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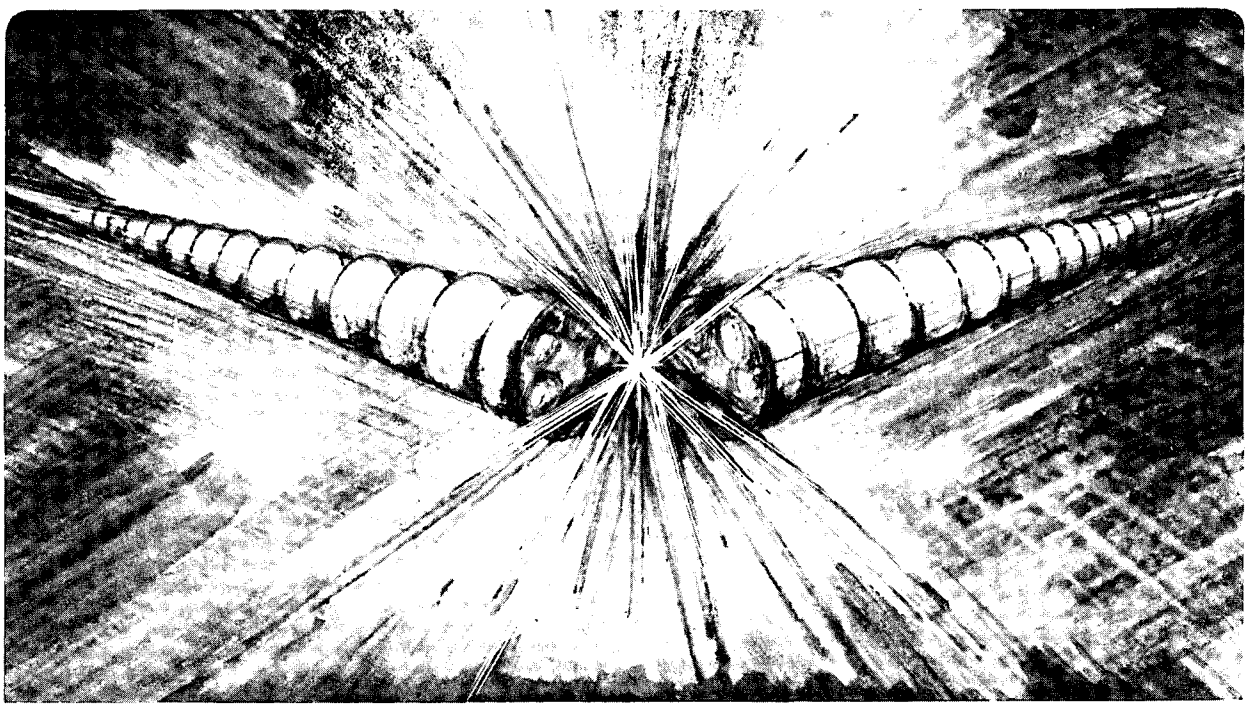

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CANONICAL DERIVATION OF THE VLASOV-COULOMB NONCANONICAL POISSON STRUCTURE*

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ABSTRACT. Starting from a Lagrangian formulation of the Vlasov-Coulomb system, canonical methods are used to define a Poisson structure for this system. Successive changes of representation then lead systematically to the noncanonical Lie-Poisson structure for functionals of the Vlasov distribution.

The Vlasov system with Coulomb interaction is not only the standard model for plasma theory, but it is also a paradigm for the understanding of noncanonical Poisson structures. The Vlasov distribution $f(\underline{r},\underline{p})$ is its dynamical field variable and has no canonical conjugate. Yet it possesses a noncanonical Lie-Poisson structure for observables $A(f)$, functionals of f:

$$\left\{A_1,A_2\right\} = \int d^3r d^3p\ f(\underline{r},\underline{p})\ [a_1(\underline{r},\underline{p}),\ a_2(\underline{r},\underline{p})], \tag{1}$$

where $a(\underline{r},\underline{p}) = \delta A/\delta f(\underline{r},\underline{p})$ is the functional derivative, and

$$[a_1,a_2] = (\partial a_1/\partial \underline{r})\ (\partial a_2/\partial \underline{p}) - (\partial a_1/\partial \underline{p})\ (\partial a_2/\partial \underline{r})$$

is the canonical bracket on functions of $(\underline{r},\underline{p})$.

Since a knowledge of the Poisson structure is essential for a Hamiltonian field theory, it is highly desirable to understand the derivation of this structure. In this paper, we present such a derivation, based on old-fashioned canonical field theory. This derivation is rigorous, and is accessible to readers ignorant of modern mathematics.

Before beginning, we survey the earlier derivations of the Poisson structure (1). In 1980, this structure was discovered by Morrison and Kaufman by manipulating Morrison's results for the Vlasov-Maxwell sytem [9]; the latter had been obtained by inspired guesswork. (The guesswork method was sytematically reviewed by Morrison in [10]. It has been extended to wave systems by McDonald and Kaufman [8].)

Independently, the structure (1) was obtained by Gibbons [3], using

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Lie-group concepts. This approach was extensively investigated by Marsden and Weinstein [7], who have applied these concepts to other systems as well.

A third approach to deriving the structure (1) was introduced by Bialynicki-Birula and Hubbard [1]. They began with the canonical structure for a system of particles, derived from it the noncanonical structure for the Klimontovich distribution, and then postulated the identical structure for the Vlasov distribution. The success of this approach led Kaufman to apply it to multi-fluid electrodynamics [4] and to waves [5].

The method adopted here is to begin with a field Lagrangian for the Vlasov-Coulomb system. Our form is a modification of the Low Lagrangian due to Dewar [2]. With an eye on the application to gyrokinetics [6], we consider one species (for simplicity) of particle, moving non-relativistically in a specified (possibly nonuniform) static magnetic field $\underline{B}(\underline{x})$, with only Coulomb interaction. At some reference time ($t = 0$), we specify the smooth Vlasov distribution f_0 of initial conditions $(\underline{r}_0, \underline{v}_0)$ in six-dimensional phase space. The dynamical field in a Lagrangian description is $\underline{R}(t;\underline{r}_0,\underline{v}_0)$, the corresponding position at any time t.

The Coulomb interaction energy C is thus $[z_0 = (\underline{r}_0, \underline{v}_0)]$

$$C = (e^2/2)\int d^6z_0 \int d^6z_0' \; f_0(z_0)f_0(z_0') \; |\underline{R}(t;z_0)-R(t;z_0')|^{-1}, \tag{2}$$

while the noninteractive field Lagrangian L_0 is

$$L_0 = \int d^6z_0 \; f_0(z_0) \; (m\dot{\underline{R}}^2/2 + (e/c)\dot{\underline{R}}\cdot\underline{A}(\underline{R}(t;z_0))), \tag{3}$$

where $\dot{\underline{R}}(t;z_0) = \partial\underline{R}(t;z_0)/\partial t$, and $\underline{B}(\underline{x}) = \nabla \times \underline{A}(\underline{x})$. We consider the total Lagrangian $L = L_0 - C$, and apply the standard canonical procedure.

The first step is to define the momentum field $\underline{\pi}(z_0)$ canonically conjugate to the position field $\underline{R}(z_0)$:

$$\underline{\pi}(z_0) = \delta L/\delta\dot{\underline{R}}(z_0) = f_0(z_0)\; \underline{P}(z_0), \tag{4}$$

where $\underline{P} = m\dot{\underline{R}} + \frac{e}{c}\underline{A}(\underline{R})$ is the usual canonical particle momentum.

The next step is to express the canonical Poisson Bracket relation for the position field and its conjugate (at equal times):

$$\left\{\underline{R}(z_0), \underline{\pi}(z_0')\right\} = \underline{I}\; \delta^6(z_0 - z_0'),$$

with other brackets vanishing. Inserting (4), we obtain the interesting relation:

$$\left\{\underline{R}(z_0), \underline{P}(z_0')\right\} = \underline{I}\; \delta^6(z_0-z_0')/f_0(z_0), \tag{5}$$

which requires that $f_0(z_0) > 0$ everywhere. The final conventional step is to form the field Hamiltonian:

$$H = \int d^6z_0\ \underline{\pi}(z_0)\cdot\underline{R}(z_0) \ - \ L$$

$$= \int d^6z_0\ f_0(z_0)\ (\underline{P}(z_0)-(e/c)\underline{A}(\underline{R}(z_0)))^2/2m \ + \ C. \tag{6}$$

We now wish to transform our representation to the Vlasov field $f(\underline{r},\underline{p};t)$:

$$f(\underline{r},\underline{p};t) = \int d^6z_0\ f_0(z_0)\ \delta^3(\underline{r} - \underline{R}(t;z_0))\delta^3(\underline{p} - \underline{P}(t;z_0)) \ . \tag{7}$$

Thus f is a function on $(\underline{r},\underline{p})$-space, and simultaneously is a functional of the fields $\underline{R}(z_0)$, $\underline{P}(z_0)$. Setting aside the former property, let us consider general functionals $G\ (\underline{R}(z_0),\underline{P}(z_0))$. The Poisson Bracket on such functionals is obtained by using its property of differentiation, and the chain rule:

$$\left\{G_1,G_2\right\} = \int d^6z_0\int d^6z_0'\ \ (\delta G_1/\delta\underline{R}(z_0))(\delta G_2/\delta\underline{P}(z_0')):\left\{\underline{R}(z_0),\underline{P}(z_0')\right\}$$

$$+\ (\delta G_1/\delta\underline{P}(z_0))(\delta G_2/\delta\underline{R}(z_0')):\left\{\underline{P}(z_0),\underline{R}(z_0')\right\}$$

$$= \int d^6z_0\ ((\delta G_1/\delta\underline{R}(z_0))\cdot(\delta G_2/\delta\underline{P}(z_0))$$

$$-\ (\delta G_1/\delta\underline{P}(z_0))\cdot(\delta G_2/\delta\underline{R}(z_0)))/f_0(z_0), \tag{8}$$

where we have used the fundamental Poisson Bracket (5).

We now may immediately apply (8) to the special functional (7), choosing $G_1 = f(z_1)$, $G_2 = f(z_2)$, where $z_j = (\underline{r}_j,\underline{p}_j)$ is a parameter for purposes of the functional differentiation in (8). After a few straightfoward steps, we obtain

$$\left\{f(z_1),f(z_2)\right\} = \int d^6z_0 f_0(z_0)\ (\partial^2/\partial\underline{r}_1\cdot\partial\underline{p}_2 - \partial^2/\partial\underline{r}_2\cdot\partial\underline{p}_1)$$

$$\times\ \delta^3(\underline{r}_1-\underline{R}(z_0))\delta^3(\underline{p}_1-\underline{P}(z_0))\delta^3(\underline{r}_2-\underline{R}(z_0))\delta^3(\underline{p}_2-\underline{P}(z_0))$$

$$= (\partial^2/\partial\underline{r}_1\cdot\partial\underline{p}_2 - \partial^2/\partial\underline{r}_2\cdot\partial\underline{p}_1)\ f(z_1)\delta^6(z_1-z_2) \tag{9}$$

Finally, we consider functionals $A(f)$, and their Poisson Bracket. We have

$$\left\{A_1,A_2\right\} = \int d^6z_1\int d^6z_2\ \ \delta A_1/\delta f(z_1)\ \ \delta A_2/\delta f(z_2)\ \ \left\{f(z_1),\ f(z_2)\right\} \ ,$$

substitute (9), and integrate by parts, obtaining (1), the desired result.

The Hamiltonian (6), (2) reads, in terms of f,

$$H(f) = \int d^6z\, f(z)\, (\underline{p} - (e/c)\underline{A}(\underline{r}))^2/2m + C,$$

$$C(f) = (e^2/2) \int d^6z \int d^6z'\, f(z)f(z')\, |\underline{r} - \underline{r}'|^{-1}.$$

This completes the canonical derivation of the Hamiltonian structure of the Vlasov-Coulomb system.

BIBLIOGRAPHY

1. I. Bialynicki-Birula and J. C. Hubbard, "Gauge-independent Canonical Formulation of Relativistic Plasma Theory," Phys. Rev. A (in press).

2. R. L. Dewar, "A Lagrangian Theory for Nonlinear Wave Packets in a Collisionless Plasma", J. Plasma Physics 7 (1972) 267-284.

3. J. Gibbons, "Collisionless Boltzmann Equations and Integrable Moment Equations," Physica 3D (1981) 503-511.

4. A. N. Kaufman, "Elementary Derivation of Poisson Structures for Fluid Dynamics and Electrodynamics," Phys. Fluids 25 (1982) 1993-1994.

5. A. N. Kaufman, "Natural Poisson Structures of Nonlinear Plasma Dynamics,"Physica Scripta T2/2 (1982) 517-521.

6. A. N. Kaufman and B. M. Boghosian, "Lie-Transform Derivation of the Gyrokinetic Hamiltonian System," this volume.

7. J. Marsden and A. Weinstein, "The Hamiltonian Structure of the Maxwell-Vlasov Equations," Physica 4D (1982) 394-406.

8. S. W. McDonald and A. N. Kaufman, "Hamiltonian Kinetic Theory of Plasma Ponderomotive Processes," in Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems, M. Tabor and Y. M. Treve, eds. (AIP Conf. Proc. 88, 1982), 117-120.

9. P. J. Morrison, "The Maxwell-Vlasov Equations as a Continuous Hamiltonian System", Phys. Lett. 80A (1980) 383-386.

10. P. J. Morrison, "Poisson Brackets for Fluids and Plasmas," in Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems, M. Tabor and Y. M. Treve, eds. (AIP Conf. Proc. 88, 1982), 13-46.

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