Submitted to Nuclear Physics

LBL-1615
Preprint c.1

DESCRIPTION OF THE NUCLEAR SURFACE BY MOMENTS

RECEIVED
LAWRENCE
RADIATION LABORATORY

Georg Süssmann

LIBRARY AND
DOCUMENTS SECTION

May 1973

Prepared for the U. S. Atomic Energy Commission
under Contract W-7405-ENG-48

For Reference

Not to be taken from this room

LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY

LBL-1615
c.1

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

DESCRIPTION OF THE NUCLEAR SURFACE BY MOMENTS*†

Georg Süssmann

Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

and

Sektion Physik der Universität München,
8046 Garching, Germany††

May 1973

Abstract

The parameters proposed by W. D. Myers to describe the radial shape dependence of the nuclear surface, are shown to be as useful as they are natural. For spherical nuclei, the central radius C, the charge radius R, the quadratic radius Q are redefined, and it is shown how they are interrelated by Myers' surface width b, flair γ_3, crookedness γ_4, and the higher shape parameter γ_5. All these quantities are calculated for some special charge distribution functions. (For symmetric distributions, possessing a symmetry center at C, the odd surface moments $b^3\gamma_3$ and $b^5\gamma_5$ vanish.) The connection of the surface moments $\Gamma_\mu = b^\mu\gamma_\mu$ with the volume moments F_κ that have been extracted by K. W. Ford and J. G. Wills from muonic atoms, is indicated.

*Work performed under the auspices of the U. S. Atomic Energy Commission.

†This work appeared previously as a draft version of UCRL-19960 (1970).

††Present address.

1. Introduction

Most of the information that has been reported about the shapes of atomic nuclei consists of empirical values of "relative" parameters. By this we mean quantities, which can be defined only in terms of some assumed parametrization that has to be adopted as close as possible to the true shape. In our case, one has to fit some "nuclear charge" distribution $\rho(r,\theta,\phi)$, which can be of electric or of baryonic nature, or could even be a one-particle potential, or something else of that kind. Well known examples are the central radius C and the diffuseness parameter a of a Fermi function $\rho_c \cdot \{1 + \exp[(r - C)/a]\}^{-1}$, adopted somehow "best", e.g. by the least squares method, to the real distribution function.

On the other hand, one sometimes encounters "absolute" parameters, which by definition have a general and exact meaning as they do not depend on a specific functional form for the density function. The most important example is the mean square radius $\langle r^2 \rangle^{1/2}$.

Other examples apply only to distribution functions $\rho(r)$, that are spherically symmetric. In this paper we shall disregard any anistropy of distribution: If there were any we would simply confine our study to the orientational average, which by construction is isotropic.

An absolute quantity of such kind is the half density radius

$$D := R_{50\%} \quad . \tag{1.1}$$

Here we have used the definition

$$R_\theta := f^{-1}(\theta) \qquad \text{for} \quad 0 < \theta < 1 \quad , \tag{1.2}$$

presupposing that the distribution function

$$f(r) := \frac{1}{\rho_c} \cdot \rho(r) \quad , \tag{1.3}$$

constructed by means of the central density

$$\rho_c := \rho(0) \quad , \tag{1.4}$$

falls monotonically and continuously from

$$f(0) = 1 \quad \text{to} \quad f(\infty) = 0 \quad . \tag{1.5}$$

This guarantees the unique existence of the inverse $f^{-1}(\theta)$ within the unit interval $0 < \theta < 1$. Therefore, D is the radius for which $\rho(D) = \frac{1}{2}\rho_c$. Another example of this kind is the surface thickness

$$t := R_{10\%} - R_{90\%} \quad . \tag{1.6}$$

This is the distance between the radii where $\rho(r)$ has the values $\frac{1}{10} \cdot \rho_c$ and $\frac{9}{10} \cdot \rho_c$, respectively.

Though being absolutely defined, in practice the quantities D and t cannot be measured directly. They have to be extracted approximately by some parametrization of the radial distribution that has been fitted somehow to the experimental results. Luckily enough, the thickness t has been reported[1] to stay rather invariant, if one replaces one parametrization by another one, that fits equally well to the observed electron scattering data. This means that t, which by definition is a theoretical invariant, proves to be practically invariant too.

Yet there are theoretical objections that can be raised against this concept of the surface thickness. The quantity t depends sensitively on small and irrelevant fluctuations one may superimpose on any accepted approximation for $\rho(r)$. Such an addition may even cause the inverse $f^{-1}(\theta)$ to lose its uniqeness. Perhaps we may explicitly exclude such a mishappening for some θ-interval including the points $\theta = \frac{1}{10}$ and $\theta = \frac{9}{10}$. But this subsidiary condition, though not wholly unplausible, looks a little bit artificial accentuating the arbitrariness of the two fractions, $\frac{1}{10}$ and $\frac{9}{10}$, chosen to construct t. A similar objection applies, of course, to the definition of D.

An even greater disadvantage of the concepts t and D is their being "punctual" functionals of the distribution $\rho(r)$. By this we mean the peculiar property that one has to select a finite number of (necessarily) isolated points from the continuous $\rho(r)$ graph, disregarding the overwhelming majority of the other points.* In the case of t the minority contains two points, in the case of D it consists only of one point.

A really good definition of the nuclear radius and the surface width has to be an "integral" one, taking into account all elements of the radial distribution of (electric or nucleonic or dynamic) charge with an appropriate weight that changes continuously if one crosses steadily the nuclear surface. A natural choice for this weight function is evidently the radial charge decrease

---

*As we know from the mathematical theory of generalized functions, proposed by P. A. M Dirac and Laurent Schwartz, a distribution f(r) need not even possess "ordinate" values f(R) at sharp points R. The only values that generally exist, are the "integral" or "mean" values $f[\phi]$ defined for a definite class of test functions $\phi(r)$ which being smooth, are not sharply localized on the r scale.

$-\rho'(r)$ or its multiple $-f'(r)$ as it may be derived from (1.3) and (1.4). Because of the boundary values (1.5) we know already that the <u>weight function</u>

$$g(r) := -f'(r) \tag{1.7}$$

is <u>normalized</u> to unity:

$$\int_0^\infty dr\ g(r) = 1 \quad . \tag{1.8}$$

Therefore, if (1.7) avoids negative values, we may be permitted to think of it as the "probability density for a nuclear particle to belong to the nuclear surface." The quoted sentence cannot and should not be taken as a provable or disprovable statement: if any, it conveys only some pictorial meaning that tries to indicate, how the function $g(r)$ is going to be used. Should $\rho(r)$ not decrease monotonically, this "probability" could become negative. Fig. 1 gives a graphical version of the idea.

This consideration suggests the center [r] of the weight function g(r) as an objective and natural concept of the nuclear radius; and it suggests the mean square width $\langle (r - [r])^2 \rangle^{1/2}$ of g(r) as an objective and natural concept of the surface width b, besides some possible constant factor which might be convenient. In this way we arrive at the integral definitions that have been proposed by W. D. Myers.[2]

We shall proceed by demonstrating the universal usefulness of these concepts. The surface width b turns out to interconnect in a quite general and simple way the various quantities that have been introduced by integral definitions, to play the roll of a "nuclear radius" R. These simple interconnections are valid as long as the condition

$$b \ll R \tag{1.9}$$

is fulfilled, according to which the nuclear surface should be thin in comparison with its own curvature radius. This is essentially the hypothesis, which W. J. Swiatecki and his coworkers[4,5] have systematically introduced into the theory of nuclear structure. A many-particle system possessing this property has been called by Swiatecki leptodermous, i.e. "of thin skin". A rather close synonym seems to be the concept of a "homogeneous, condensed system without lattice correlations" or simply the concept of a "perfect fluid".

In part I of this paper we shall develop the general theory mentioned above. Some of the results have been already given by Hahn, Ravenhall, and Hofstadter[1]) approximately and in a more implicit manner. Part II is devoted to some special distribution functions, for which the occurring integrals will be evaluated. There we shall assume some typical bell-shaped functions $g_\alpha(r)$ and proceed each time to the corresponding threshold-shaped functions $f_\alpha(r)$ by means of the formula

$$f(r) = 1 - \int_0^r dr\ g(r) = \int_r^\infty dr\ g(r) \quad , \tag{1.10}$$

which integrates (1.7) according to the boundary condition (1.5).

Part I: GENERAL THEORY

2. The charge distribution

As was mentioned in the introduction, we distinguish between at least three different meanings of "nuclear charge" distribution $\rho(r)$: the electrical density $q(r)$, the nucleonic density $n(r)$, or the one particle potential $v(r)$. For most what follows it does not matter which of these possibilities we choose.

The only exception stems from the fact, that the central nucleonic density $n_c := n(0)$ is a constant that varies very slowly if one changes Z or N or both. This is true insofar as we can neglect some perturbations, the most important of which are the following: 1. The effect of the Coulomb field against the resistance of the isospin force* that tends to make the ratio $\rho(r)$: $n(r)$ as close to Ze^2:A as possible; 2. the neutron skin; and 3. the compression exerted by the surface tension against the rigidity** of nuclear matter. These three effects and their interferences can be properly described by Myers' droplet model[3,4]). It is useful to consider a zeroth approximation where all such perturbations are disregarded in order to simplify the discussion of the questions we are treating now. Of course, we eventually could add any correction that should turn out to be important. With this understanding in mind we are assuming that n_c^{-1}, the mean volume per nucleon, is the same for all nuclei in their ground state. To convey the empirical value of this constant of nature, it is as usual as useful to tell the value of the nuclear radius constant r_c, defined by the equation

---

*This is closely related to the symmetry energy term in Weizsäcker's mass formula.

**This is sometimes called "compressibility," though it is in fact the inverse of it.

$$\frac{4\pi}{3} r_c^{\,3} := n_c^{\,-1} \quad . \tag{2.1}$$

The experimental value[6] seems to be close to $r_c = 1.13_5$ fm.

If n_c is known, the <u>central electrical density</u> q_c can immediately be calculated by the formula

$$q_c = \frac{Ze}{A} \cdot n_c \quad , \tag{2.2}$$

which again disregards the Coulomb polarization. The assumption, that $q(r)$ is constant within the nucleus, varying proportionally with Z/A for different nuclei, has been introduced into the theory of nuclear sizes by L. R. B. Elton[7] with remarkable success[8]. In fact, Elton's hypothesis improved appreciably the former, rather crude views on that subject. Now it has to be understood as the correct basis for a more refined analysis as that of Myers and Swiatecki.

The central <u>one-particle potential</u> v_o may also be connected with the central density n_o. First one has to calculate the <u>Fermi sea depth</u>

$$v_F := v_o + e_F \quad , \tag{2.3}$$

using the <u>Fermi level</u> e_F. The two depths v_o and v_F are positive, whereas e_F is negative. Then, one has to explout the equality,

$$n_c = \frac{2}{3\pi^2} \cdot \left(\frac{2m}{\hbar^2}\right)^{3/2} v_F^{\,3/2} \quad , \tag{2.4}$$

which is well known from the Thomas-Fermi method. For homogeneous nuclear matter this coincides with Hartree's method. In writing (2.4), we have assumed that N = Z, and that the one particle potential be velocity independent. Both

assumptions can, of course, be abandoned, if one is ready to accept more complicated formulae. The theory which we are now going to propose, does not depend on such assumptions or complications.

3. The Charge Moments

Let $f(r)$ now be any functions satisfying the boundary conditions (1.5). It need not be monotonic, but we assume its norm

$$F := \int_0^\infty f(r) r^2 dr \quad , \tag{3.1}$$

to be a finite, positive number. Then, for any continuous function $\psi(r)$ we define the charge mean

$$\langle \psi(r) \rangle := F^{-1} \int_0^\infty \psi(r)\ f(r)\ r^2 dr \quad , \tag{3.2}$$

provided the improper integration converges, including the cases where it diverges definitely either to $+\infty$ or to $-\infty$.

Of special interest are the power functions $\psi(r) = r^{\nu-2}$ for $\nu > -1$. We shall employ the notation

$$F_\nu := \int_0^\infty f(r)\ r^\nu\ dr = F \cdot \langle r^{\nu-2} \rangle \quad . \tag{3.3}$$

If ν becomes a natural number n, we encounter the nth charge moment F_n.

Further specializing to $n = 0$ we arrive at the central radius

$$C := \int_0^\infty f(r)\ dr = F_0 = F \cdot \langle r^{-2} \rangle \quad , \tag{3.4}$$

which fulfills the plausible condition

$$\int_0^\infty [f(r) - \theta(C - r)]\, dr = 0 \quad . \tag{3.5}$$

This means that the deviation of the real distribution $f(r)$ from the approximating rectangular distribution $\theta(C - r)$ of radius C has to vanish in the mean. Here the averaging is performed with an extra weight factor r^{-2}, applied in order to compensate the geometrical factor r^2.

This interpretation of C leads one to consider another conception of the "nuclear radius," simply by keeping the geometrical factor r^2 uncompensated. Thus we obtain the condition

$$\int_0^\infty [f(r) - \theta(R - r)]r^2\, dr = 0 \quad , \tag{3.6}$$

which implicitely defines the <u>charge radius R</u>. A more explicit definition is given by the formula

$$\frac{1}{3} R^3 := \int_0^\infty f(r)r^2\, dr = F_2 = F \quad . \tag{3.7}$$

As easily seen, R is the radius of that homogeneously charged sphere which contains the same baryonic charge A as the real charge distribution $n(r) = n_c\, f(r)$. Indeed, the <u>total baryon number</u> amounts to

$$A = \int_0^\infty 4\pi\, r^2 \cdot n(r)\, dr = 4\pi\, n_c\, F = \frac{4\pi}{3} R^3\, n_c \quad . \tag{3.8}$$

Using the nuclear radius constant r_c as introduced in (2.1), we obtain the well known relation

$$R = r_c A^{1/3} \quad . \tag{3.9}$$

The important point here is, that r_c does not depend on $A = N + Z$ nor on $\theta = N - Z$, apart from some small perturbations. For any other notion of nuclear radius the connection with the constant $r_c = 1.13_5$ fm is more involved.

A radius quantity which can be derived rather directly from electron scattering experiments is the <u>quadratic radius</u> Q, as defined by the equation

$$\frac{3}{5} Q^2 := \langle r^2 \rangle \quad . \tag{3.10}$$

Its right hand side may be evaluated from the charge moments F_4 and F_2 according to (3.3) or

$$\langle r^\kappa \rangle = \frac{F_{\kappa+2}}{F_2} \quad , \tag{3.11}$$

which is valid for all numbers $\kappa \geqslant -2$. The quadratic radius Q is the special case R_2 of the <u>κ^{th}-power radius</u>

$$R_\kappa := \left(\frac{\kappa+3}{3} \langle r^\kappa \rangle\right)^{1/\kappa} \quad . \tag{3.12}$$

K. F. Ford and J. G. Wills[9] have shown that κ and R_κ are the almost absolute charge parameters of nuclei that are measured by the muonic atoms.

The factor $\frac{3}{5}$ in (3.10) normalizes Q to be the radius of that homogenously charged sphere which would lead to the same mean square $\langle r^2 \rangle$ as the real distribution. Should this sphere contain the right nucleon number A, the homogenous density n_o would be less than the central density $n_c = n(0)$, that is assumed to be a natural constant. This "density depression" exhibits Q to be a rather artificial quantity, though the effect would not be strong for heavy nuclei.

The question as to how much the radii C,R,Q differ, will be answered in the next section.

4. The Original Surface Moments

The order of magnitude, by which different radii differ, is the width of the surface weight function $g(r)$ defined in (1.7). Because of the normalization (1.8), the averaging according to $g(r)$ needs no normalization factor ($G = 1$). We may calculate the surface mean

$$[\psi(r)] := \int_0^\infty \psi(r)\, g(r)\, dr \quad , \tag{4.1}$$

in each case where the integral is convergent.

The new averaging functional may be easily compared with the old one. By partial integration we derive the identity

$$[\psi(r)] = \int_0^\infty f(r)\cdot\psi'(r)\, dr = F\cdot\langle r^{-2}\cdot\psi'(r)\rangle \quad , \tag{4.2}$$

provided the integral exists. Inserting a power function $\psi(r) = r^\mu$ with $\mu \geqslant 0$, we obtain the integrals

$$G_\mu := \int_0^\infty g(r)\, r^\mu\, dr = [r^\mu] \quad , \tag{4.3}$$

in some analogy with (3.3). A typical condition sufficient for convergence at $r \to \infty$ reads

$$0 \leqslant g(r) \leqslant K \cdot r^{-\mu-1-\varepsilon} \quad \text{for } r > M \quad , \tag{4.4}$$

with some K and some positive ε. An exponentially decaying $g(r)$ would suffice even for the highest μ values.

If we confine μ to natural numbers m, we get the mth original surface moment $G_m = [r^m]$. Of course $G_0 = 1$, and because of (3.3) the relation $G_m = m \cdot F_{m-1}$ holds for each positive m. If all these moments are given, the weight function $g(r)$ may be reconstructed, if its Fourier transform, the character $\gamma(k) := [\exp i\, kr]$, is a whole function (i.e. holomorphic in the whole k-plane). The construction is rendered by the Fourier integral $2\pi\, g(r) = \int_{-\infty}^{\infty} dk\, \gamma(k) e^{-ikr}$ of the Taylor series $\gamma(k) = \Sigma_{m=0}^{\infty} (m!)^{-1} G_m (ik)^m$.

A comparison of (4.3) and (4.2) with (3.3) exhibits the relation

$$G_\mu = \mu \cdot F_{\mu-1} \quad , \tag{4.5}$$

valid if $\mu \geqslant 0$. By combining this and (4.3) with (3.11), we may express the former power averages by the new ones:

$$\langle r^\kappa \rangle = \frac{3}{\kappa+3} \frac{[r^{\kappa+3}]}{[r^3]} \quad , \tag{4.6}$$

for each $\kappa \geqslant -2$.

For $\mu = 1$, relation (4.5) is reduced by (3.4) and (4.3) to the identity $C = F_o$ and

$$C = [r] \qquad \text{or} \qquad C = G_1 \tag{4.7}$$

which means that the central radius equals the mean radius if this is weighted by the surface distribution $g(r)$. For $\mu = 3$, we deduce from (4.5), (4.3), and (3.7) the simple equation

$$R^3 = [r^3] \qquad \text{or} \qquad R = G_3^{1/3} \tag{4.8}$$

Finally, the relation

$$Q^2 = \frac{[r^5]}{[r^3]} \qquad \text{or} \qquad Q = \left(\frac{G_5}{G_3}\right)^{1/2} \tag{4.9}$$

may be read off from (4.6) and (3.10) for $\kappa = 2$ (or $\mu = 5$).

5. The Surficial Moments

By means of the functional (4.1) and its simplest nontrivial case (4.7) we introduce the quantity

$$\Gamma_\nu := [(r - C)^\nu] \quad , \tag{5.1}$$

for each non-negative ν that does not destroy the convergence of the infinite integration. If ν is a natural number n, this becomes the nth surficial moment

$$\Gamma_n = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} C^{n-m} G_m \quad . \tag{5.2}$$

The binomial identity $\Sigma_{n=0}^{m} (-1)^{n-m'} \binom{m}{n} \binom{n}{m'} = \delta_{mm'}$, which may be easily derived by binomial expansion of $[1 + (x - 1)]^m = x^m$, yields the inversion

$$G_m = \sum_{n=0}^{m} \binom{m}{n} C^{m-n} \Gamma_n \quad . \tag{5.3}$$

For noninteger exponents ν or μ, one may write down similar formulae with summations that extend to infinity, creating new convergence problems. We shall concentrate on natural exponents.

For the two lowest values of n the central moments are trivial:

$$\Gamma_0 = G_0 = 1 \quad , \tag{5.4}$$

and

$$\Gamma_1 = 0, \quad \text{as} \quad G_1 = C \quad , \tag{5.5}$$

according to (4.7) and (4.3).

But the next order is of special interest: The second central moment

$$\Gamma_2 = [r^2] - [r]^2 \tag{5.6}$$

gives rise to the <u>surface width</u>

$$b := \Gamma_2^{1/2} = (G_2 - G_1^2)^{1/2} \quad . \tag{5.7}$$

Evidently, this length quantity delivers the most important information on the <u>size</u> of the nuclear surface layer, whereas its <u>location</u> has already been specified by the surface radius $C = G_1$.

The higher moments add further information that specifies the <u>shape</u> of the surface layer whatever its position and/or size. Now, pure shape parameters are dimensionless quantities. Therefore we shall replace each nth central moment Γ_n with $n \geqslant 3$ by the <u>n^{th} shape coefficient</u>

$$\gamma_n := b^{-n} \Gamma_n \quad . \tag{5.8}$$

For $n < 3$, one gets only the trivial values $\gamma_0 = 1$, $\gamma_1 = 0$, and $\gamma_2 = 1$. On the other hand, the contribution of a term containing a coefficient γ_n with $n \gg 1$ may likely be neglected to the extent, that $(b/R)^n$ is a negligible number according to the hypothesis (1.9). This will be demonstrated by two typical examples, that we are going to consider.

We would like to know, how the charge radius R and the quadratic radius Q compares with the central radius C. According to (4.8), (4.9) and (4.7), the answer will be given when we shall have calculated G_3 and G_5. To this end we insert (5.4), (5.5), (5.7), and (5.8) into (5.3) in the cases $m = 3$ and $m = 5$. This procedure yields

$$R^3 = C^3 + 3Cb^2 + \gamma_3 b^3 \quad , \tag{5.9}$$

and

$$R^3Q^2 = C^5 + 10C^3b^2 + 10\gamma_3 C^2b^3 + 5\gamma_4 Cb^4 + \gamma_5 b^5 \quad , \tag{5.10}$$

which are exact equations.

In the leptodermous case (1.9), it is useful to deduce expansions in powers of the <u>skin coefficient</u>

$$\beta := \frac{b}{R} \ll 1 \quad . \tag{5.11}$$

We obtain via $b/C = \beta + \beta^3 + \frac{1}{3}\gamma_3\beta^4 + \ldots$ the following results:

$$C = R\cdot(1 - \beta^2 - \frac{1}{3}\gamma_3\cdot\beta^3 + 0 - \frac{1}{3}\gamma_3\beta^5 + \ldots) \quad , \tag{5.12}$$

the terms indicated by dots being $O(\beta^6)$, and

$$Q = R\cdot[1 + \frac{5}{2}\beta^2 + \frac{25}{6}\gamma_3\beta^3 + \frac{5}{2}(\gamma_4 - \frac{21}{4})\beta^4 + \frac{1}{2}(\gamma_5 - \frac{275}{6}\gamma_3)\beta^5 + \ldots] \tag{5.13}$$

This is essentially Elton's formula,[8] generalized from the Fermi function to arbitrary leptodermous distributions. If b/C or b/Q is used instead of b/R,

analogous expression arise. For the Fermi distribution a similar series, yet with $\gamma_3 = 0$, has been computed by Elton.[7] Our derivation shows, that the result does not depend on the choice of a special parametrization.

The approximate relations $R^3 \approx C^3 + 3Cb^2$ and $R^3Q^2 \approx C^5 + 10C^3b^2$ have been stated implicitly by Hahn, Ravenhall, and Hofstadter[1]) in their Eqs. (4) to (7), including $Q^2 \approx C^2[1 + (10^2b^2/C^2)]/[1 + (3b^2/C^2)]$, the third line after (6) if translated into our language.

The most important correction in (5.12) is due to the skin coefficient β. The qualitative estimation $C < R < Q$ confirms simple geometrical considerations: The larger r values are stronger emphasized in R than in C, and even stronger in Q than in R. But from (5.12) and (5.13) we may deduce quantitative estimations, e.g. limiting proportions as

$$\lim_{\beta\to 0} \frac{R - C}{\beta^2 R} = 1 , \quad \lim_{\beta\to 0} \frac{Q - C}{R - C} = \frac{7}{2} , \tag{5.14}$$

where $\gamma_3, \gamma_4, \gamma_5, \ldots$ are kept constant (or at least bounded).

These shape coefficients are connected with smaller corrections to $Q - C$ and $R - C$. The <u>flair</u> γ_3 is a simple measure of the shape asymmetry, being positive, if the tail of the charge distribution $f(r)$ is relatively thin and long (Fig. 1). It is not so easy to visualize the <u>crookedness</u> γ_4; but the <u>contortion</u> $\kappa_4 := \gamma_4 - 3$ may be called a measure of how much less "trapezoidal" the distribution $f(r)$ really is than an error function. The subtraction of 3 is an example of the "cumulative transformation", which switches from the original moments $G_m = b^m g_m$ through the surficial moments $\Gamma_n = b^n \gamma_n$ to the <u>cumulants</u> $K_r = b^r K_r$. These are generally defined by the expansion $\log [e^{-ikC}\gamma(k)] = \Sigma_{r=1}^{\infty}(r!)^{-1}K_r(ik)^r$. This delivers the first five of them to be $K_1 = \Gamma_1 = 0$, $K_2 = \Gamma_2 = b^2$, $K_3 = \Gamma_3$, $K_4 = \Gamma_4 - 3\Gamma_2^2$, and $K_5 = \Gamma_5 - 10\Gamma_2\Gamma_3$. Therefore the first five of the <u>deviation coefficients</u> n_n amount to $n_1 = 0$, $n_2 = 1$, $n_3 = \gamma_3$, $n_4 = \gamma_4 - 3$, and $n_5 = \gamma_5 - 10\gamma_3$. For a normal distribution all but the second of the n_n vanish.

Part II: SPECIAL DISTRIBUTION

6. Symmetric Distribution

This semi-general section is devoted to that special class of distributions, for which all shape coefficients γ_n with an odd n vanish:

$$\gamma_3 = \gamma_5 = \gamma_7 = \cdots = 0 \quad . \tag{6.1}$$

From this one may deduce, that $\kappa_3 = \kappa_5 = \kappa_7 = \cdots 0$, as $\gamma(-k)e^{ikC} = \gamma(k)e^{-ikC}$.

Another condition, evidently equivalent to (6.1), is the symmetry

$$g(C - s) = g(C + s) \quad , \tag{6.2}$$

with the understanding that

$$g(C + s) = 0 \qquad \text{if} \qquad s > C \quad . \tag{6.3}$$

This amendment stems from the fact, that in all the integrations $g(C - s)$ contributes nothing for $s > C$. That this symmetry radius C really coincides with the central radius (3.4), is easily inferred from (4.7).

If we insert the symmetry condition to (1.10), we find the corresponding condition

$$f(C - s) + f(C + s) = 1 \quad , \tag{6.4}$$

ammended by

$$f(C + s) = 0 \quad \text{if} \quad s > C \quad . \tag{6.5}$$

The half density radius (1.1) now coincides with the central radius,

$$D = C \quad , \tag{6.6}$$

as is evident from (6.4) with s = 0.

7. The Incomplete-Gamma Distribution ($I\Gamma_\lambda$)

This surface structure is shaped by a weight function proportional to $r^{\lambda-1}\, e^{-r/a}$ for some non-negative exponent $\lambda - 1$ and some length a. Comparing the normalization (1.8) with the construction

$$\Gamma(\lambda) := (\lambda - 1)! := \int_0^\infty dx\, x^{\lambda-1}\, e^{-x} \tag{7.1}$$

of the gamma or faculty function, we arrive at

$$g(r) := \frac{a^{-1}}{\Gamma(\lambda)} \left(\frac{r}{a}\right)^{\lambda-1} e^{-r/a} \quad . \tag{7.2}$$

According to (1.10) this corresponds to

$$f(r) = 1 - P(\lambda, \frac{r}{a}) \quad , \tag{7.3}$$

where use is made of the normalized incomplete gamma function

$$P(\lambda,x) := \frac{1}{\Gamma(\lambda)} \int_0^x dx\, x^{\lambda-1}\, e^{-x} \quad . \tag{7.4}$$

As $P(\lambda;0) = 0$ and $P(\lambda,\infty) = 1$, the boundary conditions (1.5) are really fulfilled.

Here it is easy to calculate the various moments. By partial integration (3.3) is evaluated to give

$$F_\nu = \frac{\Gamma(\lambda + \nu - 1)}{(\nu + 1)\Gamma(\lambda)} \; a^{\nu+1} \quad , \tag{7.5}$$

and (4.3) then yields

$$G_\mu = \frac{\Gamma(\lambda + \mu)}{\Gamma(\lambda)} \; a^\mu \; = \; [r^\mu] \quad . \tag{7.6}$$

Because of the identity $\Gamma(x + 1) = x\,\Gamma(x)$, the expressions (7.5) and (7.6) fulfill eq. (4.5). Now, (3.11) or (4.6) yield

$$\langle r^\kappa \rangle = \frac{3}{\kappa + 3} \cdot \frac{\Gamma(\lambda + \kappa + 3)}{\Gamma(\lambda + 3)} \cdot a^\kappa \quad . \tag{7.7}$$

As (4.7), (4.8), (4.9) show, we are especially interested in (7.6) with $\mu = 1$, $\mu = 3$, $\mu = 5$. The final results which are displayed in Fig. 3, read

$$C = \lambda \cdot a \quad , \tag{7.8}$$

$$R = [\lambda(\lambda + 1)(\lambda + 2)]^{1/3} \cdot a \quad , \tag{7.9}$$

$$Q = [(\lambda + 3)\cdot(\lambda + 4)]^{1/2} \cdot a \quad . \tag{7.10}$$

By means of (5.2) we may now obtain the lowest central moments Γ_n, and proceed by (5.7) and (5.8) to the surface width b and the lowest shape coefficients γ_n. In that manner we achieve

$$b = \lambda^{1/2} \cdot a \quad , \tag{7.11}$$

and

$$\gamma_3 = \frac{2}{\lambda^{1/2}} \quad , \quad \gamma_4 = 3 \cdot \frac{\lambda + 2}{\lambda} \quad , \quad \gamma_5 = 4 \cdot \frac{5\lambda + 6}{\lambda^{3/2}} \tag{7.12}$$

Hence $\kappa_3 = 2\lambda^{-1/2}$, $\kappa_4 = 6\lambda^{-1}$, $\kappa_5 = 24\lambda^{-3/2}$. The general result $\kappa_r = (r-1)!\lambda^{(2-r)/2}$ for $r = 2,3,4, \ldots$ may be easily derived via the characteristic function $\gamma(k)=(1-iak)^\lambda$.

These results may be easily checked with the eqs. (5.9) and (5.10).

It is interesting to compare the "integral" radii C, R, Q with the "punctual" radius D, and the "integral" thickness 2b with the "punctual" thickness t. To this end we have to evaluate (1.2). In the present case we need the function $X(\lambda,p)$ inverse to $P(\lambda,x)$ with respect to the second variable, the first variable being treated as a constant parameter. That means,

$$P(\lambda, X(\lambda,p)) = p \text{ and } X(\lambda, P(\lambda,x)) = x \tag{7.13}$$

by definition. Then

$$R_\theta = X(\lambda, 1-\theta) \cdot a \quad , \tag{7.14}$$

so that $D = X(\lambda, 1/2)\cdot a$ and $t = [X(\lambda, 1/10) - X(\lambda, 9/10)]\cdot a$. These results, and all the analogous ones which follow are shown graphically in Fig. 2.

8. The Generalized Error Function Distribution (GE_λ)

Here $g(r)$ is assumed to be proportional to $\exp(-|r-C|^\lambda/a^\lambda)$ with a central radius C large enough to meet the leptodermous condition

$$\left(\frac{a}{C}\right)^\lambda \ll 1 \quad . \tag{8.1}$$

If $\lambda > 1$, this permits us to completely disregard terms like $\exp(-C^\lambda/a^\lambda)$ as compared with 1, which we indicate by writing $\exp(-C^\lambda/a^\lambda) \lll 1$. After normalization we have

$$g(C + s) = \frac{1}{2}\frac{\lambda a^{-1}}{\Gamma(\lambda^{-1})} \cdot \exp\left(-\frac{|s|^\lambda}{a^\lambda}\right) \tag{8.2}$$

and

$$f(C + s) = \frac{1}{2}\left[1 - \text{sgn}s.P\left(\lambda^{-1}, \frac{|s|^\lambda}{a^\lambda}\right)\right] \tag{8.3}$$

This satisfies the symmetry condition (6.2) exactly and its amendment (6.3) accurately as long as (8.1) is fulfilled. Therefore (6.1) may be accepted, which means, that $\Gamma_3 = \Gamma_5 = \Gamma_7 = \cdots = 0$. The even moments $\Gamma_{2\ell}$ are easily calculated by (5.1). All these results are summarized by the formula

$$\Gamma_n = \begin{cases} 0 & \text{for} \quad n = 3,5,7,\cdots, \\ \Gamma(\lambda^{-1})^{-1}\, \Gamma(n\lambda^{-1} + \lambda^{-1})\, a^n & \text{for} \quad n = 0,2,4,6,\cdots, \end{cases} \tag{8.4}$$

which, of course, should be implemented by (5.5). Thus we obtain

$$b = \left[\frac{\Gamma(3\lambda^{-1})}{\Gamma(\lambda^{-1})}\right]^{1/2} a \quad , \tag{8.5}$$

and

$$\gamma_3 = 0, \quad \gamma_4 = \frac{\Gamma(\lambda^{-1})\ \Gamma(5\lambda^{-1})}{\Gamma(3\lambda^{-1})^2}\ , \quad \gamma_5 = 0 \quad . \tag{8.6}$$

These values may be inserted into (5.9) and (5.10) to yield R and Q, respectively.

The evaluation of (1.2) is easily performed delivering

$$R_\theta = C + \operatorname{sgn}(1-2\theta)\cdot X(\lambda^{-1}, |1-2\theta|)^{\lambda^{-1}} \cdot a \quad . \tag{8.7}$$

Therefore $t = 2\cdot X(\lambda^{-1}, 4/5)^{\lambda^{-1}} \cdot a$ besides $D = C$, which is a special case of (6.6). Here X is again defined by (7.13).

In the limit $\lambda \to \infty$ we arrive at the <u>trapezoidal distribution</u>, ($T = GE_\infty$) defined by

$$g(C + s) = \frac{1}{2}\, a^{-1} \cdot \theta\,(a - |s|) \quad , \tag{8.8}$$

or

$$f(C + s) = \frac{1}{2} \cdot [1 - \frac{s}{\max(a, |s|)}] \quad . \tag{8.9}$$

As $z \cdot \Gamma(z) \to 1$ for $z \to 0$, we get $\Gamma_n = (n + 1)^{-1} a^n$ for all even n, yielding

$$b = \frac{a}{\sqrt{3}}, \quad \text{and} \quad \gamma_4 = \frac{9}{5} \tag{8.10}$$

or $\kappa_4 = -6/5$. Therefore,

$$R^3 = C^3 + Ca^2, \quad R^3Q^2 = C^5 + \frac{10}{3} C^3a^2 + Ca^4 \quad . \tag{8.11}$$

Finally we have

$$R_\theta = C + (1-2\theta) \cdot a \quad , \tag{8.12}$$

of course, so that $t = \frac{8}{5} \cdot a = \frac{8}{5}\sqrt{3} \cdot b = 1.6\ a = 2.77128'129_2 \cdot b$.

For $\lambda = 2$, the simple <u>normal</u> or <u>error function distribution</u> ($\mathfrak{I} = GE_2 = I\Gamma_\infty$) comes about. Its definition is

$$g(C + s) = \frac{a^{-1}}{\pi^{1/2}} \exp\left(-\frac{s^2}{a^2}\right) \quad , \quad f(C + s) = \frac{1}{2}\left(1 - \operatorname{erf} \frac{s}{a}\right) \quad , \tag{8.13}$$

where the <u>error function</u>

$$\operatorname{erf} x := \frac{2}{\pi^{1/2}} \int_0^x e^{-x^2}\, dx = \operatorname{sgn} x \cdot P(x^2, 1/2) \tag{8.14}$$

is used. Since $\Gamma_n = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \ldots \cdot \frac{n-1}{2} \cdot a^n$ if n is even, we have

$$b = \frac{1}{\sqrt{2}} a \ , \qquad \gamma_4 = 3 \quad , \tag{8.15}$$

thus the excess κ_4 vanishes as do all the other deviations κ_n. We get

$$R^3 = C^3 + \frac{3}{2} C\, a^2 \;, \qquad R^3 \cdot Q^2 = C^5 + 5\, C^3 a^2 + \frac{15}{4} C a^4 \quad . \tag{8.16}$$

Finally,

$$R_\theta = C + \mathrm{erf}^{-1}(1-2\theta)\cdot a \quad , \tag{8.17}$$

and therefore

$$t = 2\ \mathrm{erf}^{-1}(\tfrac{4}{5})\cdot a = \sqrt{8}\ \mathrm{erf}^{-1}\ (\tfrac{4}{5})\cdot b = 1.812_4 \cdot a = 2.56_3 \cdot b \quad .$$

Putting $\lambda = 1$, we obtain the <u>Laplace distribution</u>, $(L = GE_1)$, defined by

$$g(C + s) = \frac{1}{2} a^{-1} e^{-|s|/a} \quad , \qquad f(C + s) = \theta(-s) + \frac{1}{2} \mathrm{sgn}\ s \cdot e^{-|s|/a} \quad . \tag{8.18}$$

Now $\Gamma_n = n!\ a^n$ for even n values, so that

$$b = \sqrt{2}\ a \quad , \qquad \gamma_4 = 6 \tag{8.19}$$

or $\kappa_4 = 3$, and

$$R^3 = C^3 + 6Ca^2 \quad , \qquad R^3 \cdot Q^2 = C^5 + 20C^3 a^2 + 120Ca^4 \quad . \tag{8.20}$$

Finally we find

$$R_\theta = C - \mathrm{sgn}\,(1-2\theta)\cdot \log(1 - |1 - 2\theta|)\cdot a \quad , \tag{8.21}$$

yielding

$$t = 2 \log 5 \cdot a = \sqrt{2} \log 5 \cdot b = 3.21887'58_2 \cdot a = 2.27608'89_2 \cdot b \quad .$$

9. The Fermi Distribution ($F = PL_\infty$)

This much used shape is obtained by the following construction:

$$g(C + s) := \frac{1}{4} a^{-1} (\coth \frac{s}{2a})^{-2} = a^{-1} \frac{e^{s/a}}{(1 + e^{s/a})^2} \quad , \tag{9.1}$$

which leads to the usual expression

$$f(C + s) = \frac{1}{2}(1 - \tanh \frac{s}{2a}) = \frac{1}{1 + e^{s/a}} \quad . \tag{9.2}$$

The symmetry conditions (6.2), (6.3) or (6.4), (6.5) are practically fulfilled as long as the leptodermous condition

$$2a \ll C \quad , \tag{9.3}$$

is valid, ensuring $e^{-C/a} \lll 1$ which means that $e^{-C/a}$ can be quite safely neglected if it is compared with 1. The moments become

$$\Gamma_n = \begin{cases} 0 & \text{for } n = 3,5,7,\cdots , \\ 2\cdot n!\ (1 - 2^{1-n})\ \zeta(n) a^n & \text{for } n = 2,4,6,\cdots , \end{cases} \tag{9.4}$$

besides (5.5) and (5.4). In these expressions Rieman's zeta function $\zeta(s) := \Sigma_{k=1}^{\infty} k^{-s}$ occurs which for integer arguments may be obtained from the formula

$$\zeta(n) := \frac{(2\pi)^n}{2\cdot n!} (-1)^{\frac{1}{2}n-1} B_n \qquad \text{for} \quad n = 2,4,6,\cdots , \tag{9.5}$$

where B_n denotes the nth Bernoulli number as defined for example in A. & S.[10] Since $B_2 = 1/6$ and $B_4 = -1/30$, we obtain*

---

*We would like to remark that κ_4 for the Fermi distribution has the same magnitude as for the trapezoidal distribution (which we have considered in the last section), but with opposite sign.

$$b = \frac{\pi}{\sqrt{3}}\, a \quad , \quad \gamma_4 = \frac{21}{5} \quad , \tag{9.6}$$

or $\kappa_4 = \frac{6}{5}$. Consequently,

$$R^3 = C^3 + \pi^2 C a^2 \quad , \qquad R^3 Q^2 = C^5 + \frac{10}{3}\pi^2 C^3 a^2 + \frac{7}{3}\pi^4 C a^4 \quad . \tag{9.7}$$

These equations are accurate as long as $e^{-C/a}$ is negligible, for which we use to write $e^{-C/a} \lll 1$. Finally one obtains

$$R_\theta = \log\,(\theta^{-1} - 1)\cdot a \quad , \tag{9.8}$$

which implies

$$t = 4\cdot\log 3\cdot a = 4.3^{1/2}\,\pi^{-1}\cdot\log 3\cdot b = 4.39444'9154_7\cdot a = 2.42278'6798_4\cdot b \ .$$

If we allow the parameter a to vanish, we get simply the <u>rectangular distribution</u> (R), <u>which is defined simply by</u>

$$g(r) = \delta(r - R) \quad , \quad f(r) = \theta(R - r) \quad . \tag{9.9}$$

Here $C = R = Q$ and all $\Gamma_n = 0$ so that b vanishes, whereas the shape coefficients γ_n lose all their meaning ($n \geqslant 3$). Of course, the same limit would have been achieved in the trapezoidal distribution, as given by (8.8) or (8.9), if the surface thickness had been set to zero, or for any other distribution law in the limit $b = 0$. The rectangular distribution corresponds to the simple idea of the liquid drop model.

10. The Powerlike Distribution (PL_λ)

There is a simple distribution with the property that $1 - f(r)$ would behave like $(r/D)^\lambda$ if $r \ll D$, and $f(r)$ like $(r/D)^{-\lambda}$ if $r \gg D$. This is the powerlike distribution, defined by

$$f(r) := [1 + (\frac{r}{D})^\lambda]^{-1} \quad , \tag{10.1}$$

which is equivalent to

$$g(r) = \frac{\lambda}{D} (\frac{r}{D})^{\lambda-1} [1 + (\frac{r}{D})^\lambda]^{-2} \quad . \tag{10.2}$$

The exponent should λ be positive. In order to compute the moments (3.3) or (4.3) we need the Beta function

$$B(p,q) := \int_0^1 dt\ t^{p-1}(1-t)^{q-1} = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)} = B(q,p) \quad , \tag{10.3}$$

in the special case $p = 1-q$, which is given by the identity

$$B(1-q,q) = \Gamma(q)\cdot\Gamma(1-q) = \frac{\pi}{\sin \pi q} \quad . \tag{10.4}$$

We are encountering the argument $q = (\nu + 1)/\lambda = \mu/\lambda$, if we switch from F_ν to G_μ by means of (4.5), using the substitution $t := [1 + (r/D)^\lambda]^{-1}$. The result

$$G_\mu = \frac{\pi\ \mu/\lambda}{\sin \pi\ \mu/\lambda} \cdot D^\mu = [r^\mu] \quad , \tag{10.5}$$

is valid if $\mu \leqslant \lambda$, and finite if $\mu < \lambda$. Proceeding as in the previous sections we obtain these results:

$$C = \frac{\pi/\lambda}{\sin \pi/\lambda} \cdot D \quad , \tag{10.6}$$

$$R = \left(\frac{3\pi/\lambda}{\sin 3\pi/\lambda}\right)^{1/3} \cdot D \quad , \tag{10.7}$$

$$Q = \left(\frac{5 \sin 3\pi/\lambda}{3 \sin 5\pi/\lambda}\right)^{1/2} \cdot D \quad , \tag{10.8}$$

and

$$b = \left(\frac{\tan \pi/\lambda}{\pi/\lambda} - 1\right)^{1/2} \cdot C \quad , \tag{10.9}$$

provided $\lambda > 2$, otherwise it is infinite. Similarly we could calculate the higher central moments and the shape coefficients, but we refrain from writing down the rather cumbersome results even for γ_3, γ_4 and γ_5. Yet we should like to remark that this distribution is <u>not</u> symmetric, so its γ_3 and γ_5 need not vanish. We have $R_\theta = (\theta^{-1} - 1)^{1/\lambda} \cdot D$, so that $t = (9^{1/\lambda} - 9^{-1/\lambda})D$.

If $\lambda \gg 1$, each of our quantities becomes a series in powers of δ^2, where

$$\delta := \frac{\pi}{\sqrt{3}} \frac{1}{\lambda} \ll 1 \quad . \tag{10.10}$$

We arrive at the following formulae

$$C = \left(1 + \frac{1}{2}\delta^2 + \frac{7}{40}\delta^4 + \cdots\right) \cdot D \quad , \tag{10.11}$$

$$R = \left(1 + \frac{3}{2}\delta^2 + \frac{99}{40}\delta^4 + \cdots\right) \cdot D \quad , \tag{10.12}$$

$$Q = \left(1 + 4\delta^2 + \frac{108}{5}\delta^4 + \cdots\right) \cdot D \quad , \tag{10.13}$$

and

$$b = \delta\cdot(1 + \frac{11}{10}\delta^2 + \cdots)\cdot D\,,\ \beta = \delta\cdot(1 - \frac{2}{5}\delta^2 + \cdots) \tag{10.14}$$

besides

$$\gamma_3 = \frac{24}{5}\,\delta\cdot(1 + \cdots)\quad,\qquad \gamma_4 = \frac{21}{5}\cdot(1 + \cdots)\ ,\ \gamma_5 = \ldots\quad . \tag{10.15}$$

We see that the skewness is relatively small but fails to vanish. This reflects the fact, that the distribution now is almost but not precisely symmetric.

If D, r, and λ all become large in such a manner that the quantities

$$a := \frac{D}{\lambda} \quad \text{and} \quad s := D-r\quad , \tag{10.16}$$

stay constant, the powerlike distribution goes over to a Fermi distribution with C = D. This follows from the fact that

$$(\frac{r}{D})^{\lambda} \to \exp\frac{s}{a} \quad \text{for} \quad \lambda \to \infty\quad , \tag{10.17}$$

under the stated conditions. One may easily check that in this limit (10.14), (10.15) would coincide with (9.6), (6.1). For large yet finite λ, the final equality of C with D should be replaced by the expression

$$C = (1 + \frac{\pi^2}{6}\frac{a^2}{D^2} + \frac{7\pi^4}{360}\frac{a^4}{D^4} + \cdots)\cdot D\quad , \tag{10.18}$$

which follows from (10.11), (10.10), and (10.16). Likewise (10.12) and (10.13) correspond to the two quantities that are given by (9.7).

11. The Thomas-Fermi Distribution for a Power Law Potential (TP_λ)

Here it is assumed that the nucleonic density $n(r)$, that is related to our distribution function $f(r)$ according to (1.3) and (1.4) with $\rho(r) := n(r)$, can be derived from a Thomas-Fermi model with a one particle potential that obeys a (truncated) power law. This power law potential may be formulated as follows:

$$V(R) := -v_0 \cdot [1 - (\frac{r}{R_0})^\lambda] \cdot \theta(R_0 - r) \tag{11.1}$$

with positive constants v_0, R_0, and λ. We omit any dependence on spin or isospin. The potential box is filled by neutrons and protons in equal numbers up to a common Fermi level e_F, which is, of course, bound to $-v_0 < e_F < 0$. In order to simplify the formulae to come, we use the Fermi sea depth (2.3) and introduce, according to Fig. 1,

$$R_F := (\frac{v_F}{v_0})^{1/\lambda} \cdot R_0 \quad . \tag{11.2}$$

Thus we obtain the particle density from the well known formula

$$n(r) := \frac{2}{3\pi^2} \cdot (\frac{2m}{\hbar^2})^{3/2} [e_F - v(r)]^{3/2} \tag{11.3}$$

which in our case yields $n(r) = n_c \cdot \theta(R_F - r) \cdot [1 - (r/R_F)^\lambda]^{3/2}$, where n_c is given by (2.4). This means,

$$f(r) = [1 - (\frac{r}{R_F})^\lambda]^{3/2} \cdot \theta(R_F - r) \quad . \tag{11.4}$$

Hence, $R_\theta = (1 - \theta^{2/3})^{1/\lambda} R_F$ so that $D = (1 - 2^{-2/3})^{1/\lambda} \cdot R_F$ and $t = \{[1 - (1/10)^{2/3}]^{1/\lambda} - [1 - (9/10)^{2/3}]^{1/\lambda}\} \cdot R_F$. It is not necessary to write down $g(r)$, because now the calculations are most easily done via (3.3) and (4.5), jumping over the definition (4.3) of G_μ. This quantity becomes

$$G = \frac{(\frac{3}{2})!\ (\frac{\mu}{\lambda})!}{(\frac{3}{2} + \frac{\mu}{\lambda})!} \cdot R_F^{\mu} = [r^{\mu}] \quad , \tag{11.5}$$

for any $\mu \geqslant 0$. We may notice that $(\frac{3}{2})! = \Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}$. Our results read:*

$$C = \left[\frac{(\frac{3}{2})!\ (\frac{1}{\lambda})!}{(\frac{3}{2} + \frac{1}{\lambda})!}\right] \cdot R_F \quad , \tag{11.6}$$

$$R = \left[\frac{(\frac{3}{2})!\ (\frac{3}{\lambda})!}{(\frac{3}{2} + \frac{3}{\lambda})!}\right]^{1/3} \cdot R_F \quad , \tag{11.7}$$

$$Q = \left[\frac{(\frac{5}{\lambda})!\ (\frac{3}{2} + \frac{3}{\lambda})!}{(\frac{3}{\lambda})!\ (\frac{3}{2} + \frac{5}{\lambda})!}\right]^{1/2} \cdot R_F \quad , \tag{11.8}$$

and

$$b = \left[\frac{(\frac{3}{2} + \frac{1}{\lambda})!^2\ (\frac{2}{\lambda})!}{(\frac{3}{2})!\ (\frac{3}{2} + \frac{2}{\lambda})!\ (\frac{1}{\lambda})!^2} - 1\right]^{1/2} \cdot C \quad . \tag{11.9}$$

Again we refrain from writing down the lengthy expressions for γ_3, γ_4, and γ_5 in the most general case.

The most important cases, those of large λ, may be treated by the Taylor expansion

$$\log[(a + z)!] = \log(a!) + \sum_{r=1}^{\infty} \frac{1}{r}\, \psi^{(r-1)}(1 + a) \cdot z^r \tag{11.10}$$

which uses the <u>digamma function</u> $\psi(z) := \Gamma'(z)/\Gamma(z)$ and is valid if $|z| < 1 + a$. From this and (11.5) one obtains

$$G_{\mu} = R_F^{\mu} \exp \sum_{r=1}^{\infty} \frac{\hat{k}_r}{\lambda}\ (\frac{\mu}{\lambda})^r \qquad \text{for } \lambda > \mu \geqslant 0 \tag{11.11}$$

with the coefficients $\hat{k}_r = \psi^{(r-1)}(\frac{5}{2}) - \psi^{(r-1)}(1)$ that may be evaluated by means of the zeta function:

---

*By the way, $R_{\theta} = (1 - \theta^{2/3})^{1/\lambda}\ R_F$ so that $D = (1 - 2^{-2/3})^{1/\lambda} \cdot R_F$ and $t = \{[1 - (1/10)^{2/3}]^{1/\lambda} - [1 - (9/10)^{2/3}]^{1/\lambda}\} \cdot R_F$.

$$\hat{k}_r = \begin{cases} -\left(\frac{8}{3} - \log 4\right) & \text{for } r = 1, \\ (-1)^r (r-1)!\, 2^r \left[(1+3^{-r}) - (1-2^{1-r})\, \zeta(r)\right] & \text{for } r = 2,3,4,\cdots. \end{cases} \tag{11.12}$$

The exponentiation is performed by the <u>inverse of the cumulant transformation</u> mentioned at the end of Part I, yielding

$$G_\mu = R_F \cdot \left[1 + \sum_{s=1}^{\infty} \frac{\hat{g}_s}{s!} \left(\frac{\mu}{\lambda}\right)^s\right] \qquad \text{for } \lambda > \mu \geqslant 0 \tag{11.13}$$

with the new coefficients

$$\hat{g}_s = s! \sum_{\sigma_1=0}^{s} \sum_{\sigma_2=0}^{[s/2]} \cdots \sum_{\sigma_s=0}^{1} \delta\left(s, \prod_{r=1}^{s} r\sigma_r\right) \prod_{r=1}^{s} \frac{\hat{k}_r^{\sigma_r}}{r!^{\sigma_r} \sigma_r!} , \tag{11.14}$$

where $\delta(s,t) := \delta_{st}$.

Inserting (11.13), (11.14) into (5.2), (4.7) we obtain for the surficial moments the expression

$$\Gamma_n = R_F^n \sum_{s=n}^{\infty} \frac{\gamma_s^{(n)}}{\lambda^s} , \quad \gamma_s^{(n)} = \frac{1}{s!} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \gamma_{sm}^{(n)} \tag{11.15}$$

where the modified coefficients $\gamma_{sm}^{(n)}$ are to be computed from the numbers $(n-m+m^r) \cdot \hat{k}_r$ in the same way as the previous coefficients $\hat{g}_s$ are to be computed from the numbers $\hat{k}_r$, which means that

$$\gamma_{sm}^{(n)} = s! \sum_{\sigma_1=0}^{s} \sum_{\sigma_2=0}^{[s/2]} \cdots \sum_{\sigma_s=0}^{1} \delta\left(s, \sum_{r=1}^{s} r\,{}_r\right) \prod_{r=1}^{s} \frac{(n-m+m^r)^{\sigma_r}\, \hat{k}_r^{\sigma_r}}{r!^{\sigma_r}\, \sigma_r!} . \tag{11.16}$$

Originally, the sum for Γ_n has run from $s = 0$ to ∞, but the first n terms cancel out according to the binomial identity

$$\sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} m^s = n!\, \delta_{sn} \qquad \text{for } s = 0, 1, 2, \ldots, n, \tag{11.17}$$

which may be checked by differentiating $\Sigma_{m=0}^{n} (-1)^m \binom{n}{m} = (1 - x)^n$ at $x = 1$ up to n times. The circumstance that $\gamma_0^{(n)} = \gamma_1^{(n)} = \gamma_{n-1}^{(n)} = 0$ makes Γ_n of the order of $(R_F/\lambda)^n$ or b^n only, even though $\lambda \gg 1$ and hence $R_F \gg b$. Accordingly, all the γ_n stay in the order of 1. Their magnitude is given approximately by the first nonvanishing term ($s = n$), which may be summed up by means of (11.17) too:

$$\gamma_n^{(n)} = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \gamma_{nm}^{(n)} = \sum_{\sigma=0}^{n} \sum_{\sigma=0}^{[n/2]} \cdots \sum_{\sigma=0}^{1} \delta(n, \sum_{r=1}^{n} r\sigma_r)$$

$$\times \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \Pi_{r=1}^{n} \left[m^{r\sigma_r} (\hat{k}_r - \delta_r \hat{k}_1)^{\sigma_r} / r!^{\sigma_r} \sigma_r!\right] ,$$

implying

$$\gamma_n^{(n)} = n! \sum_{\nu_1=0}^{n} \sum_{\nu_2=0}^{[n/2]} \cdots \sum_{\nu_n=0}^{1} \delta(n, \sum_{r=1}^{n} r\nu_r) \prod_{r=1}^{n} \frac{(1 - \delta_{r_1})^{\nu_r} \hat{k}_r^{\nu_r}}{r!^{\nu_r} \nu_r!} \tag{11.18}$$

Thus $\gamma_n^{(n)}$ is to be computed from the numbers $0, \hat{k}_2, \hat{k}_3, \ldots, \hat{k}_3$ in the same way as g_n is to be computed from the numbers $\hat{k}_1, \hat{k}_2, \hat{k}_3, \ldots, \hat{k}_n$.

When (11.11) is inserted into the formulae (4.7), (4.8), (4.9), one finds

$$C = R_F \cdot \exp \sum_{r=1}^{\infty} \frac{\hat{k}_r}{r!} \frac{1}{\lambda^r} = R_F \cdot \left[1 + \sum_{s=1}^{\infty} \frac{\hat{g}_s}{s!} \frac{1}{\lambda^s}\right] \tag{11.19}$$

$$R = R_F \cdot \exp \sum_{r=1}^{\infty} \frac{\hat{k}_r}{r!} \frac{3^r}{3\lambda^r} = R_F \cdot \left[1 + \sum_{s=1}^{\infty} \frac{\hat{r}_s}{s!} \frac{1}{\lambda^s} \right] \tag{11.20}$$

$$Q = R_F \cdot \exp \sum_{r=1}^{\infty} \frac{\hat{k}_r}{r!} \frac{5^r - 3^r}{2\lambda^r} = R_F \cdot \left[1 + \sum_{s=1}^{\infty} \frac{\hat{q}_s}{s!} \frac{1}{\lambda^s} \right] \tag{11.21}$$

Here $\hat{k}_r$ is to be taken from (11.12) and $\hat{g}_s$ from (11.13), whereas $\hat{r}_s$ and $\hat{g}_s$ are obviously obtained by the same expression as $\hat{g}_s$ save that $\hat{k}_r$ has been replaced by $\frac{1}{3} \cdot 3^r \cdot \hat{k}_r$ or $\frac{1}{2} \cdot (5^r - 3^r) \cdot \hat{k}_r$, respectively. Furthermore, by inserting (11.15) into (5.7) we arrive at

$$b = R_F \cdot \frac{\sqrt{\gamma_2''}}{\lambda} \cdot \left[1 + \frac{\gamma_3''}{2\gamma_2''\lambda} + \frac{4\gamma_2''\gamma_4'' - \gamma_3''^2}{8\gamma_2''^2\lambda^2} + \cdots \right] \tag{11.22}$$

using the abbreviation $\gamma_n'' := \gamma_n^{(2)}$. In order to achieve numerical results, we infer from (11.12), (11.14), and (11.18) the values

$$\begin{aligned}
\hat{k}_1 &= -\gamma_{(1)}^{(1)} &&= -\ 2 \cdot [\tfrac{4}{3} - \log 2] &&= -1.28037'2_3 \ , \\
\hat{k}_2 &= \gamma_2^{(2)} &&= +1 \cdot 2 \cdot [2 \cdot \tfrac{10}{9} - 1 \cdot \zeta(2)] &&= 1.15457'6_3 \ , \\
\hat{k}_3 &= \gamma_3^{(3)} &&= -2 \cdot 2 \cdot [4 \cdot \tfrac{28}{27} - 3 \cdot \zeta(3)] &&= -2.16790'9_8 \ , \\
\hat{k}_4 &= \gamma_4^{(4)} - 3[\gamma_2^{(2)}]^2 &&= +6 \cdot 2 \cdot [8 \cdot \tfrac{82}{81} - 7 \cdot \zeta(4)] &&= 6.27003'3_6 \ , \\
k_5 &= \gamma_5^{(5)} - 10\gamma_2^{(2)}\gamma_3^{(3)} &&= -24 \cdot 2 \cdot [16 \cdot \tfrac{244}{243} - 15 \cdot \zeta(15)] &&= -24.57251'0_1 .
\end{aligned} \tag{11.23}$$

Note that $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$ according to (9.5). Now the coefficients

$$\hat{g}_1 = \hat{k}_1 \quad , \qquad \hat{r}_1 = \hat{k}_1 \quad , \qquad \hat{q}_1 = \hat{k}_1 \quad ,$$

$$\hat{g}_2 = \hat{k}_1^2+\hat{k}_2 \quad , \qquad \hat{r}_2 = \hat{k}_1^2+3\hat{k}_2 \quad , \qquad \hat{q}_2 = \hat{k}_1^2+8\hat{k}_2 \quad , \tag{11.24}$$

$$\hat{g}_3 = \hat{k}_1^3+3\hat{k}_1\hat{k}_2+\hat{k}_3 \quad , \qquad \hat{r}_3 = \hat{k}_1^3+9\hat{k}_1\hat{k}_2+9\hat{k}_3 \quad , \qquad \hat{q}_3 = \hat{k}_1^3+24\hat{k}_1\hat{k}_2+46\hat{k}_3 \quad ,$$

and

$$\gamma_2'' = \hat{k}_2 \quad , \quad \gamma_3'' = 2\hat{k}_1\hat{k}_2+\hat{k}_3 \quad , \quad \gamma_4'' = 2\hat{k}_1^2\hat{k}_2+ \frac{3}{2}\hat{k}_2^2 + 2\hat{k}_1\hat{k}_3 + \frac{7}{12}\hat{k}_4^2 \tag{11.25}$$

may easily be computed.

If R_F, r, and λ all become large in such a way that

$$a: = \frac{R_F}{\lambda} \qquad \text{and} \qquad s: = r - C \tag{11.26}$$

or $x: = R_F - r$ stay constant, we finally arrive at the last distribution to be discussed.

12. The Thomas-Fermi Distribution for an Exponential Law Potential ($E = TP_\infty$)

In this case, the one particle potential has the following form:

$$v(r): = -v_0 \cdot [1 - \exp \frac{r-R_0}{a}] \cdot \theta(R_0 - r) \tag{12.1}$$

with positive constants v_0, R_0, and a. We now put

$$R_F: = R_0 - a \cdot \log \frac{v_0}{v_F} \tag{12.2}$$

using (2.3), and obtain from (11.3) a density leading to

$$f(r): = [1 - \exp \frac{r-R_F}{a}]^{3/2} \cdot \theta(R_F - r) \tag{12.3}$$

or $R_\theta = R_F + a \log (1 - \theta^{2/3})$, so that $D = R_F - 0.99414'559_0\ a$ and $t = 2.44811'0_4\ a$.

Using the binomial series and partial integration we may easily evaluate (4.3), (1.7), yielding

$$G_\mu = \sum_{s=0}^{\infty} (-1)^s \binom{\mu}{s} \hat{\gamma}_s a^s R_F^{\mu-s} \qquad \text{for } \mu \geqslant 0 \tag{12.4}$$

with

$$\hat{\gamma}_\sigma := \frac{3}{2} \int_0^\infty x^\sigma e^{-x} (1 - e^{-x})^{1/2} dx \qquad \text{for } \sigma \geqslant 0 \quad . \tag{12.5}$$

If μ assumes an integer value m, the summation terminates just after s = m. The upper limit of integration was originally $x = X_F := R_F/a$, but we have replaced it by ∞ assuming that $R_F >> a$ so that $e^{-X_F} <<< 1$. The two most simple cases of the integral are

$$\hat{\gamma}_0 = 1, \qquad \hat{\gamma}_1 = -k_1 = \frac{8}{3} - \log 4 = 1.28037'2_3 \quad . \tag{12.6}$$

The first holds because of normalization, and the second may also be deduced with the variable $w := (1 - e^{-x})^{1/2}$ which yields

$$\hat{\gamma}_1 = \lim_{\varepsilon \to +0} \left[2w + \frac{2}{3} w^3 - (1 + w^3) \log (1 + w) + (1 - w^3) \log (1 - w)\right]_{w=0}^{1-\varepsilon} .$$

These two values suffice to ensure by (4.7) that

$$C = R_F - \hat{\gamma}_1 \cdot a = D - 0.28622'6_7 \cdot a \quad . \tag{12.7}$$

The higher moments may be calculated by binomial expansion of the square root in (12.5) which leads to the series

$$\hat{\gamma}_\sigma = \frac{3}{2} \cdot \nu! \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{\sigma+1}} \binom{1/2}{k-1} \qquad \text{for } \sigma > 0 \tag{12.8}$$

If $\sigma \geqslant 2$ it can be evaluated rather quickly because the rest term $\Sigma_{k=k+1}^{\infty}$ may be thoroughly estimated on both sides by rather simple integrals. With K = 50, 40, 30, 20 for σ = 2, 3, 4, 5 (respectively) we obtained

$$\begin{aligned}
\hat{\gamma}_2 &= 3 \times 0.93130'98_6 = 2.79392'9_6 , \\
\hat{\gamma}_3 &= 9 \times 0.96686'16_5 = 8.70175'4_8 , \\
\hat{\gamma}_4 &= 36 \times 0.98378'13_6 = 35.41612'_9 , \\
\hat{\gamma}_5 &= 180 \times 0.99199'74_0 = 178.5595_3 .
\end{aligned} \tag{12.9}$$

For σ = 1 we had to go until K = 400 in order to obtain $\gamma_1 = \frac{3}{2} \cdot 0.85358'15_4$, thus confirming the previous result (12.6).

After inserting (12.6) and (12.9) into (12.4), and this in turn into (5.7), and after introducing the numerical constants

$$\begin{aligned}
\hat{g} &:= \hat{\gamma}_2 - \hat{\gamma}_1^2 = 1.15457'6_3 , \\
\hat{\gamma} &= \hat{g}^{1/2} = 1.07451'2_1 ,
\end{aligned} \tag{12.10}$$

we arrive at the results

$$b = \hat{\gamma} \cdot a \tag{12.11}$$

Accordingly $t/b = 2.27834'6_0$ for this distribution. Now. eqs. (5.12), (5.13), and (5.11) tell us that

$$\begin{aligned}
R &= C + \hat{g} \cdot \frac{a^2}{R_F} + O\left(\frac{a^2}{R_F}\right) \cdot a , \\
Q &= C + \frac{7}{2}\hat{g} \cdot \frac{a^2}{R_F} + O\left(\frac{a^2}{R_F^2}\right) \cdot a .
\end{aligned} \tag{12.12}$$

In order to obtain the higher order corrections we have to sum up (5.2) after the insertion of (12.4) and (12.7). Using the binomial expansion $C^{n-m} = \Sigma_{v=0}^{n-m}(-1)^v\binom{n-m}{v}R_F^{n-m-v}(\hat{\gamma}_1 a)^v$ and the identity $\binom{n}{m}\binom{n-m}{v}\binom{m}{s} = \binom{n}{s+v}\binom{s+v}{s}\binom{n-s-v}{m-s}$ we find $\Gamma_n = \Sigma_{m=0}^{n}\,\Sigma_{v=0}^{n-m}\,\Sigma_{s=0}^{m}(-1)^{n+m+s+v}\,R_F^{n-s-v}\,a^{v+s}\,\hat{\gamma}_1^v\,\hat{\gamma}_s\,\binom{n}{s+v}\,\binom{s+v}{s}\,\binom{n-v-s}{m-s}$ $= \Sigma_{s=0}^{n}\,\Sigma_{v=0}^{n-s}\,R_F^{n-s-v}\,(-a)^{s+v}\,\binom{n}{s+v}\,\binom{s+v}{s}\,\hat{\gamma}_1^v\,\gamma_s\,\Sigma_{m=s}^{n-v}\,(-1)^{n+m}\,\binom{n-v-s}{m-s}$. Now the identity $\Sigma_{j=0}^{k}(-1)^j\,\binom{k}{j} = \delta_{k0}$, taken from (11.17) as valid for all* natural k, shows with k=n=u, u=s+v, and j=m-s that $\Sigma_{m=0}^{n-v}(-1)^{n+m}\binom{n-v-s}{m-s}=(-1)^{m+s}\cdot\Sigma_{j=0}^{n-u}(-1)^j\binom{n-u}{j}=(-1)^{n+s}\delta_{nu}$. Consequently $\Gamma_n = \Sigma_{s=0}^{n}\,\Sigma_{u=s}^{n}\,R_F^{n-u}\,(-a)^u\,\binom{n}{u}\,\binom{u}{s}\,\hat{\gamma}_1^{u-s}\hat{\gamma}_s(-1)^{n+s}\delta_{nu}$ which simplifies to

$$\Gamma_n = a^n \sum_{s=0}^{n} (-1)^s \binom{n}{s} \hat{\gamma}_1^{n-s}\,\hat{\gamma}_s \quad . \tag{12.14}$$

Inserting (12.6), (12.9) into this rather compact result, and this together with (12.10) into (5.8), we find (besides $\gamma_0 = 1$, $\gamma_1 = 0$, $\gamma_2 = 1$) these values

$$\begin{aligned}
\gamma_3 &= -(\hat{\gamma}_3 - 3\hat{\gamma}_2\hat{\gamma}_1 - 2\hat{\gamma}_1^3)/\hat{\gamma}^3 &&= -\,1.74746'_0 \quad , \\
\gamma_4 &= +(\hat{\gamma}_4 - 4\hat{\gamma}_3\hat{\gamma}_1 + 6\hat{\gamma}_2\hat{\gamma}_1^2 - 3\hat{\gamma}_1^4)/\hat{\gamma}^4 &&= \;\;7.70353'_7 \quad , \\
\gamma_5 &= -(\hat{\gamma}_5 - 5\hat{\gamma}_4\hat{\gamma}_1 + 10\hat{\gamma}_3\hat{\gamma}_1^2 - 10\hat{\gamma}_2\hat{\gamma}_1^3 - 4\hat{\gamma}_1^5)/\hat{\gamma}^5 &&= -34.6296_8 \quad .
\end{aligned} \tag{12.15}$$

The deviations κ_n require more complicated formulae when expressed in terms of the integrals $\hat{\gamma}_s$.

---

*$\Sigma_{j=0}^{k}(-1)^j\,\binom{k}{j} = (1-1)^k = 0^k = \delta_{k0}$ prevails for all natural numbers k including k=0, because $0^0 = 1$ is, though often questioned or even denied, certainly a true statement.

Yet there is another method to attain the coefficients κ_n. If in (11.15) the limiting procedure $\lambda \to \infty$ with $R_F/\lambda \to a$ is performed, only the first term survives. Thus we have for the present distribution the simple result

$$\Gamma_n = a^n \gamma_n^{(n)} \quad , \tag{12.16}$$

and therefore $\gamma_n = \gamma_n^{(n)}/[\gamma_n^{(2)}]^{n/2}$ by (5.8) and (5.7). Comparison with (12.14) shows now that $\gamma_n^{(n)} = \Sigma_{s=0}^{n}(-1)^s \binom{n}{s} \hat{\gamma}_1^{n-s} \hat{\gamma}_s$, but we are at present more interested in the representation (11.18), regardless of its more complicated appearance. The reason: as it is the inverse of the cumulant transformation it may easily be inverted to yield the cumulants $K_r = a^r \kappa_r^{(r)}$ simply as

$$K_r = a^r \cdot (1 - \delta_{r1}) \cdot \hat{k}_r \quad . \tag{12.17}$$

The deviation coefficients $\kappa_r = b^{-r} K_r$ are therefore given by the formula $\kappa_r = (1-\delta_{r1}) \cdot \hat{k}_r/\hat{k}_2^{r/2}$, which (reproducing $\kappa_1=0$, $\kappa_2=1$) produces by means of (11.23) the numerical results

$$\begin{aligned} \kappa_3 &= -1.74746'_0 \quad , \\ \kappa_4 &= 4.70353'_7 \quad , \\ \kappa_5 &= -17.1550_8 \quad . \end{aligned} \tag{12.18}$$

They turn out, when the relations $\gamma_3 = \kappa_3$, $\gamma_4 = \kappa_4+3$, $\gamma_5 = \kappa_5+10\kappa_3$ are employed, to be accurately consistent with our previous results displayed in (12.15). They may be inserted into (5.9) and (5.10) in order to acquire R and Q besides C and D, which quantities have been given already in (12.7) and (12.6).

13. Numerical Results and Discussion

In Fig. 2 we have shown the ratio $v = t:2b$ of the punctual surface thickness t to the integral surface thickness 2b. This quotient depends remarkably on which distribution is chosen, being about 25% higher for the traperoidal distribution (T) than for the Laplacian one (L).

Now let us compare C and Q with R. According to eqs. (5.12) and (5.13) or (5.14) the ratios $c: = C/R$ and $q: = Q/R$ both differ from unity by quantities which have the same order of magnitude as the quantity $\sigma: = \beta^2 = (b/R)^2$. The numerical results are given in the Table and Fig. 3 for different distribution laws. In both cases we have accompanied the σ-scale by a nonlinear A-scale by assuming[6] that

$$b = 0.92_8 \text{ fm} \tag{13.1}$$

independently of A, whereas R is, as shown at (3.9), proportional to $A^{1/3}$ with $r_c = R \cdot A^{-1/3} = 1.13_5$ fm. From these empirical data we find that

$$\beta = 0.81_8 \cdot A^{-1/3} \tag{13.2}$$

and thus $\sigma = 0.66_9 \cdot A^{-2/3}$. In case of the power like distribution (PL_λ), the square radius $Q = q \cdot R$ becomes ∞ at $\gamma = \pi/\lambda = \pi/5 = 0.628_3$ which corresponds to $\sigma = 0.113_2$ and $\beta = 0.336_5$, and σ attains its greatest possible value 0.157_7 at $\gamma = \pi/\lambda = 0.858_0$ which corresponds to $\beta = 0.397_1$ and $c = 0.672_8$. This means that the leptodermicity condition $\beta \ll 1$ completely breaks down, for this distribution, at $\sigma \approx 0.1$ or $A \approx 30$. The generalized error function distribution (GE_λ) cannot be included simply into the Table and Fig. 3, because it depends, besides a and λ, on the third parameter C uncoupled from the two other ones. This consideration

does not apply to Fig. 2, because this shows a local surface property for which the value of the radius C (or R) is irrelevant. In the cases of the one-parametric distributions (T, N, F, L) the validity of all results is confined to the condition that $e^{-1/\beta} <<< 1$.

ACKNOWLEDGMENTS

The author would like to thank W. J. Swiatecki and the Nuclear Chemistry Division of the Lawrence Berkeley Laboratory for their kind hospitality, and W. D. Myers especially for discussions and help with the English manuscript.

REFERENCES

1. B. Hahn, D. G. Ravenhall, and R. Hofstadter, Phys. Rev. 101, 1131 (1956).
2. W. D. Myers, private communication (1967); W. D. Myers, Nucl. Phys. A104 (1973) 465.
3. W. D. Myers, Nucl. Phys. A145, 387-400 (1970).
4. W. D. Myers and W. J. Swiatecki, Ann. of Phys. 55, 395-505 (1969).
5. Chin-Fu Tsang, Ph.D. Thesis, Lawrence Radiation Laboratory Report UCRL-18899, (1969).
6. H. R. Collard, L. R. B. Elton, and R. Hofstadter, Handb. Phys. I/2, (Springer 1967).
7. L. R. B. Elton, Nucl. Phys. 5, 173 (1958); 8, 396 (1958).
8. L. R. B. Elton, Nucl. Sizes (Oxford University Press, London, 1961).
9. K. W. Ford and J. G. Wills, Phys. Rev. 185, 1429 (1969).
10. M. Abramowitz and F. A. Stegun: Handbook of Mathematical Functions (National Bureau of Standards 1964, Dover 1965).

FIGURE CAPTIONS

Fig. 1. Qualitative picture of a surface distribution with positive flair κ_3 and positive contortion κ_4.

Fig. 2. The punctual (10% to 90%) thickness t in units of the integral thickness 2b as a function of the distribution parameter λ for four different distribution laws: incomplete gamma (FΓ), generalized error (GE), power like (PL), and Thomas-Fermi for a power-law potential (TP). Special cases (for particular values of λ) are also indicated: normal (N), trapezoidal (T), Laplace (L), Fermi (F), and Thomas-Fermi for an exponential-law potential (E).

Fig. 3. The central radius C and the quadratic radius Q in units of the charge radius R as a function of the width-square b^2 in units of R^2, for eight different distribution laws (PL, IΓ, TP, E and L, F, N, T).

Table. The Ratios $c = C/R$ and $q = Q/R$ for Different Distribution Laws and Different Values of $\beta = b/R$.

β^2	c_{PL}	c_{IG}	$c_L=c_F=$ $=c_N=c_T$	c_{TP}	c_E	q_E	q_{TP}	q_T	q_N	q_F	q_L	q_{IG}	q_{PL}	A
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	∞
0.01	0.9898	0.9899	0.9900	0.9904	0.9906	1.0185	1.0198	1.0242	1.0245	1.0247	1.0252	1.0253	1.0269	546.6
0.02	0.9793	0.9797	0.9800	0.9810	0.9817	1.0327	1.0371	1.0468	1.0479	1.0490	1.0507	1.0511	1.0582	193.2
0.03	0.9684	0.9694	0.9700	0.9717	0.9731	1.0447	1.0533	1.0679	1.0704	1.0728	1.0765	1.0776	1.0954	105.2
0.04	0.9570	0.9589	0.9600	0.9624	0.9648	1.0553	1.0689	1.0878	1.0920	1.0962	1.1025	1.1047	1.1404	68.3
0.05	0.9452	0.9482	0.9500	0.9530	0.9568	1.0649	1.0841	0.1064	1.1128	1.1192	1.1287	1.1324	1.1964	48.9
0.06	0.9327	0.9374	0.9401	0.9436	0.9491	1.0736	1.0991	1.1239	1.1329	1.1418	1.1551	1.1609	1.2685	37.2
0.07	0.9196	0.9263	0.9301	0.9341	0.9415	1.0817	1.1139	1.1404	1.1523	1.1641	1.1816	1.1902	1.3660	29.5
0.08	0.9057	0.9151	0.9202	0.9245	0.9342	1.0892	1.1287	1.1559	1.1711	1.1861	1.2082	1.2203	1.5077	24.2
0.09	0.8908	0.9037	0.9103	0.9148	0.9271	1.0962	1.1435	1.1705	1.1893	1.2077	1.2349	1.2513	1.7388	20.2
0.10	0.8747	0.8921	0.9004	0.9050	1.9202	1.1027	1.1584	1.1843	1.2069	1.2291	1.2616	1.2832	2.2120	17.3
0.11	0.8571	0.8802	0.8905	0.8951	0.9135	1.1088	1.1734	1.1974	1.2241	1.2502	1.2884	1.3161	4.2795	15.0
0.12	0.8374	0.8681	0.8806	0.8851	0.9070	1.1144	1.1886	1.2097	1.2407	1.2710	1.3152	1.3502	∞	13.1
0.13	0.8147	0.8558	0.8708	0.8749	0.9007	1.1195	1.2039	1.2214	1.2570	1.2916	1.3419	1.3854	∞	11.7
0.14	0.7871	0.8431	0.8610	0.8646	1.8945	1.1242	1.2195	1.2324	1.2728	1.3120	1.3687	1.4219	∞	10.4
0.15	0.7492	0.8302	0.8513	0.8541	0.8886	1.1284	1.2354	1.2429	1.2883	1.3322	1.3954	1.4598	∞	9.4
0.16	———	0.8169	0.8416	0.8434	0.8828	1.1320	1.2516	1.2529	1.3035	1.3521	1.4220	1.4992	∞	8.5
0.17	———	0.8033	0.8319	0.8326	0.8772	1.1351	1.2682	1.2624	1.3183	1.3719	1.4486	1.5404	∞	7.8

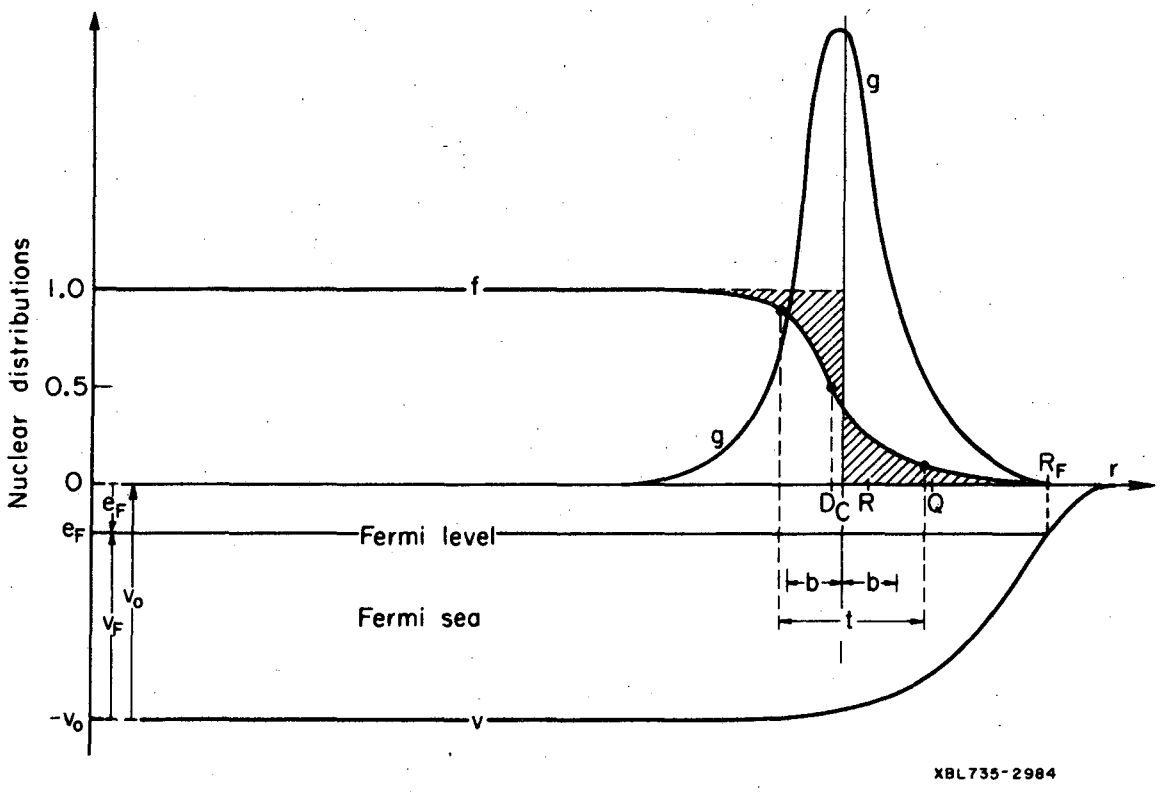
Nuclear distributions
1.0
0.5
0
e_F
e_F
v_F
v_o
$-v_o$
f
g
g
v
Fermi level
Fermi sea
D_C
R
Q
R_F
r
b
b
t
XBL735-2984

Fig. 1

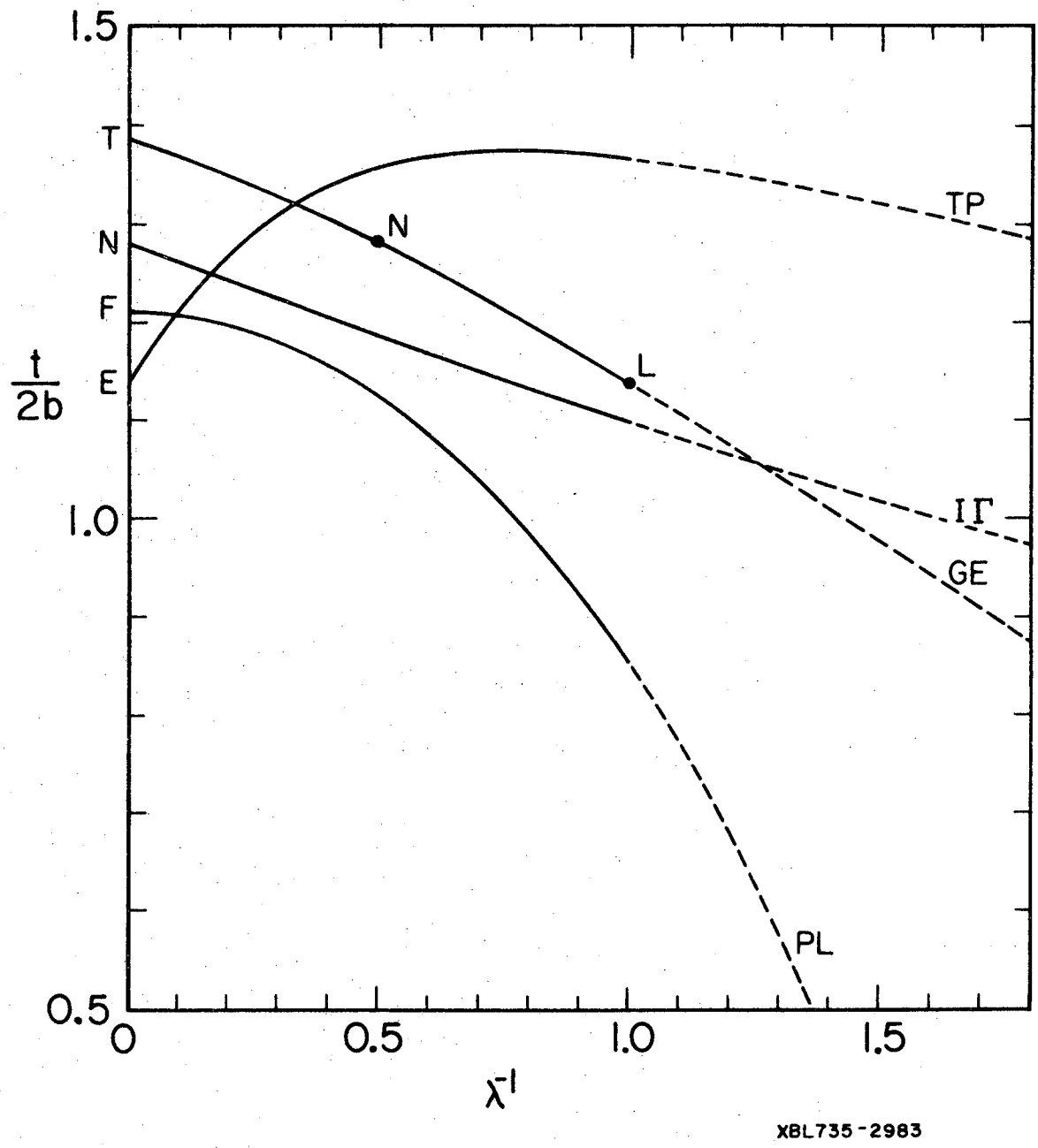
1.5
T
N
F
E
$\frac{t}{2b}$
1.0
0.5
0
0.5
1.0
1.5
λ^{-1}
N
L
TP
IΓ
GE
PL
XBL735-2983

Fig. 2

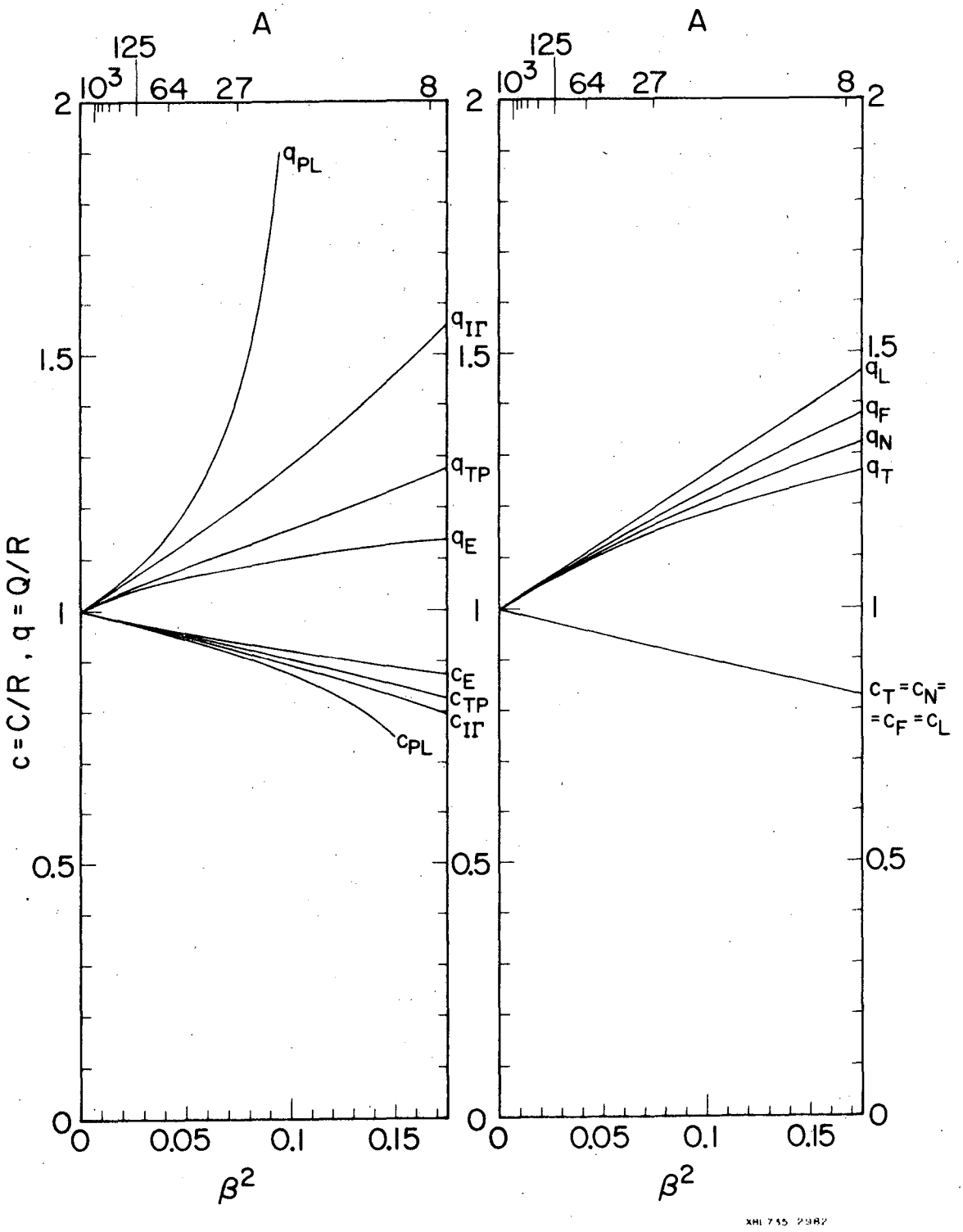
A
125
10^3
64
27
8
2
1.5
1
0.5
0
q_{PL}
$q_{I\Gamma}$
q_{TP}
q_E
c_E
c_{TP}
$c_{I\Gamma}$
c_{PL}
$c = C/R$, $q = Q/R$
0
0.05
0.1
0.15
β^2
A
125
10^3
64
27
8
2
1.5
1
0.5
0
q_L
q_F
q_N
q_T
$c_T = c_N =$
$= c_F = c_L$
0
0.05
0.1
0.15
β^2
XBL 735 2982

Fig. 3

LEGAL NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

TECHNICAL INFORMATION DIVISION
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720