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VORTEX METHODS FOR FLOWS OF VARIABLE DENSITY

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(Ph.D. Thesis)

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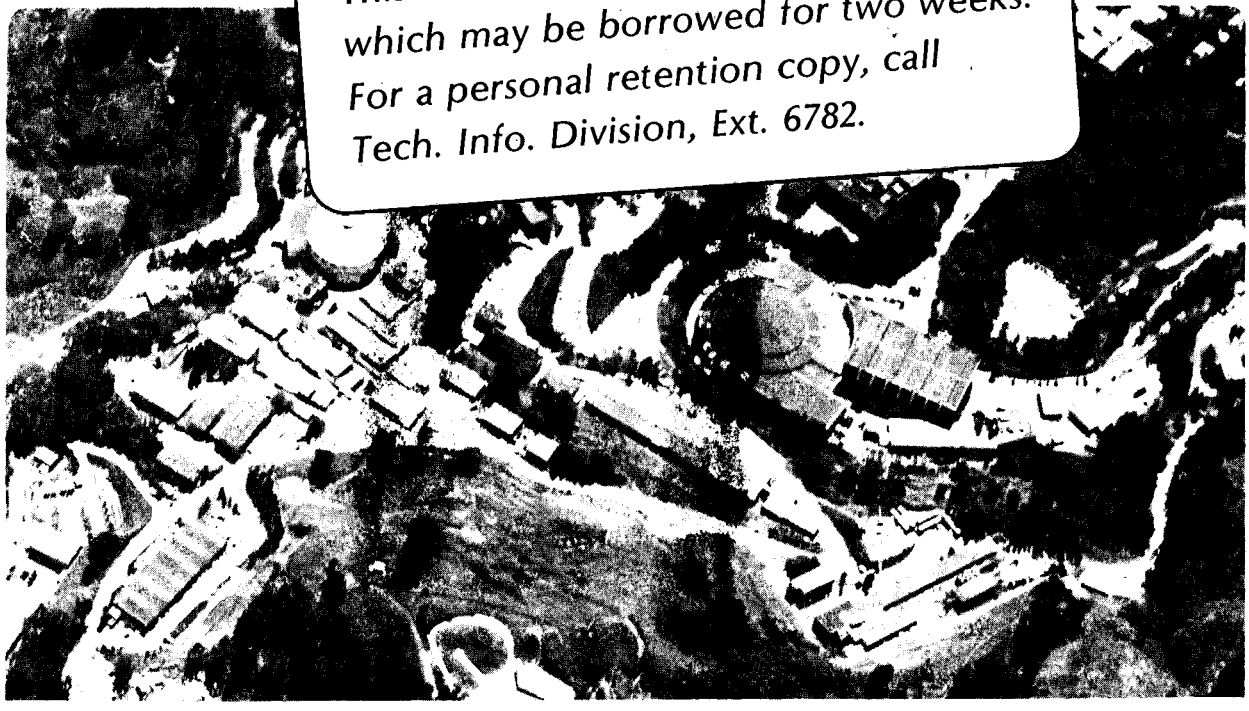

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VORTEX METHODS FOR FLOWS OF VARIABLE DENSITY

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Ph.D. Thesis

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Vortex Methods for Flows of Variable Density

Christopher Radcliff Anderson

Abstract

We present two numerical methods for calculating the motion of an incompressible, inviscid fluid of slightly varying density. The methods, both based on the vortex method developed by Chorin, are grid free and have no intrinsic source of numerical diffusion. We analyze the methods using techniques derived from the recent work on the convergence of the vortex method. We prove a convergence result for one method and prove a partial convergence result for the other method.

We present an exact solution to the equations of motion for a fluid of variable density and use this solution to test both numerical methods. The test results indicate that the methods are stable and accurate.

An application to the problem of calculating the motion of a 2-D thermal is also presented. The computational results indicate that the methods are suitable for calculating flows associated with thermal convection phenomena. In the course of our investigations of the motion of a 2-D thermal, we found significant computational evidence to suggest that a singularity develops in the flow in finite time. This singularity appears to be confined to a small set, possibly a point, and is characterized by an infinite value of vorticity there.

The two numerical methods presented here can be viewed as a specific application of a more general technique for constructing algorithms for incompressible flow problems. We discuss this general numerical technique as well as the theoretical tools that are used to analyze the numerical methods constructed with this technique.

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CONTENTS

Introduction 1

0 : Preliminaries 8

1 : Equations of Motion 12

2 : Model Problem 20

3 : Discretization of the Equations of Motion 37

4 : Analysis of Method A 41

4.1 : Error Estimates 41

4.2 : Test Problem 50

4.3 : Convergence of Modified Algorithm 60

5 : Analysis of Method B 71

5.1 : Convergence of Method B 71

5.2 : Computational Results 101

6 : Application to a 2-D Thermal 107

6.1 : Application of Method B to a 2-D Thermal 108

6.2 : Application of Method A to a 2-D Thermal 135

6.3 : Initialization of Method A for Non-rectangular Grids 147

7 : Discussion and Conclusions 152

Bibliography 157

Introduction

In this thesis we present, test, and analyze two numerical methods, both based on the vortex method developed by Chorin [13], for calculating the solution to incompressible flow problems with variable density. Previously the vortex method, as implemented in [13], has been used with success to calculate high Reynolds number incompressible fluid flow. Specific applications which have used the vortex method have included the calculation of unstable boundary layers [15], flow past heart valves [42], aerodynamic applications ([11],[40],[50]), the modelling of turbulent combustion ([27],[48]), and the driven cavity problem [49]. (For a general discussion of vortex methods see ([15],[39])) An assumption made in these studies is that the fluid is of constant density, or, in the case of combustion modelling, that the effect of variable density on the vorticity growth is negligible.

However, density variations can have an important effect on fluid motion, especially when the fluid is acted upon by an external force. A good example of this is thermal convection. In this situation the external force is gravity and density variations are those caused by temperature differences in the fluid. The motion induced by the buoyancy of the lighter fluid is certainly important and cannot be neglected.

Often the motion induced by density variations within a fluid is highly complicated and difficult to model numerically. This thesis is a result of our investigations into possible ways of incorporating within a vortex method the effects of variable density. The previous success of the vortex method in modelling complicated flows makes it a natural choice as a basis of a method for solving variable density flow problems. We mention that Meng and Thomson present an early application of the vortex method to such problems in [43].

Another aspect of this thesis is a presentation of a set of ideas which form the basis of a general numerical technique. From the vortex method [13] and recent work on the convergence of the vortex method ([5],[6],[19],[30],[31]), we have been able to abstract a procedure for constructing numerical methods for incompressible flow problems and for analyzing these methods. We will present this technique mainly by way of example. In fact, the two numerical methods that we present for variable density flows can be considered as two examples of applications of the technique.

The numerical technique that we discuss is essentially a Lagrangian one. We approximate the motion of the fluid by calculating the motion of a finite number of fluid particles and the evolution of fluid quantities (density, vorticity, etc.) associated with them. This numerical technique is different from other particle techniques, such as those described in ([21],[28],[44]). The difference is subtle, but important. In the methods that we discuss here, the computational points, or particles, are merely positions in a continuum of fluid. The fluid quantities that are associated with the computational points are the values of the fluid quantities evaluated at the point positions. For example, when we represent density by a collection of points, the points do not carry mass, but are considered to carry the continuous density evaluated at those points. Also, we calculate the motion of the computational points using the vortex method, i.e., a discretization of the vorticity-stream formulation of the equations of motion. In the particle method described in ([21],[28],[44]) the computational points represent true particles. The particles have a specific mass and velocity, and the motion of the particles is calculated using Newton's laws. To evaluate a fluid quantity such as density, one must form a local average over the computational particles.

There is an interesting similarity between these two techniques. In the numerical technique that we discuss, to evaluate a fluid quantity at points other than those whose evolution we are computing, we use a "blob" type approximation. Originally introduced in the vortex method [13] the "blob" approximation consists of representing a function by a collection of functions of small support or "blobs". In the particle techniques that are described in ([21], [28],[44]) the fluid quantities are evaluated by forming a local average over the computational particles. Recently, Gingold and Monaghan in [28] have introduced an averaging procedure, kernel estimation, to obtain more accurate estimates of fluid quantities in their particle methods.. One interpretation of the kernel estimation procedure is that it is an average over the particles assuming that the particles are of finite extent and have a definite shape. Upon closer examination one finds that kernel estimation and "blob" approximation are nearly identical. It is found that kernel estimation can be viewed as an approximation procedure for evaluating a function using the values of the function at a finite number of points. Alternatively, the "blob" approximation can be considered as a statistical estimate of a fluid quantity based on samples of the function at the computational points. The fact that kernel estimation can be considered as an approximation procedure was noticed by Monaghan in [44]. An interesting result given in [44], is that kernel estimation, and hence "blob" approximation, can be viewed as a generalization of more standard approximation techniques, i.e. polynomial or Fourier approximation.

An observation that has helped us in our investigations is the similarity between the equations of motion for an incompressible fluid of variable density in two dimensions and the equations of motion of an incompressible fluid of constant density in three dimensions. An implication of this similarity is that numerical methods for one set of equations have a direct analog

in the other set of equations. This is a fortunate occurrence, for the convergence results obtained by Beale and Majda in ([5],[6],[8]) can be used with very little modification to obtain convergence results for the methods presented here. Also, the numerical methods presented here suggest numerical methods for three dimensional fluid calculations. (In fact, it was our work on the motion of a fluid of variable density that led us to suggest the numerical method whose convergence is investigated in [8]) Another benefit of this similarity is that the numerical results presented here may prove useful in assessing the properties of the three dimensional vortex methods analyzed in [5] and [8].

This thesis is divided into eight sections. In section 0 we discuss some of our notation and prove some elementary lemmas that we need for the theoretical analysis in later sections.

Section 1 contains a presentation of the equations that we use to describe the motion of a fluid of variable density. In deriving the equations we make the assumption that the density variations in the fluid are small. (We make the Boussinesq approximation ([51],[53]).) This assumption leads to a simplification of the equations and makes their numerical approximation easier to accomplish. We will use two different formulations of the equations, and, in this section, we prove their equivalence.

In section 2 we illustrate our general numerical technique by applying it to a model problem. The model problem is that of calculating the motion of a quantity being transported by an incompressible flow. The numerical approximation for this problem and the theoretical analysis of this approximation are exemplary of the techniques to be used throughout this work. In particular, the techniques used to estimate the accuracy of the numerical approximation will be used repeatedly in later sections.

In section 3 we apply our numerical technique to the two formulations of the equations given in section 1. The result is two numerical methods which we designate method A and method B. Methods A and B represent two different approaches to the problem of calculating the motion of a fluid of variable density. Loosely speaking, the motion of the fluid is calculated by calculating the evolution of the vorticity in the fluid. In a fluid of variable density the vorticity grows where there are density gradients. Thus, to account for the growth of vorticity one needs to calculate density gradients. It is in the calculation of the density gradients that the methods differ. The first approach, that taken in method A, is to calculate the evolution of the density, and then evaluate density gradients by differentiating the resulting approximation. The second approach, that taken in method B, is to calculate the density gradients by solving an equation that describes their evolution in time. This latter approach requires an equation of evolution for the density gradients as well as a method for constructing a meaningful approximation to the density from the solution of these equations. The equation of evolution is that which is described in section 1 and we use a numerical implementation of Poisson's formula to reconstruct the density from its gradients.

We analyze method A in section 4. We begin by estimating the error in the approximations used in the method. The importance of these estimates is that they indicate how the parameters of the method might be chosen. To test various choices of the parameters, as well as its accuracy and stability we apply the method to a test problem. The test problem consists of an exact solution to the equations and is described in section 4.2. In the last part of this section we present a method which is a modification of method A and prove a convergence result for this modified method. Although the modified method has yet to be tested, we expect that the method will

behave in a way similar to method A. We present the convergence result to verify, albeit not rigorously, that the approach taken with method A is essentially sound.

We analyze method B in section 5. We begin our analysis by proving a convergence result. We use this result and the error estimates that the proof provides to select the parameters for the approximations used in the method. Then, as with method A, we apply the method to the test problem described in section 4.2. We use this test to verify the accuracy and stability of the method. These results also allow us to make a direct comparison between the two methods.

The test problem of section 4.2 is somewhat special. In particular the solution of the test problem is smooth (has many bounded derivatives) and has radial symmetry. It is not clear that the preformance of the methods on this problem are indicative of their preformance on problems which are not smooth and whose flows are highly complicated. To examine the behavior of the method when they are applied to problems of the latter type, we apply the methods to the problem of calculating the motion of a 2-D thermal. A description of the problem and the results are presented in section 6

There are two very interesting conclusions obtained from this last test. The first is that care must be taken when one uses these methods on non-smooth problems. In particular, conclusions about the methods and their implementation obtained from the results on the test problem of section 4.2 are not necessarily applicable. The other interesting conclusion is that we find significant computational evidence to suggest that the equations have a singularity. Aside from the theoretical interest in the singularity, we find it remarkable that the presence of the singularity does not destroy the computational results. The numerical methods are capable of computing reliable solutions past the time when the singularity occurs. This suggests that

such methods are suitable tools for investigating singularities of the equations of motion.

Finally, section 7 is devoted to a general discussion of the results, our conclusions about the methods, and some suggestions for future work.

0. Preliminaries

In this section we describe some of our notation. We also prove some elementary lemmas that we will need in later sections.

The symbol "^" will denote the Fourier transform, i. e. for a function f,

$$\hat{f}(\omega) = \int e^{2\pi i x \cdot \omega} f(x)dx$$

We will use $\| \cdot \|_{L^2(S)}$ and $\| \cdot \|_{L^1(S)}$ to denote the L^2 and L^1 norms of functions restricted to a set S. We will use H^s to denote the Sobolev space of order s. The symbol "C" will designate a general constant.

We say that a function Ψ is in $M^{L,p}$ (M for mollifier) if Ψ satisfies the following:

(*i*) $D^\gamma\Psi$ is integrable over R^2 for every multi-index γ, $0 \le |\gamma| \le L$

(*ii*) $\int_{R^2} \Psi(x)dx = 1$

(*iii*) $\int_{R^2} x^\beta \Psi(x)dx = 0$ for every multi–index β, $1 \le |\beta| \le p - 1$

(*iv*) $\Psi(x)$ is rapidly decreasing , i.e., there exits a constant $C_{\beta j}$ depending only on β, β a multi-index, and j, an integer, such that ,

$$|D^\beta\Psi(x)| \le C_{\beta j}(1 + |x|^2)^{-j}$$

Specific functions in this class will be given in sections 4 and 5.

The symbol Λ^h will denote the set of lattice points of a grid of uniform mesh width h,

$$\Lambda^h = \{(j_1h, j_2h) \mid (j_1, j_2) \in Z \times Z\} .$$

As in [5] we shall use the norms associated with the spaces H_h^{-1}, H_h^0, and H_h^1 to measure functions defined on Λ^h. These spaces are the discrete analogs of the Sobolev spaces H^{-1}, H^0, and H^1 respectively. The H_h^0 norm, $\| \cdot \|_{0,h}$, is defined by

$$\| f_j \|_{0,h} = (\sum_{j \varepsilon \Lambda^h} f_j^2 h^2)^{1/2}$$

with the corresponding inner product

$$(f_j, g_j)_{0,h} = \sum_{j \varepsilon \Lambda^h} f_j g_j h^2 .$$

If the function has support contained in a bounded set Ω, we define $\Omega^h = \Omega \cap \Lambda^h$ and denote the $H_{0,h}$ norm of such a function also by $\| \cdot \|_{0,h}$, i.e., we have

$$\| f_j \|_{0,h} = (\sum_{jh \varepsilon \Omega^h} f_j^2 h^2)^{1/2}$$

(It should always be clear from context whether the norm is restricted to Ω^h.) The H_h^1 norm , $\| \cdot \|_{1,h}$, is defined by

$$\| f_j \|_{1,h}^2 = \| f_j \|_{0,h}^2 + \sum_{i=1}^{2} \| D_i^+ f_j \|_{0,h}^2$$

where D_i^+ denotes the forward divided difference operator on the set Λ^h in the ith direction. The H_h^{-1} norm is defined by

$$\| f_j \|_{-1,h} \leq \sup_{w_j \varepsilon H_h^1} \frac{| (f_j, w_j)_{0,h} |}{\| w_j \|_{1,h}}$$

i. e. H_h^{-1} is the dual to H_h^1 with respect to the H_h^0 inner product.

The following four lemmas concern the norms $\| \ \|_{-1,h}$, $\| \ \|_{0,h}$, and $\| \ \|_{1,h}$, and will be used in the proofs contained in sections 4 and 5.

Lemma 0.1: If $f_j \ \varepsilon \ H_h^0$ then

$$\max_j | f_j | \leq h^{-1} \| f_j \|_{0,h} .$$

Proof: We have

$$h^2 | f_j | \leq h^2 \sum_j | f_j |^2 = \| f_j \|_{0,h}^2$$

so

$$| f_j | \leq h^{-1} \| f_j \|_{0,h}$$

for all j.

Lemma 0.2 : If $f_j \varepsilon H_h^0$ then

$$\| f_j \|_{0,h} \leq C\, h^{-1} \| f_j \|_{-1,h}$$

where the constant does not depend on h or f.

Proof: We have

$$\begin{aligned} \| f_j \|_{1,h}^2 &= \| f_j \|_{0,h}^2 + \sum_{i=1}^{2} \| D_i^+ f_j \|_{0,h}^2 \\ &\leq \| f_j \|_{0,h}^2 + \frac{8}{h^2} \| f_j \|_{0,h}^2 \\ &\leq C' h^{-2} \| f_j \|_{0,h}^2 \end{aligned}$$

where C' does not depend on h . Therefore

$$\begin{aligned} \| f_j \|_{0,h} = \frac{| (f_j, f_j)_{0,h} |}{\| f_j \|_{0,h}} &\leq \frac{h^{-1} C' | (f_j, f_j)_{0,h} |}{h^{-1} C' \| f_j \|_{0,h}} \\ &\leq h^{-1} C' \frac{| (f_j, f_j)_{0,h} |}{\| f_j \|_{1,h}} \\ &\leq h^{-1} C' \sup_{w_j \varepsilon H_h^1} \frac{| (f_j, w_j) |}{\| w_j \|_{1,h}} \\ &= h^{-1} C' \| f_j \|_{-1,h} \ . \end{aligned}$$

Lemma 0.3 : If $f_j \varepsilon H_h^0$ and $g_j \varepsilon H_h^{-1}$ then we have

$$\| f_j g_j \|_{-1,h} \leq h^{-2} \| f_j \|_{0,h} \| g_j \|_{-1,h} \ .$$

Proof: If we use lemmas 0.1 and 0.2, then

$$\begin{aligned} \| f_j g_j \|_{-1,h} &\leq \| f_j g_j \|_{0,h} \\ &\leq \max_j | f_j | \, \| g_j \|_{0,h} \\ &\leq C h^{-2} \| f_j \|_{0,h} \| g_j \|_{-1,h} \ . \end{aligned}$$

Lemma 0.4 : If $f_j \varepsilon H_h^{-1}$, f_j has support in a compact set Ω, and g is a continuously differentiable function on Ω, then

$$\| f_j g_j \|_{-1,h} \le C \| f_j \|_{-1,h}$$

where $g_j = g(jh)$.

Proof: For any $w_j \varepsilon H_h^{-1}$

$$\| g_j w_j \|_{1,h}^2 = \| g_j w_j \|_{0,h}^2 + \sum_{i=1}^{2} \| D_i^+ g_j w_j \|_{0,h}^2$$

$$\le \max_{j \varepsilon \Omega^h} | g_j |^2 \| w_j \|_{0,h}^2 + (\sum_{i=1}^{2} \max_{j \varepsilon \Omega^h} | D_i^+ g_j |^2) \| w_j \|_{0,h}^2$$

$$+ (\sum_{i=1}^{2} \| D_i^+ w_j \|_{0,h}^2) \max_{j \varepsilon \Omega^h} | g_j |^2$$

$$\le \bar{C} \| w_j \|_{1,h}^2$$

where $\bar{C}$ is independent of w. Here we have used the fact that g is continuously differentiable so that

$$\max_{j \varepsilon \Omega^h} | D_i^+ g_j | \le C'$$

where C' is independent of h. Thus

$$\| f_j g_j \|_{-1,h} = \sup_{w_j \varepsilon H_h^1} \frac{| (f_j g_j, w_j)_{0,h} |}{\| w_j \|_{1,h}} = \frac{\bar{C}}{\bar{C}} \frac{| (f_j, g_j w_j)_{0,h} |}{\| w_j \|_{1,h}}$$

$$\le \frac{\bar{C} | (f_j, g_j w_j)_{0,h} |}{\| g_j w_j \|_{1,h}}$$

$$\le \bar{C} \| f_j \|_{-1,h} .$$

1. Equations of motion

In this section we present and prove the equivalence of two sets of equations which describe the motion of an ideal inviscid incompressible fluid with small density variations.

The equations of motion in two dimensions for an inviscid fluid are

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot grad\, \rho = 0 \tag{1.1}$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot grad\, \vec{u} = -\frac{grad\, P}{\rho} + \vec{F} \tag{1.2}$$

$$div\ \vec{u} = 0\ . \tag{1.3}$$

The initial conditions are

$$\rho(x,y,0) = \rho_0(x,y) \qquad \vec{u}(x,y,0) = \vec{u}_0(x,y)$$

Here ρ is the density, $\vec{u} = (u_1,u_2)$ is the velocity, P the pressure, and $\vec{F} = (F_1,F_2)$ is the external force. We assume that the force $\vec{F}$ is conservative, i.e. curl $\vec{F} = 0$. For a derivation see ([17],[38]).

If the variations in density are small we can make the Boussinesq approximation ([51],[53]) to (1.1) - (1.3). Consider the following steady solution to equations (1.1) - (1.3) :

$$\bar{u}(x,y,t) = 0 \tag{1.4}$$

$$\bar{\rho}(x,y,t) = \rho_c \tag{1.5}$$

and $\bar{P}$ is chosen so that

$$grad\, \bar{P} = \rho_c \vec{F} \tag{1.6}$$

where ρ_c is a constant.

Define ρ', P', and $\vec{u}'$ by

$$\rho' = \rho - \bar{\rho} \qquad P' = P - \bar{P} \qquad \vec{u}' = \vec{u} - \bar{u}\ .$$

We substitute these expressions into (1.1)- (1.3) and find, after some

simplification,

$$\frac{\partial \rho'}{\partial t} + \vec{u}' \cdot grad\, \rho' = 0 \tag{1.7}$$

$$(1 + \frac{\rho'}{\rho_c})(\frac{\partial \vec{u}'}{\partial t} + \vec{u}' \cdot grad\, \vec{u}') = \frac{grad\, P'}{\rho_c} + \frac{\rho'}{\rho_c}\vec{F} \tag{1.8}$$

$$div\ \vec{u}' = 0 \tag{1.9}$$

and, initially,

$$\rho'(x,y,0) = \rho'_0(x,y) \qquad \vec{u}'(x,y,0) = \vec{u}_0(x,y)\ .$$

For small variations in the density about the state ρ_c, $\rho' \ll \rho_c$, we make the Boussinesq approximation,

$$1 + \frac{\rho'}{\rho_c} \approx 1\ . \tag{1.10}$$

Thus (1.8) becomes

$$\frac{\partial \vec{u}'}{\partial t} + \vec{u}' \cdot grad\, \vec{u}' = \frac{grad\, P'}{\rho_c} + \frac{\rho'}{\rho_c}\vec{F}\ . \tag{1.11}$$

For the sake of simplicity we will drop the primes and refer to $\vec{u}'$, $\frac{\rho'}{\rho_c}$, and P' in (1.7)-(1.11) as the velocity, density and pressure respectively. In this notation, equations (1.7), (1.8), and (1.11) now become:

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot grad\, \rho = 0 \tag{1.12}$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot grad\, \vec{u} = \frac{grad\, P}{\rho_c} + \rho\vec{F} \tag{1.13}$$

$$div\ \vec{u} = 0 \tag{1.14}$$

$$\rho(x,y,0) = \rho_0(x,y) \qquad \vec{u}(x,y,0) = \vec{u}_0(x,y)\ .$$

(1.12)-(1.14) are the equations that we shall be using.

It is convenient for numerical work to put (1.12), (1.13), and (1.14) into a vorticity-stream form. Let ξ be the vorticity, $\xi = curl(\vec{u})$, and let Ψ be the stream function, then these equations can be written as

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot grad\, \rho = 0 \tag{1.15}$$

$$\frac{\partial \xi}{\partial t} + \vec{u} \cdot grad\, \xi = curl(\rho \vec{F}) \tag{1.16}$$

$$\Delta \Psi = -\xi \qquad u_1 = \Psi_y \qquad u_2 = -\Psi_x \tag{1.17}$$

where Δ is the Laplace operator. If we use the fact that $G = \frac{1}{2\pi}\log(r)$ is the Green's function for Δ, and $r = (x^2 + y^2)^{\frac{1}{2}}$, we find

$$\Psi = G * \xi$$

where $*$ represents convolution. Thus, if we use (1.17) we obtain the following expression for the velocity in terms of the vorticity:

$$u = \frac{\partial G}{\partial y} * \xi \qquad v = -\frac{\partial G}{\partial x} * \xi$$

or

$$\vec{u} = K * \xi \tag{1.18}$$

where

$$K = \begin{pmatrix} \frac{y}{2\pi r^2} \\ \frac{-x}{2\pi r^2} \end{pmatrix}.$$

(1.18) is the " Biot-Savart " law.

We mention that the pressure P does not occur in equations (1.15)-(1.18). This is a result of using the Boussinesq approximation to (1.7)-(1.9). The elimination of the pressure from the equations greatly simplifies our computational task and is our primary motivation for using the Boussinesq approximation.

Our first numerical method, numerical method A, will be an approximation scheme based on equations (1.15),(1.16) and (1.18). Our second numerical method, method B, will be based on an alternative formulation of (1.15)-(1.18). If we make the assumption that the support of the initial

density distribution is contained in a bounded set, then we can express (1.15)-(1.18) in the following equivalent form:

$$\frac{\partial \rho_x}{\partial t} + \vec{u} \cdot grad\, \rho_x = -u_{1x}\rho_x - u_{2x}\rho_y \tag{1.19}$$

$$\frac{\partial \rho_y}{\partial t} + \vec{u} \cdot grad\, \rho_y = -u_{1y}\rho_x - u_{2y}\rho_y \tag{1.20}$$

$$\rho = G_x * \rho_x + G_y * \rho_y \tag{1.21}$$

$$\frac{\partial \xi}{\partial t} + \vec{u} \cdot grad\, \xi = curl(\rho \vec{F}) \tag{1.22}$$

$$\vec{u} = K * \xi \tag{1.23}$$

$$\rho_x(x,y,0) = \rho_{0x}(x,y) \qquad \rho_y(x,y,0) = \rho_{0y}(x,y) \qquad \xi(x,y,0) = \xi_0(x,y) \ .$$

Here ρ_x and ρ_y are the derivatives of the density and $G = \frac{1}{2\pi}\log(\vec{r})$, $\vec{r} = (x^2+y^2)^{\frac{1}{2}}$. Equation (1.19) is obtained by differentiating equation (1.15) with respect to x and using the condition that $div\ \vec{u} = 0$. Equation (1.20) is obtained similarly. Equation (1.21) is derived from Poisson's formula [36] and integration by parts,

$$\rho = G * \Delta\rho = G_x * \rho_x + G_y * \rho_y \ .$$

We use (1.21) to reconstruct the density from its gradients because it has an easily implementable numerical analogue. Numerical method B is based on (1.19)-(1.23).

We mention that the assumption on the support of ρ occurring in (1.15) translates into the assumption that the density occurring in (1.1) is uniform outside some bounded set. (We are describing the evolution of the density variation in (1.15) and not the evolution of the density.) Also, this assumption on ρ simplifies the proof that (1.15)-(1.18) are equivalent to (1.19)-(1.23). The equivalence of these equations is most likely true under less restrictive assumptions, but such a result is not needed for our work, so we do

not pursue the matter.

To show the equivalence of (1.19)-(1.23) to (1.15)-(1.18) , it is sufficient to show that (1.19)-(1.21) are equivalent to (1.15). If $\vec{u}$ and ρ are sufficiently smooth, and $div\ \vec{u} = 0$, then it is easily seen that any solution of (1.15) is also a solution of (1.19)-(1.21). The fact that a sufficiently smooth solution of (1.19)- (1.21) is a solution of (1.15) is the content of the following theorem.

Theorem 1.1

Assume that for any time t, $0 \le t < T$, $\vec{u}$ and its derivatives are continuous and bounded for $x \in R^2$ and $div\ \vec{u} = 0$. Also assume $\bar{\rho}_x$ and $\bar{\rho}_y$ have compact support and are twice continuously differentiable (in both $\vec{x}$ and t) solutions of

$$\frac{\partial \bar{\rho}_x}{\partial t} + \vec{u} \cdot grad\, \bar{\rho}_x = -u_{1x}\bar{\rho}_x - u_{2x}\bar{\rho}_y \tag{1.24}$$

$$\frac{\partial \bar{\rho}_y}{\partial t} + \vec{u} \cdot grad\, \bar{\rho}_y = -u_{1y}\bar{\rho}_x - u_{2y}\bar{\rho}_y \tag{1.25}$$

$$\bar{\rho}_x(x,y,0) = \rho_{0_x}(x,y) \tag{1.26}$$

$$\bar{\rho}_y(x,y,0) = \rho_{0_y}(x,y) \ . \tag{1.27}$$

If we define ρ by

$$\rho = G_x * \bar{\rho}_x + G_y * \bar{\rho}_y \tag{1.28}$$

where G is the fundamental solution of Laplaces equation, then

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot grad\, \rho = 0$$

and

$$\frac{\partial \rho}{\partial x} = \bar{\rho}_x \qquad \frac{\partial \rho}{\partial y} = \bar{\rho}_y \ .$$

In order to prove this result, we need the following lemma:

Lemma 1.1: Under the hypothesis of Theorem 1.1 we have

$$(\bar{\rho}_x)_y = (\bar{\rho}_y)_x$$

Proof: We derive a differential equation for the quantity $\tilde{\omega} = (\bar{\rho}_x)_y - (\bar{\rho}_y)_x$. We differentiate equation (1.24) with respect to y and find

$$\frac{\partial(\bar{\rho}_x)_y}{\partial t} = -\Big[u_{1y}(\bar{\rho}_x)_x + u_{2y}(\bar{\rho}_x)_y + u_{1xy}\bar{\rho}_x + u_{2xy}\bar{\rho}_y$$

$$+ u_1(\bar{\rho}_x)_{xy} + u_2(\bar{\rho}_x)_{yy} + u_{1x}(\bar{\rho}_x)_y + u_{2x}(\bar{\rho}_y)_y\Big] . \tag{1.29}$$

Similarly we differentiate (1.25) with respect to x and find

$$\frac{\partial(\bar{\rho}_y)_x}{\partial t} = -\Big[u_{1x}(\bar{\rho}_y)_x + u_{2x}(\bar{\rho}_y)_y + u_{1yx}\bar{\rho}_x + u_{2yx}\bar{\rho}_y$$

$$+u_1(\bar{\rho}_y)_{xx} + u_2(\bar{\rho}_y)_{yx} + u_{1y}(\bar{\rho}_x)_x + u_{2y}(\bar{\rho}_y)_x\Big] . \tag{1.30}$$

We subtract (1.30) from (1.29), and using the assumptions that $div\ \vec{u} = 0$ and equality of cross partials, we find that

$$\frac{\partial\tilde{\omega}}{\partial t} = -u_1\Big[(\bar{\rho}_x)_{xy} - (\bar{\rho}_y)_{xx}\Big] - u_2\Big[(\bar{\rho}_y)_{yy} - (\bar{\rho}_y)_{yx}\Big]$$

$$= -u_1\Big[(\bar{\rho}_x)_y - (\bar{\rho}_y)_x\Big]_x - u_2\Big[(\bar{\rho}_x)_y - (\bar{\rho}_y)_x\Big]_y$$

$$= -\vec{u}\cdot grad\,\tilde{\omega} .$$

Therefore $\tilde{\omega}$ satisfies

$$\frac{\partial\tilde{\omega}}{\partial t} + \vec{u}\cdot grad\,\tilde{\omega} = 0 \tag{1.31}$$

with initial data

$$\tilde{\omega}(x,y,0) = \rho_{0_{xy}} - \rho_{0_{yx}} = 0.$$

Using the fact that $\tilde{\omega}$ is a C^1 solution of (1.31) with vanishing initial data, one can show, using energy estimates [36,pg. 140], that $\tilde{\omega} = 0$ for all t, $0 \le t < T$.

This completes the proof. We now prove Theorem 1.

Proof: Consider $\mathbf{u}\cdot grad\,\rho$, with ρ given by (1.28),

$$\mathbf{u}\cdot grad\,\rho = u_1\left[G_x * \bar{\rho}_x + G_y * \bar{\rho}_y\right]_x + u_2\left[G_x * \bar{\rho}_x + G_y * \bar{\rho}_y\right]_y$$

$$= u_1\left[G * (\bar{\rho}_x)_{xx} + G_y * (\bar{\rho}_y)_x\right] + u_2\left[G_x * (\bar{\rho}_x)_y + G * (\bar{\rho}_y)_{yy}\right] \quad (1.32)$$

If we use Lemma 1.1 and integration by parts, (1.32) can be written as

$$= u_1\left[G * (\bar{\rho}_x)_{xx} + G * (\bar{\rho}_x)_{yy}\right] + u_2\left[G * (\bar{\rho}_y)_{xx} + G * (\bar{\rho}_y)_{yy}\right]$$

$$= u_1\bar{\rho}_x + u_2\bar{\rho}_y \quad . \quad (1.33)$$

Here we have used Poissons formula for this last simplification. We use Poisson's formula once again and rewrite (1.33) as

$$u_1\bar{\rho}_x + u_2\bar{\rho}_y = G_x * \left[u_1\bar{\rho}_x + u_2\bar{\rho}_y\right]_x + G_y * \left[u_1\bar{\rho}_x + u_2\bar{\rho}_y\right]_y$$

$$= G_x * \left[\mathbf{u}\cdot grad\,\bar{\rho}_x + u_{1x}\bar{\rho}_x + u_{2x}\bar{\rho}_y\right] +$$

$$G_y * \left[\mathbf{u}\cdot grad\,\bar{\rho}_y + u_{1y}\bar{\rho}_x + u_{2y}\bar{\rho}_y\right] .$$

If we use (1.24) and (1.25) this expression can be further simplified to

$$G_x * \frac{\partial\bar{\rho}_x}{\partial t} + G_y * \frac{\partial\bar{\rho}_y}{\partial t}$$

therefore

$$\mathbf{u}\cdot grad\,\rho = G_x * \frac{\partial\bar{\rho}_x}{\partial t} + G_y * \frac{\partial\bar{\rho}_y}{\partial t} .$$

So,

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}\left(G_x * \bar{\rho}_x\right) + \frac{\partial}{\partial t}\left(G_y * \bar{\rho}_y\right)$$

$$= G_x * \frac{\partial\bar{\rho}_x}{\partial t} + G_y * \frac{\partial\bar{\rho}_y}{\partial t}$$

$$= \mathbf{u}\cdot grad\,\rho \quad .$$

This proves the first part of the theorem. For the second part,

$$\frac{\partial \rho}{\partial x} = (G_x * \bar{\rho}_x + G_y * \bar{\rho}_y)_x$$

$$= G * (\bar{\rho}_x)_{xx} + G * (\bar{\rho}_y)_{xy} \quad .$$

If we use Lemma 1.1 and Poisson's formula, then we find

$$= G * (\bar{\rho}_x)_{xx} + G * (\bar{\rho}_x)_{yy}$$

$$= \bar{\rho}_x \quad .$$

In a similar manner one can show that $\frac{\partial \rho}{\partial y} = \bar{\rho}_y$. This completes the proof.

2. Model problem

As mentioned in the introduction, we abstract from the vortex method [13] and the analysis of the vortex method ([5],[6],[19],[30],[31]) a general numerical technique. We will apply this technique to construct and analyze numerical schemes for (1.15)-(1.18) and (1.19)-(1.23). Before we do this however, we illustrate the technique on a simpler, model, problem. We consider the problem of calculating the motion of a quantity being transported by an incompressible flow. The method we present may not be of any practical value, but our description and analysis of it will serve to reveal the key features of the general technique. Some of the theoretical results will be used in later sections.

Let $\vec{u}(x,t) = (u_1(x,t),u_2(x,t))$ be a given velocity field defined on R^2 such that $div\ \vec{u} = 0$. Let $f(x,t)$ be some quantity passively transported by the velocity field $\vec{u}$, i.e. f satisfies the partial differential equation

$$\frac{\partial f}{\partial t} + \vec{u}\cdot\nabla f = 0 \tag{2.1}$$

$$f(x,0) = f_0(x) \ . \tag{2.2}$$

We assume that the support of $f_0 \subseteq \Omega$, Ω a bounded set in R^2. The problem is to calculate an approximation to $f(x,t)$ for times $t \geq 0$.

Our starting point is to write (2.1) and (2.2) in Lagrangian form as

$$\frac{dx(\alpha,t)}{dt} = \vec{u}(x(\alpha,t)) \qquad x(\alpha,0) = \alpha \tag{2.3}$$

$$\frac{df(\alpha,t)}{dt} = 0 \qquad f(\alpha,0) = f_0(\alpha) \tag{2.4}$$

where $\alpha = (\alpha_1,\alpha_2) \in R^2$. The solution to (2.3), $x(\alpha,t)$, is the trajectory of a fluid particle which at time t=0 is located at the point α. Equation (2.4) describes the evolution of f along the particle trajectory $x(\alpha,t)$. In particular, (2.4) expresses the fact that the quantity f does not change along particle paths. For a discussion of the equivalence of (2.1) and (2.2) with (2.3)

and (2.4) , see [20, pp. 139-145].

We approximate the solutions to (2.1) and (2.2) by constructing discrete approximations to equations (2.3) and (2.4). Define the intersection of a grid of uniform mesh width h with Ω as, $\Omega^h = \Omega \cap \Lambda^h$, where Λ^h is defined as in section 0. The discrete approximation to (2.3) and (2.4) is obtained by computing the solution to the following set of O.D.E.s :

$$\frac{dx(jh,t)}{dt} = \vec{u}(x(jh,t)) \qquad x(jh,0) = jh \tag{2.5}$$

$$\frac{df(jh,t)}{dt} = 0 \qquad f(jh,0) = f_0(jh) \tag{2.6}$$

for all $jh \in \Omega^h$.

Thus the computed approximation to $f(x,t)$ for $t \geq 0$ consists of the values of the function f at the set of points $\{x(jh,t) \mid jh \in \Omega^h\}$.

What is attractive about such a scheme is that the values of the function f at the points $x(jh,t)$ are not smoothed by this process. It is for this reason that we are justified in calling the method non-diffusive. Note that we are computing the exact solution at the points $x(jh,t)$, thus the method is similar to the Random Choice Method ([14],[29]) where a solution is constructed as a superposition of locally exact solutions. We expect that the most prominent feature of the Random Choice Method, the ability to compute accurately the evolution of sharp fluid discontinuities, will also be a feature of this method. This property is not shared by conventional finite difference schemes for solving (2.1) and (2.2).

One unattractive feature of such a scheme is the difficulty approximating the function $f(x,t)$ at points other than the particle trajectories, $\{x(jh,t) \mid jh \in \Omega^h\}$, or approximating differential and integral operators applied to f. This difficulty arises because the points at which the approximate solution is computed, $x(jh,t)$, are not necessarily distributed uni-

formly in space. Interpolation or differentiation formulas tend to be computationally unstable or of low accuracy. An approximation procedure that has proven successful in overcoming this difficulty is that which is implicitly used in the vortex method ([5],[13]). We now describe this approximation scheme.

Consider a function Ψ in $M^{L,p}$, where $M^{L,p}$ is defined as in section 0. Let $\Psi_\delta = \frac{1}{\delta^2}\Psi(\frac{x}{\delta})$ where δ is a parameter to be specified later. We take as our approximation to f

$$f^h(x,t) = \sum_{ih \in \Omega^h} \Psi_\delta(x - x(ih,t))f(ih,t)h^2 \ . \tag{2.7}$$

In the vortex method this type of approximation is used to approximate the vorticity.(Although in some versions of the vortex method, the function Ψ is not necessarily chosen to be in $M^{L,p}$.) An interpretation of (2.7), due to Chorin [13], is that we are approximating $f(x,t)$ by a sum of "blobs" $\Psi_\delta(x)$ with strength $f(jh,t)h^2$ located at the points $x(jh,t)$.

An estimate of the accuracy, or consistency, of the approximation scheme (2.7) is implicit in the work of Beale and Majda ([5],[6]) or Cottet [19]. The following result is a minor modification of the consistency lemma contained in [19].

Lemma 2.1 : Consistency of the approximation

Assume the velocity field u and the function $f(x,t)$ satisfying (2.1) are sufficiently smooth for $0 \le t < T$. If f_0 has support contained in Ω, Ω a bounded set on R^2 and $\Psi(x)$ is in $M^{L,p}$ with $L > 3$ and $p > 0$, then

$$\max_{0 \le t \le T} | f^h(x,t) - f(x,t) | \le C\delta^p + C(\frac{h}{\delta})^L \tag{2.8}$$

for all $x \in R^2$. The constants depend on T,L,p,Ω, and bounds on a finite

number of derivatives of u and f.

In order to prove lemma 2.1 we need the following result which describes the error in using the trapezoidal rule for approximating integrals of functions over R^2. The proof of the following lemma is a result from [3].

Lemma 2.2 Given a function g such that g has compact support, $g \in C^2(R^2)$ and $D^\beta g \in L^1$ for $|\beta| \le m$ with $m \ge 3$, then

$$\left| h^2 \sum_{k \in Z\times Z} g(kh) - \int_{R^2} g(x)dx \right| \le Ch^m \max \left\{ \| D^m_{x_1} g \|_{L^1(R^2)} , \| D^m_{x_2} g \|_{L^1(R^2)} \right\} . \tag{2.9}$$

Proof : We use the Poisson summation formula [22] and express the sum in (2.9) as

$$h^2 \sum_{k \in Z\times Z} g(kh) = \sum_{k \in Z\times Z} \hat{g}(2\pi \frac{k}{h}) . \tag{2.10}$$

Now,

$$\hat{g}(0) = \int_{R^2} g(x)dx$$

so

$$h^2 \sum_{k \in Z\times Z} g(kh) - \int_{R^2} g(x)dx = \sum_{\substack{k \in Z\times Z \\ k \ne 0}} \hat{g}(2\pi \frac{k}{h}) . \tag{2.11}$$

For any function $f \in L^1$,

$$\max_{\omega \in R^2} |\hat{f}(\omega)| \le \int_{R^2} |f| \, dx .$$

So,

$$\max_{\omega \in R^2} |(2\pi i)^m \omega_1^m \hat{g}(\omega_1, \omega_2)| \le \int_{R^2} |D^m_{x_1} g| \, dx$$

which implies

$$|\hat{g}(\omega_1, \omega_2)| \le \frac{1}{(2\pi)^m |\omega_1|^m} \int_{R^2} |D^m_{x_1} g| \, dx .$$

Similarly,

$$|\hat{g}(\omega_1,\omega_2)| \le \frac{1}{(2\pi)^m |\omega_2|^m} \int_{R^2} |D^m_{x_2} g| \, dx \ .$$

Let

$$\overline{C} = \max \left\{ \frac{1}{(2\pi)^m} \int |D^m_{x_1} g| \, dx, \quad \frac{1}{(2\pi)^m} \int |D^m_{x_2} g| \, dx \right\}$$

then

$$|\hat{g}(\omega_1,\omega_2)| \le \frac{\overline{C}}{|\omega_1|^m} \quad ; \quad |\hat{g}(\omega_1,\omega_2)| \le \frac{\overline{C}}{|\omega_2|^m} \tag{2.12}$$

for all $(\omega_1,\omega_2) \varepsilon R^2$. Define $\|(\omega_1, \omega_2)\|_{max}$ by

$$\|(\omega_1,\omega_2)\|_{max} = \max\{|\omega_1|, |\omega_2|\}$$

Then (2.12) implies that

$$|\hat{g}(\omega_1,\omega_2)| \le \frac{\overline{C}}{\|(\omega_1,\omega_2)\|^m_{max}}$$

so that

$$\|\hat{g}(\frac{2\pi}{h}(k_1,k_2))\| \le \frac{\overline{C}}{\|\frac{2\pi}{h}(k_1,k_2)\|^m_{max}}$$

$$\le \frac{h^m}{2\pi^m} \frac{1}{\|(k_1,k_2)\|^m_{max}} \ .$$

Now, using the fact that for $m \ge 3$

$$\sum_{k \ne 0} \frac{1}{\|(k_1,k_2)\|^m_{max}} = 8 \sum_{j=1}^{\infty} \frac{1}{j^{m-1}}$$

(see [3]) we have that

$$|\sum_{k \ne 0} \hat{g}(2\pi \frac{k}{h})| \le \frac{h^m \, 8 \, \overline{C}}{(2\pi)^m} (\sum_{j=1}^{\infty} \frac{1}{j^{m-1}}) \ . \tag{2.13}$$

If we combine (2.11) and (2.13) we obtain the desired estimate. We now prove lemma 2.1.

Proof: Our goal is to estimate

$$f^h(x,t) - f(x,t) = \sum_{ih} \Psi_\delta(x - x(ih,t)) f(ih,t) h^2 - f(x,t) \quad . \tag{2.14}$$

Consider t fixed, $0 \le t \le T$. We write (2.14) as the sum of two terms

$$f^h(x,t) - f(x,t) = \left\{ \sum_{ih} \Psi_\delta(x - x(ih,t)) f(ih,t) h^2 - \int \Psi_\delta(x - x') f(x',t) dx' \right\}$$

$$+ \left\{ \int \Psi_\delta(x - x') f(x',t) dx' - f(x,t) \right\}$$

$$= \{A\} + \{B\}. \quad .$$

To estimate term A, we first change variables in the integral using the flow map $x(\alpha,t)$. Since the flow is incompressible, $|Jx(\alpha,t)| \equiv 1$, where J is the jacobian. Also we have $f(ih,t) = f(x(ih,t),t)$ so that term A can be written as

$$\sum_{ih} \Psi_\delta(x - x(ih,t)) f(x(ih,t),t) h^2 - \int \Psi_\delta(x - x(\alpha,t)) f(x(\alpha,t),t)) d\alpha \quad .$$

Thus we recognize term A as the error in using the trapezoidal rule for approximating the integral with respect to α of the function

$$g(x,\alpha,t) = \Psi_\delta(x - x(\alpha,t)) f(x(\alpha,t)\ ,\ t) \quad .$$

Under the assumptions on Ψ and f, we can apply lemma 2.2, and we find

$$|A| \le Ch^L \max \{ \| D^L_{\alpha_1} g(x,\alpha,t) \|_{L^1} ,\ \| D^L_{\alpha_2} g(x,\alpha,t) \|_{L^1} \} \quad . \tag{2.15}$$

If we use Leibnitz's rule we find,

$$D^L_{\alpha_i} g(x,\alpha,t) = \sum_{\beta_1 + \beta_2 = L} C_{\beta_1,\beta_2} D^{\beta_1}_{\alpha_i} \Psi_\delta(x - x(\alpha,t)) D^{\beta_2}_{\alpha_i} f(x(\alpha,t))$$

where C_{β_1,β_2} is a constant depending only on β_1 and β_2 and not on f. The function f has support in Ω and is smooth so we can estimate this by,

$$\| D^L_{\alpha_i} g(x,\alpha,t) \|_{L^1} = \| \sum_{\beta_1 + \beta_2 = L} C_{\beta_1,\beta_2} D^{\beta_1}_{\alpha_i} \Psi_\delta(x - x(\alpha,t)) D^{\beta_2}_{\alpha_i} f(x(\alpha,t)) \|_{L^1(\Omega)}$$

$$\leq C \sum_{0 \leq |\beta| \leq L} \| D^{\beta}_{\alpha_i} \Psi_\delta(x - x(\alpha,t)) \|_{L^1(\Omega)} \cdot \tag{2.16}$$

Since the flow is smooth and incompressible, we estimate each term in (2.16) by

$$\| D^{\beta}_{\alpha_i} \Psi_\delta(x - x(\alpha,t)) \|_{L^1(\Omega)} \leq C \sum_{|\gamma| \leq \beta} \| D^{\gamma} \Psi_\delta(x - x') \|_{L^1(\overline{\Omega})} \cdot \tag{2.17}$$

Here $\overline{\Omega}$ is a set that contains the image of Ω under the flow. Thus we must find L^1 estimates on the derivatives of the function Ψ_δ. We have the following pointwise estimates,

(i) $\quad | D^{\beta} \Psi_\delta(x) | \leq \dfrac{C}{\delta^{\beta+2}} \qquad$ *for all* $x \in R^2$

(ii) $\quad | D^{\beta} \Psi_\delta(x) | \leq \dfrac{C}{|x|^{2+\beta}} \qquad$ *for all* $x, \quad |x| > \delta$.

For (i), since Ψ is rapidly decreasing,

$$\begin{aligned} | D^{\beta} \Psi_\delta(x) | &= \frac{1}{\delta^{2+\beta}} | D^{\beta} \Psi(\frac{x}{\delta}) | \\ &\leq \frac{C}{\delta^{\beta+2}} . \end{aligned}$$

For (ii), again using the fact that Ψ is rapidly decreasing,

$$\begin{aligned} | D^{\beta} \Psi_\delta(x) | &= \frac{1}{\delta^{2+\beta}} | D^{\beta} \Psi(\frac{x}{\delta}) | \\ &\leq \frac{1}{\delta^{2+\beta}} \frac{C_{\beta,k}}{(1 + |\frac{x}{\delta}|^2)^k} \end{aligned}$$

Choosing $k > \dfrac{\beta+2}{2}$ and noting that with $|x| > \delta$ we have

$$\delta^{2+\beta}(1 + |\frac{x}{\beta}|^2)^k \geq |x|^{2+\beta}$$

it follows that

$$\frac{C_{\beta,k}}{\delta^{2+\beta}(1 + |\frac{x}{\delta}|^2)^k} \leq \frac{C}{|x|^{2+\beta}}$$

This proves (ii). Now,

$$\int_{\Omega} | D^{\beta}\Psi_{\delta}(x - x') | \, dx' = \int_{\{|x - x'| < \delta\} \cap \Omega} | D^{\beta}\Psi_{\delta}(x - x') | \, dx'$$

$$+ \int_{\{|x - x'| \geq \delta\} \cap \Omega} | D^{\beta}\Psi_{\delta}(x - x') | \, dx' \quad . \tag{2.18}$$

We use (i) to estimate the first term of (2.18) and (ii) to estimate the second term, thus

$$\int_{\Omega} | D^{\beta}\Psi_{\delta}(x - x') | \, dx' \leq \frac{C}{\delta^{\beta + 2}}(2\pi\delta^2) + C \int_{\delta}^{diameter\,(\Omega) + \delta} \frac{1}{|x|^{1+\beta}} dx$$

$$\leq \frac{C}{\delta^{\beta}} \, .$$

If we use these bounds for the terms in the sum (2.17), and use (2.15) and (2.16), then we find,

$$|A| \leq C(\frac{h}{\delta})^{L} \quad . \tag{2.19}$$

By choosing the constants large enough, the estimate (2.19) can be shown to be hold for any t, $0 \leq t \leq T$, and any $x \in R^2$, thus

$$\max_{0 \leq t \leq T} |A| \leq C(\frac{h}{\delta})^{L} \quad . \tag{2.20}$$

To estimate term B, we use the fact that a function is bounded by the L^1 norm of its Fourier transform, so

$$\max_{x \in R^2} | B | = \max_{x \in R^2} | f - f * \Psi_{\delta} |$$

$$\leq \int | \hat{f}(\omega) - \hat{f}(\omega) \cdot \hat{\Psi}_{\delta}(\omega) | \, d\omega$$

$$= \int | \hat{f}(\omega) | \; | 1 - \hat{\Psi}(\omega\delta) | \, d\omega \quad . \tag{2.21}$$

Our assumptions on Ψ imply the following:

(*i*) $\hat{\Psi}(0) = \int \Psi(x) dx = 1$

(*ii*) $D_{\omega}^{\beta}\hat{\Psi}(0) = C\int x^{\beta}\Psi(x) dx = 0 \quad \text{for} \quad 1 \leq |\beta| \leq p - 1$

$$(iii)\ \max_{\omega \in R^2} |D_\omega^\alpha \hat{\Psi}(\omega)| \le C \qquad \text{for } |\alpha| = p\ .$$

We use Taylor's theorem with remainder and properties (i), (ii), (iii) of Ψ to estimate

$$|1 - \hat{\Psi}(\delta\omega)|$$

by

$$\begin{aligned} |1 - \hat{\Psi}(\delta\omega)| &= |\hat{\Psi}(0) - \hat{\Psi}(\delta\omega)| \\ &\le \Big| \sum_{0 \le \alpha \le |p|-1} \frac{1}{\alpha!} D^\alpha \hat{\Psi}(0)(\delta\omega)^\alpha \Big| \\ &\qquad + \delta^p \sum_{|\alpha| = p} \sup_{0 \le \vartheta \le 1} |D_\omega^\alpha \hat{\Psi}(\delta\vartheta\omega)|\ |(\omega^\alpha)| \\ &\le C\delta^p\ |\omega|^p \end{aligned}$$

Therefore, assuming that f is smooth so that its Fourier transform decays sufficiently fast, the right hand side of (2.21) can be estimated by

$$\int |\hat{f}(\omega)|\ \delta^p\ |\omega|^p\ d\omega \le C\delta^p\ . \tag{2.22}$$

Again, if we choose the constants large enough, this estimate can be shown to hold for all t, $0 \le t \le T$. If we combine the estimate (2.20) for term A and (2.22) for term B, then we find

$$\max_{0 \le t \le T} \Big| \sum_{ih} \Psi_\delta(x - x(ih,t)) f(ih,t) h^2 - f(x,t) \Big| \le C\,\delta^p + C\left(\frac{h}{\delta}\right)^L . \tag{2.23}$$

This completes the proof.

The pointwise estimate (2.23) implies the following L^2 estimate,

Corollary: Under the assumptions of lemma 2.1, for any $\bar{R}$, $\bar{R}$ a bounded set,

$$\max_{0 \le t \le T} \| f^h(x,t) - f(x,t) \|_{L^2(\bar{R})} \le C\,\delta^p + C\left(\frac{h}{\delta}\right)^L . \tag{2.24}$$

From the second term in the error estimate (2.8) we see that it is a good idea to choose the parameter $\delta > h$. If we let $\delta = h^q$ for $q < 1$ then for

h sufficiently small as $h \to 0$ the error decreases.

In deriving this estimate the trajectories $x(jh,t)$ are assumed to be exact solutions to (2.5). During the course of an actual computation $x(jh,t)$ will not be computed exactly, and it is of interest to estimate the behavior of the approximation (2.7) under perturbations of the trajectories $x(jh,t)$. Such an estimate is a stability estimate for the approximation scheme (2.7). We prove the following result:

Lemma 2.3 : Stability of the approximation

Under the assumptions for the consistency lemma and the assumption that for $0 \le t \le T$,

$$\| \tilde{x}(jh,t) - x(jh,t) \|_{0,h} \le \delta h$$

where $\tilde{x}(jh,t)$ are computed trajectories, then for any bounded set $\overline{R}$,

$$\| \sum_{ih \in \Omega^h} \Psi_\delta(x - \tilde{x}(ih,t)) f(ih,t) h^2 - \sum_{ih \in \Omega^h} \Psi_\delta(x - x(ih,t)) f(ih,t) h^2 \|_{L^2(\overline{R})} \le$$
$$C\delta^{-1} \| \tilde{x}(ih,t) - x(ih,t) \|_{0,h} \quad . \tag{2.25}$$

The constants depend on T,L,p, Ω, and bounds on a finite number of derivatives of $\vec{u}$ and f.

The proof of this lemma follows closely the proof of the stability of the velocity approximation in the vortex method given in ([5],[6],[30]). To prove the stability lemma, we need the following discrete L^1 estimates for $D^\beta \Psi_\delta$, β being a multi-index,

Lemma 2.4 : With t fixed , $x_j = x(jh,t)$, we have

$$\sum_{jh \le R_0} \max_{|y_j| \le C_0\delta} | D^\beta \Psi_\delta (x - x_j + y_j) | h^2 \le \frac{C_\beta}{\delta^{|\beta|}} \qquad |\beta| \ge 1 \tag{2.26}$$

for all $x \in R^2$ and R_0 a bounded set in R^2, provided h is sufficiently small. The constants C_β depend on C_0, the derivatives of the flow and the derivatives of

the function Ψ.

Proof: We have the following two bounds on the derivatives of Ψ_δ .

$$\text{(i)} \quad |D^\beta \Psi_\delta(x)| \le \frac{C}{\delta^{2+|\beta|}} \quad \text{all } x \in R^2$$

$$\text{(ii)} \quad |D^\beta \Psi_\delta(x)| \le \frac{C}{|x|^{2+|\beta|}} \quad |x| > \delta \, , \, x \in R^2 \, .$$

For a proof of these bounds, see lemma 2.2. Let j be a multi-index, $j = (j_1, j_2)$, define

$$Q_j = [j_1 - \tfrac{1}{2}h, j_1 + \tfrac{1}{2}h] \times [j_2 - \tfrac{1}{2}h, j_2 + \tfrac{1}{2}h]$$

and

$$B_j = \{ x \in R^2 \mid x = x(\alpha, t), \alpha \in Q_j \}, \tag{2.27}$$

i. e. B_j is the image under the flow of the rectangle Q_j. Since the flow is incompressible, the area of B_j is h^2. Also, since we are assuming that the flow is sufficiently smooth, then it follows that the diameter of B_j is uniformly of order h, so that, as $h < \delta$

$$|x' - x_j| \le C_1 \delta \qquad x' \in B_j$$

for some C_1 . We assume that $C_0 > C_1$. Fix an $x \in R^2$ and let

$$J_1 = \{ j \in Z \times Z \mid jh \in R_0 \;\; |x_j - x| \le (3C_1 + 1)\delta \}$$
$$J_2 = \{ j \in Z \times Z \mid jh \in R_0 \;\; |x_j - x| > (3C_1 + 1)\delta \} \, .$$

Then

$$\cup \{ B_j \mid j \in J_1 \} \subseteq \{ |x' - x| \le (4C_1 + 1)\delta \}. \, .$$

We use the pointwise bound (i) to estimate the contribution to the sum (2.26) from points in J_1.

$$\sum_{j \in J_1} \max |D^\beta \Psi_\delta(x - x_j + y_j)| \, h^2 \le \pi[(4C_1 + 1)\delta]^2 \frac{C}{\delta^{2+|\beta|}} \le \frac{C}{\delta^{|\beta|}} \tag{2.28}$$

For the remaining part, we use the fact that the area of B_j is h^2 so that for a

fixed $x \in R^2$ we have

$$\sum_{j \in J_2} \max_{|y_j| \le C_0\delta} |D^\beta \Psi_\delta(x - x_j + y_j)| h^2 = \int_{\bigcup_{j \in J_2} B_j} G(x')dx' \tag{2.29}$$

where we are considering $G(x')$ as a step function on B_j defined by,

$$G(x') = \max_{|y_j| \le C_0\delta} |D^\beta \Psi_\delta(x - x_j + y_j)| \qquad \text{for } x' \in B_j \ .$$

If $j \in J_2$, then

$$|x - x_j + y_j| \ge (2C_1 + 1)\delta > \delta$$

so that the pointwise bound (ii) implies that

$$\max_{|y_j| \le C_0\delta} |D^\beta \Psi_\delta(x - x_j + y_j)| \le C(x - x_j + y_j)^{-|\beta|-2} \ .$$

For $x' \in B_j$

$$|x - x_j + y_j| \ge |x - x'| - 2C_1\delta$$

which implies

$$(x - x_j + y_j)^{-|\beta|-2} \le (|x - x'| - 2C_1\delta)^{-|\beta|-2} \ .$$

Therefore,

$$|G(x')| \le C(|x - x'| - 2C_1\delta)^{-|\beta|-2} \qquad \text{for } x' \in B_j \ . \tag{2.30}$$

Also, for $x' \in B_j$,

$$\begin{aligned} |x - x'| &\ge |x - x_j| - |x' - x_j| \\ &\ge (1 + 2C_1)\delta \end{aligned}$$

so that

$$\bigcup \{B_j \mid j \in J_2\} \subseteq \{x' \in R^2 \mid (1 + 2C_1)\delta \le |x - x'| \le \infty\} \ . \tag{2.31}$$

If we use (2.30) and (2.31), then the integral in (2.29) can be estimated by

$$\int_{\bigcup_{j \in J_2} B_j} |G(x')|\,dx' \le C \int_{\bigcup_{j \in J_2} B_j} (|x - x'| - 2C_1\delta)^{-|\beta|-2} dx'$$

$$\le C \int_{(1+2C_1)\delta \le |x - x'| \le \infty} (|x - x'| - 2C_1\delta)^{-|\beta|-2} dx' \ .$$

If we change to polar coordinates and let $r = |x - x'|$, then this latter integral becomes

$$C \int_{(1+2C_1)\delta}^{\infty} (r - 2C_1\delta)^{-|\beta|-2} r\, dr \ . \tag{2.32}$$

If we use the change of variables $\tilde{r} = r - 2C_1\delta$ and note that $\tilde{r} + 2C_1\delta \ge (1 + 2C_1)\tilde{r}$ for $\tilde{r} > \delta$, then (2.32) can be estimated by

$$C \int_{\delta}^{\infty} \tilde{r}^{-|\beta|-1} d\tilde{r} \le \frac{C}{\delta^{|\beta|}} \quad \text{for } |\beta| \ge 1 \ . \tag{2.33}$$

(2.28) and (2.33) imply the desired bound.

We now prove lemma 2.3.

Proof: Let $x_j = x(jh,t)$, $\tilde{x}_j = \tilde{x}(jh,t)$, $e_j = x_j - \tilde{x}_j$, and $f_j = f(jh,t)$. We need a pointwise bound for e_j, so we use lemma 0.2 and the assumptions of this lemma to find

$$\max_j |e_j| \le h^{-1} \|e_j\|_{0,h}$$

$$\le \delta \ .$$

If we use the mean value theorem along the segment from $x - x_j$ to $x - \tilde{x}_j$, then

$$\sum_{jh \in \Omega^h} \Psi_\delta(x - \tilde{x}_j) f_j h^2 - \sum_{jh \in \Omega^h} \Psi_\delta(x - x_j) f_j h^2 =$$

$$\sum_{jh \in \Omega^h} \Big[\sum_{|\beta| = 1} D^\beta \Psi_\delta(x - x_j + y_j(x)) e_j^\beta \Big] f_j h^2$$

where $\max_j |y_j(x)| \le C_1\delta$ and e_j^i is the ith component of e_j so that $e_j^\beta = e_j^{\beta_1} e_j^{\beta_2}$. We will estimate each term separately and sum the estimates.

Since the two terms have the same form we concentrate on estimating

$$\sum_{jh \,\varepsilon\, \Omega^h} D^\beta \Psi_\delta(x - x_j + y_j(x)) e_j^\beta f_j h^2 \ . \tag{2.34}$$

The other term is treated similarly. Let B_j be defined as in (2.27) of lemma 2.1, and define

$$S = \bigcup \{B_j \mid jh \,\varepsilon\, \Omega^h\} \ .$$

To estimate (2.34) we will consider $D\Psi_\delta(x - x_j + y_j(x))$ and $e_j f_j$ as step functions defined on $\bar{R} \times S$ and S respectively. Specifically, let

$$G(x,x') = D_\beta \Psi_\delta(x - x_j + y_j(x)) \qquad x \,\varepsilon\, \bar{R} \quad x' \,\varepsilon B_j$$

$$f(x') = f_j e_j \qquad\qquad x' \,\varepsilon\, B_j$$

then, since the area of each B_j is h^2,

$$\sum_{jh \,\varepsilon\, \Omega^h} D^\beta \Psi_\delta(x - x_j + y_j(x)) f_j e_j h^2 = \int_S G(x,x') f(x') dx' \ .$$

We apply generalized Young's inequality [26] to the latter integral and find;

$$\left\| \int_S G(x,x') f(x') dx' \right\|_{L^2(\bar{R})} \le \bar{C} \| f \|_{L^2(S)}$$

where $\bar{C}$ is that number such that

$$\max_{x \,\varepsilon\, \bar{R}} \int_S | G(x,x') | \, dx' \le \bar{C} \tag{2.35}$$

$$\max_{x' \,\varepsilon\, S} \int_{\bar{R}} | G(x,x') | \, dx \le \bar{C} \ . \tag{2.36}$$

To estimate (2.35) we apply lemma 2.4 with $R_0 = \bar{R}$,

$$\begin{aligned} \max_{x \,\varepsilon\, \bar{R}} \int_S | G(x,x') | \, dx' &= \max_{x \,\varepsilon\, \bar{R}} \sum_{jh \,\varepsilon\, \Omega^h} | D^\beta \Psi_\delta(x - x_j + y_j(x)) | \, h^2 \\ &\le \max_{x \,\varepsilon\, \bar{R}} \sum_{jh \,\varepsilon\, \Omega^h} \max_{|y_j| \le C_1 \delta} | D^\beta \Psi_\delta(x - x_j + y_j) | \, h^2 \\ &\le \frac{C}{\delta} \ . \end{aligned}$$

Similarly for (2.36),

$$\max_{x' \varepsilon S} \int_{\bar{R}} | G(x,x') | \, dx = \max_{x' \varepsilon S} \sum_k \int_{B_k} | G(x,x') | \, dx$$

$$= \max_{jh \varepsilon \Omega^h} \left[\sum_k \int_{B_k} | D^\beta \Psi_\delta(x - x_j + y_j(x)) | \, dx \right]$$

$$\leq \max_{jh \varepsilon \Omega^h} \sum_k | D^\beta \Psi_\delta(\tilde{x}_{jk} - x_j + y_j(\tilde{x}_{jk})) | \, h^2$$

where $\tilde{x}_{jk}$ is that point in B_k at which the continuous function $D\Psi_\delta(x - x_j + y_j(x))$, considered as a function of x, assumes it maximum value. The sum is over all k such that $B_k \cap \bar{R}$ is not empty. Thus, if we apply lemma 2.4 with $R_0 = \{\bigcup B_k \mid B_k \cap \bar{R} \neq 0\}$, then (2.36) can be bounded by

$$\leq \max_{jh \varepsilon \Omega^h} \sum_k \max_{|y_{jk}| \leq 2C_1\delta} | D^\beta \Psi_\delta(x_k - x_j + y_{jk}) | \, h^2$$

$$\leq \frac{C}{\delta}$$

Thus $\bar{C} \leq \frac{C}{\delta}$. If we use this bound on $\bar{C}$ then,

$$\| \sum_{jh \varepsilon \Omega^h} \Psi_\delta(x - \tilde{x}_j) f_j h^2 - \sum_{jh \varepsilon \Omega^h} \Psi_\delta(x - x_j) f_j h^2 \|_{L^2(\bar{R})} = \| \int_S G(x,x') f(x') dx' \|_{L^2(\bar{R})}$$

$$\leq \bar{C} \| f \|_{L^2(S)} \leq \frac{C}{\delta} \| f_j e_j \|_{0,h}$$

$$\leq C\delta^{-1} \| e_j \|_{0,h} .$$

This completes the proof.

We combine the consistency and the stability estimate and obtain a convergence proof.

Theorem 2.1 Convergence of the approximation

Assume the velocity field $\vec{u}$ and the function f are sufficiently smooth for $0 \leq t \leq T$. Assume that the support of f_0 is contained in a bounded set Ω and that $\Psi(x)$ is in $M^{L,p}$ for some $p > 0$ and $L \geq 3$. If h is sufficiently small and if $\delta = h^q$ for $q < 1$, $\| \tilde{x}(ih,t) - x(ih,t) \|_{0,h} \leq h^s$ where $s > q + 1$, then we have

for $0 \le t \le T$ and any bounded set $\overline{R}$,

$$\max_{0 \le t \le T} \| \tilde{f}^h(x,t) - f(x,t) \|_{L^2(\overline{R})} \le C\,(h^{pq} + h^{L(1-q)} + h^{s-q}) \qquad (2.37)$$

where

$$\tilde{f}^h(x,t) = \sum_{ih \in \Omega^h} \Psi_\delta(x - \tilde{x}(ih,t)) f(ih,t) h^2$$

and $\tilde{x}(jh,t)$ is a computed solution to (2.5). The assumptions on p,q,L and s imply that as $h \to 0$ the right hand side of (2.37) tends to zero. Here the constant depends on p,L,$\overline{R}$, Ω, and a finite number of derivatives of the velocity and the function f.

Proof: We have

$$\begin{aligned} \tilde{f}^h(x,t) - f(x,t) &= \{\tilde{f}^h(x,t) - f^h(x,t)\} + \{f^h(x,t) - f(x,t)\} \\ &= \{ \sum_{ih \in \Omega^h} \Psi_\delta(x - \tilde{x}(ih,t)) f(ih,t) h^2 - \sum_{ih \in \Omega^h} \Psi_\delta(x - x(ih,t)) f(ih,t) h^2 \} \\ &\quad + \{ \sum_{ih \in \Omega^h} \Psi_\delta(x - x(ih,t)) f(ih,t) h^2 - f(x,t) \} \ . \end{aligned} \qquad (2.38)$$

We estimate the first term of (2.38) using the stability lemma 2.3 and the second term using the consistency lemma 2.1. Thus

$$\begin{aligned} \| \tilde{f}^h(x,t) - f(x,t) \|_{L(\overline{R})} &\le C\delta^{-1} \| \tilde{x}(ih,t) - x(ih,t) \|_{o,h} + C(\delta^p + (\frac{h}{\delta})^L) \\ &\le C(h^{s-q} + h^{pq} + h^{L(1-q)}) \end{aligned}$$

This completes the proof.

We note that the trajectories $\tilde{x}(jh,t)$ need to be computed with increasing accuracy as $h \to 0$ to assure convergence. This is not an unreasonable requirement. We expect that the errors committed in the calculation of $\tilde{x}(jh,t)$ will be largely in the numerical solution of the O.D.E.s (2.5) and therefore uniformly of order $(\Delta t)^l$ for some $l > 1$. The requirement that

$$\| \tilde{x}(ih,t) - x(ih,t) \|_{0,h} \leq Ch^s$$

implies that we need $(\Delta t)^l \leq Ch^s$ or $\frac{(\Delta t)^l}{h^s} \leq C$. Thus, this requirement is very similar to the stability requirement for explicit finite difference schemes for the approximation of solutions of hyperbolic equations.

In summary, a numerical method for approximating the solution to (2.1) and (2.2) consists of solving the O.D.E.s (2.5) and (2.6). We construct an approximation of the solution at points other than $x(jh,t)$ using the approximation scheme (2.7). The consistency and stability lemmas combine to establish the convergence of this approximation to the exact solution.

We will be using a technique similar to that described above for the construction of approximations to the solution of (1.15-1.18) and (1.19)-(1.23). We compute the trajectories of a finite number of fluid particles and the values of the flow quantities (density, vorticity, etc.) associated with them. It will be necessary to find approximations, based on this computed information, to derivative and integral operators applied to the flow quantities. We construct the approximations by applying the particular operators to approximations of the form (2.7). For example, we use

$$f_x^h(x,t) = \sum_{jh \in \Omega^h} \frac{\partial \Psi_\delta}{\partial x}(x - x(ih,t)) f(ih,t) h^2 \quad ,$$

as an approximation to the x derivative of f. For each of these approximations one must evaluate their consistency and stability. We preform such an analysis for a variety of operators in later sections.

3. Discretization of the Equations of Motion

In this section we describe approximation schemes for (1.15)-(1.18) and (1.19)-(1.23). Our technique will be to write the equations in Lagrangian form and then discretize them in a manner analogous to that used in the model problem. We begin by discussing a scheme for (1.15)-(1.18).

In Lagrangian form equations (1.15)-(1.18) become

$$\frac{dx(\alpha,t)}{dt} = u(x(\alpha,t)) \qquad x(\alpha,0) = \alpha \tag{3.1}$$

$$\frac{d\xi(\alpha,t)}{dt} = \rho_x(\alpha,t)F_2 - \rho_y(\alpha,t)F_1 \qquad \xi(\alpha,0) = \xi_0(\alpha) \tag{3.2}$$

$$\frac{d\rho(\alpha,t)}{dt} = 0 \qquad \rho(\alpha,0) = \rho_0(\alpha) \tag{3.3}$$

$$u = K * \xi(\alpha,t) \tag{3.4}$$

where $\alpha = (\alpha_1,\alpha_2) \varepsilon R^2$. The solution to (3.1), $x(\alpha,t)$, is the trajectory of a fluid particle which at time $t = 0$ is located at the point α. Equations (3.2) and (3.3) describe the evolution of the vorticity and density along the particle trajectory $x(\alpha,t)$. For a discussion of the equivalence of (3.1)-(3.3) and (1.15)-(1.18) see [20, pp. 139-145].

Assume the support of ρ_0 is contained in Ω, where Ω is a bounded set in R^2. For a given value of h, define Ω^h as the intersection of a grid of uniform mesh width h with Ω; $\Omega^h = \Omega \cap \Lambda^h$. Let Ψ be a function in $M^{L,p}$ and for a value of δ, $\delta > 0$, define $\Psi_\delta = \frac{1}{\delta^2}\Psi(\frac{\vec{x}}{\delta})$. Our approximation to (3.1) to (3.4) consists of solving an approximation to the O.D.E.s (3.1)-(3.4) for all points in Ω^h, i.e.,

$$\frac{d\tilde{x}(jh,t)}{dt} = \tilde{u}^h(\tilde{x}(jh,t)) \qquad \tilde{x}(jh,0) = jh \tag{A.1}$$

$$\frac{d\tilde{\xi}(jh,t)}{dt} = \tilde{\rho}_x^h(jh,t)F_2 - \tilde{\rho}_y^h(jh,t)F_1 \qquad \tilde{\xi}(jh,0) = \xi_0(jh) \tag{A.2}$$

$$\frac{d\tilde{\rho}(jh,t)}{dt} = 0 \qquad \tilde{\rho}(jh,0) = \rho_0(jh) \tag{A.3}$$

for all $jh \in \Omega^h$.

The velocity $\tilde{u}^h(\tilde{x}(jh,t))$ in (A.1) is computed by

$$\tilde{u}^h(\tilde{x}(jh,t)) = \sum_{ih \in \Omega^h} (K * \Psi_\delta)(\tilde{x}(jh,t) - \tilde{x}(ih,t))\, \tilde{\xi}(ih,t)h^2 \,. \tag{A.4}$$

The derivatives of the density necessary to evaluate (A.2) are computed by

$$\tilde{\rho}_x^h(jh,t) = \sum_{ih \in \Omega^h} \frac{\partial \Psi_\delta}{\partial x}(\tilde{x}(jh,t) - \tilde{x}(ih,t))\rho(ih,t)h^2 \tag{A.5}$$

$$\tilde{\rho}_y^h(jh,t) = \sum_{ih \in \Omega^h} \frac{\partial \Psi_\delta}{\partial y}(\tilde{x}(jh,t) - \tilde{x}(ih,t))\rho(ih,t)h^2 \,. \tag{A.6}$$

The parameter δ occurring in (A.4)-(A.6) will be specified later. As in the vortex method, the function Ψ is chosen so that $K * \Psi_\delta$ may be computed explicitly. This is method A.

We remark that the expression for the velocity $\vec{u}$ is obtained by applying the kernel K to the approximation of the vorticity given by

$$\tilde{\xi}^h(x,t) = \sum_{ih \in \Omega^h} \Psi_\delta(x - \tilde{x}(ih,t))\tilde{\xi}(ih,t)h^2 \tag{3.11}$$

and the density derivatives are obtained by differentiating an approximation to the density given by

$$\tilde{\rho}^h(x,t) = \sum_{ih \in \Omega^h} \Psi_\delta(x - \tilde{x}(ih,t))\rho(ih,t)h^2 \,. \tag{3.12}$$

To construct an approximation scheme for (1.19)-(1.23) we first write these equations in Lagrangian form,

$$\frac{dx(\alpha,t)}{dt} = \vec{u}(x(\alpha,t)) \qquad x(\alpha,0) = 0 \tag{3.13}$$

$$\frac{d\xi(\alpha,t)}{dt} = \rho_x(\alpha,t)F_2 - \rho_y(\alpha,t)F_1 \qquad \xi(\alpha,0) = \xi_0(\alpha) \tag{3.14}$$

$$\frac{d\rho_x(\alpha,t)}{dt} = -u_{1x}(x(\alpha,t))\,\rho_x(\alpha,t) - u_{2x}(x(\alpha,t))\,\rho_y(\alpha,t) \tag{3.15}$$

$$\rho_x(\alpha,0) = \rho_{0_x}(\alpha)$$

$$\frac{d\rho_y(\alpha,t)}{dt} = -u_{1y}(x(\alpha,t))\,\rho_y(\alpha,t) - u_{2y}(x(\alpha,t))\,\rho_y(\alpha,t) \tag{3.16}$$

$$\rho_y(\alpha,0) = \rho_{0_y}(\alpha)$$

$$\rho(\alpha,t) = G_x * \rho_x(\alpha,t) + G_y * \rho_y(\alpha,t) \tag{3.17}$$

where we are using the same notation as that in (3.1)-(3.4). Assume the support of ξ_0, ρ_{0_x} and ρ_{0_y} are contained in some bounded set Ω. For some $h > 0$, define Ω^h by $\Omega^h = \Omega \cap \Lambda^h$. Let Ψ be in M^{L_p}, we will approximate the solutions of (3.13)-(3.17) by the solving an approximation to (3.13)-(3.17) for all points in Ω^h given by

$$\frac{d\tilde{x}(jh,t)}{dt} = \tilde{u}^h(\tilde{x}(jh,t)) \qquad \tilde{x}(jh,0) = 0 \tag{B.1}$$

$$\frac{d\tilde{\xi}(jh,t)}{dt} = \tilde{\rho}_x^h(jh,t)F_2 - \tilde{\rho}_y^h(jh,t)F_1 \qquad \tilde{\xi}(jh,0) = \xi_0(jh) \tag{B.2}$$

$$\frac{d\tilde{\rho}_y(jh,t)}{dt} = -\tilde{u}_{1x}^h(\tilde{x}(jh,t))\tilde{\rho}_y(jh,t) - \tilde{u}_{2x}^h(\tilde{x}(jh,t))\tilde{\rho}_x(jh,t) \tag{B.3}$$

$$\tilde{\rho}_x(jh,0) = \rho_{0_x}(jh)$$

$$\frac{d\tilde{\rho}_x(jh,t)}{dt} = -\tilde{u}_{1y}^h(\tilde{x}(jh,t))\tilde{\rho}_y(jh,t) - \tilde{u}_{2y}^h(\tilde{x}(jh,t))\tilde{\rho}_x(jh,t) \tag{B.4}$$

$$\tilde{\rho}_y(jh,0) = \rho_{0_y}(jh) \ .$$

The velocity $\tilde{u}^h(\tilde{x}(jh,t))$ is computed by

$$\tilde{u}^h(\tilde{x}(jh,t)) = \sum_{ih \,\varepsilon\, \Omega^h} (K*\Psi_\delta)(\tilde{x}(jh,t) - \tilde{x}(ih,t))\tilde{\xi}(ih,t)h^2 \tag{B.5}$$

and the derivatives of the velocity field in (B.3) and (B.4) are computed by

$$\tilde{u}_x^h(\tilde{x}(jh,t)) = \sum_{ih \,\varepsilon\, \Omega^h} (K * \frac{\partial \Psi_\delta}{\partial x})(\tilde{x}(jh,t) - \tilde{x}(ih,t))\ \tilde{\xi}(ih,t)h^2 \tag{B.6}$$

$$\tilde{u}_y^h(\tilde{x}(jh,t)) = \sum_{ih \,\varepsilon\, \Omega^h} (K * \frac{\partial \Psi_\delta}{\partial y})(\tilde{x}(jh,t) - \tilde{x}(ih,t))\ \tilde{\xi}(ih,t)h^2 \ . \tag{B.7}$$

The density is approximated by

$$\rho(x,t) = \sum_{ih \,\varepsilon\, \Omega} (G_x * \Psi_\delta)(x - \tilde{x}(ih,t))\tilde{\rho}_x(ih,t)h^2 \tag{B.8}$$

$$+ \sum_{ih \,\varepsilon\, \Omega} (G_y * \Psi_\delta)(x - \tilde{x}(ih,t))\tilde{\rho}_y(ih,t)h^2 \ .$$

This is method B.

We note that the approximation to the velocity derivatives given by (B.6) and (B.7) are obtained by differentiating the approximation to the velocity (B.5). Also, the density in approximation (B.8) is obtained by using the approximations to the density derivatives

$$\tilde{\rho}_x^h(x,t) = \sum_{jh \,\varepsilon\, \Omega^h} \Psi_\delta(x - \tilde{x}(jh,t))\tilde{\rho}_x(jh,t)h^2$$

$$\tilde{\rho}_y^h(x,t) = \sum_{jh \,\varepsilon\, \Omega^h} \Psi_\delta(x - \tilde{x}(jh,t))\tilde{\rho}_y(jh,t)h^2$$

in the formula (3.17).

The accuracy of these schemes is dependent upon the initial grid spacing h, the smoothing parameter δ and the function Ψ. In sections 4 and 5 we will consider error estimates, similar to those found for the model problem, for these schemes. These estimates will be used as a guide for choosing the parameters.

4. Analysis of Method A

To implement method A we must choose h, the initial mesh width, the function Ψ used in (A.4)-(A.6), and the smoothing parameter δ. (We note that it is not necessary to choose the same function Ψ and parameter δ for (A.4) and for (A.5)-(A.6).) To guide us in our choice of these parameters we consider error estimates for the velocity and density derivative approximations. The stability, as well as the accuracy of the method for various choices of the parameters will be investigated by applying the method to a test problem. This test problem consists of an exact solution to equations (1.15) - (1.18) and is described in section 4.2. We are not able to prove convergence of method A, but if we modify the method, we can prove convergence of the modified algorithm. The modified algorithm and the convergence result are presented in section 4.3. Although we have not yet tested the modified algorithm, we expect computational results similar to those obtained with method A. Thus, we present the convergence results to demonstrate the validity of the general approach.

4.1. Error Estimates

We are interested in error estimates for the velocity approximation (A.4) and the density derivative approximation (A.5)-(A.6). The error estimates contained in [6] and [19] are applicable to (A.4). The statement and proof of the estimate that we use is a slight modification of the results in [6] and in [19].

Lemma 4.1.1 : Consistency of velocity approximation

Assume that the velocity field $u(x,t)$ is sufficiently smooth for $x \in R^2$, $0 \le t \le T$, and that the initial vorticity has support contained in Ω, Ω a bounded set. Let $\Omega^h = \Omega \cap \Lambda^h$. Also assume $\Psi \in M^{L,p}$ with $L \ge 3$ and $p > 0$,

then for $u^h(x,t)$ defined by

$$u^h(x,t) = \sum_{ih \,\varepsilon\, \Omega^h} K * \Psi_\delta(x - x(ih,t))\, \xi(ih,t) h^2$$

we have the estimate

$$\max_{0 \le t \le T} | u^h(x,t) - u(x,t) | \le C\delta^p + C(\frac{h}{\delta})^L \delta \tag{4.1.1}$$

where the constants depend on a finite number of derivatives of the flow, Ω, T, p and L.

Proof: Fix an $x \,\varepsilon\, R^2$ and t, $0 \le t \le T$. Let $K * \Psi_\delta = K_\delta$. We begin by writing the error as a sum of two terms,

$$\begin{aligned} u^h(x,t) - u(x,t) &= \{u^h(x,t) - \int K_\delta(x - x')\xi(x')dx'\} \\ &\qquad + \{\int K_\delta(x - x')\xi(x')dx' - u(x,t)\} \\ &= \{A\} + \{B\} \ . \end{aligned}$$

To estimate term A we change variables in the integral using the flow map $x(\alpha,t)$: $R^2 \to R^2$ (i. e. the solution of (3.1) in section 3). Since the flow is incompressible, the jacobian of this transformation is identically 1, so that term A can be written as

$$\sum_{ih \,\varepsilon\, \Omega^h} K_\delta(x - x(ih,t))\xi(ih,t)h^2 - \int K_\delta(x - x(\alpha,t))\xi(x(\alpha,t),t)d\alpha \tag{4.1.2}$$

Since $\xi(ih,t) = \xi(x(ih,t),t)$, then (4.1.2) can be viewed as the error in using the trapezoidal rule for integrating the function

$$g(x,\alpha,t) = K_\delta(x - x(\alpha,t))\xi(x(\alpha,t),t)$$

with respect to α over R^2.

We apply lemma 2.2 in section 2 to (4.1.2) and find

$$|A| \le Ch^L \max\{\| D^L_{\alpha_1} g(x,\alpha,t)\|_{L^1}, \| D^L_{\alpha_2} g(x,\alpha,t)\|_{L^1}\} . \tag{4.1.3}$$

Thus we must estimate

$$\| D^L_{\alpha_i} g(x,\alpha,t)\|_{L^1}.$$

If we use Leibnitz's rule, then

$$D^L_{\alpha_i} g(x,\alpha,t) = \sum_{|\beta_1|+|\beta_2|=L} C_{\beta_1,\beta_2} D^{\beta_1}_{\alpha_i} K_\delta(x - x(\alpha,t)) D^{\beta_2}_{\alpha_i} \xi(x(\alpha,t),t) \tag{4.1.4}$$

The function $\xi(x(\alpha,t),t)$ is smooth and has support contained in Ω so that (4.1.4) can be estimated by

$$C \sum_{0 \le |\beta| \le L} \| D^\beta_{\alpha_i} K_\delta(x - x(\alpha,t))\|_{L^1(\Omega)} . \tag{4.1.5}$$

Since the flow $x(\alpha,t)$ is smooth and incompressible, we can estimate integrals over R^2 at time 0 by integrals over R^2 at time t. Thus each term in (4.1.5) can be bounded by

$$C \sum_{0 \le |\beta| = L} \| D^\beta K_\delta(x - x')\|_{L^1(\overline{\Omega})} \tag{4.1.6}$$

where the constant depends on a finite number of derivatives of the flow and $\overline{\Omega}$ is a set in R^2 containing the image of Ω under the map $x(\alpha,t)$.

To estimate (4.1.6) we need point-wise estimates for the function K_δ . The estimates given in [5] are suitable. If we make the appropriate modifications to account for the change in dimension, then for any multi-index $\beta = (\beta_1,\beta_2)$,

(i) $|K_\delta(x)| \le C\delta^{-1-|\beta|}$ all $x \in R^2$

(ii) $|K_\delta(x)| \le C|x|^{-1-|\beta|}$ all $|x| > \delta, x \in R^2$.

For the proofs of these estimates see the proof of lemma 5.1 in [5] .

For a given multi-index β, then

$$\int_{\Omega} | D^{\beta}K_{\delta}(x - x') | \, dx' = \int_{\{|x - x'| \le \delta\} \cap \Omega} | D^{\beta}K_{\delta}(x - x') | \, dx'$$

$$+ \int_{\{|x - x'| > \delta\} \cap \Omega} | D^{\beta}K_{\delta}(x - x') | \, dx' \ . \qquad (4.1.7)$$

If we use the estimate (i) for the first term in (4.1.7) we find

$$\int_{\{|x - x'| \le \delta\} \cap R_1} | D^{\beta}K_{\delta}(x - x') | \, dx' \le \pi\delta^2 \ C\delta^{-1-|\beta|}$$

$$\le C\delta^{1-|\beta|} \ . \qquad (4.1.8)$$

To estimate the second term in (4.1.7) we use (ii),

$$\int_{\{|x - x'| > \delta\} \cap \Omega} | D^{\beta}K_{\delta}(x - x') | \, dx' \le C \int_0^{2\pi} \int_{\delta}^{\text{diamater}(\Omega) + \delta} r^{-|\beta|}$$

$$\le \begin{cases} C \ \textit{diamater}\,(\overline{\Omega}) + \delta & \beta = 0 \\ C \log |\delta| & |\beta| = 1 \\ C\delta^{1-|\beta|} & |\beta| > 1 \end{cases} \qquad (4.1.9)$$

We combine (4.1.8), (4.1.9) with (4.1.4)-(4.1.6) and find

$$\| D^{L}_{\alpha_i} g\,(x,t,\alpha) \|_{L^1} \le C\delta^{1-L}$$

thus,

$$|A| \le Ch^{L}\max\{ \| D^{L}_{\alpha_i} g\,(x,t,\alpha) \|_{L^1}, \ \| D^{L}_{\alpha_i} g\,(x,t,\alpha) \|_{L^1} \}$$

$$\le C\,h^{L}\ \delta^{1-L} = C(\frac{h}{\delta})^{L}\delta \ . \qquad (4.1.10)$$

To estimate term B, we have

$$B = K * \Psi_{\delta} * \xi - K * \xi \ .$$

We use the fact that a function is bounded by the L^1 norm of its Fourier transform, so

$$\sup_{x \in R^2} | K * \Psi_{\delta} * \xi - K * \xi | \le \int (| \hat{K}(\omega)\, \hat{\Psi}_{\delta}(\omega)\, \hat{\xi}(\omega) | \ - \ | \hat{K}(\omega)\, \hat{\xi}(\omega) |) \, d\omega \ .$$

$$\leq \int |\hat{K}(\omega)|\ |\hat{\xi}(\omega)|\ |1-\hat{\Psi}(\delta\omega)|\ d\omega$$

We now proceed as in [6]. As in the proof of lemma 2.1 of section 2, if Ψ is in $M^{L,p}$, then

$$|1-\hat{\Psi}(\delta\omega)| \leq C\delta^p\ |\omega|^p$$

Assuming that ξ is sufficiently differentiable,then

$$|\hat{\xi}(\omega)| \leq \frac{C}{|1+\omega|^{p+2}}$$

for some constant C, and noting that the Fourier transform of K satisfies

$$|\hat{K}| \leq \frac{1}{|\omega|}$$

we have that

$$\int |\hat{K}(\omega)|\ |\hat{\xi}(\omega)|\ |1-\hat{\Psi}(\omega\delta)|\ d\omega \leq \int \frac{1}{|\omega|}\frac{C|\omega\delta|^p}{|1+\omega|^{p+2}}d\omega$$

$$\leq C\delta^p$$

Thus,

$$|B| \leq C\delta^p \quad . \tag{4.1.11}$$

By choosing the constants large enough the two estimates (4.1.11) and (4.1.10) can be shown to hold for all $x \in R^2$ and t, $0 \leq t \leq T$, so that

$$\max_{0\leq t\leq T}\ |u^h(x,t)-u(x,t)| \leq C\delta^p + C(\frac{h}{\delta})^L\delta$$

for all $x \in R^2$. This completes the proof.

An estimate of the accuracy of the density derivatives is contained in the following lemma.

Lemma 4.1.2 : Consistency of the density derivatives

Assume that the velocity field $u(x,t)$ and the density $\rho(x,t)$ are sufficiently smooth for $x \in R^2$, $0 \le t \le T$. Assume that the density has support contained in a bounded set Ω and Ψ is in $M^{L,p}$ with $L \ge 3$ and $p > 0$. Let $\Omega^h = \Omega \cap \Lambda^h$. Then for $\rho_x^h(x,t)$ and $\rho_y^h(x,t)$ defined by

$$\rho_x^h(x,t) = \sum_{ih \in \Omega^h} \frac{\partial \Psi_\delta}{\partial x}(x - x(ih,t))\rho(ih,t)h^2$$

$$\rho_y^h(x,t) = \sum_{ih \in \Omega^h} \frac{\partial \Psi_\delta}{\partial y}(x - x(ih,t))\rho(ih,t)h^2$$

we have the estimates

$$\max_{0 \le t \le T} |\rho_x(x,t) - \rho_x^h(x,t)| \le C\delta^p + C(\frac{h}{\delta})\delta^{-1} \tag{4.1.12}$$

$$\max_{0 \le t \le T} |\rho_y(x,t) - \rho_y^h(x,t)| \le C\delta^p + C(\frac{h}{\delta})\delta^{-1}$$

where the constants depend upon a finite number of derivatives of the flow, a finite number of derivatives of $\rho(x,t)$, and on T, p and L.

Proof: The proof of this result follows very closely that of lemma 2.1 in section 2. We will only prove the consistency of the x derivative, the consistency of the y derivative is proven in a similar manner.

We begin by writing the error as the sum of two terms,

$$\rho_x^h(x,t) - \rho_x(x,t) = \{\rho_x^h - \frac{\partial \Psi_\delta}{\partial x} * \rho\} + \{\frac{\partial \Psi_\delta}{\partial x} * \rho - \rho_x\}$$

$$= \{A\} + \{B\} .$$

For term A, we change variables in the convolution integral using the map $x(\alpha,t) : R^2 \to R^2$. Since the flow is incompressible, the jacobian of this transformation is identically 1, so term A can be written as

$$\sum_{ih} \frac{\partial \Psi_\delta}{\partial x}(x - x(ih,t))\rho(ih,t)h^2 - \int \frac{\partial \Psi_\delta}{\partial x}(x - x(\alpha,t))\rho(x(\alpha,t))d\alpha .$$

Now $\rho(ih,t) = \rho(x(ih,t),t)$ so that we recognize this difference as the error in using the trapezoidal rule to integrate the function

$$g(x,\alpha,t) = \frac{\partial \Psi_\delta}{\partial x}(x - x(\alpha,t))\rho(x(\alpha,t))$$

with respect to α over R^2.

As in the proof of lemma 2.1, we use lemma 2.2, and bounds on the derivatives of the function $\frac{\partial \Psi_\delta}{\partial x}$ to find

$$\max_{0 \le t \le T} |A| \le C(\frac{h}{\delta})^L \delta^{-1} \quad . \tag{4.1.13}$$

To estimate term B, we rewrite this term as

$$B = \frac{\partial \Psi_\delta}{\partial x} * \rho - \rho_x = \Psi_\delta * \rho_x - \rho_x \quad . \tag{4.1.14}$$

The estimation of the right hand side of (4.1.14) is identical to the estimate of term B in lemma 2.1, i. e. identify ρ_x with ρ. Thus we have the estimate

$$\max_{0 \le t \le T} | \Psi_\delta * \rho_x - \rho_x | \le C\delta^p \quad . \tag{4.1.15}$$

(4.1.13) and (4.1.15) together imply the estimate (4.1.12).

In our choice of the parameters we will consider h to be arbitrary and then seek choices of the other parameters so that as $h \to 0$ the error decreases. It is clear from the estimates (4.1.1) and (4.1.12) that we should choose our functions Ψ in $M^{L,p}$ for some L and p. In our computations we will choose functions for which L is arbitrarily large (Ψ is infinitely differentiable and rapidly decreasing). Thus we need only choose an appropriate value of p. From both of the estimates (4.1.1) and (4.1.12), we conclude that choosing p larger should decrease the error. It might be thought that choosing p as large as possible is the best choice. There are two reasons why this may not be true. First, the functions Ψ for which p is large tend to be expensive to evaluate, hence the increase in accuracy may not be justified in view of the

increased cost. Secondly, the increased accuracy is dependent upon the flow being increasingly smooth. In particular, the size of the constant multiplying the δ^p term in (4.1.1) is proportional to the decay of the Fourier transform of ξ and in (4.1.12) is proportional to the decay of the Fourier transform of ρ. For flows which are not sufficiently smooth, i.e. their transforms do not decay sufficiently fast, then we expect that there is a finite p_0 such that for $p \geq p_0$, increasing the value of p will not increase the accuracy.

The other parameter we must choose for each approximation is δ. If we let $\delta = h^q$, then in order for the second term in each error estimate to decrease, we see that it is necessary that $q < 1$. What precise value of $q < 1$ is not clear. To make the $C\delta^p$ term in each error small, we should choose $\delta = h^q$ with q near one, i.e. $q = 1 - \varepsilon$ for some small $\varepsilon > 0$. Unfortunately this is the worst choice in view of the error associated with the second error terms of the form $(\frac{h}{\delta})^L \delta^{-k}$. Ideally, one would like to balance the errors in each of the terms, but without detailed knowledge of the constants, this is difficult. It may appear that if Ψ is in $M^{L,p}$ and L is arbitrarily large the error in the second term may be considered negligible for any $q < 1$. This is an erroneous conclusion, for the L appearing in (4.1.1) and (4.1.12) can only be taken arbitrarily large assuming the quantities $\mathfrak{u}$, $x(\alpha,t)$, ξ, and ρ are infinitely differentiable. If the flow is not infinitely differentiable, then the value of L appearing in the estimates (4.1.1) and (4.1.12) can only be taken as large as the maximum number of derivatives of the vorticity for (4.1.1) and only as large as the maximum number of derivatives of the flow or the density for (4.1.12). (See the estimate of term (4.1.4) in lemma 4.1.1 and (4.1.13) in lemma 4.1.2.) Thus if the flow and associated quantities are not sufficiently smooth, there is good reason to choose $\delta = h^q$, with q somewhat less than 1. (i.e. q nearer to .5 than to 1) We remark that such a conclusion

is further substantiated by the results on the accuracy of the vortex method presented in [46].

Our conclusions based on the error estimates are

(1) Choose the function Ψ in $M^{L,p}$ for some L and p

(2) Choosing the parameter p larger should decrease the error

(3) Choose $\delta = h^q$ with q near 1. If the flow or associated quantities are not too smooth, then one should choose a smaller q.

4.2. Test Problem

Given the force function

$$\vec{F} = 2t\ \bar{F}\ (\frac{-y}{r^2}, \frac{x}{r^2})$$

where $r^2 = (x^2 + y^2)$, t is the time variable, and $\bar{F}$ is a constant, then a C^7 solution to (1.15)-(1.18) with initial conditions

$$\rho(x,y,0) = \begin{cases} \frac{\alpha}{16} - \frac{\alpha}{16}(1 - \frac{r^2}{\alpha})^8 & r^2 \le \alpha \\ 0 & r^2 > \alpha \end{cases} \tag{4.2.1}$$

$$\xi(x,y,0) = 0 \tag{4.2.2}$$

is

$$\rho(x,y,t) = \rho(x,y,0)$$

$$u_1(x,y) = \begin{cases} \frac{-y\ \rho(x,y,0)}{r^2}\bar{F}\ t^2 & r^2 \le \alpha \\ \frac{-y(\frac{\alpha}{16})\bar{F}t^2}{r^2} & r^2 > \alpha \end{cases} \tag{4.2.3}$$

$$u_2(x,y) = \begin{cases} \frac{x\ \rho(x,y,0)}{r^2}\bar{F}\ t^2 & r^2 \le \alpha \\ \frac{x(\frac{\alpha}{16})\bar{F}t^2}{r^2} & r^2 > \alpha \end{cases} \tag{4.2.4}$$

The smoothness of the density follows from the definition of $\rho(x,y,0)$. For $r^2 \le \alpha$, the term $\rho(x,y,0)$ occurring in (4.2.3) and (4.2.4) is a polynomial in r^2 with no constant term, so $\frac{\rho(x,y,0)}{r^2}$ is a polynomial in r^2. Thus the velocities are C^∞ for $r^2 < \alpha$. The non-smoothness of the velocity occurs at the points for which $r^2 = \alpha$.

The flow is a radially symmetric body of fluid rotating about the origin. What makes the flow interesting is that fluid particles at different distances from the origin move at different rates, i.e. there is local shear. Typically, flows with shear present the greatest challenge to Lagrangian schemes [45]

and we therefore believe that this problem is a nontrivial test of the method.

Since $\rho(x,y,0)$ does not have compact support we need to modify our numerical procedure slightly. If we define $\bar{\rho}$ by

$$\bar{\rho} = \frac{\alpha}{16} - \rho$$

then $\bar{\rho}$ has compact support and satisfies the same equation that ρ does. To apply method A, we proceed as the method is described in section 3, but we compute the evolution of $\bar{\rho}$ instead of ρ. Using an approximation of $\bar{\rho}$, one easily obtains an approximation of ρ and of the derivatives of ρ.

In our computations we choose α in (4.2.1)-(4.2.4) to be .5 and $\bar{F} = 10.0$. In figure 4.2.1 we show the density distribution $\rho(x,y,0)$ as a function of r. Figure 4.2.2 shows the distribution of the points in Ω^h for an $h = .0886$ at time $t = 0$. The positions of the points in Ω^h at times $t = 0.0$, $t = 1.0$ and $t = 1.6$ are depicted in figures 4.2.3. To help illustrate the nature of the flow, in figure 4.2.4 we present the positions at times 0, 1.0 and 1.6 of those points in Ω^h whose initial coordinate was less than zero at time $t = 0$. (i.e. we are presenting the evolution of the left half of the body of fluid).

The solutions of the ordinary differential equations (A1)-(A3) were calculated using 4th order Runge-Kutta. Our time step, .1, was sufficiently small so that a decrease in the time step did not significantly effect the results. The space step h was allowed to take on three values, .0823, .0726, and .0628, corresponding to 200, 300 and 400 computational points respectively.

The functions Ψ used in the approximations (A.4) and (A.5)-(A.6) were chosen to be in $M^{L,p}$ and one of

$$(p = 2\,,\ L = \infty) \qquad \Psi(r) = \frac{e^{-r^2}}{\pi} \tag{4.2.5}$$

$$(p = 4\,,\ L = \infty) \qquad \Psi(r) = \frac{1}{\pi}\left(2e^{-r^2} - \frac{1}{2}e^{-\frac{r^2}{2}}\right) \tag{4.2.6}$$

$$(p = 6\,,\ L = \infty) \qquad \Psi(r) = \frac{1}{\pi}\left(8e^{-r^2} - \frac{27}{4}e^{-\frac{3}{4}r^2} + e^{-\frac{r^2}{2}}\right) \tag{4.2.7}$$

where $r^2 = (x^2 + y^2)$. These functions are suggested by Beale and Majda in [7].

To estimate the accuracy of our numerical approximation for (1.15)-(1.18) with the initial conditions (4.2.1)-(4.2.2) we compared the computed velocity (A.5) to the exact velocity (4.3.3). We measured the relative error in the first component of the velocity in the discrete L^2 norm, i.e. we used

$$\frac{\left[\sum_{ih\in\Omega^h} |\,\tilde{u}_1^h(\tilde{x}(ih,t)) - u_1(\tilde{x}(ih,t))\,|^2 h^2\right]^{\frac{1}{2}}}{\left[\sum_{ih\in\Omega^h} |\,u_1(\tilde{x}(ih,t))\,|^2 h^2\right]^{\frac{1}{2}}}$$

where $\tilde{u}_1^h$ is the computed velocity, u_1 is the exact velocity (4.2.3) and $\tilde{x}(ih,t)$ are the computed trajectories. These errors were computed at each time step, but we only present the results for times $t = 1.0$ and $t = 1.6$. These times correspond to a maximum point rotation of $\frac{\pi}{2}$ and 2π radians respectively. At time $t = 1.0$, the initial point distribution is still relatively organized (see figure 4.2.3) and we expect the errors measured at this time are indicative of the short time error. At time $t = 1.6$ the organized point distribution is lost, and we believe that the errors measured at this time are indicative of long time error. We consider both types of errors because they behave differently with respect to the parameters.

In all our computational experiments it was found that the error grew at a rate which was independent of the number of time steps taken, i.e. the method is stable. Computational experiments also indicated that the accuracy of the method is not particularly sensitive to the choice of the function Ψ and δ used in the velocity approximation (A.4). Therefore, for the velocity

approximation we chose $\delta = h^{.85}$ and Ψ to be of type (4.2.6) above. The largest changes in the errors occurred when the parameters of the density derivative approximation (A.5)-(A.6) were varied. To examine this behavior, we let $\delta = h^q$ for q = .95, .85, .75 and at the same time varied the function Ψ in (A.5)-(A.6) over the three types (4.2.5)-(4.2.7). The results of these computations are given in tables 4.2.1 and 4.2.2.

The results indicate that the method is convergent. Reducing the value of h while the other parameters were fixed reduced the error in each case. The effect of using a Ψ for which p was larger is a beneficial one. This effect is also quite dramatic. For example, at t=1.6 with h = .0886 and $\delta = h^{.85}$, the error is 19 % for p = 2 and 1.7 % for p = 6.

An unexpected result is that the optimal choice of the parameters (that choice which yielded the smallest errors) depends upon the length of time that one desires to compute the solution. For example, as seen in tables 4.3.1 and 4.3.2 the optimal choice at t = 1.0 is not the optimal choice at t = 1.6. It appears that for short time, choosing $\delta = h^q$ with q = .95 and p = 6 is best, while for longer times, choosing q = .85 and p = 6 is the best choice. The major difference between the approximations at these times is the organization of the computational points that enter into them. Thus we believe that the change in time of the optimal choice is dependent upon computational point organization. We conclude that when the computational points are disorganized it is better to choose a large smoothing parameter.

We also measured the error in the positions of the trajectories $x(ih,t)$. It was found that the errors behaved in much the same way as the velocity. Thus we believe that the velocity errors give a reliable indication of the accuracy of the solution. In general the computational results indicated that the method is convergent and stable.

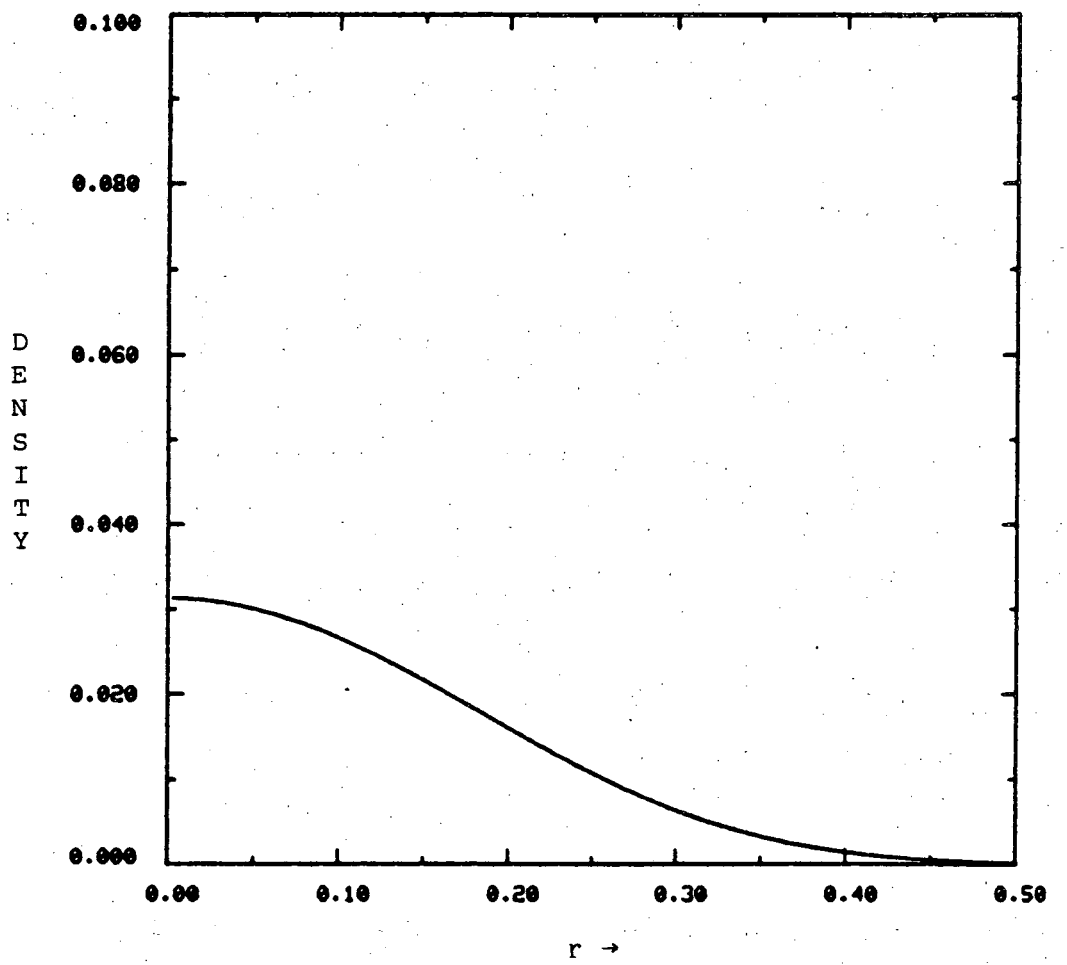


Figure 4.2.1
Density as a Function of r

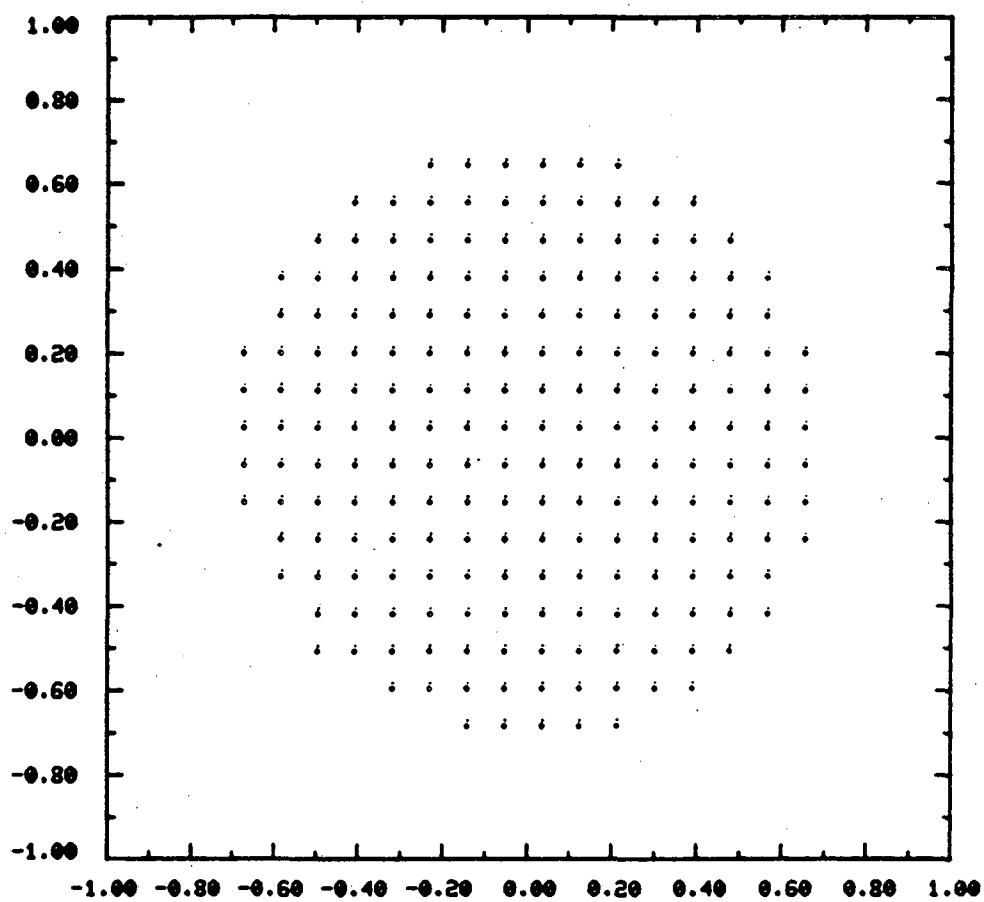


Figure 4.2.2

Distribution of Computational Points in Ω^h

Figure 4.2.3

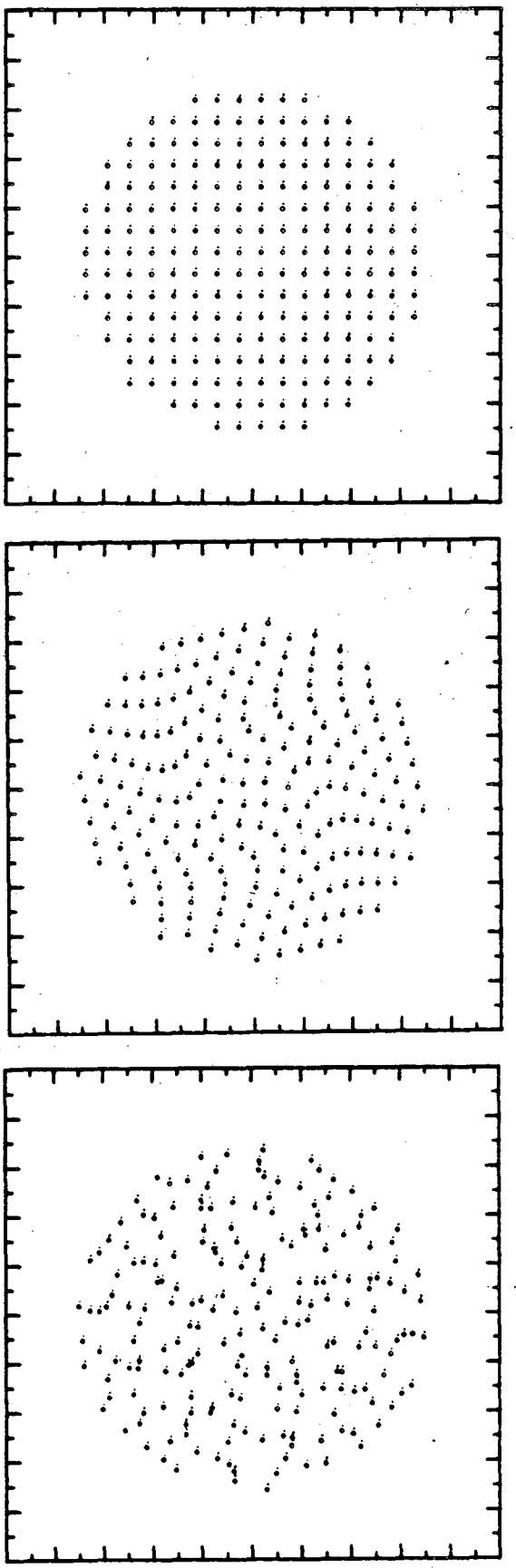
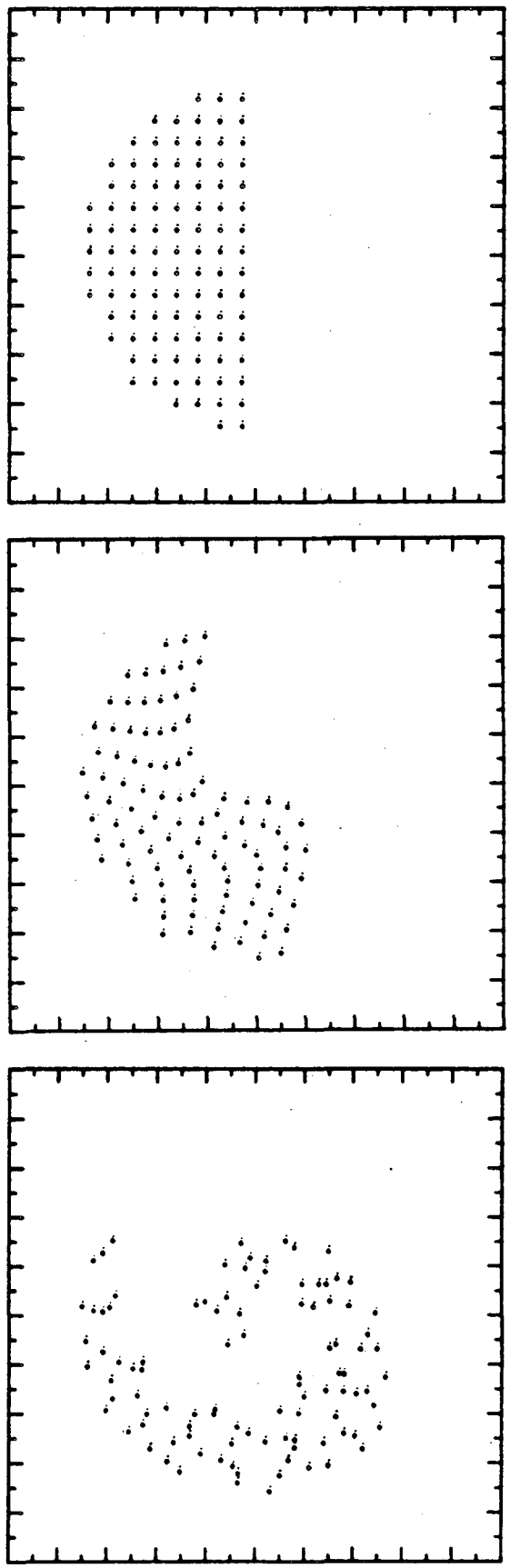


Figure 4.2.4



t = 0.0

t = 1.0

t = 1.6

Relative Velocity Error (in %) h = .0886			
	p=2	p=4	p=6
$\delta = h^{.95}$	18.0	5.5	2.5
$\delta = h^{.85}$	26.0	10.0	3.5
$\delta = h^{.75}$	37.0	18.0	7.2
$\delta = h^{.65}$	50.0	31.0	15.0

Relative Velocity Error (in %) h = .0723			
	p=2	p=4	p=6
$\delta = h^{.95}$	12.0	2.9	1.3
$\delta = h^{.85}$	19.0	5.8	1.7
$\delta = h^{.75}$	30.0	12.0	3.8
$\delta = h^{.65}$	43.0	23.0	9.9

Relative Velocity Error (in %) h = .0628			
	p=2	p=4	p=6
$\delta = h^{.95}$	9.9	1.8	0.84
$\delta = h^{.85}$	16.0	3.9	1.0
$\delta = h^{.75}$	25.0	9.0	2.4
$\delta = h^{.65}$	38.0	18.0	7.0

Table 4.2.1

Error in Velocity at t = 1.0

Relative Velocity Error (in %) h = .0886			
	p=2	p=4	p=6
$\delta = h^{.95}$	18.0	5.8	4.0
$\delta = h^{.85}$	26.0	10.0	3.6
$\delta = h^{.75}$	37.0	18.0	7.3
$\delta = h^{.65}$	50.0	31.0	15.0

Relative Velocity Error (in %) h = .0723			
	p=2	p=4	p=6
$\delta = h^{.95}$	13.0	3.1	2.4
$\delta = h^{.85}$	19.0	5.8	1.7
$\delta = h^{.75}$	30.0	12.0	3.7
$\delta = h^{.65}$	43.0	23.0	9.9

Relative Velocity Error (in %) h = .0628			
	p=2	p=4	p=6
$\delta = h^{.95}$	10.0	2.1	2.5
$\delta = h^{.85}$	16.0	3.7	0.96
$\delta = h^{.75}$	25.0	8.8	2.1
$\delta = h^{.65}$	38.0	18.0	6.9

Table 4.2.2

Error in Velocity at t = 1.6

4.3. Convergence of the Modified Algorithm

We are not able to prove the convergence of method A. The difficulty arises in obtaining estimates for the density derivative approximations (A.5)-(A.6). However, if we compute the derivatives in the following Lagrangian fashion, this difficulty is overcome and we are able to obtain a convergence proof.

If $x(\alpha,t) = (x_1(\alpha,t), x_2(\alpha,t))$ is the solution to (3.1) , then we use

$$\frac{\partial\rho}{\partial x_1}\Big|_{x(ih,t)} \approx \frac{D^h x_2}{\partial\alpha_2}(ih,t)\frac{\partial\rho_0}{\partial\alpha_1}(ih) - \frac{D^h x_2}{\partial\alpha_1}(ih,t)\frac{\partial\rho_0}{\partial\alpha_2}(ih) \tag{4.3.1}$$

$$\frac{\partial\rho}{\partial x_2}\Big|_{x(ih,t)} \approx -\frac{D^h x_1}{\partial\alpha_2}(ih,t)\frac{\partial\rho_0}{\partial\alpha_1}(ih) + \frac{D^h x_1}{\partial\alpha_1}(ih,t)\frac{\partial\rho_0}{\partial\alpha_2}(ih) \tag{4.3.2}$$

as approximations to the derivatives of ρ. Here $\frac{D^h}{\partial\alpha_i}$ is a finite difference approximation to $\frac{\partial}{\partial\alpha_i}$.

To derive approximations (4.3.1) and (4.3.2), consider the identity

$$\begin{pmatrix} \frac{\partial\rho}{\partial\alpha_1} \\ \frac{\partial\rho}{\partial\alpha_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial\alpha_1} & \frac{\partial x_2}{\partial\alpha_1} \\ \frac{\partial x_1}{\partial\alpha_2} & \frac{\partial x_2}{\partial\alpha_2} \end{pmatrix} \begin{pmatrix} \frac{\partial\rho}{\partial x_1} \\ \frac{\partial\rho}{\partial x_2} \end{pmatrix} . \tag{4.3.3}$$

Since the flow is incompressible

$$\frac{\partial x_1}{\partial\alpha_1}\frac{\partial x_2}{\partial\alpha_2} - \frac{\partial x_2}{\partial\alpha_1}\frac{\partial x_1}{\partial\alpha_2} \equiv 1$$

and we can invert the matrix of partials (4.3.3) to obtain

$$\begin{pmatrix} \frac{\partial\rho}{\partial x_1} \\ \frac{\partial\rho}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_2}{\partial\alpha_2} & -\frac{\partial x_2}{\partial\alpha_1} \\ -\frac{\partial x_1}{\partial\alpha_2} & \frac{\partial x_1}{\partial\alpha_1} \end{pmatrix} \begin{pmatrix} \frac{\partial\rho}{\partial\alpha_1} \\ \frac{\partial\rho}{\partial\alpha_2} \end{pmatrix} . \tag{4.3.4}$$

Now, the density is a solution of the equations

$$\frac{d\rho}{dt}(\alpha,t) = 0 \qquad \rho(\alpha,0) = \rho_0(\alpha) \tag{4.3.5}$$

$$\frac{dx}{dt}(\alpha,t) = u(x(\alpha,t)) \qquad x(\alpha,0) = \alpha \tag{4.3.6}$$

where ρ_0 is the initial density distribution. (4.3.5) and (4.3.6) imply that $\rho(x(\alpha,t),t) = \rho_0(\alpha)$. Therefore

$$\frac{\partial\rho}{\partial\alpha_i}(\alpha,t) = \frac{\partial\rho}{\partial\alpha_i}(x(\alpha,t),t) = \frac{\partial\rho_0}{\partial\alpha_i}(\alpha) \tag{4.3.7}$$

for $i = 1,2$. Thus (4.3.4) can be written as

$$\frac{\partial\rho}{\partial x_1} = \frac{\partial x_2}{\partial\alpha_2}\frac{\partial\rho_0}{\partial\alpha_1} - \frac{\partial x_2}{\partial\alpha_1}\frac{\partial\rho_0}{\partial\alpha_2} \tag{4.3.8}$$

$$\frac{\partial\rho}{\partial x_2} = -\frac{\partial x_1}{\partial\alpha_2}\frac{\partial\rho_0}{\partial\alpha_1} + \frac{\partial x_1}{\partial\alpha_1}\frac{\partial\rho_0}{\partial\alpha_2}. \tag{4.3.9}$$

If we replace the derivatives $\frac{\partial x_i}{\partial\alpha_j}$ by finite differences we obtain (4.3.1) and (4.3.2). The reason for using (4.3.1) and (4.3.2) instead of a finite difference approximation to the formula

$$\frac{\partial\rho}{\partial x_j} = \frac{\partial\rho}{\partial\alpha_1}\frac{\partial\alpha_1}{\partial x_j} + \frac{\partial\rho}{\partial\alpha_2}\frac{\partial\alpha_2}{\partial x_j}$$

is that the terms $\frac{\partial\alpha_i}{\partial x_j}$ are not easily approximated by finite differences.

We approximate the solution of (3.1) - (3.4) by solving the following O. D. E.'s.

$$\frac{d\tilde{x}}{dt}(ih,t) = \tilde{u}^h(ih,t) \qquad \tilde{x}(ih,0) = ih \tag{A'.1}$$

$$\frac{d\tilde{\xi}}{dt}(ih,t) = F_2\frac{\partial\tilde{\rho}}{\partial x_1}(ih,t) - F_1\frac{\partial\tilde{\rho}}{\partial x_2}(ih,t) \qquad \tilde{\xi}(ih,0) = \xi_0(ih) \tag{A'.2}$$

$$\frac{d\tilde{\rho}}{dt}(ih,t) = 0 \qquad \tilde{\rho}(ih,0) = \rho_0(ih) \tag{A'.3}$$

where

$$\tilde{u}^h(ih,t) = \sum_{jh}(K * \Psi_\delta)(\tilde{x}(ih,t) - \tilde{x}(jh,t))\tilde{\xi}(jh,t)h^2 \tag{A'.4}$$

and the density derivatives in (A'2) are computed by

$$\frac{d\tilde{\rho}}{\partial x_1}(ih,t) = \frac{D^h\tilde{x}_2}{\partial\alpha_2}(ih,t)\frac{\partial\rho_0}{\partial\alpha_1}(ih) - \frac{D^h\tilde{x}_2}{\partial\alpha_1}(ih,t)\frac{\partial\rho_0}{\partial\alpha_2}(ih) \tag{A'.5}$$

$$\frac{\partial\tilde{\rho}}{\partial x_2}(ih,t) = -\frac{D^h\tilde{x}_1}{\partial\alpha_2}(ih,t)\frac{\partial\rho_0}{\partial\alpha_1}(ih) + \frac{D^h\tilde{x}_1}{\partial\alpha_1}(ih,t)\frac{\partial\rho_0}{\partial\alpha_2}(ih) \ . \tag{A'.6}$$

Here we are using the notation of section 3. This is method A'.

As mentioned in the introduction there is a great similarity between equations (3.1) - (3.4) and Euler's equations in three dimensions. This similarity also extends to numerical methods for each set of equations. One benefit of this similarity is that much of the analysis in [5] for the 3-D vortex method presented there is applicable to method A' without substantial modification. Thus to prove the convergence of method A' we will be following closely the proof of convergence presented in [5].

Before we give a precise statement of the convergence result, we need the following definitions.

Definition: We say that $\frac{D^h}{\partial\alpha_i}$ is rth order accurate if for any sufficiently smooth function of compact support

$$\left\| \frac{D^h f}{\partial\alpha_i}(jh) - \frac{\partial f}{\partial\alpha_i}(jh) \right\|_{0,h} \le Ch^r \tag{4.3.10}$$

where C depends on a finite number of derivatives of $f(x)$. We assume that the constant C appearing in (4.3.10) has the following bound:

$$C \le C' \left\| \frac{\partial^{r+1} f}{(\partial\alpha_i)^r} \right\|_{L^2(R)} \tag{4.3.11}$$

where the set R contains the support of f, and C' does not depend on f.

Definition: We say that $\frac{D^h}{\partial\alpha_i}$ is stable if $\frac{D^h}{\partial\alpha_i}$ are uniformly bounded map-

pings from H_h^0 to H_h^{-1} i. e. there exists a fixed constant C so that for $h \le h_0$,

$$\| \frac{D^h f_j}{\partial \alpha_j} \|_{-1,h} \le C \| f_j \|_{0,h} \quad . \tag{4.3.12}$$

For a characterization of a set of difference approximations which are stable, we have the following result from [5].

Proposition 4.3.1 Given a multi-index l, let T^l denote grid translation in the direction of l. Then, every difference operator of the form

$$\frac{D^h}{\partial \alpha} = \frac{1}{h} \sum_{|l| \le l_0} a_l(h) T^l$$

with $| a_l(h) | \le C$ satisfies the stability condition (4.3.12).

Proof: See the proof of proposition 1 in [5].

We now give a statement of our convergence result.

Theorem 4.3.1 : Convergence of method A'

Assume that the velocity field $u(x,t)$ and density $\rho(x,t)$ are sufficiently smooth for $x \in R^2$, $0 \le t \le T$, and that the initial vorticity and density have support contained in Ω. Define Ω^h by $\Omega^h = \Omega \cap \Lambda^h$. Also assume

(1) Ψ is in $M^{L,p}$ for some L and p satisfying (3) below.

(2) The difference operators $\frac{D^h}{\partial \alpha_i}$ in (A'.5)-(A'.6) are rth order accurate and stable in the sense of (4.3.12). The value r is chosen so $r > 3$.

(3) We choose $\delta = h^q$ with $q < 1$ and L, p, q are chosen so that $L \ge 3$, $pq > 3$ and $L(q - 1) + q > 3$.

Then, if we compute $\tilde{x}(ih,t)$, $\tilde{\xi}(ih,t)$, and $\tilde{\rho}(ih,t)$ as solutions of (A'.1) - (A'.3) for $ih \in \Omega^h$, we have the following error bounds:

$$(i) \max_{0 \le t \le T} \{ \| \tilde{x}(ih,t) - x(ih,t) \|_{0,h} + \| \tilde{\xi}(ih,t) - \xi(ih,t) \|_{-1,h} \} \le C(h^{pq} + h^{L(q+1)})$$

$$(ii) \max_{0 \le t \le T} \| \tilde{u}^h(ih,t) - u(ih,t) \|_{0,h} \le C(h^{pq} + h^{L(q+1)})$$

$$(iii) \max_{0 \le t \le T} \| \tilde{u}^h(x,t) - u(x,t) \|_{L^2(\bar{R})} \le C(h^{pq} + h^{L(q+1)})$$

$$(iv) \max_{0 \le t \le T} \| \tilde{\rho}^h(x,t) - \rho(x,t) \|_{L^2(\bar{R})} \le C(h^{(p-1)q} + h^{Lq})$$

where $\bar{R}$ is a bounded set in R^2. Here the constants depend on T, p, L, Ω, $\bar{R}$, and a finite number of derivatives of the flow and the density.

The proof of this theorem depends on a consistency and stability estimate for the velocity approximation. The consistency estimate that we need is that of lemma 4.1, whose hypothesis are satisfied under the assumptions of this theorem. The Main Stability Lemma in [5], with the obvious modifications to account for the change in dimension and the kernel K, is applicable to our problem. Since the necessary modifications are so slight, we do not prove the following result but we refer the reader to [5] for a proof.

Lemma 4.3.1 : Stability of velocity approximation

Assume the hypothesis of theorem 4.3.1. Provided that

$$\max_{0 \le t \le T} \| \tilde{x}(ih,t) - x(ih,t) \|_{0,h} \le h^3$$

for some $T^* \le T$, we have for $0 \le t \le T^*$ the estimates

$$\| \tilde{u}^h(\tilde{x}(ih,t)) - u^h(x(ih,t)) \|_{0,h} \le C(\| \tilde{x}(ih,t) - x(ih,t) \|_{0,h} + \| \tilde{\xi}(ih,t) - \xi(ih,t) \|_{-1,h}) \tag{4.3.13}$$

$$\| \tilde{u}^h(x,t) - u^h(x,t) \|_{L^2(\bar{R})} \le C(\| \tilde{x}(ih,t) - x(ih,t) \|_{0,h} + \| \tilde{\xi}(ih,t) - \xi(ih,t) \|_{-1,h}) \tag{4.3.14}$$

where $\tilde{u}^h$ and $\tilde{u}^h$ are given by

$$\tilde{u}^h(x,t) = \sum_{jh \in \Omega^h} K_\delta(x - \tilde{x}(jh,t))\tilde{\omega}(jh,t)h^2$$

and

$$u^h(x,t) = \sum_{jh \,\varepsilon\, \Omega^h} K_\delta(x - x(jh,t))\omega(jh,t)h^2 \ .$$

We will also need the following result for differential inequalities due to Hale [32].

Lemma 4.3.2 : Suppose $g(t,y)$ is continuous on an open connected set Ω in R^2 and that the initial value problem for the scaler equation

$$\dot{y} = g(t,y)$$

has a unique solution. If $x(t)$ is a continuous n-vector function with continuous first derivative on [a,b] such that $|x(0)| \le y(0)$, $(t,x(t)) \subseteq \Omega$ for $a \le t \le b$ and $|x(t)| \le g(t,|x(t)|)$ for $a \le t \le b$, then

$$|x(t)| \le y(t) \qquad \text{on } a \le t \le b \ .$$

Proof: See the proof of the corollary 6.3 in [32].

We now prove theorem 4.3.1.

Proof: Consider the error in the particle positions

$$e_i(t) = \tilde{x}(ih,t) - x(ih,t)$$

and the error in the vorticity

$$v_i(t) = \tilde{\xi}(ih,t) - \xi(ih,t) \ .$$

As in [5] we will obtain a differential inequality for

$$\| e_i \|_{0,h} + \| v_i \|_{-1,h} \ .$$

From the ordinary differential equations (3.1) and (A'.4), we conclude

$$\begin{aligned} \dot{e}_i &= \tilde{u}^h(\tilde{x}(ih,t)) - u(x(ih,t)) \qquad (4.3.15) \\ &= \{\tilde{u}^h(\tilde{x}(ih,t)) - u^h(x(ih,t))\} + \{u^h(x(ih,t)) - u(x(ih,t))\} \ . \end{aligned}$$

If we use the consistency lemma, lemma 4.1.1, then the second term of

(4.3.15) is estimated by

$$\| u^h(x(ih,t)) - u(x(ih,t)) \|_{0,h} \le C(\delta^p + (\frac{h}{\delta})^L \delta) \quad . \tag{4.3.16}$$

We make the assumption that

$$| e_i |_{0,h} + | v_i |_{-1,h} < h^3 \tag{4.3.17}$$

for $0 \le t \le T^*$ with $T^* \le T$. We use the stability lemma, lemma 4.3.2 and estimate the first term of (4.3.15) by

$$| u^h(\tilde{x}(ih,t)) - u^h(x(ih,t)) |_{0,h} \le C(\| e_i \|_{0,h} + | v_i \|_{-1,h}) \quad .$$

Thus, for $0 \le t \le T^*$ we have

$$| \dot{e}_i(t) \|_{0,h} \le C(| e_i |_{0,h} + | v_i |_{-1,h} + \delta^p + (\frac{h}{\delta})^L \delta) \quad .$$

To estimate $\dot{v}_i$ we use (3.3) and (A'.5)-(A'.6),

$$\begin{aligned} \dot{v}_i &= (\tilde{F}_2\tilde{\rho}_x(ih,t) - \tilde{F}_1\tilde{\rho}_y(ih,t)) - (F_2\rho_x(ih,t) - F_1\rho_y(ih,t)) \\ &= \{(\tilde{F}_2\tilde{\rho}_x(ih,t) - F_2\rho_x(ih,t))\} - \{(\tilde{F}_1\tilde{\rho}_y(ih,t) - F_1\rho_y(ih,t))\} \quad . \end{aligned} \tag{4.3.18}$$

where we are using $\tilde{F}$ to indicate that the force function is being evaluated at the computed trajectories. We will show how to estimate the first term of (4.3.18), the second term is estimated in a similarly.

$$\begin{aligned} \tilde{F}_2\tilde{\rho}_x(ih,t) - F_2\rho_x(ih,t) &= \{[\tilde{F}_2 - F_2]\rho_x(ih,t)\} + \{[\tilde{F}_2 - F_2][\tilde{\rho}_x(ih,t) - \rho_x(ih,t)]\} \\ &\quad + \{F_2[\tilde{\rho}_x(ih,t) - \rho_x(ih,t)]\} \\ &= \{A\} + \{B\} + \{C\} \quad . \end{aligned}$$

Since F_2 is continuously differentiable and ρ_x is smooth, for term A we find

$$\begin{aligned} \| (\tilde{F} - F_2)\rho_x(ih,t) \|_{-1,h} &\le C | \tilde{F}_2 - F_2 |_{0,h} \\ &= C \| F_2(\tilde{x}(ih,t)) - F_2(x(ih,t)) |_{0,h} \\ &\le C \sup_{x \in \bar{\Omega}} | \nabla F_2 | \, | \tilde{x}(ih,t) - x(ih,t) |_{0,h} \\ &\le C | e_i |_{0,h} \end{aligned}$$

where $\bar{\Omega}$ is a compact set in R^2 containing the points $x(ih,t)$, $\tilde{x}(ih,t)$ for

t, $0 \le t \le T$.

To estimate term B we use lemma 0.3, and estimate $\| \tilde{F} - F \|_{0,h}$ as in the estimate for term A.

$$\begin{aligned} \| (\tilde{F}_2 - F_2)(\tilde{\rho}_x(ih,t) - \rho_x(ih,t)) \|_{-1,h} &\le Ch^{-2} \| \tilde{F}_2 - F_2 \|_{0,h} \| \tilde{\rho}_x(ih,t) - \rho_x(ih,t) \|_{-1,h} \\ &\le C\, h^{-2} \| e_i \|_{0,h} \| \tilde{\rho}_x(ih,t) - \rho_x(ih,t) \|_{-1,h} \\ &\le C \| \tilde{\rho}_x(ih,t) - \rho_x(ih,t) \|_{-1,h} \quad . \qquad (4.3.19) \end{aligned}$$

For an estimate of the right hand side of (4.3.19) we use (A'.5),

$$\begin{aligned} \tilde{\rho}_x(ih,t) - \rho_x(ih,t) = & \\ & \left[\frac{D^h \tilde{x}_2}{\partial \alpha_2}(ih,t) \frac{\partial \rho_0}{\partial \alpha_1}(ih) - \frac{D^h \tilde{x}_2}{\partial \alpha_1}(ih,t) \frac{\partial \rho_0}{\partial \alpha_2}(ih) \right] \\ & - \left[\frac{\partial x_2}{\partial \alpha_2}(ih,t) \frac{\partial \rho_0}{\partial \alpha_1}(ih) - \frac{\partial x_2}{\partial \alpha_1}(ih,t) \frac{\partial \rho_0}{\partial \alpha_2}(ih) \right] \\ = & \left[\frac{D^h \tilde{x}_2}{\partial \alpha_2}(ih,t) - \frac{\partial x_2}{\partial \alpha_2}(ih,t) \right] \frac{\partial \rho_0}{\partial \alpha_1}(ih) \qquad (4.3.20) \\ & - \left[\frac{D^h \tilde{x}_2}{\partial \alpha_1}(ih,t) - \frac{\partial x_2}{\partial \alpha_1}(ih,t) \right] \frac{\partial \rho_0}{\partial \alpha_2}(ih) \quad . \end{aligned}$$

We show how to estimate the first term of (4.3.20), the second term is handled similarly.

$$\begin{aligned} & \left\| \left[\frac{D^h \tilde{x}_2}{\partial \alpha_2}(ih,t) - \frac{\partial x_2}{\partial \alpha_2}(ih,t) \right] \frac{\partial \rho_0}{\partial \alpha_1}(ih) \right\|_{-1,h} \qquad (4.3.21) \\ & \le \left\| \left[\frac{D^h \tilde{x}_2}{\partial \alpha_2}(ih,t) - \frac{D^h x_2}{\partial \alpha_2}(ih,t) \right] \frac{\partial \rho_0}{\partial \alpha_1}(ih) \right\|_{-1,h} \\ & + \left\| \left[\frac{D^h x_2}{\partial \alpha_2}(ih,t) - \frac{\partial x_2}{\partial \alpha_2}(ih,t) \right] \frac{\partial \rho_0}{\partial \alpha_1}(ih) \right\|_{-1,h} \end{aligned}$$

Since $\frac{\partial \rho_0}{\partial \alpha_1}$ is continuously differentable, we apply lemma 0.4 to the first term on the right hand side of (4.3.21), followed by the stability criterion (4.3.12),

$$\left\| \left[\frac{D^h \tilde{x}_2}{\partial \alpha_2}(ih,t) - \frac{D^h x_2}{\partial \alpha_2}(ih,t) \right] \frac{\partial \rho_0}{\partial \alpha_1}(ih) \right\|_{-1,h} \le C \left\| \left[\frac{D^h \tilde{x}_2}{\partial \alpha_2}(ih,t) - \frac{D^h x_2}{\partial \alpha_2}(ih,t) \right] \right\|_{-1,h}$$

$$\le C \| \tilde{x}_2(ih,t) - x_2(ih,t) \|_{0,h}$$

$$\le C \| e_i \|_{0,h} \quad . \tag{4.3.22}$$

In this last estimate we are extending $\tilde{x}_2(ih,t) - x_2(ih,t)$ to be zero outside Ω^h. This extension does not effect the estimate since $\frac{\partial \rho_0}{\partial \alpha_1}$ has support in Ω^h. For the second term on the right hand side of (4.3.21) we use lemma 0.4 and the accuracy estimate (4.3.10).

$$\left\| \left[\frac{D^h x_2}{\partial \alpha_2}(ih,t) - \frac{\partial x_2}{\partial \alpha_2}(ih,t) \right] \frac{\partial \rho_0}{\partial \alpha_1}(ih) \right\|_{-1,h} \le C \left\| \frac{D^h x_2}{\partial \alpha_2}(ih,t) - \frac{\partial x_2}{\partial \alpha_2}(ih,t) \right\|_{0,h}$$

$$\le Ch^r \quad . \tag{4.3.23}$$

Thus (4.3.22) and (4.3.23) imply

$$\| B \|_{-1,h} \le C(\| e_i \|_{0,h} + h^r) \quad . \tag{4.3.24}$$

For term C, since F_2 is continuously differentiable, this term can be estimated in a manner similar to that for term B,

$$\| F_2(\tilde{\rho}_x(ih,t) - \rho_x(ih,t) \|_{-1,h} \le \| \tilde{\rho}_x(ih,t) - \rho_x(ih,t) \|_{-1,h}$$

$$\le C(h^r + \| e_i \|_{0,h}) \quad . \tag{4.3.25}$$

Therefore, combining the estimates of terms A, B, and C, we have

$$\| \dot{v}_i \|_{-1,h} \le C(\| e_i \|_{0,h} + h^r) \quad . \tag{4.3.26}$$

If we add the estimates for $\dot{e}_i$ and $\dot{v}_i$ then we have

$$\| \dot{e}_i \|_{0,h} + \| \dot{v}_i \|_{-1,h} \le C_0(\| e_i \|_{0,h} + \| v_i \|_{-1,h} + h^r + \delta^p + (\frac{h}{\delta})^L \delta) \quad . \tag{4.3.27}$$

Consider the set of functions $\{e_i(t)\}$ and $\{v_i(t)\}$ as forming the components of a vector valued function $E(t) : [0,T^*] \to R^{2m}$, where m is the number of points in Ω^h,

$$E(t) = (e_1(t), \cdots, e_m(t), v_1(t), \cdots, v_m(t)) \ .$$

Then (4.3.27) implies that

$$|\dot{E}(t)| \le C_0(E(t) + h^r + \delta^p + (\frac{h}{\delta})^L\delta)$$

$$E(0) = 0 \ .$$

Here we are using the norm

$$|\vec{x}| = |(x_1 \cdots x_{2m})| = \|\{x_i\}_{i=1}^m\|_{0,h} + |\{x_j\}_{j=m+1}^{2m}|_{-1,h}$$

on R^{2m}. We apply lemma 4.3.2 to E(t) and find, $|E(t)| \le y(t)$, where $y(t)$ is a solution of

$$\dot{y}(t) = C_0(\delta^p + (\frac{h}{\delta})^L\delta + h^r + y) \ .$$

Therefore, for $0 \le t \le T^*$,

$$\|e_i\|_{0,h} + \|v_i\|_{-1,h} \le C_1(\delta^p + (\frac{h}{\delta})^L\delta + h^r) \tag{4.3.28}$$

where the constant C_1 depends on C_0 and T, but not on T^*. We now remove the assumption (4.3.17). Our assumptions on δ, p, L, and r imply

$$\delta^p + (\frac{h}{\delta})^L\delta + h^r = h^{pq} + h^{L(1-q)+q} + h^r \le 3h^{3+\varepsilon}$$

for $h < 1$ and some $\varepsilon > 0$. Choose h_0 small enough so that for $h < h_0$,

$$C_1(\delta^p + (\frac{h}{\delta})^L\delta + h^r) \le \frac{h^3}{2} \ .$$

Let T^* be the first time less than T such that

$$|e_i|_{0,h} + \|v_i|_{-1,h} \ge h^3 \ . \tag{4.3.29}$$

By (4.3.28) we have that

$$\| e_i \|_{0,h} + \| v_i \|_{-1,h} \le \frac{h^3}{2} \tag{4.3.30}$$

for $0 \le t \le T^*$. But (4.3.30) contradicts (4.3.29) at T^*, so T^* must be equal to T. Thus, if $h \le h_0$

$$\| e_i \|_{0,h} + \| v_i \|_{-1,h} \le C(h^{pq} + h^{L(1-q)+q} + h^r) \tag{4.3.31}$$

for $0 \le t \le T$. The first estimate (i) is now proven. To estimate (ii), we write

$$\tilde{u}^h(ih,t) - u(ih,t) = \{\tilde{u}^h(ih,t) - u^h(ih,t)\} + \{u^h(ih,t) - u(ih,t)\} \ .$$

The first term is estimated using the stability lemma 4.3.1 and the second using consistency estimate, lemma 4.1.1. Therefore,

$$\begin{aligned} \| \tilde{u}^h(ih,t) - u(ih,t) \|_{0,h} &\le C(\| e_i \|_{0,h} + \| v_i \|_{-1,h} + h^r) + C(h^{pq} + h^{L(1-q)+q}) \\ &\le C(h^{pq} + h^{L(1-q)+q} + h^r) \ . \end{aligned}$$

Similarly for (iii), we use the stability estimate (4.3.14) and consistency estimate (4.1.1).

$$\begin{aligned} \| \tilde{u}^h(x,t) - u(x,t) \|_{L^2(\mathbb{R})} &= \| \tilde{u}^h(x,t) - u^h(x,t) \|_{L^2(\mathbb{R})} + \| \hat{u}^h(k,t) + u(x,t) \|_{L^2(\mathbb{R})} \\ &\le C(\| e_i \|_{0,h} + \| v_i \|_{-1,h} + h^r) + C(h^{pq} + h^{L(1-q)+q}) \\ &\le C(h^{pq} + h^{L(1-q)+q} + h^r) \ . \end{aligned}$$

The convergence of the density approximation follows immediately from (4.3.31) and theorem 2.1 in section 2. This completes the proof.

5. Analysis of method B

We begin our analysis of method B by proving a convergence result. As in the proof for method A' in section 4, we will exploit the similarity between this method and a corresponding method for 3-D vortex motion and use much of the analysis contained in [5] and [8]. This convergence result is somewhat weak in the sense that we must assume that the method is accurate of order 2 + 2q for $0 \le q < 1$. Specifically we must assume that the component approximations used in the method have error estimates which can be bounded by Ch^{2+2q} for some constant C which does not depend on h. To investigate the necessity of this assumption in practice, as well as to assess the behavior of the method when the parameters Ψ and δ are varied, we test the method on the problem described in section 4.2. The results of these computations are discussed in section 5.2.

5.1. Convergence of method B

A precise statement of the convergence result for method B is the following:

Theorem 5.1

Assume that the velocity field $u(x,t)$ and the density $\rho(x,t)$ are sufficiently smooth for $0 \le t \le T$ and $x \in R^2$. Assume that the support of the initial vorticity and the support of the initial density are contained in some bounded set Ω, and that the force $\vec{F} = (F_1, F_2)$ is conservative (curl $\vec{F} = 0$) and is continuously differentiable. If we compute $\tilde{x}(ih,t)$, $\tilde{\xi}(ih,t)$, $\tilde{\rho}_x(ih,t)$, $\tilde{\rho}_y(ih,t)$, and $\tilde{\rho}^h(ih,t)$ for $ih \in \Omega^h$ according to (B.1)-(B.4) and (B.8), and we choose

(1) $\Psi \in M^{L,p}$ for some L and p which satisfy (3) below

(2) $\delta = h^q \quad q < 1$

(3) we choose a combination of p, q, and L in (1) and (2) such that

$$pq > 2 + 2q \text{ and } L(1-q) > 2 + 2q$$

then for $h \le h_0$ we have

$$(i) \quad \max_{0 \le t \le T} (\| \tilde{x}(ih,t) - x(ih,t) \|_{0,h} + \| \tilde{\xi}(ih,t) - \xi(ih,t) \|_{-1,h}$$
$$+ \| \tilde{\rho}_x(ih,t) - \rho_x(ih,t) \|_{-1,h} + \| \tilde{\rho}_y(ih,t) - \rho_y(ih,t) \|_{-1,h})$$
$$\le C(h^{pq} + h^{L(1-q)})$$

$$(ii) \quad \max_{0 \le t \le T} \| \tilde{u}^h(ih,t) - u(ih,t) \|_{0,h} \le C(h^{pq} + h^{L(1-q)})$$

$$(iii) \quad \max_{0 \le t \le T} \| \tilde{u}^h(x,t) - u(x,t) \|_{L^2(\bar{R})} \le C(h^{pq} + h^{L(1-q)})$$

$$(iv) \quad \max_{0 \le t \le T} \| \tilde{\rho}^h(ih,t) - \rho(ih,t) \|_{0,h} \le C(h^{pq} + h^{L(1-q)})$$

$$(v) \quad \max_{0 \le t \le T} \| \tilde{\rho}^h(x,t) - \rho(x,t) \|_{L^2(\bar{R})} \le C(h^{pq} + h^{L(1-q)})$$

where $\bar{R}$ is a bounded set in R^2. The constants depend on T, p, L, $\bar{R}$, and bounds on a finite number of derivatives of the flow and the density. We mention that we are considering the norm $\| \cdot \|_{0,h}$ in (ii) to be restricted to the set Ω^h.

To prove this theorem we will need consistency and stability estimates for the approximations used in (B.1)-(B.7). One consistency estimate that we need is lemma 4.1.1 in section 4, and the other is an estimate of the consistency of the approximations (B.6)-(B.7) to the derivatives of the velocity. This estimate is the content of the following lemma.

Lemma 5.1.1 : Assume that the velocity field $u(x,t)$ is sufficiently smooth for $x \in R^2$, $0 \le t \le T$, and that the initial vorticity has support contained in a bounded set Ω. Also assume Ψ is in $M^{L,p}$ with $L \ge 3$ and $p > 0$. If $(x_1, x_2) = (x, y)$ and we compute $u^h_{x_i}$ by

$$u^h_{x_i}(x,t) = \sum_{ih \,\varepsilon\, \Omega^h} (K * \frac{\partial \Psi_\delta}{\partial x_i})\,(x - x(ih,t))\xi(ih,t)h^2 \tag{5.1.1}$$

then we have the estimates,

$$\max_{0 \le t \le T} | u^h_{x_i}(x,t) - u_{x_i}(x,t) | \le C\,\delta^p + C\,(\frac{h}{\delta})^L \tag{5.1.2}$$

for i=1,2. Here the constants depend on a finite number of derivatives of the flow, Ω, T, p, and L.

Proof: Since the approximation (5.1.1) is of the same form as the approximation to the velocity (B.5), the proof of this lemma is a minor modification of the of the proof of lemma 4.1.1. In particular we obtain a proof if we identify $K * \frac{\partial \Psi_\delta}{\partial x_i}$ with $K * \Psi_\delta$ in lemma 4.1.1. We remark that the loss of a factor of delta in the second term of (5.1.2) is due to the increased singularity of $K * \frac{\partial \Psi_\delta}{\partial x_i}$.

We need two stability estimates. The first is lemma 4.3.1 and the second is a stability estimate of the approximations occurring in the right hand side of (B.3)-(B.4). To prove this latter stability estimate we need several technical lemmas. For the most part, these lemmas concern bounds on integral operators whose kernels are related to $K * \Psi_\delta$ ($= K_\delta$) and its derivatives. Since many of the results presented here are taken directly from [5] and need little modification, we will refer the reader to the corresponding results in [5] for many of the proofs.

We need the following discrete and continuous L^1 estimates for $D^\beta K_\delta$, β being a multi-index.

Lemma 5.1.2 : With time t fixed and $x_j = x(jh,t)$, we have

$$\sum_{jh \,\varepsilon\, R_0} \max_{|y_j| \le \delta} |D^\beta K_\delta(x - x_j + y_j)|\, h^2 \le \begin{cases} C & \beta = 0 \\ C\log|\delta| & \beta = 1 \\ C\delta^{1-|\beta|} & \beta > 1 \end{cases} \tag{5.1.3}$$

and

$$\sum_{ih \,\varepsilon\, R_0} \max_{|y_j| \le \delta} |D^\beta K_\delta(x_i - x + y_i)|\, h^2 \le \begin{cases} C & \beta = 0 \\ C\log|\delta| & \beta = 1 \\ C\delta^{1-|\beta|} & \beta > 1 \end{cases} \tag{5.1.4}$$

for all $|x| < \bar{R}$, provided h (and thus δ) are small enough. The constants depend on R_0, C_0 and bounds for the derivatives of the flow.

Proof: See the proof of lemma 3.2 in [5].

A continuous L^1 estimate that we use is,

Lemma 5.1.3: There exists a universal constant C, so that

$$\int_{|x| \le R} |K_\delta(x)|\, dx \le CR \quad .$$

Proof: See the proof of lemma 3.3 in [5].

An L^2 estimate for $D^\beta K_\delta$ we need is

Lemma 5.1.4: For a multi-index β, $|\beta| = 1$, we have for all $f \,\varepsilon\, H^0$

$$\| (D^\beta K_\delta) * f \|_0 \le C \| f \|_0 \quad .$$

Proof: see the proof of lemma 3.5 in [5].

To obtain estimates on the convolution sums involving $D^\beta K_\delta$, we shall use the following two results which are closely related to the generalized Young's inequality [26].

Lemma 5.1.5 (a): Let $K(x,x')$ be a function defined on $S_1 \times S_2$, where S_1 and S_2 are bounded subsets of R^2. If we define the operator $K : H^0(S_2,R) \to H^0(S_1,R)$ by

$$(Kf)(x) = \int_{S_2} K(x,x')f(x')dx' \qquad x \in S_1$$

then

$$\| Kf \|_{H^0(S_1)} \leq \| K \| \, \| f \|_{H^0(S_2)}$$

where $\| K \|$ is the smallest number satisfying

$$\int_{S_2} | K(x,x') | \, dx' \leq \| K \| \qquad \int_{S_1} | K(x,x') | \, dx \leq \| K \|$$

for all $x \in S_1$ and $x' \in S_2$.

Lemma 5.1.5 (b) : With S_1 and S_2 as in 5.1.5 (a) and $H_h^0(S) = H_h^0(\Lambda^h \cap S)$. Define the operator $K : H_h^0(S_2,R) \to H_h^0(S_1,R)$ by

$$(Kf)(ih) = \sum_{jh \in S_2} K(ih,jh)f(jh)h^2$$

then

$$\| Kf \|_{H_h^0(S_1)} \leq \| K \| \, \| f \|_{H_h^0(S_2)}$$

where $\| K \|$ is the smallest number so that

$$\sum_{jh \in S_2} | K(ih,jh) | \, h^2 \leq \| K \|, \qquad \sum_{ih \in S_1} | K(ih,jh) | \, h^2 \leq \| K \|$$

for all i,j with $ih \in S_1$ and $jh \in S_2$. Here $\| \cdot \|_{H_h^0(S_2)}$ and $\| \cdot = \|_{H_h^0(S_1)}$ are the norm $\| \cdot \|_{0,h}$ restricted to the set S_2 and S_1 respectively.

Proof: See the proof of lemma 3.1.a and 3.1. b in [5] or lemma 0.10 in [26].

We combine lemma 5.1.2 and 5.1.5 to obtain the following discrete L^2 estimates.

Lemma 5.1.6: Let $f_j \in H_h^0$, and assume the support of f_j is contained in R_0, a bounded set in R^2. Let β be a multi-index, $|\beta| \geq 1$, then if $| y_{ij} | \leq C_0\delta$,

$$\| \sum_{jh \in R^0} D^\beta K_\delta(x_i - x_j + y_{ij}) f_j h^2 \|_{0,h} \leq C\delta^{1-|\beta|} \| f_j \|_{0,h} \, .$$

Proof: By lemma 5.1.5 (b) it is sufficient to show

$$\max_{ih\,\varepsilon\,\Omega^h} \sum_{jh\,\varepsilon\,R_0} | D^\beta K_\delta(x_i - x_j + y_{ij}) | h^2 \le C\delta^{1-|\beta|} \tag{5.1.5}$$

and

$$\max_{jh\,\varepsilon\,R_0} \sum_{ih\,\varepsilon\,\Omega^h} | D^\beta K_\delta(x_i - x_j + y_{ij}) | h^2 \le C\delta^{1-|\beta|} \ . \tag{5.1.6}$$

For (5.1.5), we use lemma 5.1.2,

$$\max_{ih\,\varepsilon\,\Omega^h} \sum_{jh\,\varepsilon\,R_0} | D^\beta K_\delta(x_i - x_j + y_{ij}) | h^2 \le \max_{ih\,\varepsilon\Omega^h} \sum_{jh\,\varepsilon\,R_0} \max_{|y_{ij}| \le \delta} | D^\beta K_\delta(x_i - x_j + y_{ij}) | h^2$$

$$\le C\delta^{1-|\beta|} \ .$$

The estimate for (5.1.6) follows similarly.

In the next lemma we use a partition of the fluid domain given by the following:

If $j = (j_1, j_2)$ is an integer multi-index, let

$$Q_j = \{x \,\varepsilon\, R^2 \,|\, x \,\varepsilon\, [\, j_1 - \frac{h}{2}, j_1 + \frac{h}{2}] \times [\, j_2 - \frac{h}{2}, j_2 + \frac{h}{2}]\} \ .$$

The Q_j's partition R^2. Define

$$B_j = \{x \,\varepsilon\, R^2 \,|\ \ x = x(\alpha, t) \ \ \alpha \,\varepsilon\, Q_j\}.$$

i. e. B_j is the image under the flow of the rectangle Q_j. Since the flow is incompressible, it is measure preserving, thus the area of each B_j is h^2. For a function f, defined on the set of points $x(jh, t)$, we will often associate a step function $f(x)$ defined by

$$f(x) = f_j \qquad x \,\varepsilon\, B_j \ .$$

With this construction

$$\| f \|_{H^0(R^2)} = | f |_{0,h}$$

The following lemma is closely related to lemma 5.1.6.

Lemma 5.1.7:

Assume $f \varepsilon H_h^0$ and the support of f is contained in a compact set Ω. Let S_1 be a bounded set in R^2 and $S_2 = \{\bigcup B_j \mid jh \varepsilon \Omega\}$. If β is a multi-index then

$$\| \sum_j D^\beta K_\delta(x - x_j) f_j h^2 \|_{H^0(S_1)} \leq \left\{ \begin{array}{ll} C & \beta = 0 \\ C\log|\beta| & \beta = 1 \\ C\delta^{1-|\beta|} & \beta > 1 \end{array} \right\} \| f \|_{0,h} \quad . \tag{5.1.7}$$

Proof: Define $D^\beta K_\delta(x,x')$ on $S_1 \times S_2$ and f on S_2 by

$$D^\beta K_\delta(x,x') = D^\beta K_\delta(x - x_j) \qquad x \varepsilon S_1 \;\; x' \varepsilon B_j$$

$$f(x') = f_j \qquad\qquad x' \varepsilon B_j$$

then,

$$\| \sum_j D^\beta K_\delta(x - x_j) f_j h^2 \|_{H^0(S_1)} = \| \int_{S_1} D^\beta K(x,x') f(x') dx' \|_{H^0(S_1)} \quad . \tag{5.1.8}$$

We apply lemma 5.1.5 (a) to the right hand side of (5.1.8), and find

$$\| \int_{S_2} D^\beta K(x,x') f(x') dx' \|_{H^0(S_1)} \leq \bar{C} \| f \|_{H^0(S_2)} = \bar{C} \| f_j \|_{0,h} \tag{5.1.9}$$

where $\bar{C}$ is that constant for which

$$\sup_{x \varepsilon S_1} \int_{S_2} | D^\beta K(x,x') | \, dx' \leq \bar{C} \tag{5.1.10}$$

and

$$\sup_{x' \varepsilon S_2} \int_{S_1} | D^\beta K(x,x') | \, dx \leq \bar{C} \quad . \tag{5.1.11}$$

For (5.1.10), since the B_j partition S_2 and have area h^2, for a given $x \varepsilon S_1$ then,

$$\int_{S_2} | D^\beta K_\delta(x,x') | \, dx' = \sum_{jh \varepsilon \Omega} \int_{B_j} | D^\beta K_\delta(x,x') | \, dx'$$

$$= \sum_{jh \,\varepsilon\, \Omega} | D^\beta K_\delta(x - x_j) | h^2 \ . \tag{5.1.12}$$

We apply lemma 5.1.2 to (5.1.12) and find

$$\max_{x \,\varepsilon\, S_1} \sum_j | D^\beta K_\delta(x - x_j) | h^2 \le \begin{cases} C & \beta = 0 \\ C\log|\delta| & \beta = 1 \\ C\delta^{1-|\beta|} & \beta > 1 \end{cases} . \tag{5.1.13}$$

For (5.1.11), we again use the fact that B_j partitions the plane. Let $S_1^h = \{j \mid B_j \cap S_1 \neq 0\}$. For a given $x' \,\varepsilon\, S_2$,

$$\begin{aligned} \int_{S_1} | D^\beta K_\delta(x,x') | \, dx &= \sum_{j \,\varepsilon\, S_1^h} \int_{B_j} | D^\beta K_\delta(x,x') | \, dx \\ &\le \sum_{j \,\varepsilon\, S_1^h} \max_{x \,\varepsilon\, B_j} | D^\beta K_\delta(x - x_j) | h^2 \\ &\le \sum_{j \,\varepsilon\, S_1^h} \max_{|y_{ij}| \le C_0\delta} | D^\beta K_\delta(x_i - x_j + y_{ij}) | h^2 \\ &\le \begin{cases} C & \beta = 0 \\ C\log|\delta| & \beta = 1 \\ C\delta^{1-|\beta|} & \beta > 1 \end{cases} . \end{aligned} \tag{5.1.14}$$

Thus (5.1.13), (5.1.14) and (5.1.9)-(5.1.11) combine to give the estimate (5.1.7).

We measure the error in the vorticity and density gradients in the H_h^{-1} norm so we need a version of lemma 5.1.6 for integral operators from H_h^{-1} to H_h^0.

Lemma 5.1.8: With the notation used in lemma 5.1.5(b), suppose $S_2 = \{x' \mid |x'| \le R_0\}$. For $f \,\varepsilon\, H_h^{-1}$ with support of f contained in $\{x \mid |x| \le R_0\}$ then,

$$\| Kf \|_{0,h} \le C (| K | + | D_j^h K |) | f \|_{-1,h}$$

with the norms $| K |$, $| D_j^h K \|$ as in lemma 5.1.5, and D_j^h is the forward difference quotient with respect to the index j.

Proof: See the proof of lemma 3.4 in [5].

When the integral operators have a special form, then the result of lemma 5.1.8 can be strengthened. This is the content of the next lemma.

Lemma 5.1.9 : For $f \varepsilon H_h^{-1}$ with support of f contained in a bounded set Ω, then

$$\| \sum_j K_\delta(x_i - x_j) f_j h^2 \|_{0,h} \leq C \mid f_j \mid_{-1,h} .$$

Proof: This estimate is that of the term $v_i^{(2)}$ in Main Stability Lemma of [5] with the appropriate modifications to account for the change in dimension and the kernel K.

We now prove the stability estimate for the approximations in (B.4) and (B.5).

Lemma 5.1.10:

Define

$$e_i = \tilde{x}(ih,t) - x(ih,t)$$
$$v_i = \tilde{\xi}(ih,t) - \xi(ih,t)$$
$$r_{x_i} = \tilde{\rho}_x(ih,t) - \rho_x(ih,t)$$
$$r_{y_i} = \tilde{\rho}_y(ih,t) - \rho_y(ih,t). \quad .$$

Under the hypothesis of theorem 5.1.1, if for t, $0 \leq t \leq T^*$, $T^* \leq T$,

$$\| e_i \|_{0,h} + \mid v_i \|_{-1,h} + \mid r_{x_i} \mid_{-1,h} + \| r_{y_i} \mid_{-1,h} \leq h^{2+2q} \qquad (5.1.15)$$

then, letting $(x_1, x_2) = (x, y)$,

$$\mid \tilde{u}^h_{1x_i}(\tilde{x}(ih,t))\tilde{\rho}_x(ih,t) - u^h_{1x_i}(x(ih,t))\rho_x(ih,t) \|_{-1,h} \leq$$
$$C(\mid e_i \mid_{0,h} + \| v_i \mid_{-1,h} + \| r_{x_i} \|_{-1,h} + \mid r_{y_i} \|_{-1,h})$$
$$(5.1.16)$$

$$\| \tilde{u}^h_{2x_i}(\tilde{x}(ih,t))\tilde{\rho}_y(ih,t) - u^h_{2x_i}(x(ih,t))\rho_y(ih,t) \|_{-1,h} \le$$
$$C(\| e_i \|_{0,h} + \| v_i \|_{-1,h} + \| r_{x_i} \|_{-1,h} + \| r_{y_i} \|_{-1,h}) \tag{5.1.17}$$

for i = 1,2. Here the constants depend on T, p, L, and a finite number of derivatives of the flow and the density, but not on T^*.

Proof: The proof of this lemma is based on a proof of the stability of the vorticity growth terms presented in [8]. We will only prove the estimate

$$\| \tilde{u}^h_{1x}\tilde{\rho}_x - u^h_{1x}\rho_x \|_{0,h} \le C(\| e_i \|_{0,h} + \| v_i \|_{-1,h} + \| r_{x_i} \|_{-1,h} + \| r_{y_i} \|_{-1,h}) \tag{5.1.18}$$

i.e. the estimate (5.1.16) for $i = 1$. The remaining estimates are done in a similar manner. Let $x_i = x(ih,t)$, $\tilde{x}_i = \tilde{x}(ih,t)$, $\xi_i = \xi(ih,t)$, $\tilde{\xi}_i = \tilde{\xi}(ih,t)$, $\tilde{\rho}_{x_i} = \tilde{\rho}_x(ih,t)$, $\tilde{\rho}_{y_i} = \tilde{\rho}_y(ih,t)$, and $K_\delta = K * \Psi_\delta$. Also let $K * \frac{\partial \Psi_\delta}{\partial x} = \frac{\partial K}{\partial x} * \Psi_\delta = D_x K_\delta$. We begin by decomposing the left hand side of (5.1.18) into four terms,

$$\begin{aligned}
\tilde{u}^h_{1x}(ih,t)\,\tilde{\rho}_x(ih,t) - u^h_{1x}(ih,t)\,\rho_x(ih,t) &= \\
&\sum_j D_x K_\delta^1(\tilde{x}_i - \tilde{x}_j)\tilde{\xi}_j h^2 \tilde{\rho}_{x_i} - \sum_j D_x K_\delta^1(x_i - x_j)\xi_j h^2 \rho_{x_i} \\
&= \{\sum_j [D_x K_\delta^1(\tilde{x}_i - \tilde{x}_j) - D_x K_\delta^1(x_i - \tilde{x}_j)]\xi_j \rho_{x_i} h^2\} \\
&+ \{\sum_j [D_x K_\delta^1(x_i - \tilde{x}_j) - D_x K_\delta^1(x_i - x_j)]\xi_j \rho_{x_i} h^2\} \\
&+ \{\sum_j D_x K_\delta^1(x_i - x_j)[\tilde{\xi}_j \tilde{\rho}_{x_i} - \xi_j \rho_{x_i}] h^2\} \\
&+ \{\sum_j [D_x K_\delta^1(\tilde{x}_i - \tilde{x}_j) - D_x K_\delta^1(x_i - x_j)][\tilde{\xi}_j \tilde{\rho}_{x_i} - \xi_j \rho_{x_i}] h^2\} \\
&= T_1 + T_2 + T_3 + T_4 \ .
\end{aligned}$$

Here K_δ^1 is the first component of K_δ . For convenience we will drop the superscript and refer to K_δ^1 as K_δ. Similarly we will drop the subscript on D and refer to D_x as D. The estimate that we seek will follow from the triangle inequality and individual estimates of T_1 to T_4.

We begin by estimating T_1. Since ρ_x is continuously differentiable, we apply lemma 0.3 to find

$$\| T_1 \|_{-1,h} = \| \sum_j [DK_\delta(\tilde{x}_i - \tilde{x}_j) - DK_\delta(x_i - \tilde{x}_j)]\xi_j h^2 \rho_{x_i} \|_{-1,h}$$

$$\leq \| \sum_j [DK_\delta(\tilde{x}_i - \tilde{x}_j) - DK_\delta(x_i - \tilde{x}_j)]\xi_j h^2 \|_{-1,h} \quad . \tag{5.1.19}$$

We apply the mean value theorem along the segment from $\tilde{x}_i - \tilde{x}_j$ to $x_i - \tilde{x}_j$ and estimate the right hand side of (5.1.19) by

$$\| \sum_j \sum_{|\beta| = 1} D^\beta DK_\delta(x_i - \tilde{x}_j + y_i) e_i \xi_j h^2 \|_{-1,h} \tag{5.1.20}$$

where $\max|y_i| \leq \max|e_i|$. To estimate (5.1.20) we estimate both terms separately and sum the estimates. Since each term has the same form we concentrate on estimating,

$$\| \sum_j D^2 K_\delta(x_i - x_j + y_{ij}) e_i \xi_j h^2 \|_{-1,h} \tag{5.1.21}$$

where we let $y_{ij} = y_i + (x_j - \tilde{x}_j)$. We employ Taylor's theorem with remainder to estimate (5.1.21).

$$\| \sum_j D^2 K_\delta(x_i - x_j + y_{ij}) e_i \xi_j h^2 \|_{-1,h} \leq \| \sum_j D^2 K_\delta(x_i - x_j) e_i \xi_j h^2 \|_{-1,h}$$

$$+ \| \sum_j D^3 K_\delta(x_i - x_j + z_{ij}) y_{ij} e_i \xi_j h^2 \|_{-1,h}$$

$$\leq T_1^{(1)} + T_1^{(2)}$$

where $\max|z_{ij}| \leq \max|y_{ij}| \leq 2\max|e_i|$. We have that $\| T_1^{(1)} \|_{-1,h} \leq \| T_1^{(1)} \|_{0,h}$, so that $T_1^{(1)}$ can be estimated by

$$\| T_1^{(1)} \|_{-1,h} \leq \| \sum_j D^2 K_\delta(x_i - x_j) e_i \xi_j h^2 \|_{0,h}$$

$$\leq \max_i | \sum_j D^2 K_\delta(x_i - x_j) \xi_j h^2 | \, \| e_i \|_{0,h} \quad . \tag{5.1.22}$$

To estimate (5.1.22),

$$|\sum_j D^2K_\delta(x_i - x_j)\xi_j h^2| \le |\sum_j D^2K_\delta(x_i - x_j)\xi_j h^2 - \int D^2K_\delta(x_i - x')\xi(x',t)dx'|$$
$$+ |\int D^2K_\delta(x_i - x')\xi(x',t)dx'| \quad . \tag{5.1.23}$$

We employ arguments similar to those used in lemma 4.1.1 to estimate the first term in (5.1.23) and find

$$|\sum_j D^2K_\delta(x - x_j)\xi_j h^2 - \int D^2K_\delta(x - x')\xi(x',t)dx'| \le C(\frac{h}{\delta})^L\delta^{-1} \quad .$$

(See the estimate of term A in lemma 4.1.1.) Also, if we use lemma 5.1.3 then we can estimate the second term of (5.1.23) by

$$|\int D^2K_\delta(x - x')\xi(x',t)dx'| = |\int K_\delta(x - x')D^2\xi(x',t)dx'|$$
$$\le \max|D^2\xi| \int_{\tilde{R}} |K_\delta(x - x')|\, dx'$$
$$\le C\tilde{R} \le C$$

where $\tilde{R}$ is the diameter of the set containing the support of the vorticity for $0 \le t \le T$. Thus,

$$\| T_1^{(1)} \| \le C(1 + (\frac{h}{\delta})^L\delta^{-1}) \| e_i \|_{0,h} \quad . \tag{5.1.24}$$

To estimate $T_1^{(2)}$, we apply lemma 0.1 and lemma 5.1.6 to find

$$\| T_1^{(2)} \|_{-1,h} \le \| T_1^{(2)} \|_{0,h} \le h^{-1} \| \sum_j D^3K_\delta(x_i - x_j + z_{ij})y_{ij}\xi_j h^2 \|_{0,h} \| e_i \|_{0,h}$$
$$\le h^{-1} (C \max_{ij}|y_{ij}| \ \delta^{-2}) \| e_i \|_{0,h}$$
$$\le Ch^{-1}\delta^{-2}\max_i|e_i| \ \| e_i \|_{0,h} \quad . \tag{5.1.25}$$

The estimates (5.1.24) and (5.1.25) imply

$$\| T_1^{(1)} \|_{-1,h} \le C (h^{-1}\delta^{-2}\max_i|e_i| + (\frac{h}{\delta})^L\delta^{-1}) \| e_i \|_{0,h} \quad .$$

The assumption (5.1.15) and the hypothesis of theorem 5.1 of this lemma imply

$$(h^{-1}\delta^{-2}\max_i | e_i | + (\frac{h}{\delta})^L\delta^{-1}) \le (h^{-2}\,\delta^{-2}\, \| e_i \|_{0,h} + (\frac{h}{\delta})^L\delta^{-1}) \le C$$

where the constant is independent of h. Thus,

$$\| T_1^{(1)} \|_{-1,h} \le C \| e_i \|_{0,h} \ .$$

To estimate T_2, we use the mean value theorem from $x_i - \tilde{x}_j$ to $x_i - x_j$,

$$\begin{aligned} T_2 &= \sum_j [DK_\delta(x_i - \tilde{x}_j) - DK_\delta(x_i - x_j)]\xi_j \rho_{x_i} h^2 \\ &= \sum_j \sum_{|\beta| = 1} D^\beta DK_\delta(x_i - x_j + y_{ij}) e_j \xi_j \rho_{x_i} h^2 \end{aligned} \tag{5.1.26}$$

where $\max| y_{ij} | \le 2 \max| e_j |$. To estimate (5.1.26) we estimate both terms separately and sum the estimates. Since the terms both have the same form we concentrate on estimating

$$\sum_j D^2K_\delta(x_i - x_j + y_{ij}) e_j \xi_j \rho_{x_i} h^2 \ . \tag{5.1.27}$$

We apply Taylor's theorem with remainder to (5.1.27) and find

$$\begin{aligned} \sum_j D^2K_\delta(x_i - x_j + y_{ij}) e_j \xi_j \rho_{x_i} h^2 &= \sum_j D^2K_\delta(x_i - x_j) e_j \xi_j \rho_{x_i} h^2 \\ &\quad + \sum_j D^3K_\delta(x_i - x_j + w_{ij}) e_j y_{ij} \xi_j \rho_{x_i} h^2 \\ &= T_2^{(1)} + T_2^{(2)} \end{aligned}$$

with $\max| w_{ij} | \le \max| y_{ij} | \le 2\max | e_j |$. We first estimate $T_2^{(1)}$. Since ρ_x is continuously differentiable,

$$\| \sum_j D^2K_\delta(x_i - x_j) e_j \xi_j \rho_{x_i} h^2 \|_{-1,h} \le \| \sum_j D^2K_\delta(x_i - x_j) e_j \xi_j h^2 \|_{-1,h} \ . \tag{5.1.28}$$

Now, by the chain rule, if $\alpha(x,t) = (\alpha_1(x,t), \alpha_2(x,t))$ is the inverse of the flow map $x(\alpha,t)$, we have,

$$\frac{\partial}{\partial x} = \sum_k \frac{\partial}{\partial \alpha_k} \frac{\partial \alpha_k}{\partial x}$$

so $D^2K_\delta(x_i - x_j) = \sum_k \frac{\partial}{\partial \alpha_k} DK_\delta(x(\alpha,t) - x_j) \cdot \frac{\partial \alpha_k}{\partial x}$. The functions $\frac{\partial \alpha_k}{\partial x}$ are

smooth so we can estimate the right hand side of (5.1.28) by

$$\| \sum_j D^2 K_\delta(x_i - x_j) e_j \xi_j h^2 \|_{-1,h} \le C \| \sum_{k=1}^{2} \sum_j D_{\alpha_k} D K_\delta(x(ih,t) - x_j) e_j \xi_j h^2 \|_{-1,h} \tag{5.1.29}$$

where the constant depends on bounds for the derivatives of $\frac{\partial \alpha_k}{\partial x}$. Let D_α^h be a finite difference approximation to the derivative which is rth order accurate in the sense of (4.3.10) and stable in the sense of (4.3.11). (We are suppressing the subscript k on α for clarity) We can estimate the terms on the right hand side of (5.1.29) by an estimate of the form,

$$\begin{aligned} \| \sum_j D_\alpha D K_\delta(x_i - x_j) e_j \xi_j h^2 \|_{-1,h} \le & \| \sum_j D_\alpha D K_\delta(x(ih,t) - x_j) e_j \xi_j h^2 \\ & - \sum_j D_\alpha^h D K_\delta(x(ih,t) - x_j) e_j \xi_j h^2 \|_{-1,h} \\ & + \| \sum_j D_\alpha^h D K_\delta(x(ih,t) - x_j) e_j \xi_j h^2 \|_{-1,h} . \end{aligned} \tag{5.1.30}$$

The first term of (5.1.30) is estimated using the accuracy criterion for the difference formula lemma 4.3.11,

$$\begin{aligned} & \| \sum_j D_\alpha D K_\delta(x(ih,t) - x_j) e_j \xi_j h^2 - \sum_j D_\alpha^h D K_\delta(x(ih,t) - x_j) e_j \xi_j h^2 \|_{-1,h} \\ & \le \| \sum_j D_\alpha D K_\delta(x(ih,t) - x_j) e_j \xi_j h^2 - \sum_j D_\alpha^h D K_\delta(x(ih,t) - x_j) e_j \xi_j h^2 \|_{0,h} \\ & \le C h^r \max_{\alpha \in \Omega} | \sum_j D_\alpha^{r+1} D K_\delta(x(\alpha,t) - x_j) e_j \xi_j h^2 | . \end{aligned} \tag{5.1.31}$$

To estimate the right hand side of (5.1.31) we apply Sobolov's lemma,

$$\begin{aligned} \max_{\alpha \in \Omega} | \sum_j D_\alpha^{r+1} D K_\delta(x(\alpha,t) - x_j) e_j \xi_j h^2 | & \le \| \sum_j D_\alpha^{r+1} D K_\delta(x(\alpha,t) - x_j) e_j \xi_j h^2 \|_{H^2(\Omega)} \\ & \le \Big(\sum_{0 \le |\beta| \le 2} \| \sum_j D_\alpha^{r+|\beta|+1} D K_\delta(x(\alpha,t) - x_j) e_j \xi_j h^2 \|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}} . \end{aligned} \tag{5.1.32}$$

Since the flow is smooth, we can replace α derivatives by x derivatives, so that the terms in (5.1.32) can be estimated by

$$\sum_{0 \le |\beta| \le 0} \| \sum_j D_\alpha^{1+r+|\beta|} DK_\delta(x(\alpha,t) - x_j) e_j \xi_j h^2 \|_{L^2(\Omega)} \le$$

$$C \sum_{1 \le |\beta| \le r+4} | \sum_j D^\beta K_\delta(x - x_j) e_j \xi_j h^2 |_{L^2(\bar{\Omega})}$$

where $\bar{\Omega}$ is a bounded set in R^2 containing the image of Ω under the map $x(\alpha,t)$. We apply lemma 5.1.7 to each term in the sum and find that

$$\sum_{1 \le |\beta| \le r+4} \| \sum_j D^\beta K_\delta(x - x_j) e_j \xi_j h^2 |_{L^2(\bar{\Omega})} \le C \left(\log|\delta| + \sum_{k=1}^{r+4} \frac{1}{\delta^{k-1}} \right) | e_i |_{0,h} . \tag{5.1.33}$$

Choose r so that $\frac{r}{r+3} > q$ (which is possible because $q < 1$), then $\frac{h^r}{\delta^{r+3}} < C$, where the constant is independent of h. Thus using (5.1.31) and (5.1.33) we have that

$$Ch^r \max_{\alpha \in \Omega} | \sum_j D_\alpha^{r+1} DK_\delta(x(\alpha,t) - x_j) e_j \xi_j h^2 | \le C \frac{h^r}{\delta^{r+3}} | e_i |_{0,h}$$

$$\le C \| e_i \|_{0,h} . \tag{5.1.34}$$

We apply the stability assumption on the finite difference approximation to the derivative, (4.3.12), to the second term of (5.1.30) and find

$$| \sum_j D_\alpha^h DK_\delta(x(\alpha_i,t) - x_j) e_j \xi_j h^2 \|_{-1,h} \le \| \sum_j DK_\delta(x_i - x_j) e_j \xi_j h^2 \|_{0,h} . \tag{5.1.35}$$

Let B_j and Q_j be defined as in lemma 5.1.7. To estimate the right hand side of (5.1.35) we consider $e_j \xi_j$ and $DK_\delta(x_i - x_j)$ as step functions on

$$S = \bigcup \{ B_j : jh \in \Omega \} .$$

We define

$$K(x,x') = DK_\delta(x_i - x_j) \quad \text{for } x \in B_i \text{ and } x' \in B_j$$

$$f(x') = e_j \xi_j \qquad \text{for } x' \in B_j .$$

Thus, since the area of B_j is h^2,

$$\sum_j DK_\delta(x_i - x_j)e_j\xi_j h^2 = \int_S K(x,x')f(x')dx' \ .$$

Also, since the flow is measure preserving,

$$\| \int_S K(x,x')f(x')dx' \|_{H^0(S)} = \| \sum_j DK_\delta(x_i - x_j)\xi_j h^2 \|_{0,h}$$

and

$$\| f \|_{H^0(S)} = \| e_j\xi_j \|_{0,h} \ .$$

Thus an estimate of K and f will yield the desired estimate. We write K as

$$K(x,x') = K_1(x,x') + K_2(x,x')$$

where

$$K_1(x,x') = DK_\delta(x - x')$$

$$K_2(x,x') = DK_\delta(x_i - x_j) - DK_\delta(x - x') \ .$$

Therefore

$$\| \int_S K(x - x')f(x')dx' \|_{H^0(S)} \le \| \int_S K_1(x,x')f(x')dx' \|_{H^0(S)} + \| \int_S K_2(x,x')f(x')dx' \|_{H^0(S)} \ . \qquad (5.1.36)$$

The first term is $DK_\delta * f$, and so can be estimated using lemma 5.1.4. We find

$$\| K_1 * f \|_0 \le C \| f \|_0 \le \max_j | \xi_j | \ \| e_i \|_{0,h} \le C \| e_i \|_{0,h} \ .$$

If we use the mean value theorem and lemma 5.1.2 then we have the following L^1 estimates of K_2,

$$\sup_{x' \in S} \int_S | K_2(x,x') | \, dx \le \sup_{x' \in S} [\sum_i \max_{|y_{ij}| \le C_0\delta} | D^2K_\delta(x_i - x + y_{ij}) | h^2] \cdot \delta$$
$$\le C\delta^{-1} \cdot \delta \le C$$

and

$$\sup_{x \in S} \int_S | K_2(x,x')dx' | \le \sup_{x \in S} [\sum_j \max_{|y_{ij}| \le C_0\delta} | D^2K_\delta(x - x_j + y_{ij}) | h^2] \cdot \delta$$
$$\le C\delta^{-1} \cdot \delta \le C \ .$$

Thus, to estimate the second term of (5.1.36) we use lemma 5.1.5 (a) and we find,

$$\| \int K_2(x,x')f(x')dx' \|_{H^0(S)} \le C \| f \|_{H^0(S)} \le C \max_j |\xi_j| \, \| e_i \|_{0,h} \le C \| e_i \|_{0,h} \quad . \tag{5.1.37}$$

If we combine the estimate (5.1.34) for the first term of (5.1.30) and (5.1.37) for the second term we find

$$\| T_2^{(1)} \|_{-1,h} \le C \| e_i \|_{0,h} \quad .$$

For $T_2^{(2)}$ we use lemma 5.1.6 to find

$$\begin{aligned} \| \sum_j D^3K_\delta(x_i - x_j + w_{ij})e_i y_{ij} \xi_j \rho_{x_i} h^2 \|_{-1,h} &\le \| \sum_j D^3K_\delta(x_i - x_j + w_{ij})e_i y_{ij} \xi_j h^2 \|_{0,h} \\ &\le C\, \delta^{-2} \max_{ij} |y_{ij}| \max_j |\xi_j| \, \| e_i \|_{0,h} \\ &\le C \| e_i \|_{0,h} \quad . \end{aligned}$$

In this last estimate we have used the fact that $\max |y_{ij}| \le 2 \max |e_i| \le h^{-1} \| e_i \|_{0,h} \le h^{2q}$. The estimates for $T_2^{(1)}$ and $T_2^{(2)}$ imply

$$\| T_2 \|_{-1,h} \le C \| e_i \|_{0,h} \quad .$$

We estimate T_3 as follows:

$$\begin{aligned} \sum_j DK_\delta(x_i - x_j)[\tilde{\xi}_j \tilde{\rho}_{x_i} - \xi_j \rho_{x_i}]h^2 &= \{\sum_j DK_\delta(x_i - x_j) v_j \rho_{x_i} h^2\} \\ &\quad + \{\sum DK_\delta(x_i - x_j) v_j r_{x_i} h^2\} \\ &\quad + \{\sum DK_\delta(x_i - x_j) \xi_j r_{x_i} h^2\} \\ &= T_3^{(1)} + T_3^{(2)} + T_3^{(3)} \end{aligned}$$

where v_j and r_{x_i} are defined as in the statement of this lemma. For $T_3^{(1)}$, we change the x derivative to an α derivative using the chain rule. (See the estimate of (5.1.28)) Since the flow and ρ_{x_i} are smooth,

$$\| T_3^{(1)} \|_{-1,h} = \| \sum_j DK_\delta(x(ih,t) - x_j) v_j \rho_{x_i} h^2 \|_{-1,h}$$
$$\leq C \| \sum_j D_\alpha K_\delta(x(ih,t) - x_j) v_j \rho_{x_i} h^2 \|_{-1,h}$$
$$\leq C \| \sum_j D_\alpha K_\delta(x_i - x_j) v_j h^2 \|_{-1,h} \quad . \tag{5.1.38}$$

Let D_α^h be an rth order finite difference approximation to the derivative that is stable in the sense of (4.3.12). We estimate (5.1.38) by

$$C \| \sum_j D_\alpha K_\delta(x(ih,t) - x_j) v_j h^2 \|_{-1,h}$$
$$\leq C \| \sum_j D_\alpha K_\delta(x(ih,t) - x_j) v_j h^2 - \sum_j D_\alpha^h K_\delta(x(ih,t) - x_j) v_j h^2 \|_{-1,h}$$
$$+ C \| D_\alpha^h K_\delta(x(ih,t) - x_j) v_j h^2 \|_{-1,h} \quad . \tag{5.1.39}$$

The first term, the error in the finite difference approximation, is estimated like the corresponding term (5.1.31) in the estimate for T_2. Using arguments similar to those employed there, we find

$$\| \sum_j D_\alpha K_\delta(x_i - x_j) v_j h^2 - \sum_j D_\alpha^h K_\delta(x_i - x_j) v_j h^2 \|_{-1,h}$$
$$\leq Ch^r \Big(\sum_{1 \leq |\beta| \leq r+4} \| \sum_j D^\beta K_\delta(x - x_j) v_j h^2 \|_{H^0(\overline{\Omega})} \Big) \quad . \tag{5.1.40}$$

Here $\overline{\Omega}$ is a set containing the points x_j for $0 \leq t \leq T$. We apply lemma 5.1.7 to each term in the sum on the right hand side of (5.1.40) and find

$$Ch^r \Big(\sum_{1 \leq |\beta| \leq r+4} \| \sum_j D^\beta K_\delta(x - x_j) v_j h^2 \|_{H^0(\overline{\Omega})} \leq Ch^r \left[\sum_{k=1}^{r+5} \frac{1}{\delta^k} \right] \| v_j \|_{-1,h}$$
$$\leq Ch^r \delta^{-(r+5)} \| v_j \|_{-1,h}$$
$$\leq C \| v_j \|_{-1,h} \tag{5.1.41}$$

where, in making the last estimate, we have chosen r so large so that $\frac{r}{r+5} > q$.

For the second term of (5.1.39), we apply the stability condition (4.3.12) and find

$$\| \sum_j D_\alpha^h K_\delta(x(ih,t) - x_j) v_j \rho_{x_i} h^2 \|_{-1,h} \le \| \sum_j D_\alpha^h K_\delta(x(ih,t) - x_j) v_j h^2 |_{-1,h}$$

$$\le \| \sum_j K_\delta(x_i - x_j) v_j h^2 |_{0,h} \quad . \qquad (5.1.42)$$

Here we have extended v_j to be zero outside of Ω^h . This does not affect the estimate since ρ_{x_i} has support in Ω^h . We estimate (5.1.42) using lemma 5.1.9 and find

$$\| \sum_j K_\delta(x_i - x_j) v_j h^2 |_{0,h} \le C \| v_j |_{-1,h} \quad . \qquad (5.1.43)$$

(5.1.42) and (5.1.43) combine to yield

$$\| T_3^{(1)} \|_{-1,h} \le C \| v_j |_{-1,h} \quad .$$

To estimate $T_3^{(2)}$, we use lemma 0.3 and lemma 5.1.8 and the assumption (5.1.15),

$$| T_3 |_{-1,h} = \| \sum_j DK_\delta(x_i - x_j) v_j r_{x_i} h^2 |_{-1,h} \le Ch^{-2} \| r_{x_i} \|_{-1,h} \| \sum_j DK_\delta(x_i - x_j) v_j h^2 \|_{0,h}$$

$$\le Ch^{-2}\delta^{-1} | r_{x_i} |_{-1,h} | v_i |_{-1,h}$$

$$\le C | v_i |_{-1,h} \quad .$$

The function $\sum_j DK_\delta(x - x_j)\xi_j h^2$ is a continuously differentiable function such that

$$\max_{x \in R^2} | \sum_j D^2 K_\delta(x - x_j)\xi_j h^2 | \le C$$

independent of δ . (See the estimate of (5.1.23) in $T_1^{(1)}$.) Thus, if we use lemma 0.4, then

$$\| T_3^{(3)} |_{-1,h} \le | \sum_j DK_\delta(x_i - x_j)\xi_j r_{x_i} |_{-1,h}$$

$$\le C \| r_{x_i} \|_{-1,h} \quad .$$

Combining the estimates on $T_3^{(1)}$, $T_3^{(2)}$, and $T_3^{(3)}$, we find

$$\| T_3 \|_{-1,h} \le C(\| r_{x_i} \|_{-1,h} + \| v_i \|_{-1,h}) .$$

It remains to estimate T_4. We first apply the mean value theorem along the segment from $\tilde{x}_i - \tilde{x}_j$ to $x_i - x_j$.

$$\begin{aligned}
&\sum_j [DK_\delta(\tilde{x}_i - \tilde{x}_j) - DK_\delta(x_i - x_j)][\tilde{\xi}_j \tilde{\rho}_{x_i} - \xi_j \rho_{x_i}] h^2 \\
&\qquad = \sum_j [DK_\delta(x_i - x_j + (e_i + e_j)) - DK_\delta(x_i - x_j)][\tilde{\xi}_j \tilde{\rho}_{x_i} - \xi_j \rho_{x_i}] h^2 \\
&\qquad = \sum_{|\beta| = 1} \sum_j D^\beta DK_\delta(x_i - x_j + y_{ij})[\tilde{\xi}_j \tilde{\rho}_{x_i} - \xi_j \rho_{x_i}](e_i + e_j) h^2
\end{aligned} \tag{5.1.44}$$

where $\max_{ij} | y_{ij} | \le 2 \max_i | e_i |$. We will estimate both terms in (5.1.44) separately and sum the estimates. Since each of the terms has the same form we will concern ourselves with estimating

$$\sum_j D^2 K_\delta(x_i - x_j + y_{ij})[\tilde{\xi}_j \tilde{\rho}_{x_i} - \xi_j \rho_{x_i}] (e_i + e_j) h^2 . \tag{5.1.45}$$

Consider

$$\begin{aligned}
\tilde{\xi}_j \tilde{\rho}_{x_i} - \xi_j \rho_{x_i} &= (\tilde{\xi}_j - \xi_j)\rho_{x_i} + (\tilde{\xi}_j - \xi_j)(\tilde{\rho}_{x_i} - \rho_{x_i}) + \xi_j(\tilde{\rho}_{x_i} - \rho_{x_i}) \\
&= v_j \rho_{x_i} + v_j r_{x_i} + \xi_j r_{x_i} \\
&= \{a_1\} + \{a_2\} + \{a_3\} .
\end{aligned}$$

We estimate (5.1.45) by estimating

$$\sum_{l=1}^{3} \sum_j D^2 K_\delta(x_i - x_j + y_{ij})(e_i + e_j) a_l h^2 . \tag{5.1.46}$$

We first estimate the terms in (5.1.46) containing e_j

$$\sum_{l=1}^{3} \sum_j D^2 K_\delta(x_i - x_j + \cdot) e_j a_l h^2 . \tag{5.1.47}$$

The first term of (5.1.47), l= 1, is estimated by

$$\| \sum_j D^2K_\delta(x_i - x_j + y_{ij})e_j a_1 h^2 \|_{-1,h} \le \| \sum_j D^2K_\delta(x_i - x_j + y_{ij})e_j a_1 h^2 |_{0,h}$$
$$\le C\,\delta^{-1} \max_j |v_j| \; |e_i|_{0,h}$$
$$\le C\,h^{-2}\delta^{-1} |v_j\|_{-1,h} |e_i|_{0,h} \quad (5.1.48)$$

Here we have used the estimate $\max_j |v_j| \le h^{-1}\|v_i\|_{0,h} \le Ch^{-2}\|v_i\|_{-1,h}$ and lemma 5.1.6 to estimate $| \sum_j D^2K_\delta(x_i - x_j + y_{ij})e_j v_i h^2 \|_{0,h}$. The second and third terms of (5.1.47) are estimated similarly. For the second term,

$$\| \sum_j D^2K_\delta(x_i - x_j + y_{ij})e_j a_2 h^2 \|_{-1,h} \le \| \sum_j D^2K_\delta(x_i - x_j + y_{ij})e_j a_2 h^2 |_{0,h}$$
$$\le C\,\delta^{-1} \max_j |v_j| \; \max_j |r_{x_j}| \; \|e_j|_{0,h}$$
$$\le Ch^{-4}\delta^{-2} |r_{x_i}|_{-1,h} \; \|v_j|_{-1,h} \; |e_j|_{0,h}$$
$$(5.1.49)$$

while for the third term,

$$\| \sum_j D^2K_\delta(x_i - x_j + y_{ij})e_j a_3 h^2 \|_{-1,h} \le \sum_j D^2K_\delta(x_i - x_j + y_{ij})e_j a_3 h^2 |_{0,h}$$
$$\le C\,\delta^{-1}\max |r_{x_i}| \; \|e_j\|_{0,h}$$
$$\le C\,h^{-2}\delta^{-1} |r_{x_i}|_{-1,h} \|e_j|_{0,h} \quad (5.1.50)$$

Under the assumption (5.1.15) then each of the estimates (5.1.48)-(5.1.50) can be bounded by $C|e_i|_{0,h}$ where the constant is independent of h. For the terms of (5.1.46) that contain e_i,

$$| \sum_j D^2K_\delta(x_i - x_j + y_{ij})(\sum_{l=1}^{3} a_l)e_i h^2 \|_{-1,h} \le \| \sum_j D^2K_\delta(x_i - x_j + y_{ij})(\sum_{l=1}^{3} a_l)e_i h^2 |_{0,h}$$
$$\le h^{-1}\|e_i\|_{0,h} \sum_{l=1}^{3} | \sum_j D^2K_\delta(x_i - x_j + y_{ij})a_l h^2 \|_{0,h} \quad .$$
$$(5.1.51)$$

We estimate each term in the sum on the right hand side of (5.1.51) individually. For the first term, l=1, we use the fact that ρ_{x_i} is a smooth function,

lemma 5.1.8 and the mean value theorem to find,

$$\| \sum_j D^2K_\delta(x_i - x_j + y_{ij})v_j\rho_{x_i}h^2 \|_{-1,h} \le C \| \sum_j D^2K_\delta(x_i - x_j + y_{ij})v_j \|_{0,h}$$

$$\le \delta^{-2} \| v_j \|_{-1,h} \quad (5.1.52)$$

Similarly, for the second term

$$\| \sum_j D^2K_\delta(x_i - x_j + y_{ij})a_2h^2 \|_{-1,h} \le C \| D^2K_\delta(x_i - x_j + y_{ij})v_j r_{x_i}h^2 \|_{0,h}$$

$$\le C \max_i | r_{x_i} | \ \| v_j \|_{-1,h} \ \delta^{-2}$$

$$\le C\delta^{-2}h^{-2} \| r_{x_i} \|_{-1,h} \| v_j \|_{-1,h} \quad (5.1.53)$$

and for the third term, we use lemma 5.1.8,

$$\| \sum_j D^2K_\delta(x_i - x_j + y_{ij})a_1h^2 \|_{-1,h} \le C \| D^2K_\delta(x_i - x_j + y_{ij})r_{x_i}h^2 \|_{0,h}$$

$$\le C\delta^{-2} \| r_{x_i} \|_{-1,h} \quad (5.1.54)$$

Under the assumption (5.1.15) then we can combine (5.1.52)-(5.1.54) with (5.1.51) to find

$$\| \sum_j D^2K_\delta(x_i - x_j + y_{ij})(\sum_{l=1}^{3} a_l)e_i \|_{0,h} \le C(\| r_{x_i} \|_{-1,h} + \| v_i \|_{-1,h}) .$$

If we combine the estimate of (5.1.47) with the estimate of (5.1.51) we have

$$\| T_4 \|_{-1,h} \le C(\| r_{x_i} \|_{-1,h} + \| r_{y_i} \|_{-1,j} + \| e_i \|_{0,h} + \| v_i \|_{-1,h}) .$$

If we combine the estimates of T_1, T_2, T_3, and T_4 we obtain the estimate (5.1.44). This completes the proof.

The next two results concern the stability and consistency of the density approximation (B.9).

Lemma 5.1.11 : Stability of density approximation

Under the assumptions of theorem 5.1, and if

$$\| e_i \|_{0,h} + | \rho_{x_i} |_{-1,h} + \| r_{y_i} |_{-1,h} \le h^{2+2q} \tag{5.1.55}$$

for $0 \le t \le T^*$, then

$$\| \tilde{\rho}^h(ih,t) - \rho^h(ih,t) \|_{0,h} \le C \, (\| e_i \| + \| r_{x_i} \|_{-1,h} + | r_{y_i} \|_{-1,h}) \tag{5.1.56}$$

$$\| \tilde{\rho}^h(x,t) - \rho^h(x,t) |_{L^2(\bar{R})} \le C(\| e_i \|_{0,h} + | r_{x_i} |_{-1,h} + \| r_{y_i} \|_{-1,h}) \tag{5.1.57}$$

where $\bar{R}$ is a bounded set in R^2.

Proof : Let $x_i = x(ih,t)$, $\tilde{x}_i = \tilde{x}(ih,t)$. We have

$$\begin{aligned}
&\tilde{\rho}^h(ih,t) - \rho^h(ih,t) = \\
&\quad \{ \sum_j (G_x * \Psi_\delta)(\tilde{x}_i - \tilde{x}_j)\tilde{\rho}_x(jh,t)h^2 + \sum_j (G_y * \Psi_\delta)(\tilde{x}_i - \tilde{x}_j)\tilde{\rho}_y(jh,t)h^2\} \\
&\quad \{\sum_j (G_x * \Psi_\delta)(x_i - x_j)\rho_x(jh,t)h^2 + \sum_j (G_y * \Psi_\delta)(x_i - x_j)\rho_y(jh,t)h^2\} \\
&\quad = \{\sum_j (G_x * \Psi_\delta)(\tilde{x}_i - \tilde{x}_j)\tilde{\rho}_x(ih,t)h^2 - \sum_j (G_x * \Psi_\delta)(x_i - x_j)\rho_x(ih,t)h^2\} \\
&\quad + \{\sum_j (G_y * \Psi_\delta)(\tilde{x}_i - \tilde{x}_j)\tilde{\rho}_y(ih,t)h^2 - \sum_j (G_y * \Psi_\delta)(x_i - x_j)\rho_y(ih,t)h^2\} .
\end{aligned} \tag{5.1.58}$$

If we identify the pair $\rho_x(ih,t)$, $\tilde{\rho}_x(ih,t)$ and the pair $\rho_y(ih,t)$, $\tilde{\rho}_y(ih,t)$ with the pair $\xi(ih,t)$, $\tilde{\xi}(ih,t)$, then the estimate of each term in (5.1.58) is identical to the estimate of the stability of the velocity approximation given in lemma 4.3.1. Thus, using the arguments of that lemma, we find

$$| \tilde{\rho}^h(ih,t) - \rho^h(ih,t) |_{0,h} \le C \, (\| e_i \|_{0,h} + \| r_{x_i} \|_{-1,h} + | r_{y_i} \|_{-1,h}) .$$

Similarly, the continuous stability estimate (5.1.57), follows from the arguments for the estimate (4.1.14) of lemma 4.3.1.

Lemma 5.1.12 : Consistency of density approximation

Under the hypothesis of theorem 5.1 we have

$$\max_{0 \le t \le T} | \rho^h(x,t) - \rho(x,t) | \le C\,(\delta^p + (\frac{h}{\delta})^L \delta\ ) \tag{5.1.59}$$

for all $x \in R^2$.

Proof: Consider

$$\rho^h(x,t) - \rho(x,t) =$$

$$\sum_j (G_x * \Psi_\delta)(x - x(jh,t))\rho_x(ih,t)h^2 + \sum_j (G_y * \Psi_\delta)(x - x(jh,t))\rho_y(jh,t)h^2$$

$$- \rho(x,t)$$

$$= \{\sum_j (G_x * \Psi_\delta)(x - x(jh,t))\rho_x(jh,t)h^2 - \int (G_x * \Psi_\delta)(x - x')\rho_x(x',t)dx'\}$$

$$+ \{\sum_j (G_y * \Psi_\delta)(x - x(jh,t))\rho_y(jh,t)h^2 - \int (G_y * \Psi_\delta)(x - x')\rho_y(x',t)dx'\}$$

$$+ \{\int (G_x * \Psi_\delta)(x - x')\rho_x(x',t)dx' + \int (G_y * \Psi_\delta)(x - x')\rho_y(x',t)dx'$$

$$- \rho(x,t)\}$$

$$= \{A\} + \{B\} + \{C\}\ .$$

As in the proof of lemma 4.1.1, terms A and B can be viewed as errors in approximating continuous integrals using the trapezoidal rule. If we use arguments similar to those employed in lemma 4.1.1 we find,

$$\max_{0 \le t \le T} | A | \le C\,(\frac{h}{\delta})^L \delta \tag{5.1.60}$$

and

$$\max_{0 \le t \le T} | B | \le C\,(\frac{h}{\delta})^L \delta\ . \tag{5.1.61}$$

To estimate term C, since ρ has compact support and is sufficiently smooth we can use Poisson's formula to express ρ in terms of its derivatives,

$$\rho(x,t) = G_x * \rho_x + G_y * \rho_y\ .$$

Thus,

$$C = \{\int (G_x * \Psi_\delta)(x - x')\rho_x(x',t)dx' - G_x * \rho_x\}$$

$$+ \{\int (G_y * \Psi_\delta)(x - x')\rho_y(x',t)dx' - G_y * \rho_y\} \ .$$

Each of these terms can be estimated using arguments similar to those used for the estimate of term B in lemma 4.1.1. If we employ those arguments, then we find,

$$| C | \leq C\delta^p \ . \tag{5.1.62}$$

By choosing the constants large enough, the bounds (5.1.60)-(5.1.62) can be shown to hold for all t, $0 \leq t \leq T$. Thus the two bounds imply the result (5.1.59).

We are now ready to prove theorem 5.1.

Proof of Theorem: Define

$$\begin{aligned} e_i(t) &= \tilde{x}(ih,t) - x(ih,h) \\ v_i(t) &= \tilde{\xi}(ih,t) - \xi(ih,t) \\ r_{x_i}(t) &= \tilde{\rho}_x(ih,t) - \rho_x(ih,t) \\ r_{y_i}(t) &= \tilde{\rho}_y(ih,t) - \rho_y(ih,t) \ . \end{aligned}$$

We proceed as in [5] and derive a differential inequality for the quantity

$$\| e_i \|_{0,h} + \| v_i \|_{-1,h} + \| r_{x_i} \|_{-1,h} + \| r_{y_i} \|_{-1,h} \ .$$

Assume that

$$\| e_i \|_{0,h} + \| v_i \|_{-1,h} + \| r_{x_i} \|_{-1,h} + \| r_{y_i} \|_{-1,h} \leq h^{2+2q} \tag{5.1.63}$$

for t, $0 \leq t \leq T^*$. We use equations (3.13) and (B.1) and find for $\dot{e}_i$,

$$\begin{aligned} \dot{e}_i &= \tilde{u}^h(ih,t) - u(ih,t) \\ &= [\tilde{u}^h(ih,t) - u^h(ih,t)] + [u^h(ih,t) - u(ih,t)] \ . \end{aligned}$$

Under the assumption (5.1.63) we can apply the stability lemma, lemma 4.3.2 to the first term and find

$$\| \tilde{u}^h(ih,t) - u^h(ih,t) \|_{0,h} \le C \left(\| e_i \|_{0,h} + \| v_i \|_{-1,h} \right) .$$

We use the consistency lemma, lemma (4.1.1), for the second term,

$$\| u^h(ih,t) - u(ih,t) \|_{0,h} \le C \left(\delta^p + (\frac{h}{\delta})^L \delta \right) .$$

Thus for $0 \le t \le T^*$

$$\| \dot{e}_i \|_{0,h} \le C(\delta^p + (\frac{h}{\delta})^L \delta + \| e_i \|_{0,h} + \| v_i \|_{0,h}) . \tag{5.1.64}$$

If we use $\tilde{F}$ to denote F evaluated at the computational points and equations (3.14) and (B.2), then for $\| \dot{v}_i \|^{-1,h}$ we have,

$$\begin{aligned} \dot{v}_i &= (\tilde{F}_2 \tilde{\rho}_x^h(ih,t) - \tilde{F}_1 \tilde{\rho}_y^h(ih,t)) - (F_2 \rho_x(ih,t) - F_1 \rho_y(ih,t)) \\ &= (\tilde{F}_2 \tilde{\rho}_x^h(ih,t) - F_2 \rho_x(ih,t)) - (\tilde{F}_1 \tilde{\rho}_y^h(ih,t) - F_1 \rho_y(ih,t)) . \end{aligned} \tag{5.1.65}$$

We will show how to estimate the first term in (5.1.65), the second term is done similarly.

$$\begin{aligned} \tilde{F}_2 \tilde{\rho}_x^h(ih,t) - F_2 \rho_x(ih,t) &= (\tilde{F}_2 - F_2) \rho_x(ih,t) + \tilde{F}_2 (\tilde{\rho}_x^h(ih,t) - \rho_x(ih,t)) \\ &= \{(\tilde{F}_2 - F_2) \rho_x(ih,t)\} + \{(\tilde{F}_2 - F_2)(\tilde{\rho}_x^h(ih,t) - \rho_x(ih,t))\} \\ &\quad + \{F_2 (\tilde{\rho}_x^h(ih,t) - \rho_x(ih,t))\} \\ &= \{A\} + \{B\} + \{C\} . \end{aligned}$$

To estimate term A, F is continuously differentiable so that

$$\begin{aligned} \| \tilde{F}_2 - F_2 \| &= \| F_2(\tilde{x}(ih,t)) - F_2(x(ih,t)) \|_{0,h} \le \sup_{x \in \bar{\Omega}} | \nabla F_2 | \, \| \tilde{x}(ih,t) - x(ih,t) \|_{0,h} \\ &\le C \| \tilde{x}(ih,t) - x(ih,t) \|_{0,h} \end{aligned} \tag{5.1.66}$$

where $\bar{\Omega}$ is a compact set containing the points $x(ih,t)$, $\tilde{x}(ih,t)$ for $0 \le t \le T^*$. If we use inequality (5.1.66) and the assumption that ρ_x is continuously differentiable then we find

$$\| (\tilde{F}_2 - F_2)\rho_x(ih,t) \|_{-1,h} \le \| (\tilde{F}_2 - F_2)\rho_x(ih,t) \|_{0,h}$$

$$\leq C \| \widetilde{F}_2 - F_2 \|_{0,h}$$
$$\leq C \| e_i \|_{0,h} \ .$$

For the second term, B, we first apply lemma 0.3 and then estimate $| \widetilde{F}_2 - F_2 |_{0,h}$ as in the estimate for term A.

$$\begin{aligned} | (\widetilde{F}_2 - F_2)(\widetilde{\rho}_x^h(ih,t) - \rho_x(ih,t)) \|_{-1,h} &\leq Ch^{-2} \| \widetilde{F}_2 - F_2 \|_{0,h} \| \widetilde{\rho}_x^h(ih,t) - \rho_x(ih,t) |_{-1,h} \\ &\leq Ch^{-2} \| e_i \|_{0,h} \| r_{x_i} \|_{-1,h} \\ &\leq C | e_i |_{0,h} \ . \end{aligned}$$

Here we make the last estimate using the assumption (5.1.63). Since F_2 is continuously differentiable, term C can be estimated using lemma 0.4,

$$\| F_2 (\widetilde{\rho}_x^h(ih,t) - \rho_x(ih,t)) \|_{-1,h} \leq C \| r_{x_i} \|_{-1,h} \ .$$

If we combine the estimates for terms A, B, and C, and estimate the second term of (5.1.65) in a similar manner, then for $0 \leq t \leq T^*$,

$$\| \dot{v}_i \|_{-1,h} \leq C(| e_i |_{0,h} + \| r_{x_i} \|_{-1,h} + \| r_{y_i} |_{-1,h}) \ . \tag{5.1.67}$$

It remains to estimate $\| \dot{r}_{x_i} \|_{-1,h}$ and $\| \dot{r}_{y_i} |_{-1,h}$. We use equations (3.15) and (B.3) to estimate $| \dot{r}_{x_i} |_{-1,h}$.

$$\begin{aligned} \dot{r}_{x_i} &= -\widetilde{u}_{1x}^h(ih,t)\widetilde{\rho}_x^h(ih,t) - \widetilde{u}_{2x}^h(ih,t)\widetilde{\rho}_y^h(ih,t) \\ &\qquad + u_{1x}(ih,t)\rho_x(ih,t) + u_{2x}(ih,t)\rho_y(ih,t) \\ &= \{u_{1x}(ih,t)\rho_x(ih,t) - \widetilde{u}_{1x}^h(ih,t)\widetilde{\rho}_x^h(ih,t)\} \\ &\qquad + \{u_{2x}(ih,t)\rho_y(ih,t) - \widetilde{u}_{2x}^h(ih,t)\widetilde{\rho}_y^h(ih,t)\} \ . \end{aligned} \tag{5.1.68}$$

To estimate the first term of (5.1.68), we rewrite it as,

$$\begin{aligned} u_{1x}(ih,t)\rho_x(ih,t) - \widetilde{u}_{1x}^h(ih,t)\widetilde{\rho}_x^h(ih,t) &= \{[u_{1x}(ih,t) - u_{1x}^h(ih,t)]\rho_x(ih,t)\} \\ &\quad + \{u_{1x}^h(ih,t)\rho_x(ih,t) - \widetilde{u}_{1x}^h(ih,t)\widetilde{\rho}_x(ih,t)\} \\ &= \{A\} + \{B\} \ . \end{aligned}$$

We use the pointwise estimate of lemma 5.1.1 and lemma 0.4 to estimate

term A, and find,

$$\| [\, u_{1x}(ih,t) - u^h_{1x}(ih,t)\,]\, \rho_x(ih,t) \|_{0,h} \le C\,(\delta^p + (\frac{h}{\delta})^L)\ .$$

We use lemma 5.1.10 to estimate term B,

$$\| u^h_{1x}(ih,t)\rho_x(ih,t) - \tilde{u}^h_{1x}(ih,t)\tilde{\rho}_x(ih,t) \|_{-1,h} \le C(\| e_i \|_{0,h} + \| r_{x_i} \|_{-1,h} + \| v_i \|_{-1,h})$$

The estimate for the second term of (5.1.68) is carried out in a similar manner. Therefore,

$$\| \dot{r}_{x_i} \|_{-1,h} \le C(\| e_i \|_{0,h} + \| r_{x_i} \|_{-1,h} + \| v_i \|_{-1,h} + \delta^p + (\frac{h}{\delta})^L)\ . \qquad (5.1.69)$$

We also find that

$$\| \dot{r}_{y_i} \|_{-1,h} \le C(\| e_i \|_{0,h} + \| r_{x_i} \|_{-1,h} + \| v_i \|_{-1,h} + \delta^p + (\frac{h}{\delta})^L)\ . \qquad (5.1.70)$$

If we add the inequalities (5.1.64), (5.1.67), (5.1.69), and (5.1.70), then we find

$$\begin{aligned} &\| \dot{e}_i \|_{0,h} + \| \dot{v}_i \|_{-1,h} + \| \dot{r}_{x_i} \|_{-1,h} + \| \dot{r}_{y_i} \|_{-1,h} \\ &\qquad \le C(\| e_i \|_{0,h} + \| v_i \|_{-1,h} + \| r_{x_i} \|_{-1,h} + \| r_{y_i} \|_{-1,h} + \delta^p + (\frac{h}{\delta})^L)\ . \end{aligned} \qquad (5.1.71)$$

We now employ arguments like those used in the proof of method A'. From the differential inequality (5.1.71) and lemma 4.3.2 it follows that for $0 \le t \le T^*$

$$\| e_i \|_{0,h} + \| v_i \| + \| r_{x_i} \|_{-1,h} + \| r_{y_i} \|_{-1,h} \le y(t)$$

where $y(t)$ is a solution of

$$\dot{y}(t) = C_0(y(t) + \delta^p + (\frac{h}{\delta})^L) \qquad y(0) = 0\ .$$

Thus

$$\| e_i \|_{0,h} + \| v_i \|_{-1,h} + \| r_{x_i} \|_{-1,h} + \| r_{y_i} \|_{-1,h} \le C_1(\delta^p + (\frac{h}{\delta})^L)$$

$$= C_1(h^{pq} + h^{L(1-q)}) \ . \tag{5.1.72}$$

The estimate (5.1.72) holds for all times such that (5.1.63) is satisfied. The constant C_1 depends on T and C_0, but not on T^*. We now remove the assumption (5.1.63).

We are assuming that p, q, and L are chosen so that $h^{pq} + h^{L(1-q)} \le h^{2+2q+\varepsilon}$ for some $\varepsilon > 0$. Thus, there exists an h_0 so that for $h < h_0$

$$C_1(h^{pq} + h^{Lq}) \le \frac{h^{2+2q}}{2} \ .$$

Let T^* be the first time such that

$$\| e_i(t) \|_{0,h} + \| v_i(t) \|_{-1,h} + \| r_{x_i} \|_{-1,h} + \| r_{y_i} \|_{-1,h} \ge h^{2+2q} \ . \tag{5.1.73}$$

If $T^* \le T$, then (5.1.72) holds and

$$\| e_i \|_{0,h} + \| v_i \|_{-1,h} + \| r_{x_i} \|_{-1,h} + \| r_{y_i} \|_{-1,h} \le \frac{h^{2+2q}}{2} \tag{5.1.74}$$

for $0 \le t \le T^*$. But at T^* (5.1.74) contradicts (5.1.73), so T^* must be greater than or equal to T. Thus for $h \le h_0$ and for $0 \le t \le T$,

$$\| e_i \|_{0,h} + \| v_i \|_{-1,h} + \| r_{x_i} \|_{-1,h} + \| r_{y_i} \|_{-1,h} \le C(h^{pq} + h^{L(1-q)}) \ . \tag{5.1.75}$$

This is estimate (i).

Using (5.1.75) we will obtain the other estimates (ii)-(v). To obtain (ii), we use (5.1.75), lemma 4.3.1, and the consistency lemma, lemma 4.1.1,

$$\begin{aligned} \| \tilde{u}^h(ih,t) - u(ih,t) \|_{0,h} &\le C(\| \tilde{u}^h(ih,t) - u^h(ih,t) \|_{0,h} + \| u^h(ih,t) - u(ih,t) \|_{0,h} \\ &\le C(\| e_i \|_{0,h} + \| v_i \|_{-1,h}) + C(\delta^p + (\frac{h}{\delta})^L \delta) \\ &\le C(h^{pq} + h^{L(1-q)}) \ . \end{aligned}$$

The continuous velocity error, (iii), can be estimated using (5.1.75), the

continuous stability estimate of lemma 4.3.1, (4.3.14), and the pointwise consistency estimate of lemma 4.1.1,

$$\begin{aligned}\| \tilde{u}^h(ih,t) - u(ih,t) \|_{L^2(\mathbb{R})} &\le \| \tilde{u}^h(ih,t) - u^h(ih,t) \|_{L^2(\mathbb{R})} + \| u^h(ih,t) - u(ih,t) \|_{L^2(\mathbb{R})} \\ &\le C(\| e_i \|_{0,h} + \| v_i \|_{-1,h}) + C(\delta^p + (\frac{h}{\delta})^L \delta) \\ &\le C(h^{pq} + h^{L(1-q)}) \ .\end{aligned}$$

For (iv), we rewrite the error as,

$$\| \tilde{\rho}^h(ih,t) - \rho(ih,t) \|_{0,h} \le \| \tilde{\rho}^h(ih,t) - \rho^h(ih,t) \|_{0,h} + \| \rho^h(ih,t) - \rho(ih,t) \|_{0,h} \ .$$

We estimate the first term using lemma 5.1.11 and the second term using lemma 5.1.12. Therefore

$$\begin{aligned}\| \tilde{\rho}^h(ih,t) - \rho(ih,t) \|_{0,h} &\le C(\| e_i \|_{0,h} + \| v_i \|_{-1,h} + \| r_{x_i} \|_{-1,h} + \| r_{y_i} \|_{-1,h}) \\ &\quad + C(\delta^p + (\frac{h}{\delta})^L \delta) \\ &\le C(h^{pq} + h^{L(1-q)}) \ .\end{aligned}$$

The estimate of the continuous density (v) is done in a similar manner. This completes the proof.

5.2. Computational Results

In this section we describe the computational results when method B is applied to the test problem described in section 4.2. We did not attempt to rigorously check the error estimates of theorem 5.1, but instead, sought to answer the following questions:

Is the method stable?

Is the requirement of high order accuracy in the convergence proof necessary in practice?

How does the error behave when the parameters h, Ψ, and δ are varied?

How does the method compare with the results of method A?

Our reason for not checking rigorously the error estimates of theorem 5.1 is that the test problem does not satisfy the assumptions of the theorem. In particular, the external force function $\vec{F}$ is not continuously differentiable. This situation can be remedied, and we plan to do a detailed analysis of the error estimates of theorem 5.1 in the future.

The parameters to be chosen are h, the initial mesh width, δ, the smoothing parameter, and the function Ψ used in (B.5)-(B.8). We allowed h to take on three values, h = .0886, h = .0728, and h = .0626 , corresponding to 200, 300, and 400 points respectively. The error estimates of theorem 5.1 suggest that we should choose $\delta = h^q$ for some $q < 1$. We therefore let $\delta = h^q$ for q = .95, q = .85, and q = .75. The functions Ψ used in the approximations (B.5)-(B.8) were chosen to be in $M^{L,p}$ and one of

$$(p = 2\,,\ L = \infty) \qquad \Psi(r) = \frac{e^{-r^2}}{\pi} \tag{5.2.1}$$

$$(p = 4\,,\ L = \infty) \qquad \Psi(r) = \frac{1}{\pi}\left(2e^{-r^2} - \frac{1}{2}e^{-\frac{r^2}{2}}\right) \tag{5.2.2}$$

$$(p = 6\,,\ L = \infty) \qquad \Psi(r) = \frac{1}{\pi}\left(8e^{-r^2} - \frac{27}{4}e^{-\frac{3}{4}r^2} + e^{-\frac{r^2}{2}}\right) . \qquad (5.2.3)$$

These functions are suggested by Beale and Majda in [7]. The integration of the ordinary differential equations (B.1)-(B.4) was preformed using fourth order Runge-Kutta. The time step, $\Delta t = .1$, was chosen sufficiently small so that a decrease in the time step did not significantly effect the results.

As in the tests on method A, we measure the relative error in the first component of the velocity. This error is computed by

$$\frac{\left[\sum_{ih \in \Omega^h} |\tilde{u}_1^h(\tilde{x}(ih,t)) - u_1(\tilde{x}(ih,t))|^2 h^2\right]^{\frac{1}{2}}}{\left[\sum_{ih \in \Omega^h} |u_1(\tilde{x}(ih,t))|^2 h^2\right]^{\frac{1}{2}}}$$

where $\tilde{u}_1^h$ is the computed velocity , u_1^h is the exact velocity and $\tilde{x}(ih,t)$ are the computed point positions. We present in tables 5.2.1 and 5.2.2 the errors for the various choices of the parameters at times t = 1.0 and t = 1.6. These times correspond to a maximum point rotation of $\frac{\pi}{4}$ and 2π radians respectively. (See figures 4.2.3 and 4.2.4.)

We found that the errors grew at a rate which was independent of the number of time steps, i.e. the method is stable. The requirement of the convergence proof, that the method be at least accurate of order h^{2+2q}, does not appear to be necessary in practice. For every choice of the parameters for which this assumption is not satisfied, the computational results indicate that the method is still convergent. We are pleased with this result for it demonstrates that the method is more robust than the convergence proof, theorem 5.1, indicates.

Although the error estimates (i)-(v) of theorem 5.1 are not strictly applicable, they indicate some of the features of the behavior of the error.

In particular, the estimates indicate that choosing the function Ψ with p larger should decrease the error. This behavior is observed in the results. Also, like method A, the change in the error was very significant. For example, at time t = 1.6 with h = .0626 and $\delta = h^{.75}$, the error for p = 2 was 8 % while for p = 6 it was .78 %.

It was found that the smallest errors at time t = 1.0 are obtained using $\delta = h^{.95}$ and $p = 6$, while the smallest errors at time t = 1.6 are obtained using $\delta = h^{.75}$ and $p = 6$. Thus it appears necessary to choose a larger δ to insure long time accuracy. Since the major difference in the approximations at these two times is the distribution of the computational points, we conclude that one should choose a larger smoothing parameter when the computational points become disorganized. This same conclusion was drawn for method A, and we remark that quite similar behavior is observed in accuracy tests for the vortex method [46].

The conclusion that one needs to choose a relatively large δ to insure long time accuracy appears to be a characteristic feature of these methods. Unfortunately, this conclusion raises the question "How large should δ be ?". We do not believe that the relative size of delta that we found for this problem is universal, i.e., we believe that the q in $\delta = h^q$ is problem dependent. To overcome the difficulty in choosing the appropriate δ we have devised an empirical procedure for determining it. This procedure is described and tested in section 6.1.

In comparison with method A, the errors associated with method B were much smaller. For example, using the optimal parameters, at time t = 1.6, the error with method A is 7.3 % while the error for method B is .29 %. A possible explanation for this result is the following: To calculate the solutions to (1.12)-(1.14) in the vorticity- stream formulation we must evaluate density derivatives. In method A we evaluate these derivatives by differentiating the

density approximation (3.12). In general, differentiating an approximation leads to a loss in accuracy. This loss of accuracy can be theoretically estimated. If one compares the accuracy estimates for the density derivative approximations (A.5) and (A.6) (see estimate (4.1.12)) with the accuracy estimates for the density approximation (3.12) (see estimate (2.8)) one finds that the accuracy of the density derivatives is a factor of δ^{-1} larger than that for the density. In method B, the density derivatives are primary variables, i.e. not derived from another computed quantity, so the error introduced by numerically differentiating the density approximation is not present.

We conclude from these computations that method B is stable and accurate. The method appears to be more robust than theorem 5.1 indicates. It also appears that method B is the more accurate of the two methods. The convergence results, although not strictly applicable, indicate to some extent the behavior of the error. In particular, the conclusions based on the error estimates, i.e. letting $\delta = h^q$ and using functions Ψ in $M^{L,p}$ with p large, seem to be valid.

Relative Velocity Error (in %) h = .0886			
	p=2	p=4	p=6
$\delta = h^{.95}$	5.3	0.82	0.25
$\delta = h^{.85}$	8.3	1.8	0.32
$\delta = h^{.75}$	12.0	3.9	1.0

Relative Velocity Error (in %) h = .0723			
	p=2	p=4	p=6
$\delta = h^{.95}$	3.7	0.40	0.14
$\delta = h^{.85}$	6.1	1.0	0.12
$\delta = h^{.75}$	9.8	2.4	0.49

Relative Velocity Error (in %) h = .0628			
	p=2	p=4	p=6
$\delta = h^{.95}$	2.8	0.24	0.10
$\delta = h^{.85}$	4.8	0.67	0.061
$\delta = h^{.75}$	8.1	1.7	0.29

Table 5.2.1

Error in Velocity at t = 1.0

Relative Velocity Error (in %) h = .0886			
	p=2	p=4	p=6
$\delta = h^{.95}$	5.7	3.7	5.4
$\delta = h^{.85}$	8.3	2.7	3.5
$\delta = h^{.75}$	12.0	3.9	2.1

Relative Velocity Error (in %) h = .0723			
	p=2	p=4	p=6
$\delta = h^{.95}$	3.9	2.4	3.7
$\delta = h^{.85}$	6.0	1.5	2.1
$\delta = h^{.75}$	9.7	2.3	1.1

Relative Velocity Error (in %) h = .0628			
	p=2	p=4	p=6
$\delta = h^{.95}$	3.0	1.6	2.7
$\delta = h^{.85}$	4.8	1.0	1.4
$\delta = h^{.75}$	8.0	1.6	0.78

Table 5.2.2

Error in Velocity at t = 1.6

6. Application to a 2-D Thermal

In this section we use method A and method B to calculate the motion of a 2-D or 'line' thermal. A 2-D thermal is that object which was initially a cylindrical body of buoyant fluid having its axis of symmetry perpendicular to the gravitational force and which has moved under the effect of gravity. We model the motion of a line thermal by computing the solution to equations (1.15)-(1.18) with initial data of the form

$$\rho_0(x,y) = \begin{cases} \rho_1 & \text{for} \quad r \le \bar{r} \\ 0 & \text{for} \quad r > \bar{r} \end{cases} \tag{6.1.1}$$

$$\xi_0(x,y) = 0 \tag{6.1.2}$$

Here $r = (x^2 + y^2)^{\frac{1}{2}}$ and ρ_1 is a constant such that $\rho_1 \le 0$. For the external force, $\vec{F}$, we use $\vec{F} = -(\,0\,,\,g\,)$ where g is the gravitational constant. In all of our calculations we use $\rho_1 = -.1$, $g = 10.0$ and $\bar{r} = .5$.

Due to the presence of gravity, the circular region of lighter fluid will rise in time. The problem is to calculate the motion of this lighter fluid. For a more detailed description of line thermals and of theoretical and experimental studies of their motion see ([25],[47],[52],[53]). An earlier numerical study of the motion of a 2-D thermal is presented in [43].

We are interested in this problem for two reasons. The first is to gain an understanding of the behavior of the methods when they are applied to more realistic problems, i.e. problems closer to those which our numerical methods are intended to be useful for. The motion of a 2-D thermal is a good test problem for it contains the key features of convective motion and yet is simple in the sense that the initial conditions are easily described and there are no external boundaries.

The second reason we consider this problem is to obtain an understanding of the nature of the solutions of equations (1.15)-(1.18) with data of the form (6.1.1)-(6.1.2). We remark that this problem can also be considered

one of calculating the motion of a vortex sheet [4] whose strength changes in time. From equation (1.22) we see that vorticity can only grow where there are density gradients. In this problem the density gradient is non-zero only on the interface separating the light and heavy fluid. (The density gradient is a distribution with support on the interface.) Thus, the vorticity will only be non-zero on the interface, i.e. the interface is a vortex sheet. We hope that some of the conclusions derived from this investigation will lead to a better understanding of the motion of vortex sheets in general. We mention that the evolution of an infinite vortex sheet is being studied by Krasny [37] using methods similar to those employed here.

We will first present the results obtained with method B. In section 6.2 we will present the results obtained with method A.

6.1. Application of Method B to 2-D Thermal

To implement method B we must choose an initial distribution for the computational points and the initial conditions for the O.D.E.s (B.2)-(B.4). As the method is described in section 3, the computational points are chosen to be those in $\Omega^h = \Omega \cap \Lambda^h$, where Ω is a set containing the support of the initial density derivatives and the initial vorticity. The initial conditions for the O.D.E.s (B.2)-(B.4) are chosen to be $\rho_{0_x}(ih)$, $\rho_{0_y}(ih)$ and $\xi_0(ih)$ where $ih \in \Omega^h$. Unfortunately, we are not able to follow this procedure because the density derivatives of the initial conditions are distributions. We proceed as follows:

We initially distribute the computational points uniformly with spacing h along the interface $\{(x,y) \mid x^2 + y^2 = \bar{r}^2\}$, i.e. if α_k are the computational points we let

$$\alpha_k = \bar{r}(\cos(kh)\,,\,\sin(kh)) \qquad \text{for} \quad k = 1, \cdots N$$

where

$$h = \bar{r}\ \frac{2\pi}{N}.$$

See figure 6.1.1.

To select initial conditions for the O.D.E.s describing the evolution of the density derivatives, we set the density derivatives equal to a multiple of the normal to the interface of the initial data. We use

$$\tilde{\rho}_x(\alpha_k,0) = \rho_{0_x}(\alpha_k) = \gamma\cos(\alpha_k)$$

$$\tilde{\rho}_y(\alpha_k,0) = \rho_{0_y}(\alpha_k) = \gamma\sin(\alpha_k)$$

where the constant γ is chosen so that when we construct the density using (B.6) the total mass of the lighter fluid is identical to the mass of the lighter fluid in the initial condition (6.1.1). Specifically, we choose γ so that

$$\begin{aligned}\int_{|x|\le \bar{r}^2} \tilde{\rho}^h(x,0)dx &= \gamma \int_{|x|\le \bar{r}} \sum_{k=1}^{N} G_x(x-\alpha_k)\cos(\alpha_k)dx \\ &\quad + \gamma \int_{|x|\le \bar{r}} \sum_{k=1}^{N} G_y(x-\alpha_k)\sin(\alpha_k)dx \\ &= -\rho_1\,\pi\,\bar{r}^2\ .\end{aligned}$$

Here we have absorbed into γ the weight factor h^2 normally occurring in the sum (B.5). For the initial conditions of the vorticity O.D.E.s (B.2) we use

$$\tilde{\xi}(\alpha_k,0) = \tilde{\xi}_0(\alpha_k) = 0 \qquad \text{for } k = 1,\cdots,N\ .$$

There is a second technique for constructing suitable initial conditions for the numerical method. This consists of modifying the initial conditions (6.1.1)-(6.1.2) so that the procedure described in section 3 will work. Specifically, one would first smooth the initial data and then approximate the solution of this smoothed problem. The usefulness of such an approach has been demonstrated in conjunction with the Fourier method [41]. A disadvantage of using this latter technique is that it uses more computational points than the former. This economy is our primary reason for using the first technique.

The other parameters that need to be selected are the function Ψ and the parameter δ. We chose the function Ψ used in (B.6) and (B.8) to be

$$\Psi(r) = \frac{1}{\pi} e^{-r^2} \tag{6.1.3}$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$. This choice corresponds to $\Psi \varepsilon M^{L,p}$ for p = 2 and $L = \infty$.

The last parameter to be specified is the smoothing parameter δ. We found that the proper choice of δ was a critical factor in obtaining consistent numerical results. Our first, and unsuccessful, technique was to choose δ in a manner suggested by the error estimates of section 5. We let $\delta = h^q$ for some $q < 1$. With these choices we then solved the O.D.E.s (B.1)-(B.4) using fourth order Runge-Kutta with a step size of $\Delta t = .1$. We monitored the quality of the computation by computing the mass of the thermal at each time step. Typically, for the values of q that we tried, q = .95, q = .75, and q = .5, it was found that there was a time $\bar{T}$, depending on q, such that the solution failed to conserve mass for $t \geq \bar{T}$. Furthermore, decreasing h, (increasing the number of computational points) did not improve the results. The results of the calculation for $\delta = h^{.75}$ are shown in figures 6.1.2 and 6.1.3. In figure 6.1.2 we present a graph of the mass as a function of time for different values of h. In figure 6.1.3 we present the position of the interface at time t = 3.0 for the values of h used in figure 6.1.2. (For clarity we have connected the computational points with straight line segments). We see that as $h \to 0$ the results are not consistent.

An explanation of these results lies in an analysis of the manner in which we are approaching the solution of equations (1.15)-(1.18) with the numerical approximation. We expect that as $h \to 0$ and $\delta \to 0$ the numerical approximation will converge to the solution. A-priori there is no reason to assume that approaching the limit $h = 0$ and $\delta = 0$ using the relation $\delta = h^q$

is appropriate for this problem. This conclusion is based on the results of section 5 which depend on the assumption that the flow and associated quantities are smooth. This assumption is clearly not satisfied in our problem. We believe that the inconsistent numerical results are occurring because the limiting process is not being carried out in the correct fashion.

To avoid this difficulty we use the the following procedure. We fix a value of δ, say δ_1, and then determine an h_1, so that for all $h < h_1$ the computational results do not significantly change for $0 \le t \le \tilde{T}$. ($\tilde{T}$ is selected arbitrarily). We then select another δ, say δ_2 with $\delta_2 < \delta_1$. Again, we find an h_2 such that for $h < h_2$ the results do not significantly change over $0 \le t \le \tilde{T}$. We continue in this manner and thus construct a sequence (h_1,δ_1) , (h_2,δ_2) , $\cdots$ (h_i,δ_i). Our hope is that as the sequence $(h_i,\delta_i) \to (0,0)$ the corresponding numerical solutions approach the solutions of (1.15)-(1.18) with data (6.1.1)-(6.1.2). One can view the above procedure as empirically determining the appropriate approach to the limit.

This procedure worked well. For the values of δ_i that we tried it was found that there was always an h_i so that for $h \le h_i$ the approximation procedure converged. Our assessment of convergence was obtained by considering three features of the computation,

(1) The conservation of mass of the lighter fluid for $0 \le t \le \tilde{T}$.

(2) The convergence of the length of the interface.

(3) The convergence of the position of the interface.

An example of the convergence behavior for $\delta = .07$ is illustrated in table 6.1.1 and figure 6.1.4. Table 6.1.1 shows the variation with respect to h at time $\tilde{T} = 3.0$ of the mass and of the arc length. Figure 6.1.4 shows the position of the interface at time $\tilde{T} = 3.0$ as h is varied. We mention that the time step, $\Delta t = .1$ was chosen small enough so that it had an insignificant

effect on the results.

In figures 6.1.5 to 6.1.11 we present the results of the computation for $0 \le t \le 3.0$ and for values of δ = .1, .09, .08, .07, .06, .05, .04. Each of these figures is the converged solution (with respect to h) corresponding to that particular δ. We limited ourselves to $\delta \ge .04$ because for smaller δ's the number of points necessary to obtain convergence was very large (> 800). In figure 6.1.11a we present a close-up of the solution corresponding to δ = .04 at time t = 3.0.

From the computational results we see that as $\delta \to 0$ the solutions approach a limit solution. This result raises two interesting questions:

What is the nature of the limiting solution ?

Is this limiting solution the solution of (1.15)-(1.18) with initial data (6.1.1)-(6.1.2)?

We first discuss the nature of the limiting solution. The most prominent feature of the solutions as $\delta \to 0$ is the behavior of the "curl" of the thermal. (See figure 6.1.11a.) As δ tends to zero, these curls have an increasing number of winds. Associated with this phenomena we find that the arc length of the interface is increasing. In figure 6.1.12 we present a plot of the length of the interface at time t = 3.0 as a function of δ. We fit a curve of the form $l(\delta) = A\delta^{-k} + B$ and found that $k = -.105$. Thus we conclude that as $\delta \to 0$ the interface becomes infinite in length. The curve $l(\delta)$ is the solid line in figure 6.1.12. This suggests that a singularity is occurring in the limiting solution.

To understand the nature of this singularity we also considered the growth in time of the length of the interface. In figure 6.1.13 we plot the arclength verses time for the values of δ tested. What we observe is that up to time $\hat{T} \approx 1$ the arclength of the solution is independent of δ. After that

time the arclength grows linearly in time at a rate which is inversely proportional to δ. We conjecture that the time $\hat{T}$ is a critical time, the time at which a singularity occurs in the flow. Before time $\hat{T}$ the limit solution is well behaved, after that time we expect that the interface of the limit solution becomes infinite.

We are also interested in the 'size' of the singularity. We note that as $\delta \to 0$ the solutions agree over an increasingly larger part of the thermal. This is illustrated in figures 6.1.14 where we have superimposed the solutions for $\delta = .1$ and $\delta = .09$ and the solutions for $\delta = .05$ and $\delta = .04$. These results suggest that the limit solution is well represented by the finite δ solution except for in the 'eye' of the curl. We conclude that the support of the singularity is small, possibly confined to one point.

The behavior of the vorticity is illustrated in figure 6.1.15. In this figure we present perspective views of the absolute value of the vorticity at time t = 3.0 for solutions corresponding to $\delta = .1$, $\delta = .07$ and $\delta = .04$. The plots were obtained by evaluating at the grid points of a 90×90 grid the approximation of the vorticity given by (3.11). This information was then used in the perspective plotting routine in [23]. From the results we see that, like the arc length, the magnitude of the vorticity increases as the smoothing parameter $\delta \to 0$. We also note that the largest growth of the vorticity is concentrated in the "eye" of the curl. To illustrate the growth of the vorticity as $\delta \to 0$ we present in figure 6.1.17 a graph of the magnitude of the vorticity as a function of δ. We fit a curve of the form $\upsilon\ (\delta) = C + D\,\delta^{-k}$ and found that $k = -2.028$. Thus we expect that as $\delta \to 0$ the vorticity becomes unbounded.

Based on these results we conclude that the limiting solution is a singular one. The singularity most likely occurs at a point and is characterized by an infinite value of vorticity there. The interface of the limit solution also

appears to be infinite. Other features of the solution deserve further study, in particular a more detailed look at the structure of the singularity and how the singularity evolves in time.

The second question, the relation between the solutions obtained as a limit of our numerical solutions and the solutions of (1.15)-(1.18) with data (6.1.1)-(6.1.2), is a difficult question to answer. One must be cautious, as there are examples of numerical schemes which are consistent and convergent, but converge to erroneous solutions. (See [33]). We are not able to answer this question, but we propose the following investigation to understand the limiting process: When we fix δ and let $h \to 0$ we expect that we are converging to the solution of

$$\frac{\partial \rho_x}{\partial t} + \vec{u}_\delta \cdot grad\rho_x = -(u_{1\delta})_x \rho_x - (u_{2\delta})_x \rho_y \tag{6.1.4}$$

$$\frac{\partial \rho_y}{\partial t} + \vec{u}_\delta \cdot grad\rho_y = -(u_{1\delta})_y \rho_x - (u_{2\delta})_y \rho_y \tag{6.1.5}$$

$$\rho = G_x * \Psi_\delta * \rho_x + G_y * \Psi_\delta * \rho_y \tag{6.1.6}$$

$$\frac{\partial \xi}{\partial t} + \vec{u}_\delta \cdot grad\xi = \rho_y F_1 - \rho_x F_2 \tag{6.1.7}$$

$$\vec{u}_\delta = K * \Psi_\delta * \xi \tag{6.1.8}$$

If the solutions of (1.15)-(1.18) are sufficiently smooth then this convergence can be proven - in fact the result follows immediately from theorem 5.1. We call the equations (6.1.4)-(6.1.8) the delta equations associated with (1.15)-(1.18). The question of the limit of the numerical solutions as $\delta \to 0$ (assuming that convergence with respect to h has been achieved) reduces to the question of the behavior as $\delta \to 0$ of the solutions of the delta equations. An investigation of this type is very similar to the study of the " model " problem for finite difference equations ([34],[54]). Certainly a satisfactory solution to this problem concerning the delta equations will aid in the understanding of the limiting process.

We have not carried out this investigation, but we note that our computed results can be used to ascertain properties of the solutions of the delta equations. In particular we see that the gross features of the flow, the distance traveled by the thermal and its general shape are well represented by relatively large δ. This result has several implications. The first is that it indicates that one need not use an excessively small delta (and hence many computational points) to obtain the general features of the solution. Also the solutions of the delta problem appear to be non-diffusive. If the solutions were diffusive we would expect that that the distance traveled by the thermal to be dependent upon δ. Another observation is that even though the limit solution is quite complicated, this limiting solution does not contaminate the finite delta solution. An example of a situation where this does not hold is in the use of Fourier methods to solve problems whose solutions have jump discontinuities. If we consider an approximate solution to be one consisting of a finite number of modes, then, due to Gibbs phenomena, the existence of a jump discontinuity will contaminate the approximate solution. These observations indicate that the delta-equations have many desirable approximation features.

We remark that the procedure for determining h and δ used above can be used for other problems. For example, in the test problem of section 4 and section 5, we let $\delta = h^q$. This choice was based on error estimates for the approximations used in the methods. Other than the fact that q should be chosen such that $0 \leq q < 1$, the precise value of q was not specified by the error estimates. One could use the procedure described in this section to determine the appropriate q. To do this one would fix a value of δ, say δ_1, and then find an h_1 such that for $h \leq h_1$ the computed solution didn't change. One would then select another δ_2, with $\delta_2 < \delta_1$, and determine an h_2 such that for $h < h_2$ the solution didn't change. Continuing in this manner a

set of values (h_1,δ_1), (h_2,δ_2), $\cdots$, (h_i,δ_i) are determined. By fitting a curve of the form $\delta = h^q$ to these points, an appropriate value of q could be found.

This completes our discussion of the application of method B to compute the motion of a 2-D thermal. In the next section we describe the results of applying method A to the same problem.

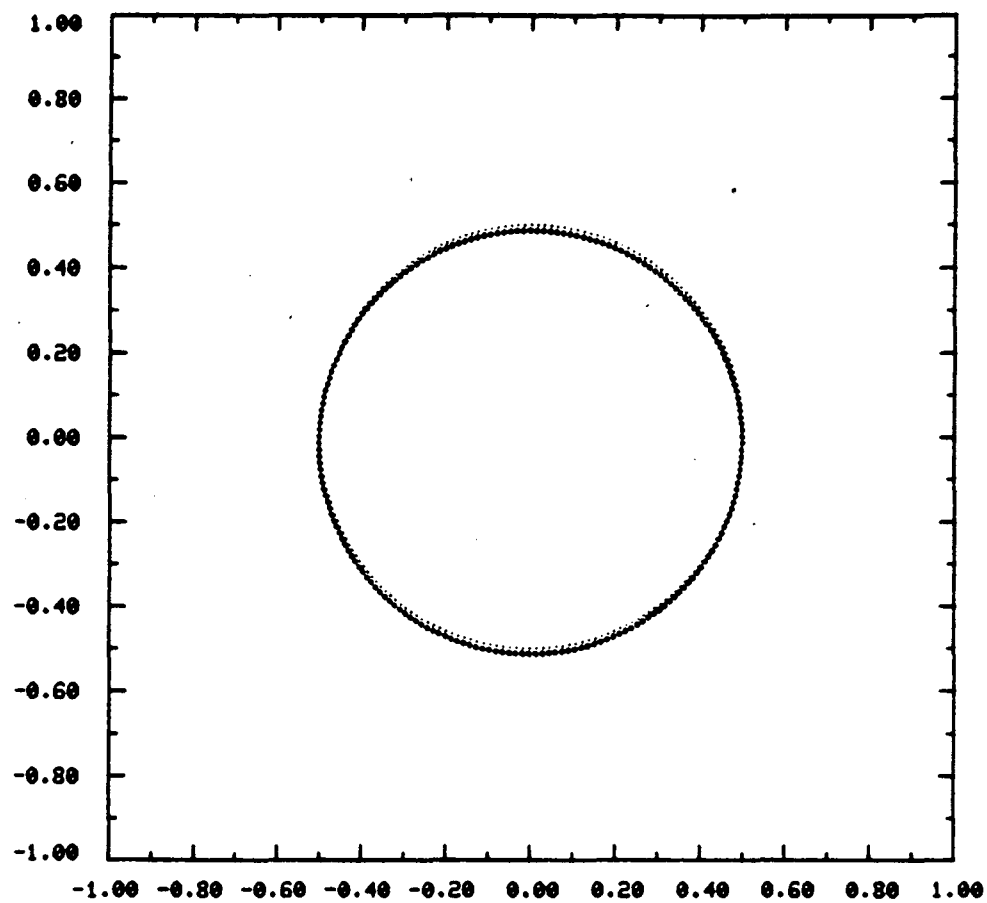
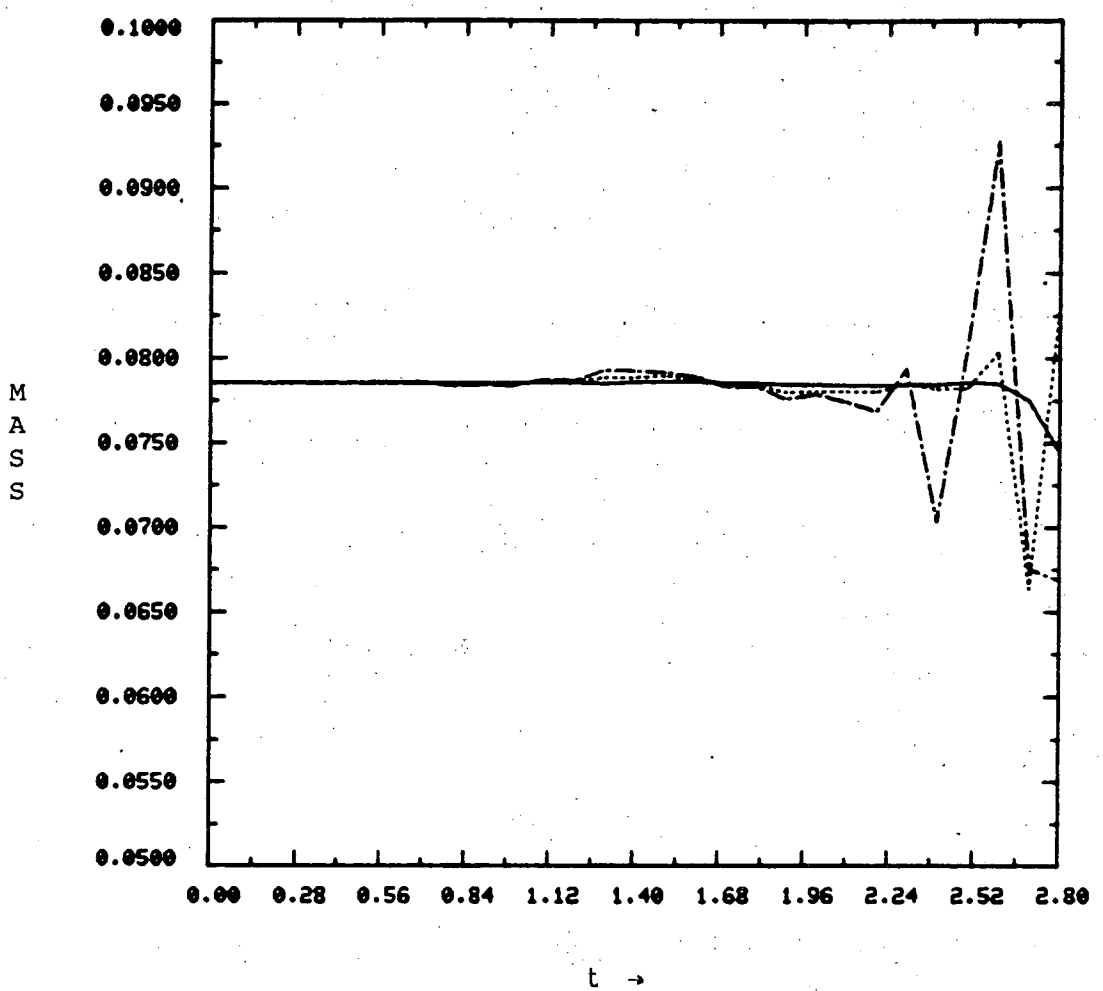


Figure 6.1.1

Distribution of Computational Points for Method B



N = 200 ——— N = 300 ····· N = 400 —·—·—·—

Figure 6.1.2

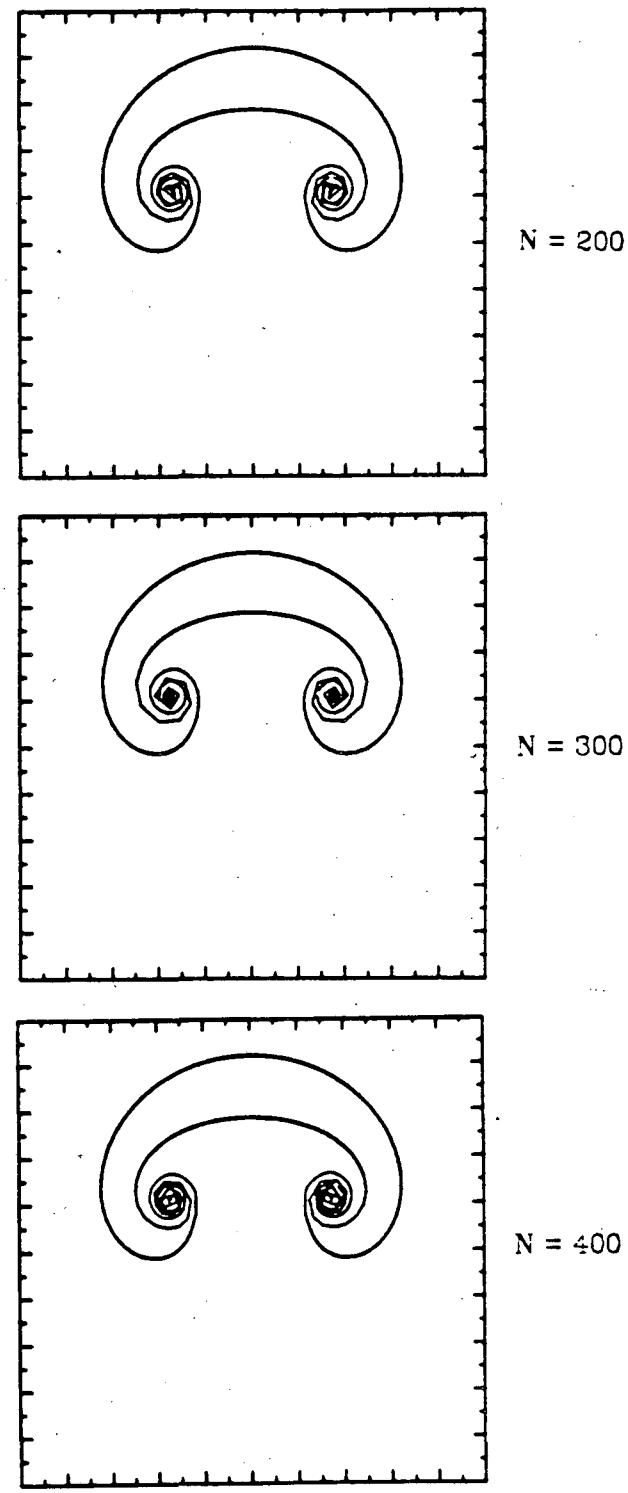
Conservation of Mass in Time

$\delta = h^{.75} \qquad h = \frac{\pi}{N}$

Figure 6.1.3

Interface Position at t = 3.0

$\delta = h^{.75} \quad h = \frac{\pi}{N}$



N	Mass	Arc length
150	7.641×10^{-2}	9.768
250	7.855×10^{-2}	9.824
350	7.854×10^{-2}	9.839
450	7.854×10^{-2}	9.846

Table 6.1.1

Fixed Delta Convergence at t = 3.0

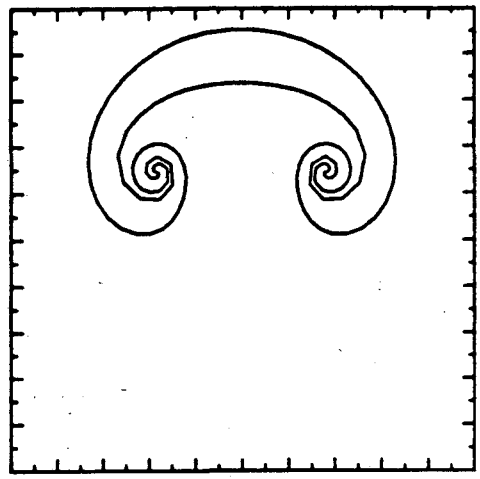
$\delta = .07$

Initial Mass = 7.8539×10^{-2}

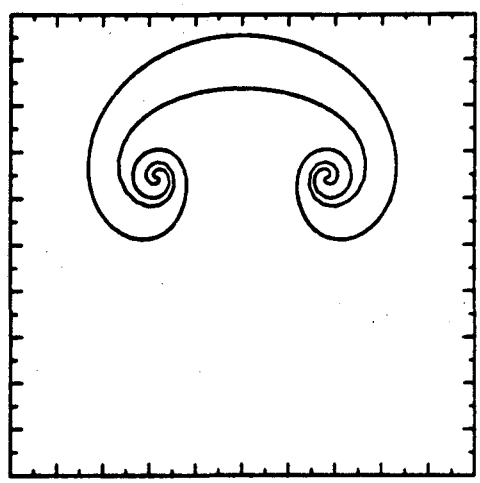
Figure 6.1.4

Interface Position at t = 3.0

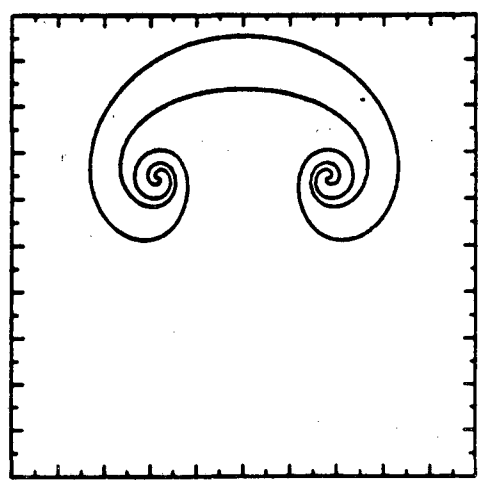
$\delta = .07 \qquad h = \frac{\pi}{N}$



N = 150



N = 250



N = 350

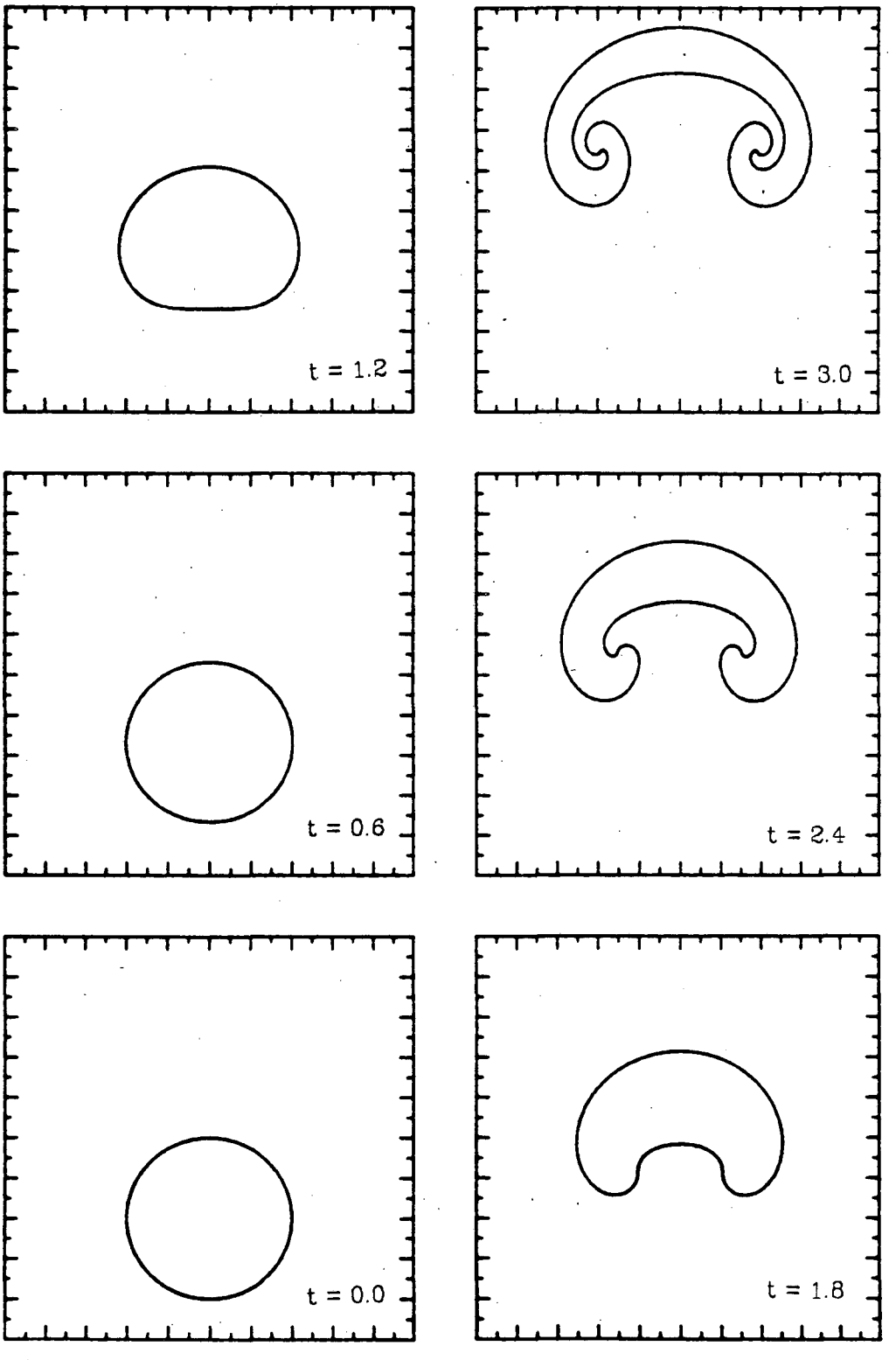


Figure 6.1.5

Interface Position as a Function of Time

$\delta = .1$

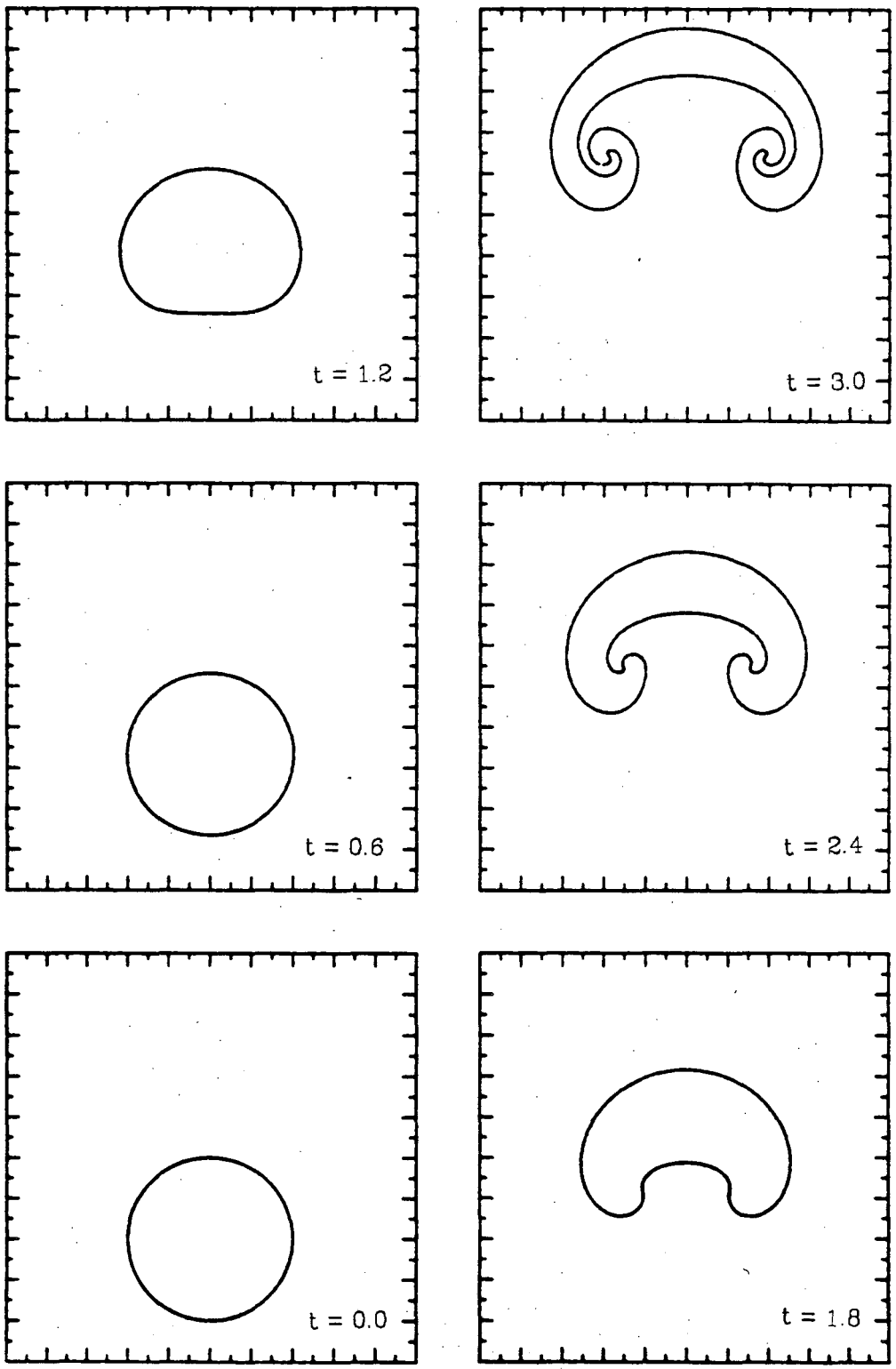


Figure 6.1.6

Interface Position as a Function of Time

$\delta = .09$

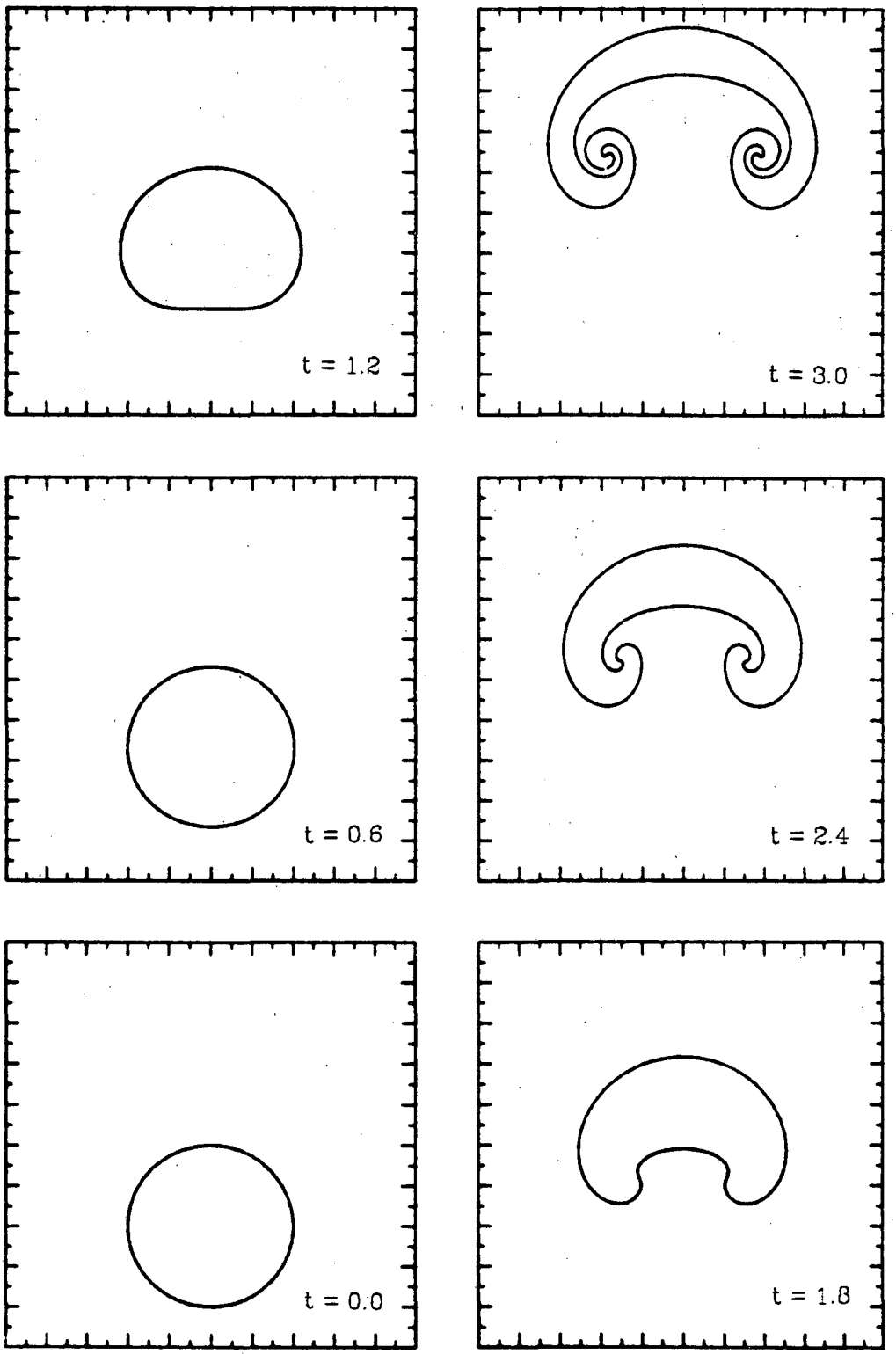


Figure 6.1.7

Interface Position as a Function of Time

δ = .08

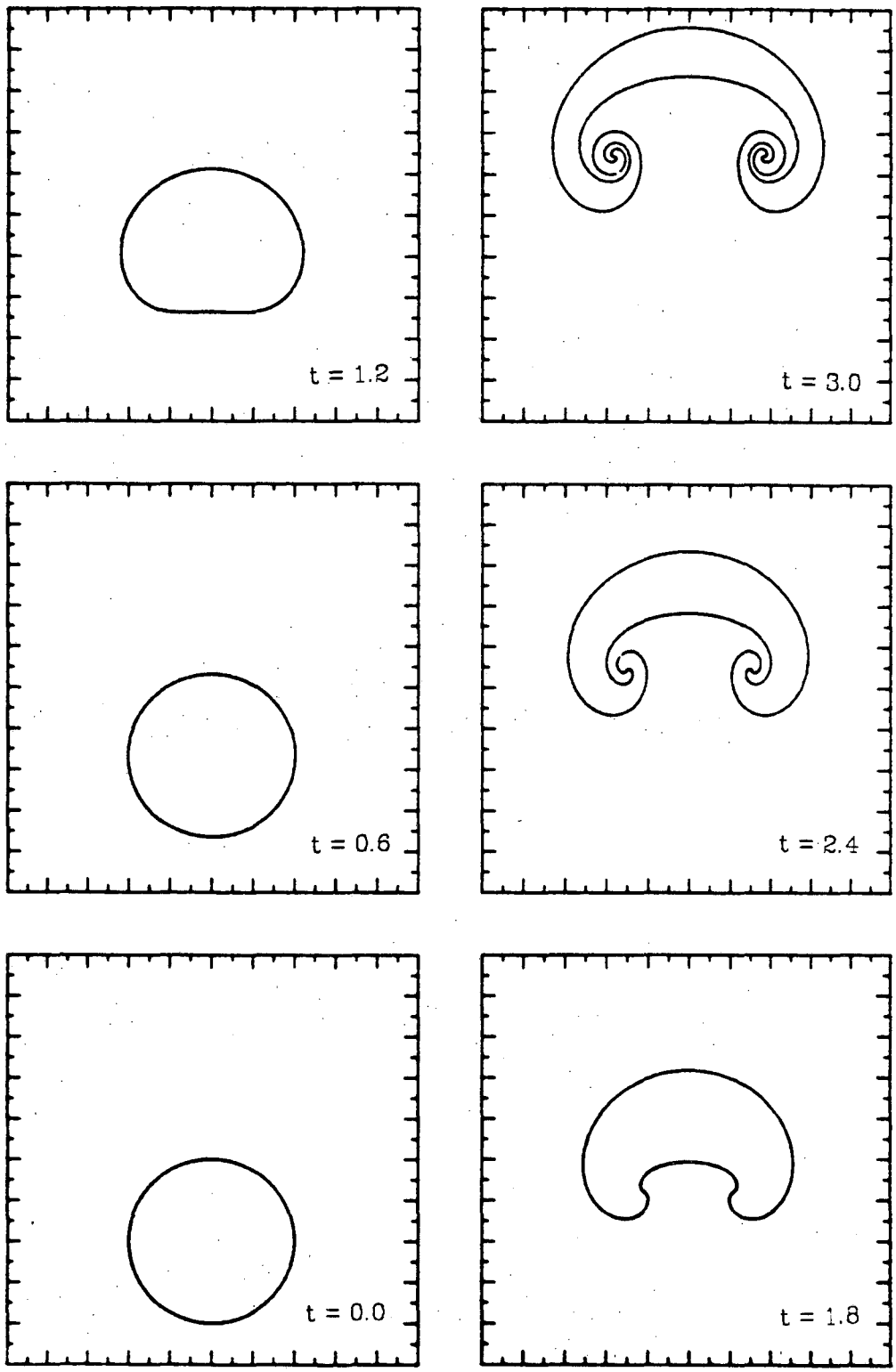


Figure 6.1.8

Interface Position as a Function of Time

$\delta = .07$

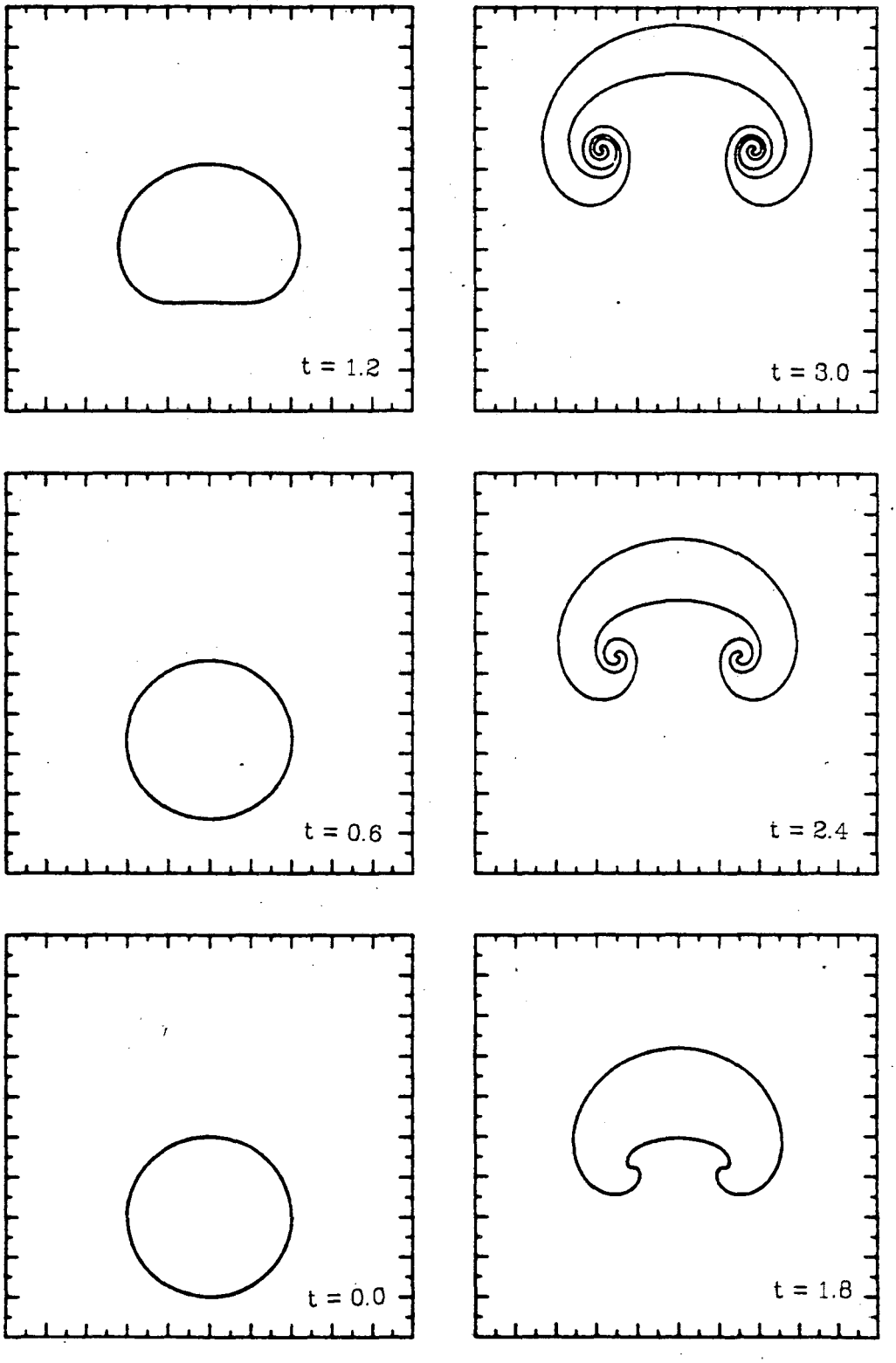


Figure 6.1.9

Interface Position as a Function of Time

$\delta = .06$

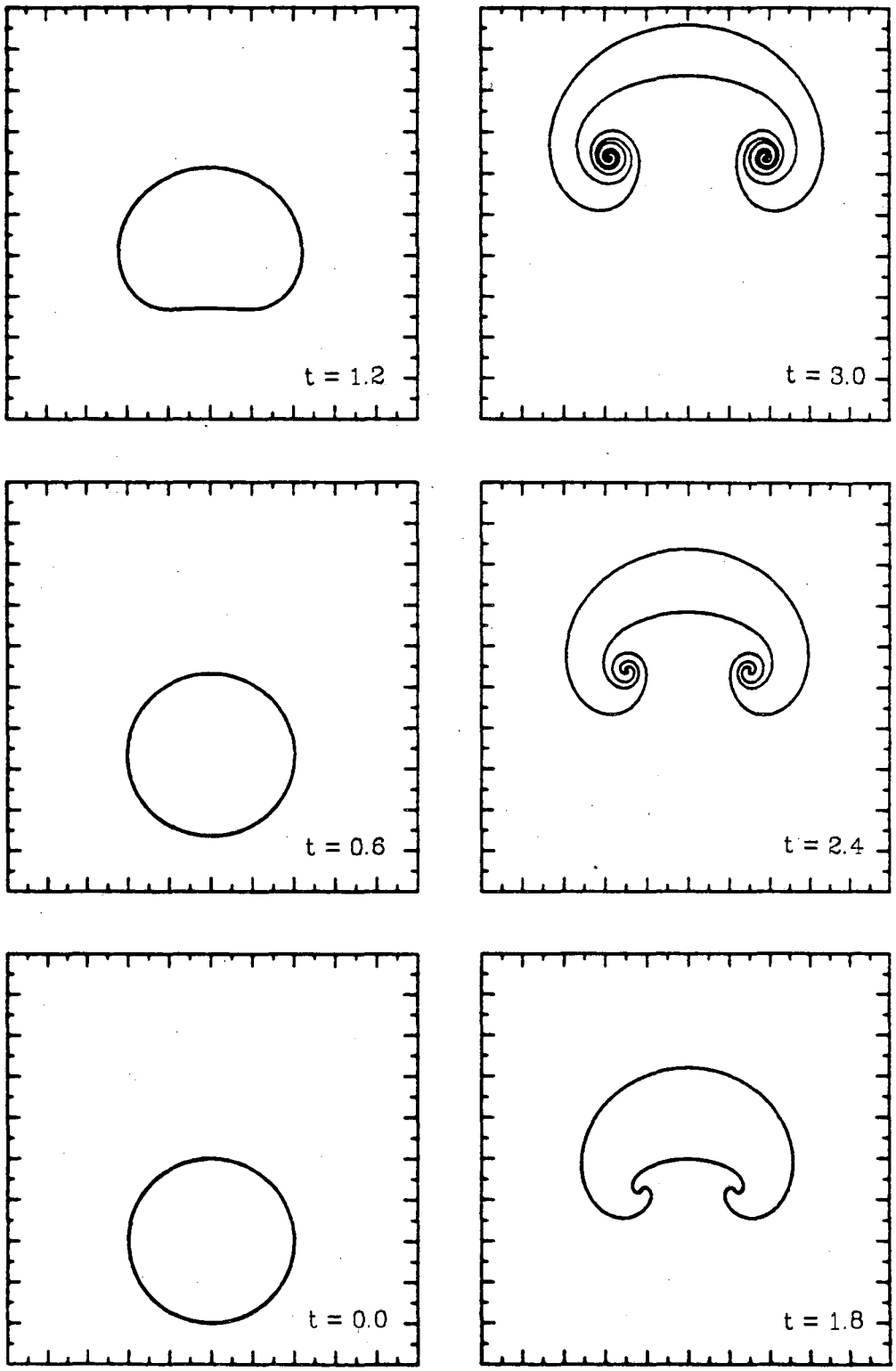


Figure 6.1.10

Interface Position as a Function of Time

$\delta = .05$

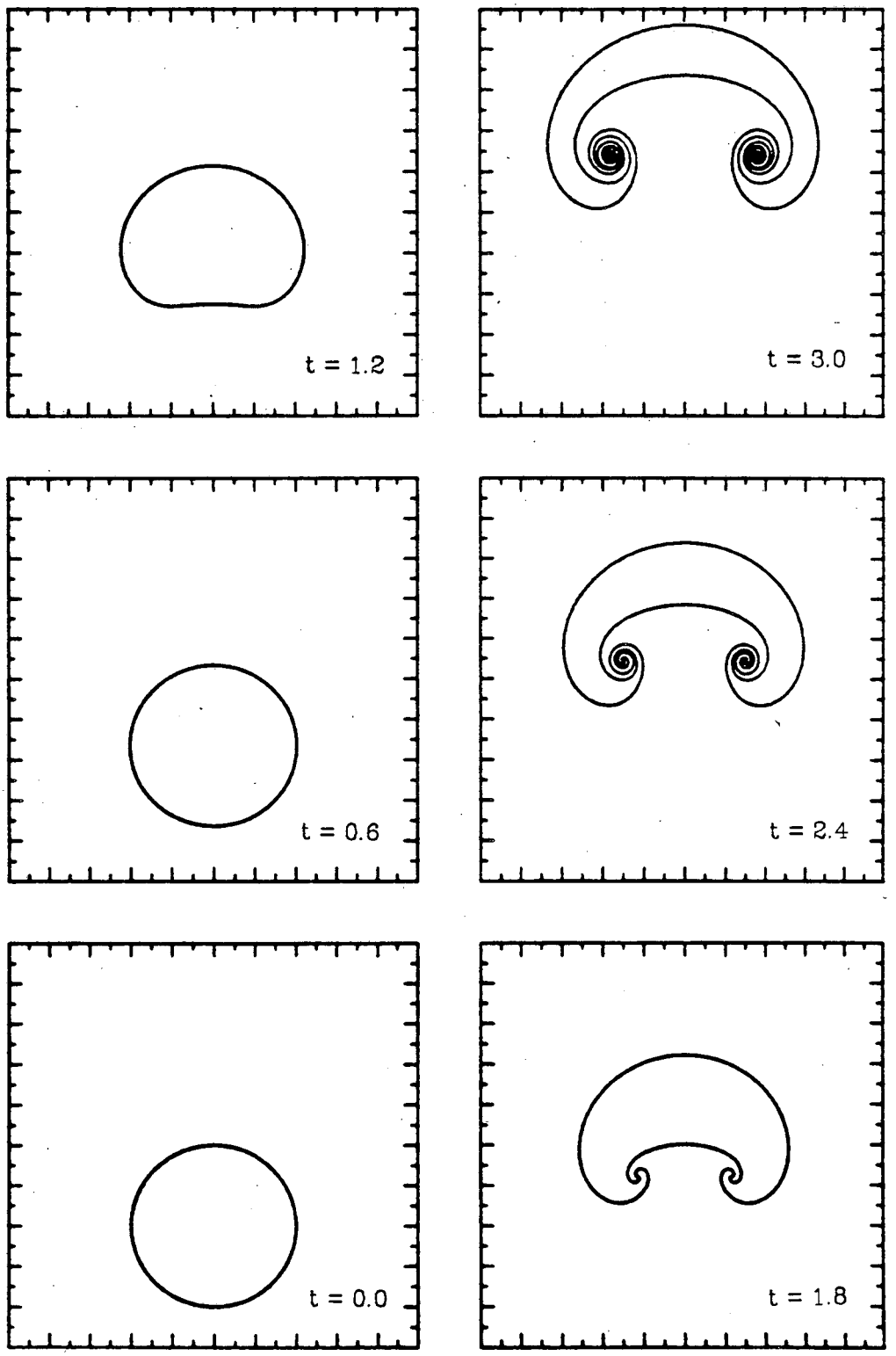


Figure 6.1.11

Interface Position as a Function of Time

$\delta = .04$

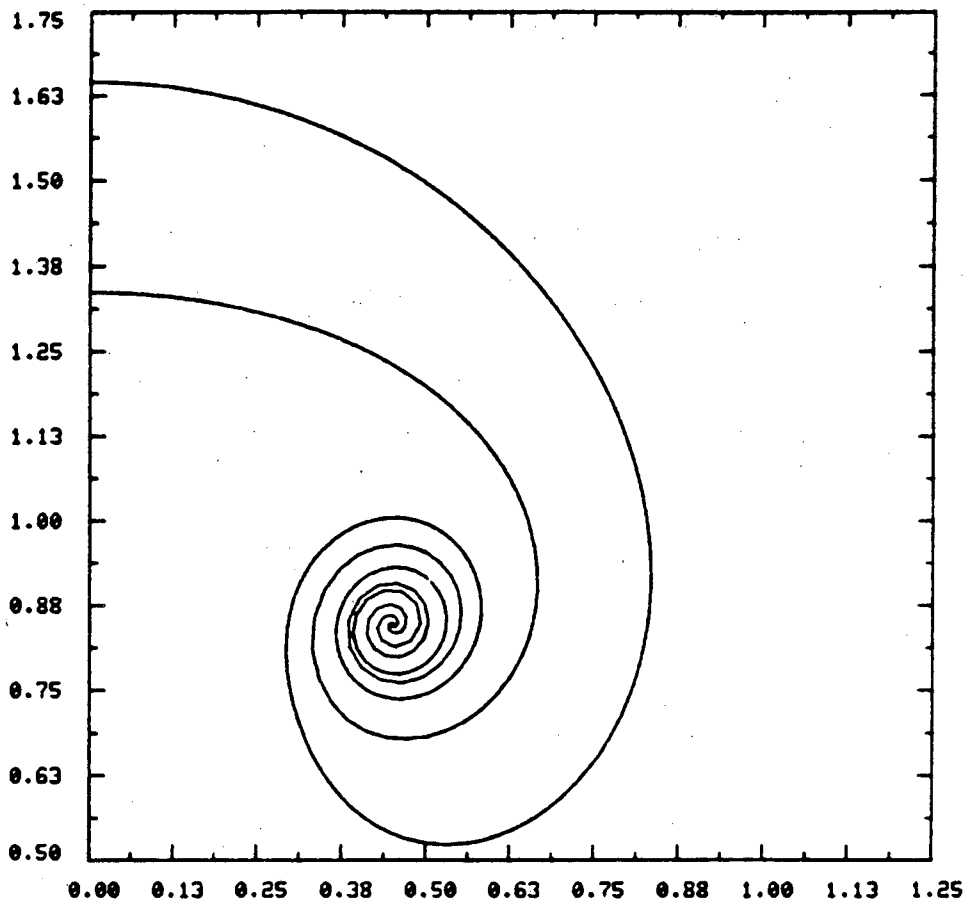


Figure 6.1.11a

Interface Position at t = 3.0

δ = .04

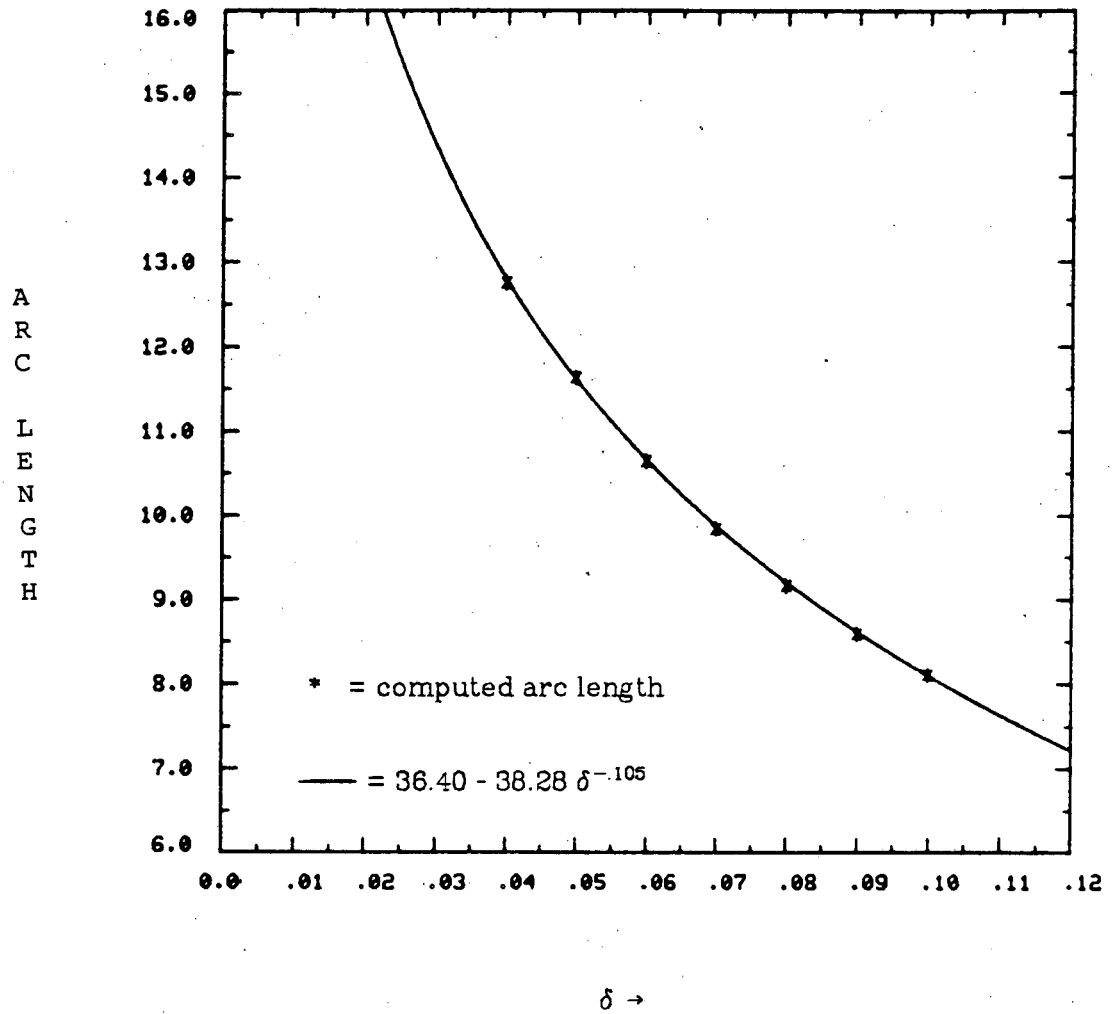


Figure 6.1.12

Arc Length as a Function of δ at t = 3.0

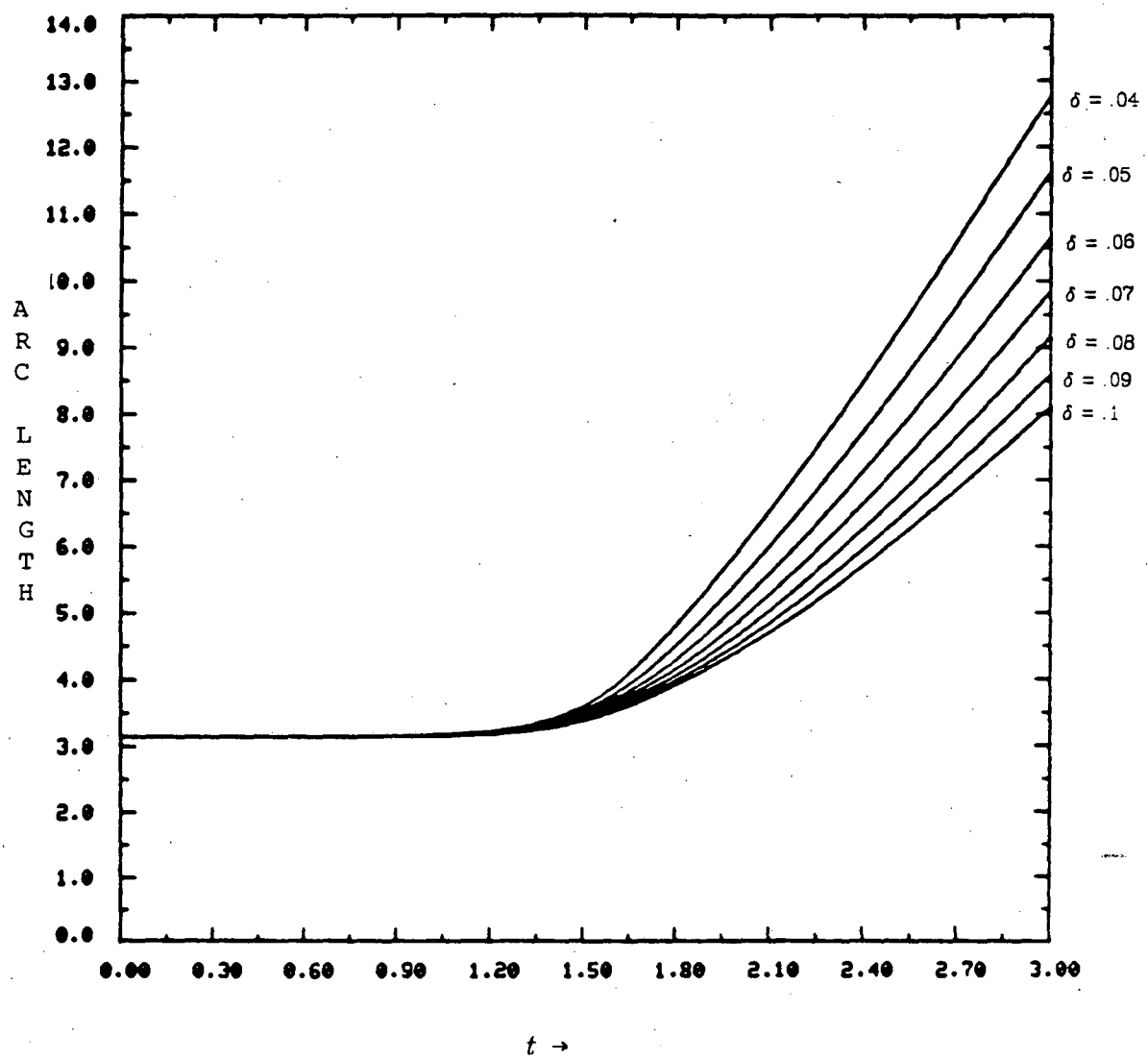
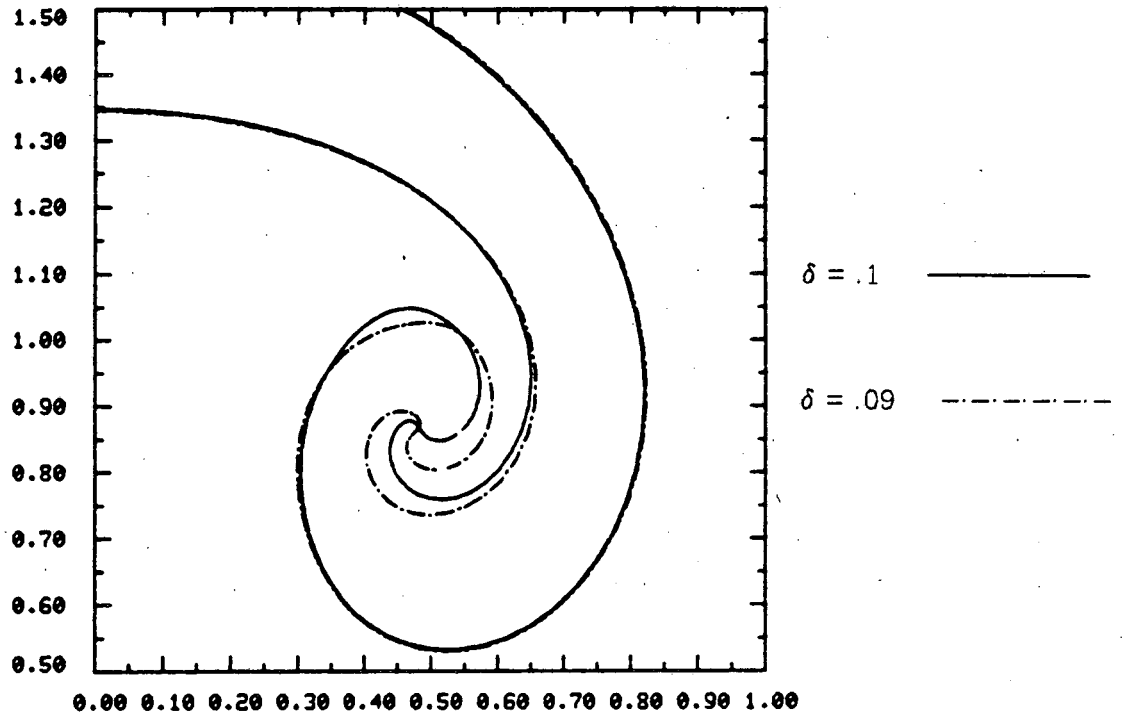


Figure 6.1.13

Arc Length as a Function of Time



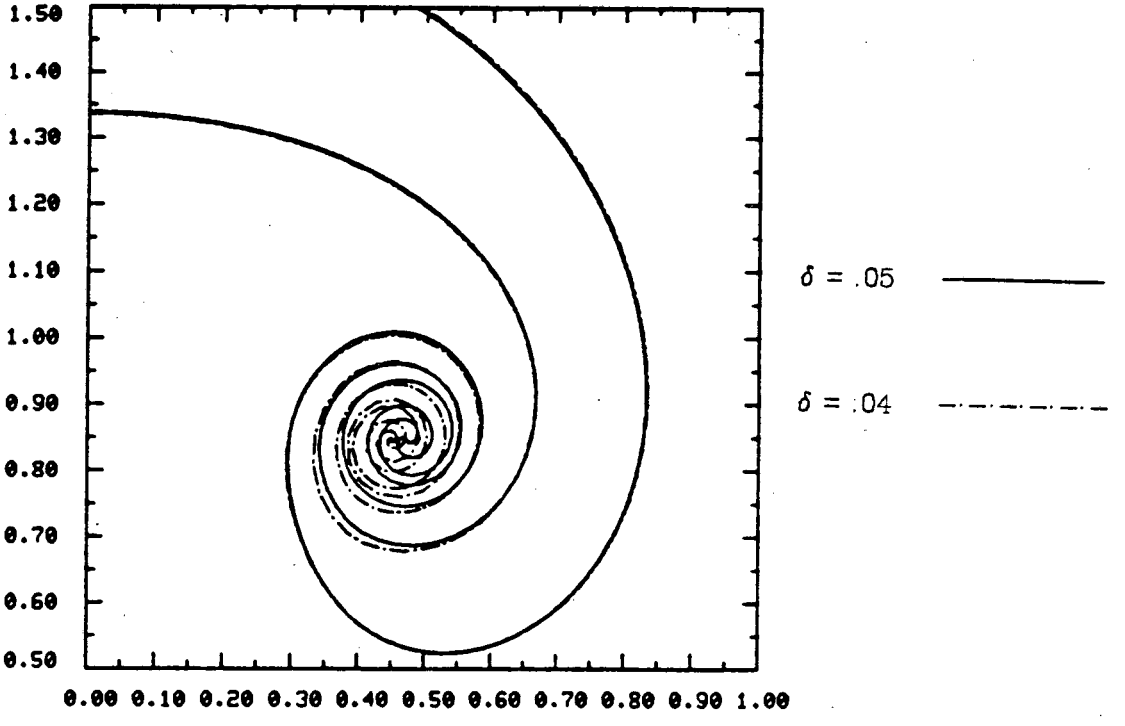


Figure 6.1.14

Interface Position at t = 3.0

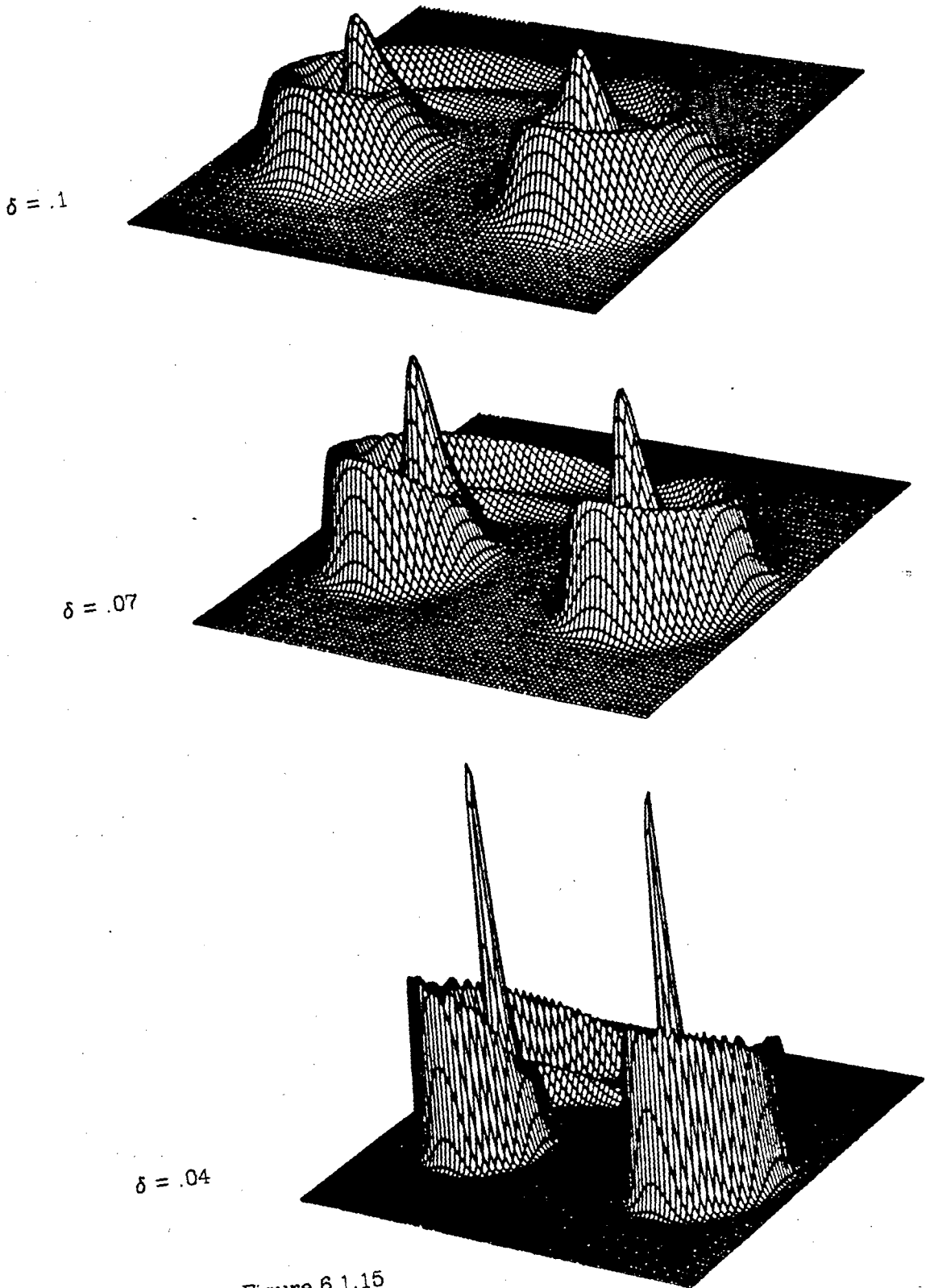


Figure 6.1.15

Absolute Value of Vorticity at t = 3.0

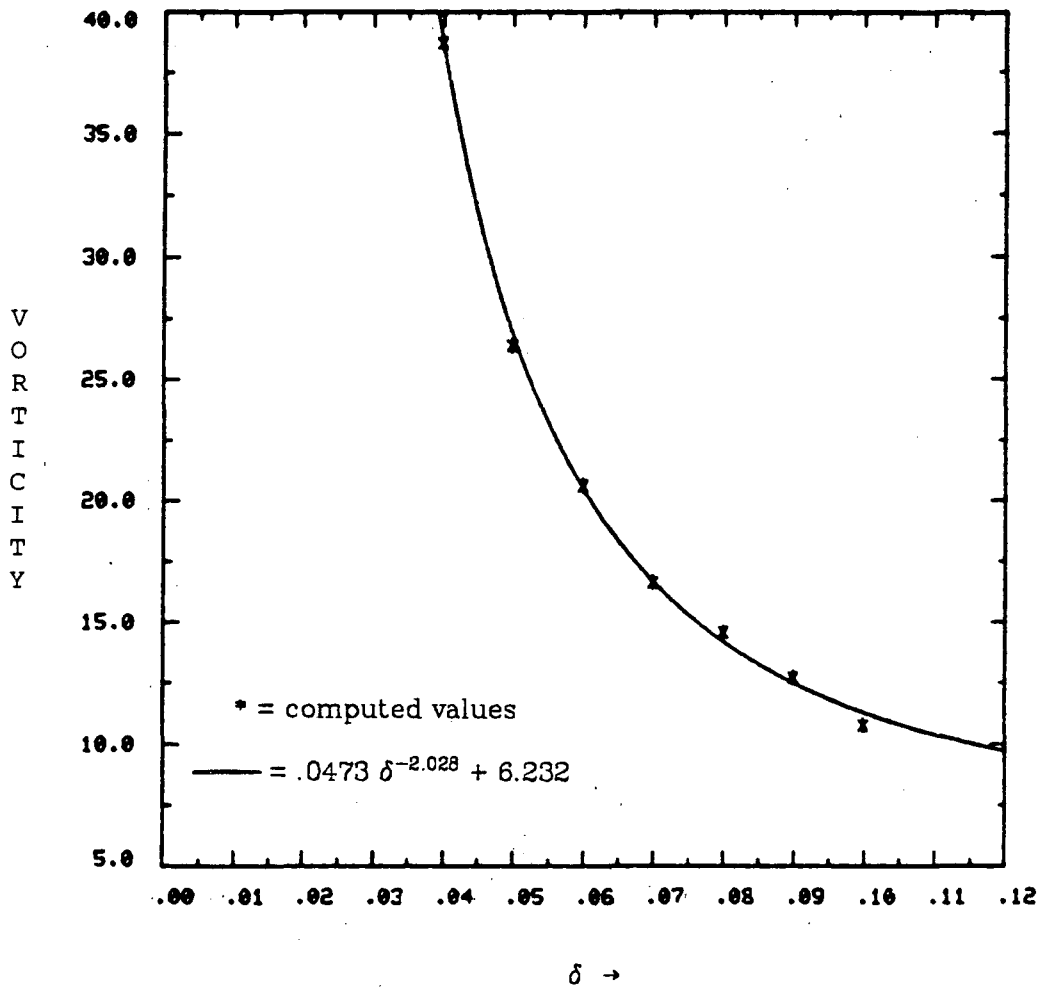


Figure 6.1.16

Magnitude of Vorticity as a Function of δ

6.2. Application of method A to a 2-D Thermal

In this section we discuss the application of method A to the problem of calculating the motion of a 2-D thermal. As in section 6.1 we calculate this motion by solving equations (1.15)-(1.18) with initial conditions (6.1.1)-(6.1.2). Our hope is that the results obtained with method A will be comparable to those obtained with method B.

We encounter two difficulties in implementing method A. The first difficulty is that which was caused by the initialization procedure for the method as described in section 3. In that procedure, the initial computational points are distributed on the intersection of the support of the initial density with the nodes of a rectangular grid, Ω^h. The initial density distribution of the 2-D thermal is radially symmetric, but for the values of h (the initial mesh width) that we were able to use, this symmetry was not well represented by the distribution of computational points. In figure 6.2.1 we show the initial distribution of computational points when distributed according to the procedure of section 3. Since the motion of a 2-D thermal is unstable, the effect of this rather coarse approximation to the initial data was very significant. Although the resulting approximate solution with this discretization may be closer to the actual physical phenomena, where perturbations on the boundary are to be expected, it is unfortunately further from the computational results obtained with method B. To facilitate a comparison between the two methods we choose to distribute the computational points in a way which preserved the initial radial symmetry. An accurate implementation of this initialization requires some care, and in section 6.3 we describe a suitable procedure. In figure 6.2.2 we show the distribution of computational points obtained using the results of section 6.3.

The second difficulty is more fundamental, having to do with the approximations made in method A. From equation (A.2) we see that the

vorticity grows where the derivatives of $\rho^h(x,t)$ are non-zero. Consider the approximation of method A at time t = 0. If we initialize as described in section 3 (with a radially symmetric distribution of computational points) then this dictates putting computational points only where the density is non-zero. Thus we place points inside a circle about the origin of radius $\bar{r}$. If we consider the derivatives of $\rho^h(x,t)$ based on this distribution of computational points, we find that they are non-zero outside the circle of radius $\bar{r}$. Since the vorticity grows in proportion to density derivatives, we find that there will be vorticity growth outside the circle of radius $\bar{r}$. In our approximation procedure, there are no computational points outside this circle and so the vorticity growth will not be approximated. The essence of the problem is that the support of the density (which determines the placement of computational points) and the support of the density derivative approximation (which determines the vorticity growth) do not coincide.

In our initial attempts to use method A to calculate the motion of a 2-D thermal this problem was not noticed. The result was that the solutions moved only about two-thirds as fast as those calculated with method B. Thus the problem can have significant effects. We remark that for problems with smooth solutions this difficulty does not appear. In this case the support of the density and density derivative approximation will be very close. It is for this reason that the difficulty was not detected in the test problem of section 4.2.

Our solution to this problem is to first smooth the initial data (6.1.1) and then apply method A. The initial smoothing that we use is

$$\tilde{\rho}_0(x',y') = \frac{1}{\pi\gamma^2}\int e^{\frac{-r^2}{\gamma^2}}\rho_0(x,y)dxdy \tag{6.2.1}$$

where $r^2 = (x'-x)^2 + (y'-y)^2$ and γ is a parameter to be chosen. We use this procedure because we expect that the solutions of (1.15)-(1.18) with

smooth initial data will still be smooth. Thus, as in the test problem of section 4.2, we expect the support of the density and the density derivative approximation will coincide. This reasoning is not rigorous, but the computational results indicate that there is some merit to it.

With the above considerations taken into account we then approximated the motion of a 2-D thermal using method A. We chose the functions in (B.6) and (B.8) to be

$$\Psi(r) = \frac{1}{\pi}(2e^{-r^2} - \frac{1}{2}e^{-\frac{r^2}{2}}) \tag{6.2.2}$$

which corresponds to $\Psi \varepsilon M^{L,p}$ for $L = \infty$ and $p = 4$. (We assumed that the solutions are smooth so that a higher order cut-off will increase the accuracy of the computation, cf. the results of section 4.) For the smoothing of the initial data we use $\gamma = .1$ in (6.2.1). The profile of this initial density distribution is shown in figure 6.2.3. We choose this γ because it yields an initial density approximation which is close to the initial density approximation obtained with method B with $\delta = .1$. Similarly, for the smoothing parameter δ in (A.4)-(A.6) we fixed the value of δ at .1. Again, the reason for this choice is to match the parameters with those of method B with $\delta = .1$. With these choices we then solved the O.D.E.s (A.1)-(A.3) with $\Delta t = .1$ for several different numbers of points N.

For the calculations with N > 400 the computed solutions appeared to have converged. In figures 6.2.4 we present the positions of the computational points for times between t = 0.0 and t = 3.0 for N = 638. One noticeable feature of these figures is the 'tail' of the thermal that forms. Upon closer inspection, it is found that the points in the tail represent very little density, and so give a misleading representation of the position of the thermal. To represent the position and shape of the thermal more accurately we present contour plots of the density in figure 6.2.5. These contours were

obtained by evaluating the approximate density $\rho^h(x,t)$ of (3.12) on a rectangular grid and then using the contour plotting routine given in [23]. To facilitate a comparison between these results and the results obtained with method B we present in figure 6.2.6 the contours associated with the solution which is depicted in figure 6.1.5.

Comparing figures 6.2.5 and 6.2.6 we see that method A agrees rather well with method B. In particular, the general shape of the thermal and the distance traveled are the same. It is difficult to compare the resolution of the two methods from the contour plots, but we note that method B gives sharper contours. It is clear from the results depicted in figure 6.1.5 that method B is superior in resolving the fine scale features of the solution. This superiority is certainly expected, because in the implementation of method A we are forced to smooth the problem and thus the solution has no sharply defined interface. Although we have presented the results for only one choice of the parameters, other computational experiments suggest that these results are indicative of the results that would be obtained with other choices of the parameters.

It was found that the calculation of the motion of the thermal can be continued for a much longer time with method A than with method B. As an example of this we present in figure 6.2.7 the results obtained with method A for the above parameter choices and for times up to 6.0. Using an equivalent parameter choice and number of computational points, method B yielded meaningful results up to time t = 4.0. We explain this difference in ability by considering the nature of the approximations made in each of the methods. The success of method B depends on an accurate calculation of the density derivatives. For the problem of the motion of a 2-D thermal, this requirement translates into an accurate calculation of the motion of the interface. The most prominent feature of the 2-D thermal is that the inter-

face becomes increasingly complicated as time progresses. Thus to compute this motion accurately, method B needs a large number of computational points. If there are an insufficient number of points then the results quickly become meaningless. (In this problem this was characterized by a failure to conserve mass.) In method A, the density derivatives are computed by differentiating a smoothed approximation to the density. In essence, the density derivatives that enter into the calculation are smoothed or averaged density derivatives. Thus, the presence of a very complicated distribution of density derivatives will not necessarily adversely affect the computation.

The difference in methods can also be explained by considering the associated delta-equations. For method B they are given by (6.1.4)-(6.1.8) and for method A they are given by

$$\frac{\partial \rho}{\partial t} + \vec{u}_\delta \cdot \text{grad}\, \rho = 0 \tag{6.2.3}$$

$$\frac{\partial \xi}{\partial t} + \vec{u}_\delta \cdot \text{grad}\, \xi = \text{curl}(\rho_\delta \vec{F}) \tag{6.2.4}$$

$$\rho_\delta = \Psi_\delta * \rho \tag{6.2.5}$$

$$\vec{u}_\delta = K * \Psi_\delta * \xi \tag{6.2.6}$$

If we compare equations (6.2.4) and (6.1.7), then the difference in the two methods becomes clear. In method A, the vorticity growth depends upon the smoothed density derivatives, while in method B the vorticity growth depends upon the actual density derivatives. For problems which have very complicated distributions of density derivatives, the approximation used in method A may be more appropriate.

Although one can continue the calculation longer with method A, we are faced with the question of the validity of the results. This is a difficult question, and we have not yet arrived at a satisfactory answer. We remark that

the solution which is depicted in figure 6.2.7 has some of the features of experimentally observed thermals. In particular, it is observed [52] that thermals split into two counter-rotating structures and that the bulk of mass of the thermal is concentrated in the rear of the thermal. This behavior is clearly seen in figure 6.2.7. More extensive computations should allow for other comparisons to be made. Another approach to the assessment of the computational results is to analyze the associated delta-equations (6.2.3)-(6.2.6). For a fixed delta as $h \to 0$ the computed solutions can be expected to converge to the solutions of the delta-equations. (The reason for considering this type of convergence is that it is easier to obtain it computationally, cf. the corresponding results in section 6.1 for method B) Thus, understanding the relation between the delta-equations and the equations (1.15)-(1.18) will help illuminate the nature of the numerical approximations that are constructed with method A.

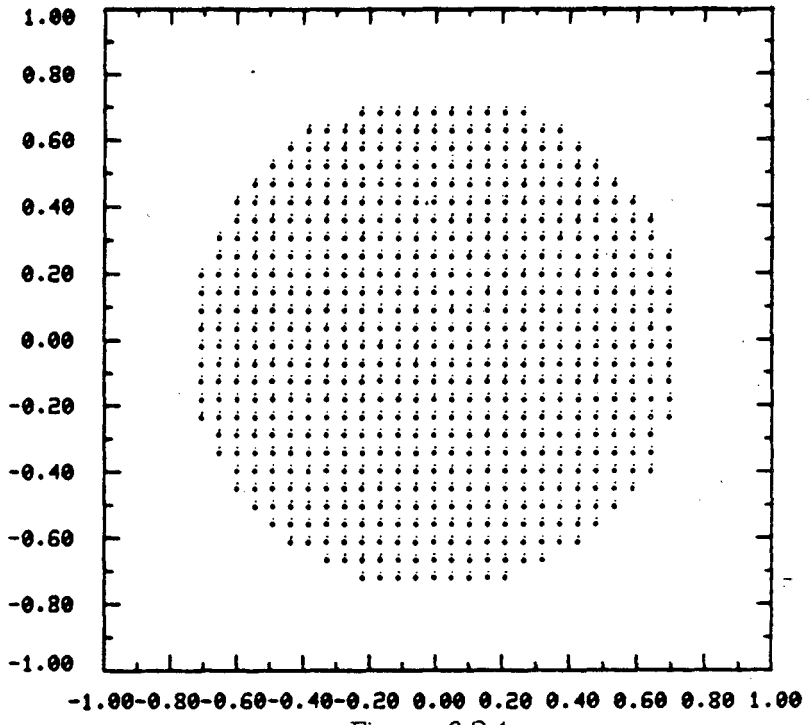


Figure 6.2.1

Initial Distribution of Computational Points

Rectangular Grid

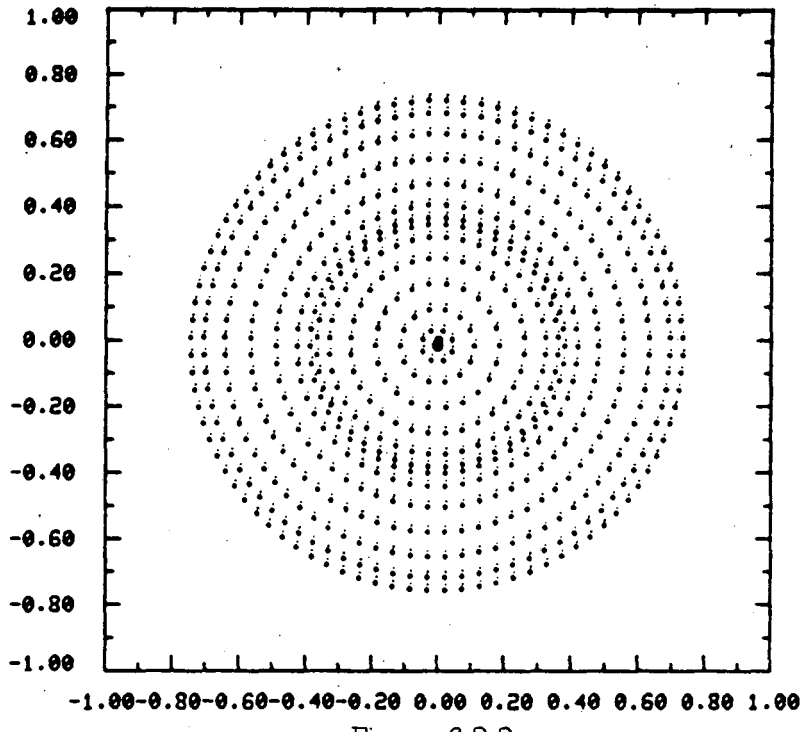


Figure 6.2.2

Initial Distribution of Computational Points

Using Results of Section 6.3

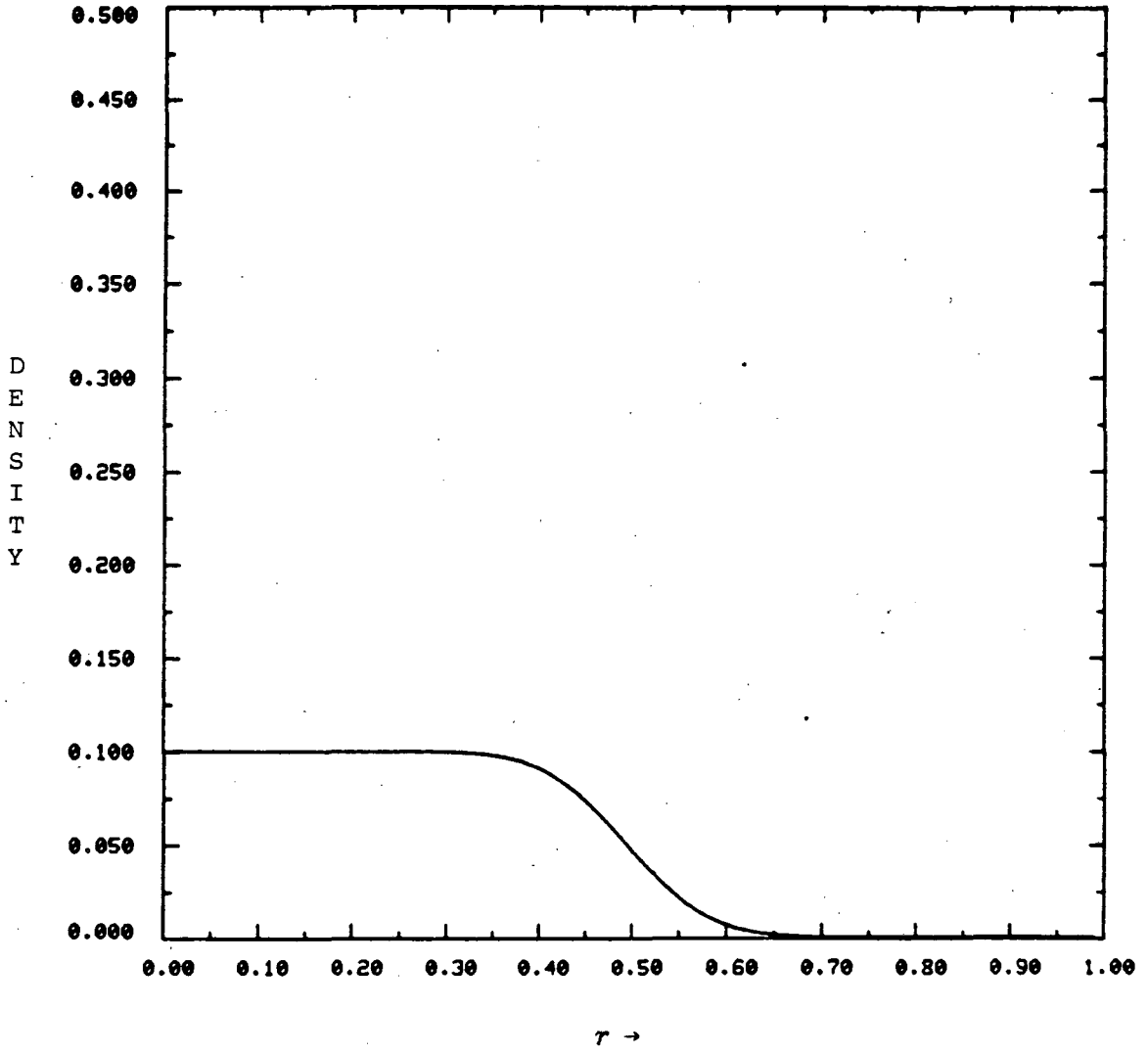


Figure 6.2.3

Initial Density as a Function of Radius

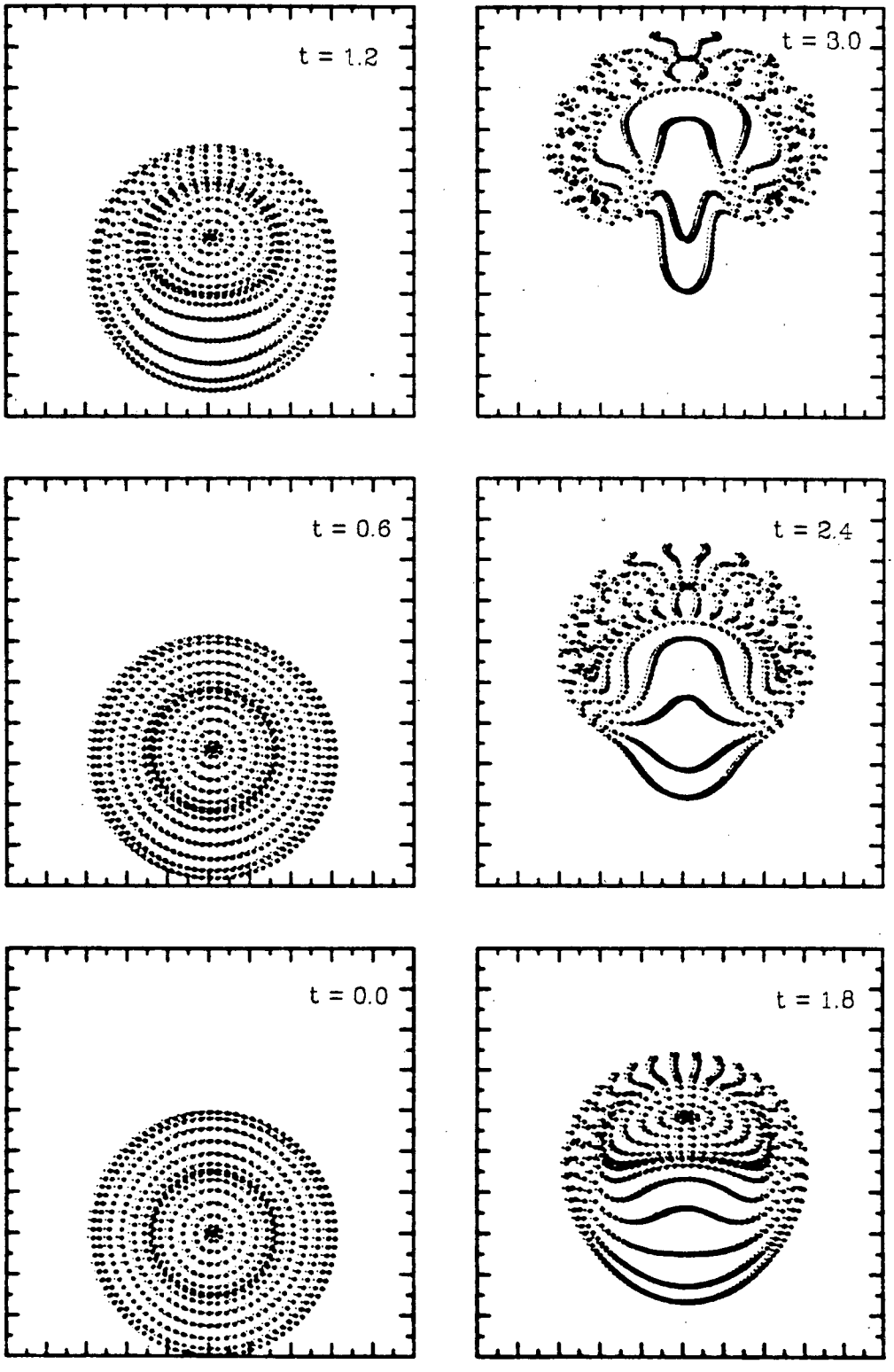


Figure 6.2.4

Position of Computational Points

Method A

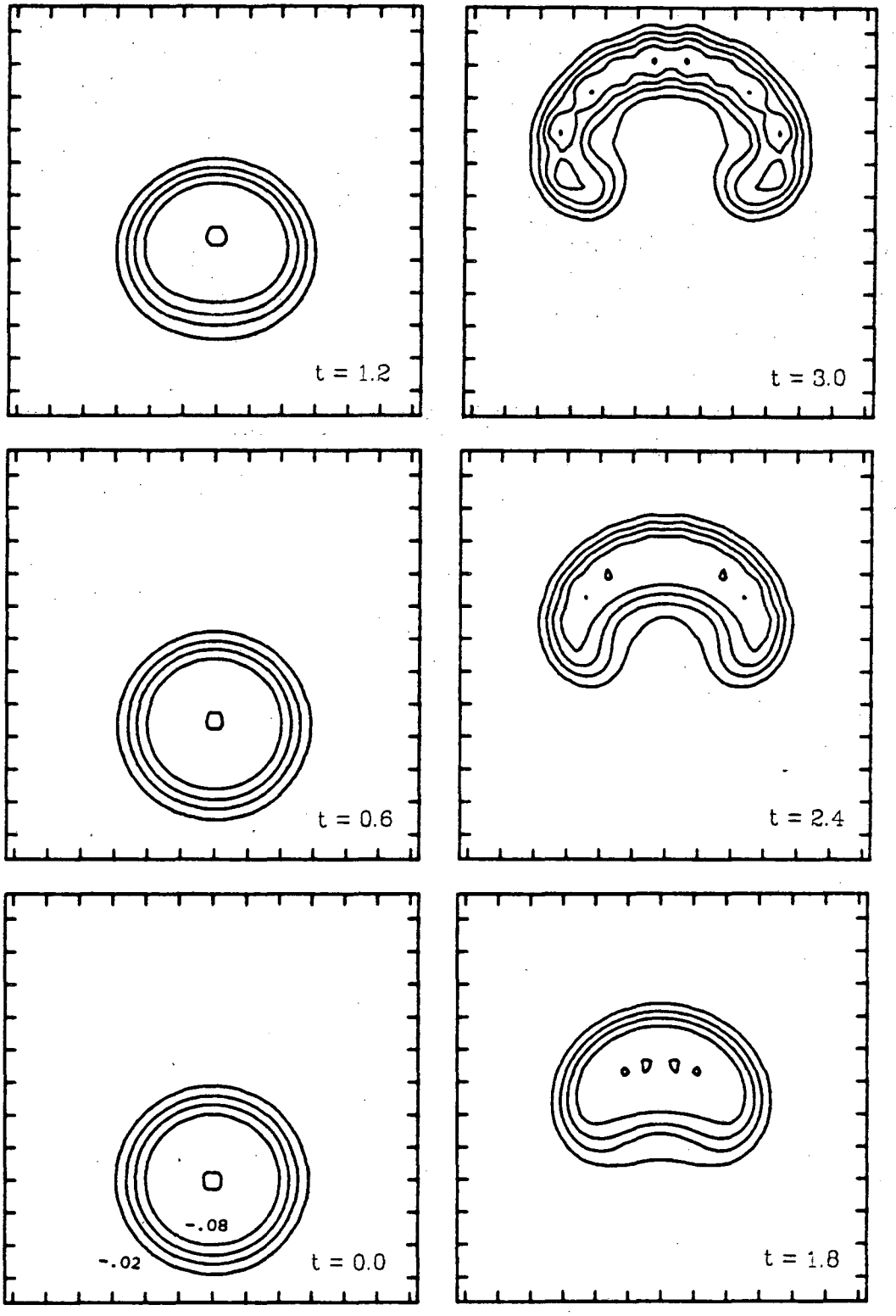


Figure 6.2.5

Contours of Density Method A

Increments of -.02 starting at -.02

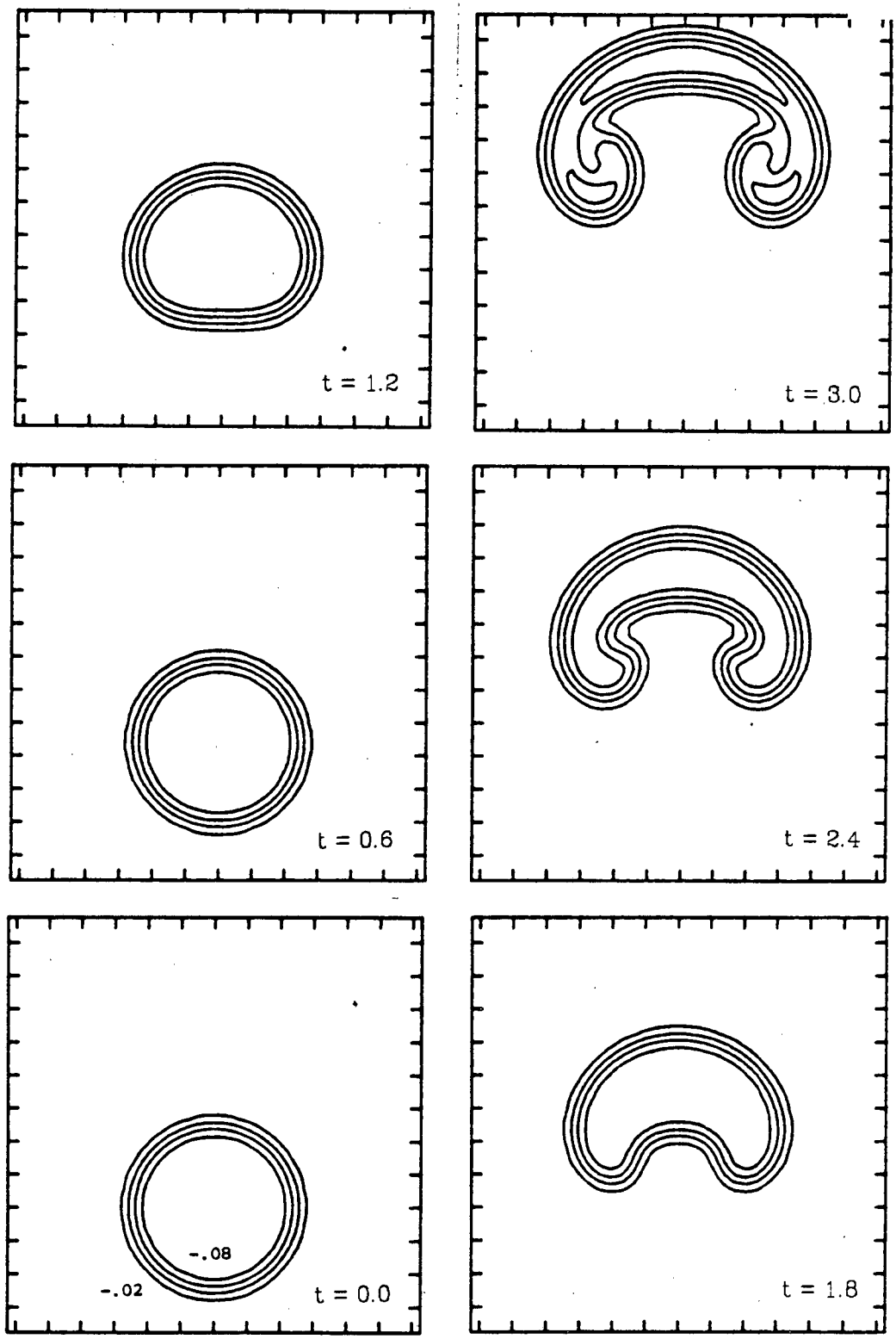


Figure 6.2.6

Contours of Density Method B

Increments of -.02 starting at -.02

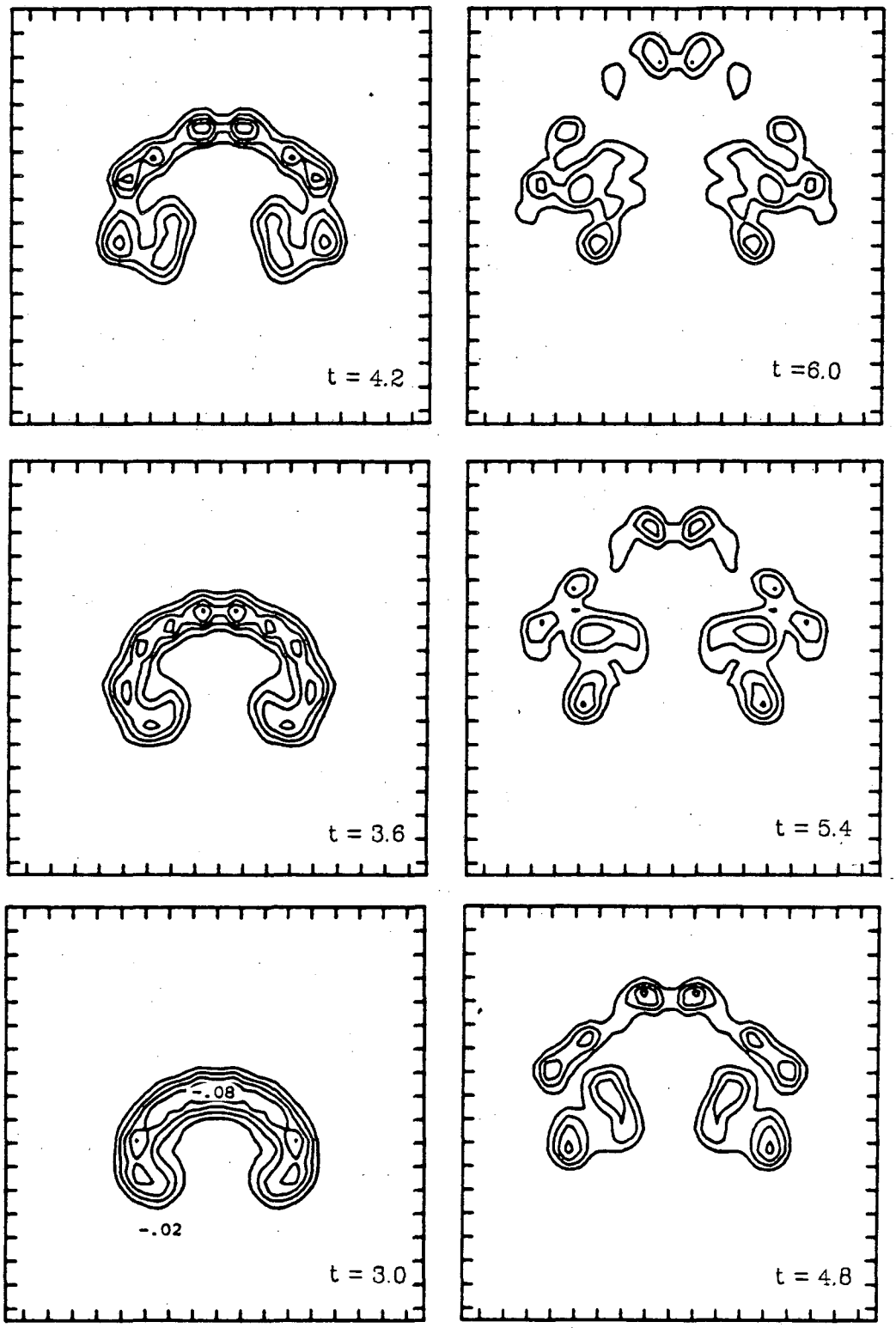


Figure 6.2.7

Contours of Density Method A

Increments of -.02 starting at -.02

6.3. Initialization of method A for non-rectangular grids

In this section we discuss an initialization procedure for equations (A.1)-(A.4) when one wants to use computational points other than those which are initially distributed on the rectangular grid Ω^h. The computational points will usually be distributed in a manner dictated by the geometry of the initial conditions as well as accuracy considerations. To simplify the presentation we discuss the procedure for the numerical scheme used in the model problem of section 2. The application of the technique to (A.1)-(A.5) follows easily.

Consider the approximation used in the model problem of section 2,

$$f^h(x,t) = \sum_{ih \in \Omega^h} \Psi_\delta(x - x(ih,t))\, f(ih,t)\, h^2 \tag{6.3.1}$$

where $x(ih,t)$ and $f(ih,t)$ are solutions to

$$\frac{dx(ih,t)}{dt} = u(x(ih,t)) \qquad x(ih,0) = ih \tag{6.3.2}$$

$$\frac{df(ih,t)}{dt} = 0 \qquad f(ih,t) = f_0(ih) \tag{6.3.3}$$

Thus the computational points $x(ih,t)$ are those that originate on the rectangular grid $\Omega^h = \Omega \cap \Lambda^h$. Let $\{\alpha_i\}$ denote another set of computational points, not necessarily distributed on a rectangular grid. With this change, the appropriate modification of formulas (6.3.1)-(6.3.3) is

$$f^h(x,t) = \sum_{\alpha_i} \Psi_\delta(x - x(\alpha_i,t))\, f(\alpha_i,t) u_i \tag{6.3.4}$$

where $x(\alpha_i,t)$ and $f(\alpha_i,t)$ are solutions to

$$\frac{dx(\alpha_i,t)}{dt} = u(x(\alpha_i,t)) \qquad x(\alpha_i,0) = \alpha_i \tag{6.3.5}$$

$$\frac{df(\alpha_i,t)}{dt} = 0 \qquad f(\alpha_i,0) = f_0(\alpha_i) \tag{6.3.6}$$

and w_i are weights that depend upon α_i. To determine the weights w_i and in part determine the α_i we will consider an error estimate for (6.3.4). As in the estimate of (6.3.1) that is given in section 2, we write the error in using (6.3.4) as a sum of two errors,

$$\begin{aligned} f^h - f &= (f^h - f * \Psi_\delta) + (f * \Psi_\delta - f) \\ &= \{A\} + \{B\} \end{aligned}$$

The error term A can be viewed as the error in approximating the integral in $f * \Psi_\delta$ by a discrete sum, while term B can be viewed as the error in approximating the function f by f convolved with the approximate delta function Ψ_δ. The error term B is analyzed as in section 2 and is found to not depend on the choice of computational points. It is in term A that the choice of computational points makes a difference. If we use the map $x(\alpha,t)$ to change variables in the integral in term A we find,

$$\begin{aligned} A &= \sum_{\alpha_i} \Psi_\delta(x - x(\alpha_i,t))\, f(\alpha_i,t)\, w_i - \int \Psi_\delta(x - x')f(x',t)dx' \\ &= \sum_{\alpha_i} \Psi_\delta(x - x(\alpha_i,t))f(\alpha_i,t)w_i - \int \Psi_\delta(x - x(\alpha,t))f(x(\alpha,t),t)d\alpha \end{aligned}$$

Since $f(\alpha_i,t) = f(x(\alpha_i,t),t)$, we recognize term A as the error in using an integration formula of the type

$$\sum_{\alpha_i} g(\alpha_i)w_i \tag{6.3.7}$$

to approximate the integral over R^2 with respect to α of the function

$$g(\alpha) = \Psi_\delta(x - x(\alpha,t))f(\alpha,t) \quad .$$

From this we conclude that w_i should correspond to integration weights and α_i to integration points of the approximation (6.3.7). Any integration

formula of type (6.3.7) will work, so we find that there are many possible choices of the α_i's and w_i's. We use this freedom to choose the computational points in a way which makes better use of the structure of a particular problem.

As an example of the use to the above analysis we discuss a possible choice of the computational points when the initial distribution $f_0(x,y)$ is radially symmetric. For the model problem, since f is transported by the flow, we have that $f(\alpha,t) = f_0(\alpha)$. Thus to make the error in term A small, we seek integration points and weights so that

$$\int \Psi_\delta(x - x(\alpha,t)) f_0(\alpha) d\alpha \tag{6.3.8}$$

is well approximated by a discrete sum of the form (6.3.7). Let $\bar{R}$ denote that radius such that $f_0(\alpha) = 0$ for $|\alpha| > \bar{R}$. (We are assuming that f_0 has finite support). We rewrite (6.3.8) using polar coordinates and find,

$$\int \Psi_\delta(x - x(\alpha,t)) f_0(\alpha) d\alpha = \int_0^{\bar{R}} \int_0^{2\pi} \Psi_\delta(x - x((r,\vartheta),t) f_0(r)\, r\, dr\, d\vartheta \tag{6.3.9}$$

The reason for writing (6.3.8) in polar coordinates so that when we approximate the iterated integral in (6.3.9) with one dimensional integration schemes, the resulting integration formula will be radially symmetric. We approximate the integral on the right hand side of (6.3.9) using Gaussian quadrature for the integration in the r coordinate and using the trapezoidal rule for the integral in the ϑ coordinate. Specifically, let q_l and r_l, $l=1, \cdots K$, denote gaussian quadrature weights and points for an integral over the interval $[0, \bar{R}]$. (See [1].)

Let $h_\vartheta(l) = \dfrac{2\pi}{N(l)}$ where $N(l)$ is some number depending on l, then

$$\int_0^{\bar{R}}\int_0^{2\pi}\Psi_\delta(x - x((r,\vartheta))f(r)\,rdrd\vartheta = \sum_{l=1}^{K}\sum_{j=1}^{N(l)}\Psi_\delta(x - x((r_l,jh_\vartheta(l)),t)f_0(r_l)\,r_l\,q_l\,h_\vartheta \tag{6.3.10}$$

In the notation for (6.3.7) we are using points $\{\alpha_i\}$ given by,

$$(r_l\cos(jh_\vartheta(l))\,,\,r_l\sin(jh_\vartheta(l)) \qquad l = 1,\cdots K \qquad j = 1,\cdots,N(l) \tag{6.3.11}$$

and weights

$$w_{l,j} = r_l q_l h_\vartheta(l) \tag{6.3.12}$$

In figure 6.3.1 we present the distribution of computational points given by the above procedure when $\bar{R}^2 = .5$ The significance of using the choice (6.3.10) is that the computational points are distributed in concentric circles about the origin. For radially symmetric data we expect that this distribution will have smaller errors than if we had used the trapezoidal rule to approximate (6.3.8). The accuracy of using (6.3.10) can be analyzed using error estimates for the resulting integration formula. We do not pursue this approach here, but leave such an analysis for future work.

The advantage of using (6.1.10) was demonstrated in the calculation of section 6.2. In that calculation the use of computational points that were not radially symmetric had a harmful effect on the results. We also tested the distribution of computational points by using these points in the calculation of section 4.2. For an equivalent number of points, the errors in using (6.1.10) were smaller than using points initially distributed on a rectangular grid. In particular for the number of points approximately equal to 300, the relative error in the velocity was 5.9% using initialization (6.1.10), while the error was 12% using the initialization of section 4.3.

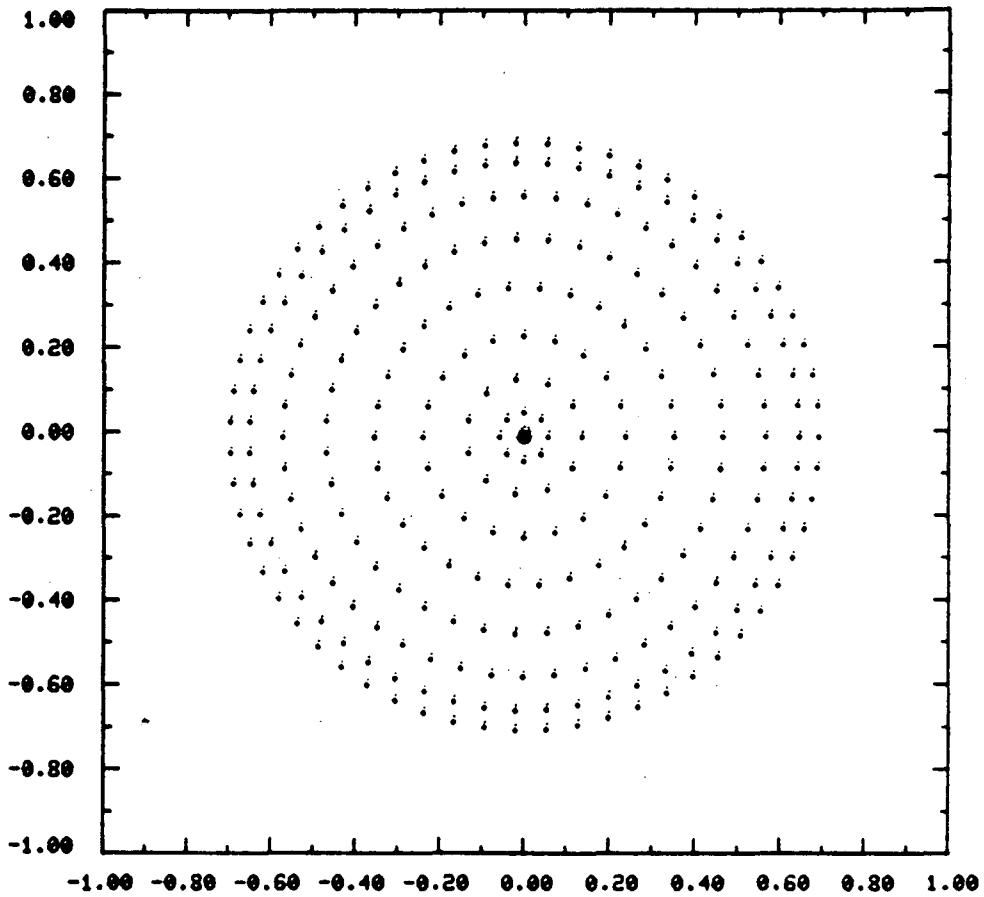


Figure 6.3.1

Initial Computational Point Distribution Using (6.3.11)

7. Discussion and Conclusions

We have presented two numerical methods for calculating the motion of an incompressible fluid of sightly varying density. We have also presented, mostly by way of example, what we believe to be a general technique for constructing and analyzing numerical algorithms for calculating the solutions of incompressible flow problems. The methods are grid free and, in view of the results of section 6, are capable of representing very complicated fluid motion. The methods give "smoothed" approximations to the solutions of the fluid equations, but this smoothing does not accumulate and contaminate the numerical approximation. It is this property that is the distinctive advantage of these methods over finite difference methods.

We have also shown that the theoretical techniques developed for the vortex method ([5],[6],[8],[19],[30]) can be used to analyze more general numerical methods. In particular, we have used these techniques to obtain convergence results for the methods proposed here. Also, our use of these theoretical techniques in sections 4 and 5 to help select parameters for the numerical methods demonstrate their power and utility. This utility is also demonstrated in section 6.3 where we use the theoretical techniques to devise the proper numerical procedure when the initial computational points are on a non-rectangular grid.

Of the two methods, method B appears to be superior. It was more accurate on the test problem of section 4.2 and was much better at resolving the fine scale features of the motion of a 2-D thermal. Although method A is less accurate than Method B, it has the property of being able to calculate the long time motion of the 2-D thermal with fewer points.

As for the efficiency of the methods, they both require $O(n^2)$ operations for each time step, where n is the number of computational points. Compared to grid techniques which usually require $O(m)$ or $O(m \log m)$, where m is the number of grid points, the methods presented here may seem to be inefficient. Such a conclusion is not necessarily valid since one typically needs much fewer computational points using these methods than with grid techniques to resolve the fine scale structures of the flow. This fact is illustrated by figure 7.1. In this figure we have superimposed a 33×33 grid over the computational results obtained with method B using an equivalent number of points. Clearly the fine scale features of this solution would not be resolved with a finite difference method using such a grid. We mention that the operation count of the methods presented here may be reduced if one uses particle-grid techniques such as Cloud In Cell ([18]) or its variants ([2],[10],[12],[35]). (For a review of particle-grid methods see [24].) Although using such algorithms leads to a loss of accuracy in the calculation, their use can reduce the amount of computational labor enormously.

Another aspect of the implementation of these methods is the choice of the function Ψ and the smoothing parameter δ. We believe that for smooth problems, the conclusions drawn from the theoretical analysis are reliable. In particular, using Ψ in $M^{L,p}$ with $p > 2$ is beneficial. Also, letting $\delta = h^q$ for $0 < q < 1$, where h is the initial mesh width, seems a proper choice. For non-smooth problems the results of section 6.1 indicate that these choices may not be appropriate. For method B, the results obtained with Ψ in $M^{L,p}$ for large p were not substantially better. Also using the relation $\delta = h^q$ to approach the limit $(\delta,h) \to (0,0)$ gave unsatisfactory computational results. We would recommend determining the relation between h and δ by the empirical procedure described in section 6.1. We remark that the

empirical determination of the relation between h and δ may also be of use in the vortex method, especially when the smoothness of the solution is not known.

We have mainly concentrated on the numerical methods, rather than on the problems which they are designed to solve. For example, our interest in the motion of a 2-D thermal was primarily to investigate the numerical methods rather than the behavior of the thermal. The next logical step is to concentrate on the physical phenomena. We hope that these methods will be useful in obtaining an understanding of the flows associated with thermal convection. One possible application may be in the study of turbulent heat transfer.

There are other directions for future work. We believe that method A can be improved. We are currently investigating another method, similar to method A, that is more accurate and avoids the problem in placing computational points that was described in section 6.2. We are also investigating ways to incorporate the effects of diffusion, surface tension, and of solid boundaries into the methods.

We find the existence of the singularity of the equations, as revealed by the computational results of method B, very intriguing. This singularity deserves more study. In particular, the similarity between two dimensional flows of variable density and three dimensional flows of constant density suggests that there may be some relation between the singularity obtained here and that obtained by Chorin in [16]. We also mention that preliminary comparisons indicate that the results obtained here are consistent with the results obtained by Krasny [37] for the singularity occurring in the motion of an infinite vortex sheet of constant strength.

As for the general technique presented here, a comparison needs to be made between results obtained with the methods presented here and those obtained with other particle techniques such as those described in ([28],[44]). Lastly, the behavior of the delta-equations (6.1.4)-(6.1.8) and (6.2.3)-(6.2.6) as $\delta \to 0$ have yet to be investigated. The results of such an investigation should illuminate the properities of the approximate solutions that are constructed using these methods.

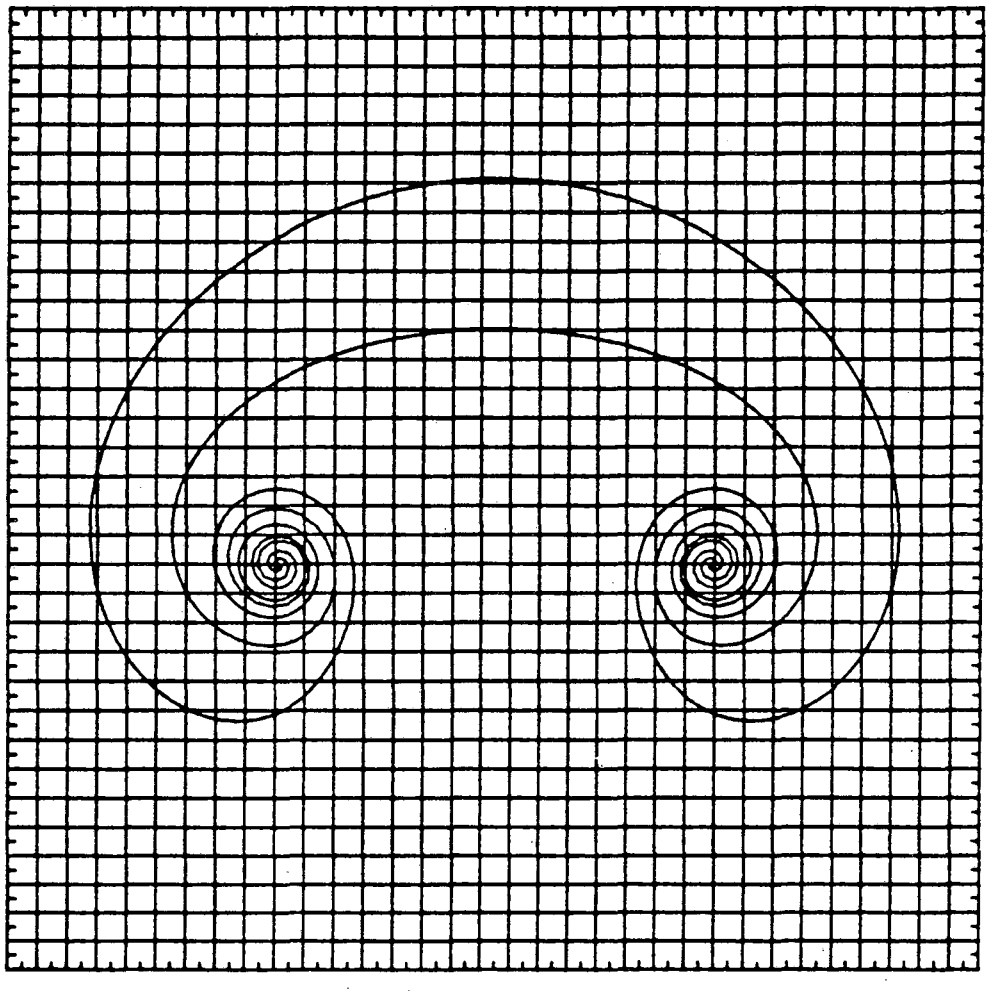


Figure 7.1

Interface Position at t = 3.0

δ = .04

On 33 × 33 Grid

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