the missing proofs the reader can see [9, 10, 11].

Now we consider the family of integral operators given by $(AA^*)_n$. Since

$$\frac{C_n^1(\cos\theta)}{C_n^1(1)} = \frac{\sin(n + 1)\theta}{(n + 1)\sin\theta}$$

we are -- by a stroke of luck -- in the case considered by Slepian in [12].

Slepian found that a remarkable phenomenon holds true for this integral operator, to wit: a second order differential operator exists which has the same eigenfunctions as $(AA^*)_n$. This is in the same spirit of the pioneering papers of Whittaker [13] and Ince [14].

In the same paper Slepian considers the matrix

$$\frac{\sin 2\pi\alpha(i - j)}{\pi(i - j)}$$

and shows that a tridiagonal matrix exists which commutes with it. This holds for any value of the parameters n,α .

In summary, the computation of the singular values and singular vectors of PP^* can be reduced to a managable numerical problem since we are in the presence of the remarkable miracle described above. For the actual implementation of this program in the tomography context see [15, 16].

3. COMMUTING INTEGRAL AND DIFFERENTIAL OPERATORS

The phenomenon mentioned above has appeared before. Already in Ince [14] we see examples of this situation.

In the sixties Slepian, Landau and Pollak came up with a remarkable series of papers dealing with the problem of time and band limited functions [17-20]. At the core of the paper is the fact that if A denotes the finite Fourier transform

$$(Af)(\lambda) = \int_{-T}^{T} e^{i\lambda t} f(t)\, dt \ , \ \lambda \in [-\Omega,\Omega]$$

then the integral operator given by A^*A , namely

$$(kf)(t) = \int_{-T}^{T} \frac{\sin \Omega(t - s)}{t - s} f(s)\, ds \qquad t \in [-T,T]$$

admits a second order differential operator commuting with it, namely

$$(Df)(t) = ((T^2 - t^2)f')' - \Omega^2 t^2 f \ .$$

Slepian proved that this same result holds if R^1 is replaced by R^n [20], and then in [12] -- mentioned earlier in connection with tomography -- extended this fact for

functions defined on the circle or on the integers. We become interested in this problem and tried to extend its range. In [21] one finds an extension to the case of the nth roots of unity and then in [22] one finds a classification of all Toeplitz matrices which have a tridiagonal matrix, <u>with single spectrum</u>, in its commutator.

In [23] this property is seen to hold for the Hilbert matrix and in [24] one finds that the property (essentially) holds for the Discrete Fourier Transform matrix.

Another example not reported before is given by the matrix

$$G_{m,n} = \frac{\sin(n - m)\theta}{n - m} - \frac{\sin(n + m)\theta}{n + m} \qquad 1 \leq n,m \leq N$$

which comes about as the Gram matrix for the functions $\sin n\xi$, $n \leq N$.

It is not hard to see that in this case a commuting tridiagonal matrix with single spectrum is given by

$$T_{ii} = 2(i^2 - 1) \cos \theta$$

$$T_{i,i+1} = T_{i+1,i} = (2 + N)(N - 1) - (2 + i)(i - 1)$$

It is a bit harder to see that in order for the commutativity property mentioned above to hold for a matrix of the form

$$\frac{\sin(n - m)\theta}{n - m} + \alpha \frac{\sin(n + m)\theta}{n + m}$$

one needs $\alpha = \pm 1$ or the previously known case $\alpha = 0$ [12]. These three instances correspond to the Gram matrices for the functions $\sin n\xi$, $\cos n\xi$, $e^{in\xi}$.

Other examples of Gram matrices with this property have been given in [25].

In trying to deal with the "limited angle" problem in R^3, we were led to consider the extension of Slepian's [12] results to the case of S^2, or more generally S^n. This was found to be true and is reported, along with other results in [26].

Since the property holds for R^n and S^n it is only natural to expect that it should hold for hyperbolic space H^n. Indeed in [26] the result is shown to hold for H^3, see also [27] for a much more transparent proof.

And yet the result is not true for any H^n, $n \geq 2$, except when $n = 3$ [28].

The eigenfunctions for the radial part of the Laplacian in R^n are given by

$$\phi(\lambda,r) = \frac{J_{\frac{n-2}{2}}(\lambda r)}{(\lambda r)^{\frac{n-2}{2}}}$$

We make here the remark that as functions of λ these functions satisfy a second order differential equation of the form

$$D_\lambda \phi(\lambda,r) = \Theta(r)\, \phi(\lambda,r) \qquad (1)$$

The eigenfunctions fo the Laplacian for H^3 are

$$\phi(\lambda,r) = \frac{\sin \lambda r}{\lambda \sin hr}$$

and we have

$$\frac{1}{\lambda^2} \frac{d}{d\lambda}\left(\lambda^2 \frac{d}{d\lambda} \phi(\lambda,r)\right) = -r^2\phi(\lambda,r)$$

It is now a fact that for any $n \neq 3$, $n \geq 2$, the radial eigenfunctions of the Laplacian for H^n do not satisfy an

equation of the form (1).

This fact and other evidence from [25] made us consider equation (1) an important ingredient in the existence of a differential operator of "band and time limiting." More on this in the next section.

4. DIFFERENTIAL EQUATIONS IN THE SPECTRAL PARAMETER

Given a second order differential operator

$$L \equiv -\frac{d^2}{dx^2} + V(x)$$

denote by $\phi(\lambda,x)$ a family of eigenfunctions of L, $L\phi = \lambda^2\phi$. The previous considerations lead us to the following

<u>Question</u>. For what $V(x)$ does there exist a differential operator D_λ such that

$$D_\lambda\, \phi(\lambda,x) = \Theta(x)\, \phi(\lambda,x)$$

We have attacked this question in collaboration with H. Duistermaat [29].

We give below some few examples of this situation leaving a more complete discussion for [29].

(a) Take $V(x,t) = \dfrac{6x(x^3 - 2t)}{(x^3 + t)^2}$,

then if $\phi(\lambda,x) = \dfrac{(\lambda^2x^3 + 3i\lambda x^2 - 3x + \lambda^2 t)}{\lambda^2(x^3 + t)}\, e^{i\lambda x}$

we've

$$\left(-\frac{d^2}{dx^2} + V(x,t)\right)\phi(\lambda,x) = \lambda^2\phi(\lambda,x)$$

and

$$\left(\left(\frac{d}{d\lambda} - \frac{2}{\lambda}\right)\left(\frac{d}{d\lambda} + \frac{2}{\lambda}\right)\right)^2 \phi - 4it\,\frac{d\phi}{d\lambda} = (x^4 + 4tx)\phi$$

(b) Take $V(x,t,s) = \dfrac{12x^{10} + 324sx^5 + 450t^2x^4 + 300t^3x + 162s^2}{(x^6 + 5tx^3 + 9sx - 5t^2)^2}$

then for an appropriate family of eigenfunctions $\phi(\lambda,x)$ we've

$$\left[i\left(\frac{d}{d\lambda} - \frac{3}{\lambda}\right)\left(\frac{d}{d\lambda} - \frac{2}{\lambda}\right)\left(\frac{d}{d\lambda} - \frac{1}{\lambda}\right)\frac{d}{d\lambda}\left(\frac{d}{d\lambda} + \frac{1}{\lambda}\right)\left(\frac{d}{d} + \frac{2}{\lambda}\right)\left(\frac{d}{d\lambda} + \frac{3}{\lambda}\right)\right]\phi +$$

$$+ \frac{63}{2}s\left(\frac{d}{d\lambda} - \frac{1}{\lambda}\right)\left(\frac{d}{d\lambda} + \frac{1}{\lambda}\right)\phi - 35it^2\frac{d}{d\lambda}\phi$$

$$+ \frac{35}{4}t\left[\left(\left(\frac{d}{d\lambda} - \frac{2}{\lambda}\right)\left(\frac{d}{d\lambda} + \frac{2}{\lambda}\right)\right)^2 - \frac{36}{\lambda^4}\right]\phi$$

$$= \left(x^7 + \frac{35}{4}tx^4 - \frac{63}{2}sx^2 - 35t^2x\right)\phi$$

We make the observation that the function $V(x,t)$ in example (a) is a solution of the Korteweg-deVries equation. Also $V(x,t,s)$ in example (b) can be written in the form

$$e^{tX_1+sX_2}\left(\frac{12}{x^2}\right)$$

where X_1, X_2 denote the Kdv flow and the fifth order differential operator given by $[(-L^{3/2})_+, L]$ and $[(-L^{5/2})_+, L]$ respectively, (up to scaling). See [30 - 35].

As one may suspect this works for the whole "hierarchy." More interesting still, not all examples are obtained this way.

(c) The simplest example which is not obtained in this fashion is the following.

Take

$$V(x,t) = \frac{15x^4 - 18tx^2 - t^2}{4x^6 + 8tx^4 + 4t^2x^2}$$

with eigenfunctions

$$\phi(x,k) = -\sqrt{x}\ J_2(xk) + \frac{2t}{(t + x^2)kx}\sqrt{x}\ J_1(xk)$$

One then has

$$\left[\left(-\partial_K^2 + \frac{15}{4k^2}\right)^2 + 2t\left(-\partial_K^2 - \frac{1}{4k^2}\right)\right]\phi = (x^4 + 2tx^2)\phi$$

It is interesting to notice that for the first family of examples one has

$$\Theta'(x) = \tau(x) \quad \text{where}$$
$$V(x) = -2(\log \tau(x))''$$

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