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BUCKLEY-LEVERETT EQUATION

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TEST PROBLEM FOR THE TWO-DIMENSIONAL BUCKLEY-LEVERETT EQUATION[1]

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Test Problem for the Two-dimensional Buckley-Leverett Equation

Norman Albright

ABSTRACT

A test problem and its analytically derived solution for the two-dimensional Buckley-Leverett equation for two phase immiscible flow in a porous medium are described. The test problem is the five-spot configuration of water flooding of a petroleum reservoir with the total velocity given by potential flow. The solution is obtained by means of a coordinate transformation.

INTRODUCTION

The simultaneous flow of two incompressible, immiscible fluids through a porous medium can be described by the Buckley-Leverett equation [1, pp. 19-22]. When capillary pressure effects are small or absent, this equation is hyperbolic in nature. For the purely hyperbolic case (zero capillary pressure) it is well known that, in general, solutions develop discontinuities in finite time, even for smooth initial data. These discontinuities correspond to propagating fronts between the two fluids. If a small, but non-zero, amount of capillary pressure is present then the fronts that are developed will not be perfectly sharp, but will correspond to a large change in fluid saturation over a small, but non-zero, distance.

The representation of such discontinuities, or near discontinuities, usually causes difficulty for conventional numerical methods. The development of new numerical methods is aided by test problems with analytically derived solutions that have such discontinuities. A suitable such test problem for the two-

dimensional Buckley-Leverett equation is the five-spot configuration of water flooding of a petroleum reservoir with the total velocity given by potential flow.

THE TEST PROBLEM

The Buckley-Leverett equation for the saturation of the flow of two immiscible, incompressible liquids through a homogeneous porous medium in the absence of capillary pressure and gravitational effects, in a region free of sources or sinks, is

$$\frac{\partial s}{\partial t} = -\frac{df}{ds}\mathbf{v}\cdot\nabla s\ , \tag{1}$$

where $\mathbf{v}=(v_x,v_y)$ is the total velocity, and $s(x,y;t)$ and $f(s)$ are the saturation and fractional flow of the wetting liquid respectively. Typically $f(s)$ has the S-shape shown in Figure 1, which for this test problem is given by the model formula

$$f(s) = \frac{s^2}{s^2 + \alpha(1-s)^2}, \tag{2}$$

where α is the ratio of the viscosities of the wetting to the non-wetting fluids.

The standard diagonal-geometry quarter five-spot configuration on the unit square is chosen, with a unit source at (0,0) and sink at (1,1), as shown in Figure 2. The total velocity is taken to be potential flow defined by

$$\mathbf{v} = \nabla\psi\ , \tag{3}$$

where ψ satisfies

$$\Delta\psi = \delta(x)\,\delta(y) - \delta(x-1)\,\delta(y-1)\ , \tag{4}$$

with no-flow conditions

$$\frac{\partial\psi}{\partial n} = 0 \tag{5}$$

on the edges of the square. This $\mathbf{v}$ is the actual velocity in a five-spot configuration for the case of constant saturation, for which ψ is proportional to the negative of the pressure.

The boundary conditions on s are

$$\frac{\partial s}{\partial n} = 0 \tag{6}$$

on the edges of the square, through which there is no flow, and $s = 1$ at the source. The initial condition is $s = 0$ everywhere except at the source.

SOLUTION

The solution to this problem is obtained by means of a transformation to orthogonal coordinates (φ,η), where $\eta(x,y) = constant$ are the flow lines, $\psi(x,y) = constant$ are the velocity equipotentials, and φ is a function of ψ that is chosen to have finite range. In these coordinates the equation for the saturation can be solved by quadrature. To facilitate this, a second transformation is made to a coordinate $\xi(\varphi,\eta)$, which is chosen so that the saturation is a function of ξ and t only, and the characteristics of (1) are straight lines in the ξ–t plane. The solution to the test problem consists of two parts: the calculation of s as a function of ξ and t and the calculation of ξ as a function of x and y.

The velocity potential ψ approaches $-\infty$ at the source and $+\infty$ the sink. For convenience of numerical calculation a function $\varphi(\psi)$ is defined that is equal to 0 at the source and 1 at the sink. The saturation can be expressed as a function of φ, η, and t. In these coordinates (1) becomes

$$\frac{\partial s}{\partial t} = -\frac{df}{ds} v^2 \frac{d\varphi}{d\psi} \frac{\partial s}{\partial \varphi}, \tag{7}$$

where $v = v(\varphi,\eta)$. Let $\psi(\varphi)$ denote the inversion of $\varphi(\psi)$, and $\psi'(\varphi)$ denote $d\psi/d\varphi$. Define

$$H(\varphi,\eta) = \psi'(\varphi) / v^2(\varphi,\eta) \tag{8}$$

and

$$\xi(\varphi,\eta) = \int_0^{\varphi} H(\sigma,\eta)\, d\sigma\,. \tag{9}$$

The function $\varphi(\psi)$ must be chosen so that H is bounded at the source and sink and positive everywhere. Then ξ equals 0 at the source and increases along each flow line. If the saturation is expressed as a function of ξ, η, and t, the saturation equation simplifies to

$$\frac{\partial s}{\partial t} = -\frac{df}{ds}\frac{\partial s}{\partial \xi}. \tag{10}$$

In the $\xi-\eta$ coordinate system, the saturation equation is independent of η. Since, in addition, the initial condition for this problem is independent of η, the solution is a function of ξ and t only, $s = s(\xi,t)$. The solution is a shock front followed by an expansion wave. The height of the shock front is

$$s_0 = \sqrt{\frac{\alpha}{1+\alpha}}. \tag{11}$$

The height is approximately 0.577 for $\alpha = 0.5$. The speed of the shock front is

$$a_0 = 0.5 + 0.5\sqrt{\frac{1+\alpha}{\alpha}}.$$

For $\alpha = 0.5$ the speed is approximately 1.366. In front of the shock, that is, for $\xi > a_0 t$, the saturation is 0, for $\xi = a_0 t$ the saturation equals s_0, and behind the shock, where $\xi < a_0 t$, the saturation is given by the solution of the equation

$$a(s) = \xi / t\,, \tag{12}$$

where

$$a(s) = \frac{df(s)}{ds} = \frac{2\alpha s(1-s)}{(s^2 + \alpha(1-s)^2)^2}. \tag{13}$$

Eq. (12) has four solutions, which are functions of the ratio ξ/t. The saturation is the solution that equals 1 for $\xi = 0$.

The diagonal $y = x$ is a flow line. The wetting liquid flowing along this line reaches the sink before that flowing on other paths; this is called breakthrough. Let ξ_1 be the limit of $\xi(\varphi,\eta)$ as φ approaches 1 for η corresponding to this diagonal. Then the time of breakthrough is given by

$$t_b = \xi_1 / a_0 . \tag{15}$$

The value of ξ_1 is independent of α; so the time of breakthrough depends on α only through the shock speed a_0. For our problem the value of ξ_1 is approximately 2.912, and for $\alpha = 0.5$ the time of breakthrough is approximately 2.132.

THE FUNCTION $\varphi(\psi)$

The velocity potential ψ is determined only up to an additive constant. Let the constant be chosen so that the ψ is zero on the diagonal $y = 1 - x$.

The function $\varphi(\psi)$ is determined by two conditions: that H be finite at the source and sink, and that φ have specified values at the source and sink, which are taken to be 0 and 1 respectively. Near the source

$$\psi \approx \frac{1}{4\pi} \log r^2 ,$$

so the square of the velocity is

$$v^2 \approx (2\pi r)^{-2} , \tag{16}$$

which can be written as a function of ψ as

$$v^2 \approx \frac{e^{-4\pi\psi}}{4\pi^2} . \tag{17}$$

Near the sink

$$\psi \approx \frac{-1}{4\pi} \log [(x-1)^2 + (y-1)^2] \tag{18}$$

and the square of the velocity is

$$v^2 \approx \frac{e^{4\pi\psi}}{4\pi^2}. \tag{19}$$

The function $\varphi(\psi)$ must be chosen so that $d\varphi/d\psi$ is proportional to $1/v^2$ near the source and sink, thus $d\varphi/d\psi$ must be proportional to $e^{4\pi\psi}$ near the source and proportional to $e^{-4\pi\psi}$ near the sink. A suitable choice for φ is

$$\varphi = \frac{e^{4\pi\psi}}{1 + e^{4\pi\psi}}. \tag{20}$$

Then φ is 0 at the source, 1 at the sink, and 0.5 on the diagonal $y = 1 - x$. φ can be inverted to give $\psi(\varphi)$.

$$\psi = \frac{1}{4\pi} \log\left(\frac{\varphi}{1-\varphi}\right) \tag{21}$$

and

$$\psi' = 1/\left[4\pi\varphi(1-\varphi)\right]. \tag{22}$$

THE FLOW LINES

The no-flow boundary conditions on ψ and the locations of the source at (0,0) and the sink at (1,1) imply that one flow line is composed of the lower and right-hand edges of the square and another flow line is composed of the left-hand and upper edges. The function $\eta(x,y)$ is constant along a flow line and is taken to be 0 on the lower and right-hand edges and 1 on the left-hand and upper edges. The function η is determined by these boundary conditions and the equation

$$\Delta\eta = 0. \tag{23}$$

By symmetry the diagonal $y=x$ corresponds to $\eta = 0.5$. The function $\eta(x,y)$ is multivalued at the source and sink.

COORDINATE TRANSFORMATION

The calculation of ξ as a function of x and y must be performed numerically. The steps are:

(i) Let (x_i, y_j) be a point on a mesh that is uniform on the unit square in x,y space. Solve (2) and (23) for ψ and η respectively on this mesh. This gives $\psi(x_i, y_j)$ and $\eta(x_i, y_j)$.

(ii) Calculate $\varphi(x_i, y_j)$ for each point on this mesh.

(iii) Interpolate φ and η smoothly on the x,y mesh. This gives $\varphi(x,y)$ and $\eta(x,y)$ for any point (x,y) on the unit square.

(iv) Let (φ_k, η_m) be a point on a mesh that is uniform on the unit square in φ,η space. Using the interpolations for $\varphi(x,y)$ and $\eta(x,y)$ calculate the point (x_{km}, y_{km}) that satisfies the pair of equations:

$$\begin{aligned} \varphi(x_{km}, y_{km}) &= \varphi_k \\ \eta(x_{km}, y_{km}) &= \eta_m \end{aligned} \tag{24}$$

x_{km} and y_{km} are the x,y coordinates corresponding to the point (φ_k, η_m) on the φ,η mesh.

(v) Let φ_x and φ_y denote $\partial\varphi(x,y)/\partial x$ and $\partial\varphi(x,y)/\partial y$ respectively. Using the interpolation for $\varphi(x,y)$ calculate φ_x and φ_y at the point (x_{km}, y_{km}).

(vi) Evaluate H on the φ,η mesh using

$$H(\varphi_k, \eta_m) = 1/\left[\psi'(\varphi_k)\left[\varphi_x^2(x_{km}, y_{km}) + \varphi_y^2(x_{km}, y_{km})\right]\right] \tag{25}$$

(vii) Interpolate H in φ for each value of η_m on the φ,η mesh. This gives $H(\varphi, \eta_m)$. Using this interpolation integrate H in φ to obtain $\xi(\varphi_k, \eta_m)$ for each point on the φ,η mesh.

(viii) Interpolate ξ on the φ,η mesh. This gives $\xi(\varphi,\eta)$. Using this interpolation, calculate the value of ξ at the point $(\varphi(x_i, y_j), \eta(x_i, y_j))$ for each point on the

x,y mesh. From these values a standard contour plot routine will generate contours on the x,y plane of $\xi = constant$.

The solution $s(\xi,\eta,t)$ was computed at times t = 0.5, 1.0, 1.5, 2.0, and 2.2. The contours on the x,y plane of s = 0.577, 0.6, 0.7, and 0.8 for these times are shown in Figures 3(a)-(e). The contour for s = 0.577 depicts the position of the discontinuity, in front of which s = 0. The contours were drawn using subroutine CONREC from the National Center for Atmospheric Research (NCAR) graphics package. (The small oscillations of some of the contours are due to the effects of the interpolation from a discrete grid.)

ACKNOWLEDGEMENT

The author would like to thank Paul Concus for acquainting him with this problem and for numerous discussions concerning it.

REFERENCE

[1] Peaceman, D. W.: "Fundamentals of Numerical Reservoir Simulation", Elsevier, Amsterdam-Oxford-New York (1977).

FIGURE CAPTIONS

Fig. 1 - Fractional flow as a function of saturation

Fig. 2 - Quarter configuration of the five-spot problem

Fig. 3A - Saturation contours 0.577, 0.6, 0.7, and 0.8 at t = 0.5.

Fig. 3B - Saturation contours 0.577, 0.6, 0.7, and 0.8 at t = 1.0.

Fig. 3C - Saturation contours 0.577, 0.6, 0.7, and 0.8 at t = 1.5.

Fig. 3D - Saturation contours 0.577, 0.6, 0.7, and 0.8 at t = 2.0.

Fig. 3E - Saturation contours 0.577, 0.6, 0.7, and 0.8 at t = 2.2.

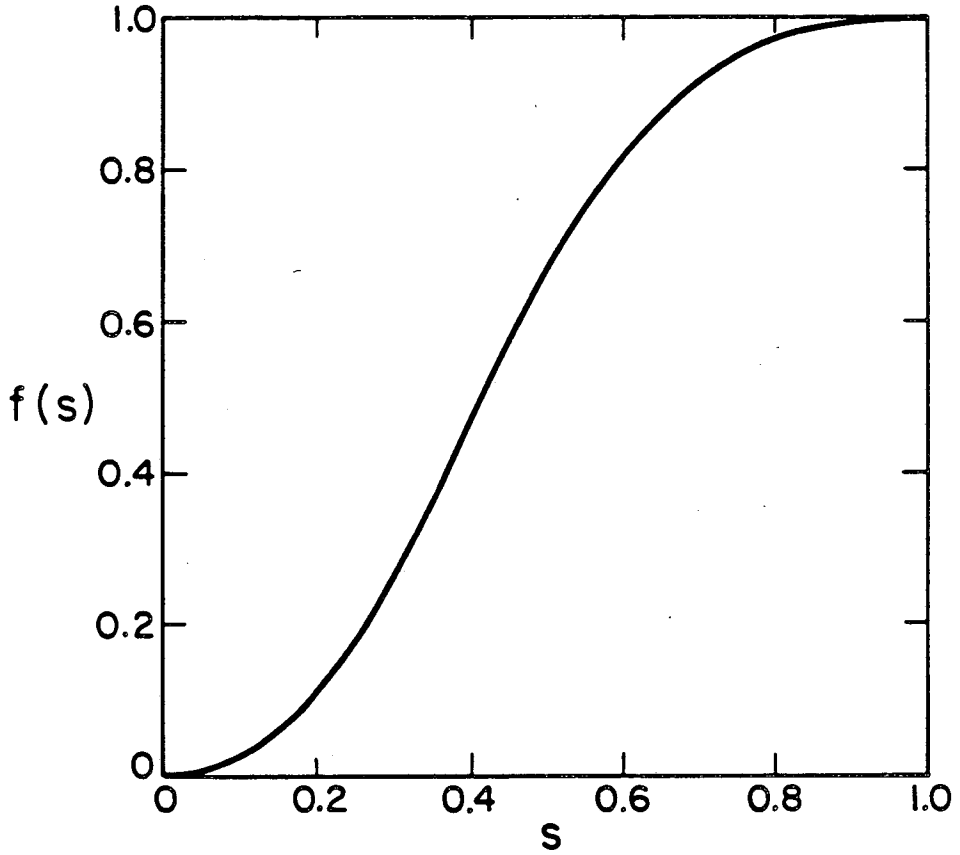

Figure 1

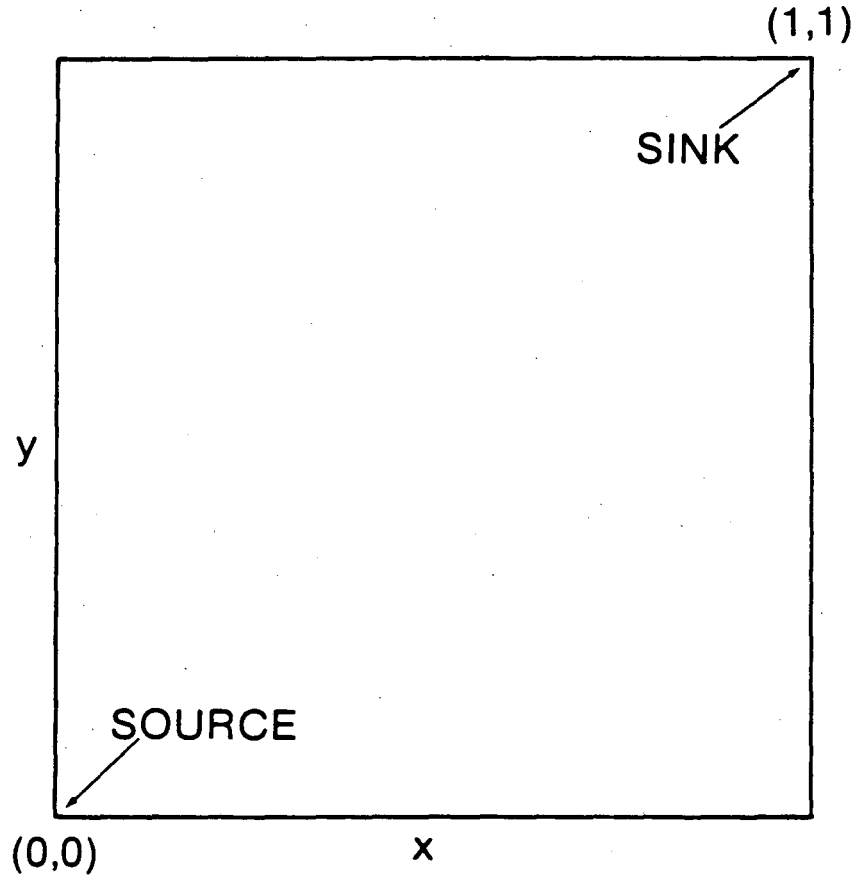

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Figure 2

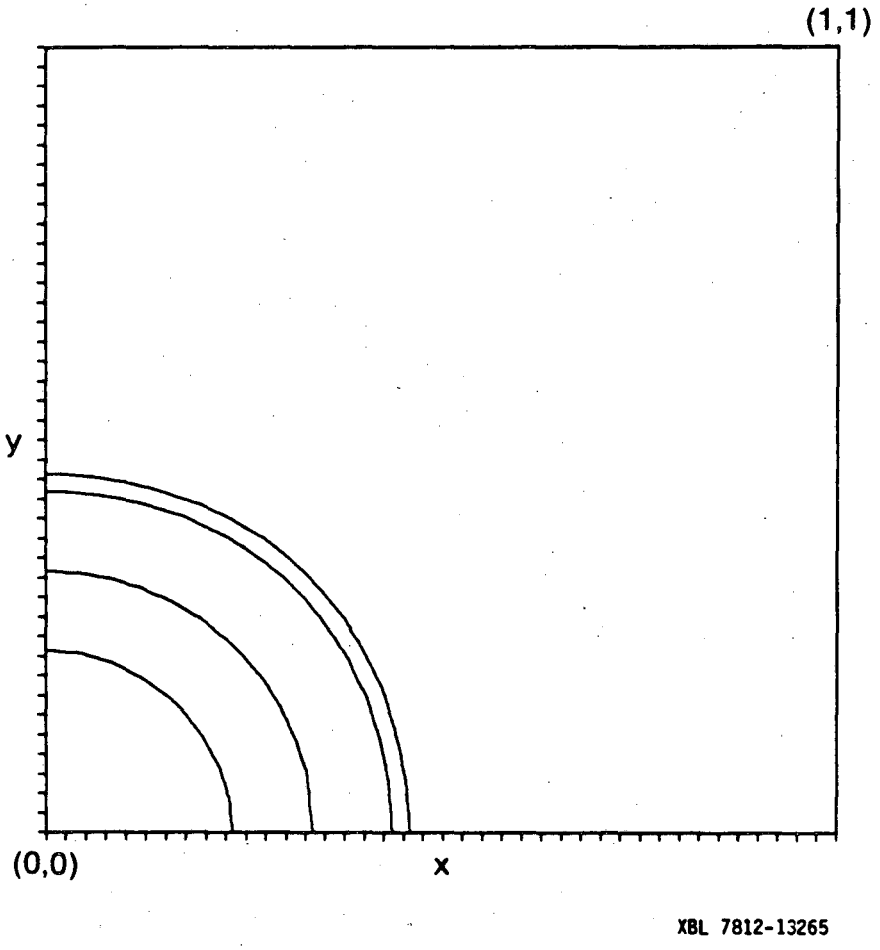

Figure 3a

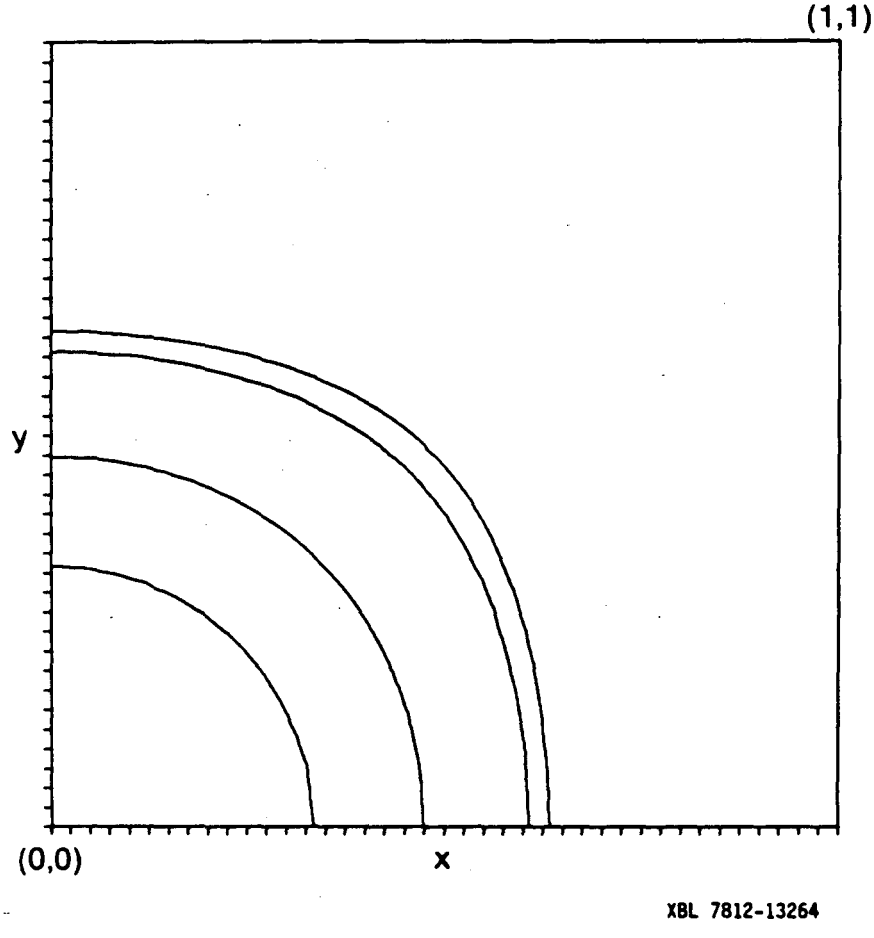

Figure 3b

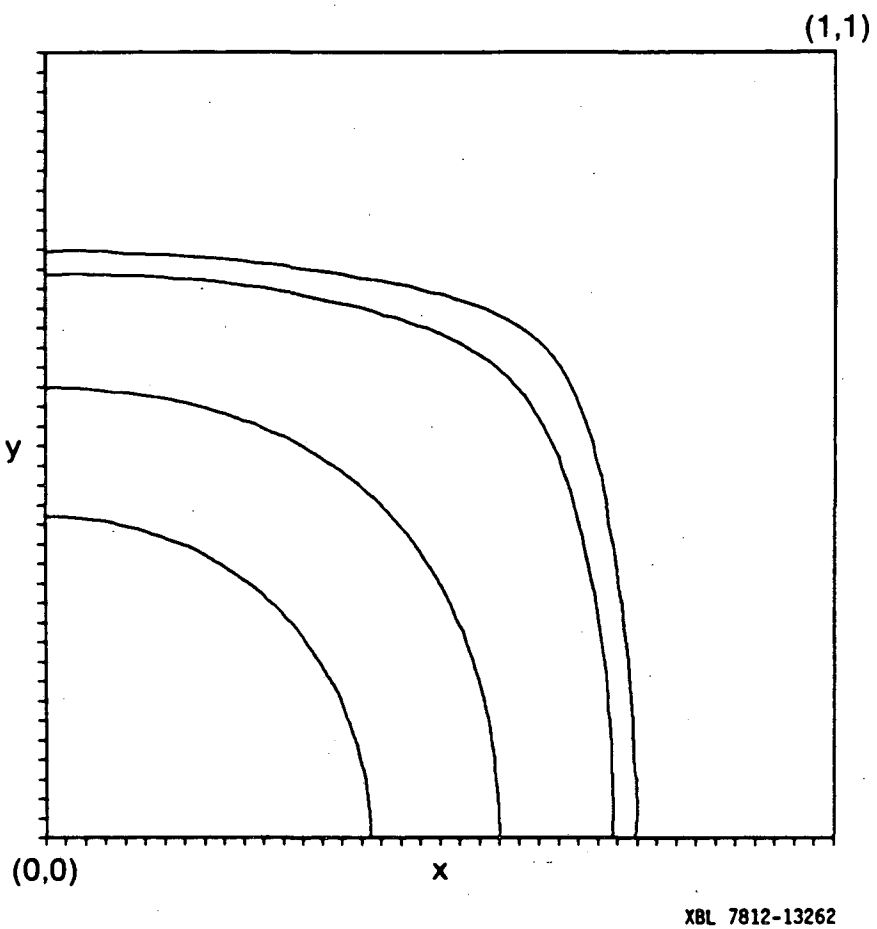

Figure 3c

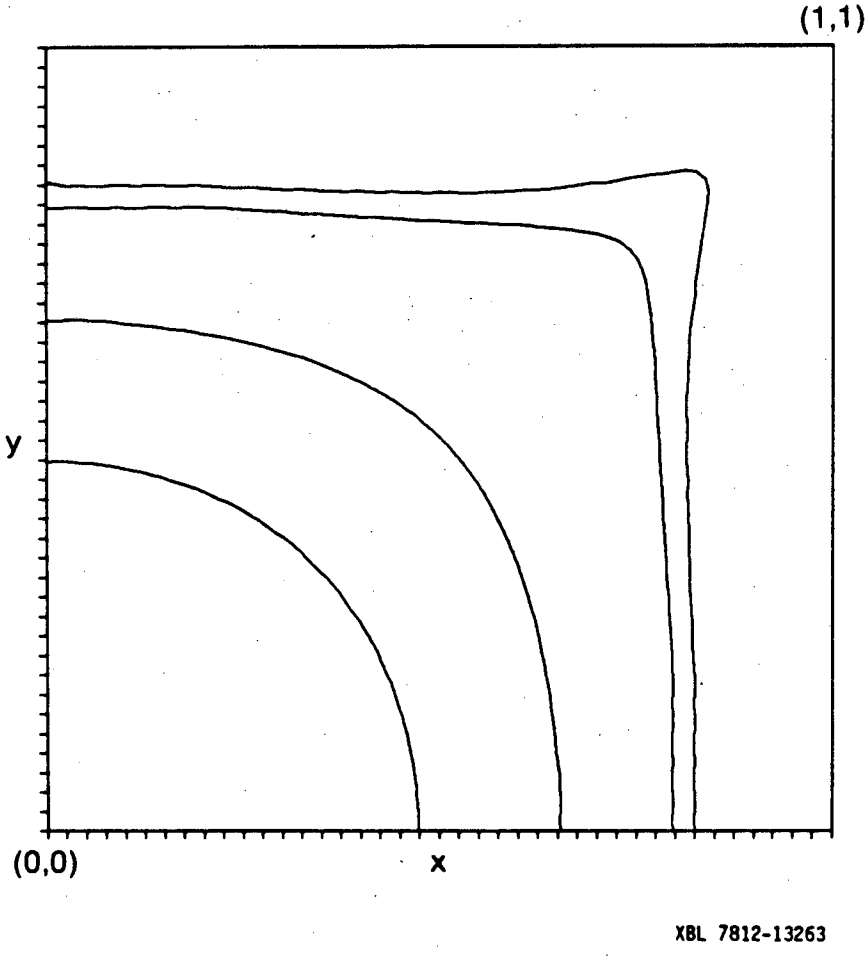

Figure 3d

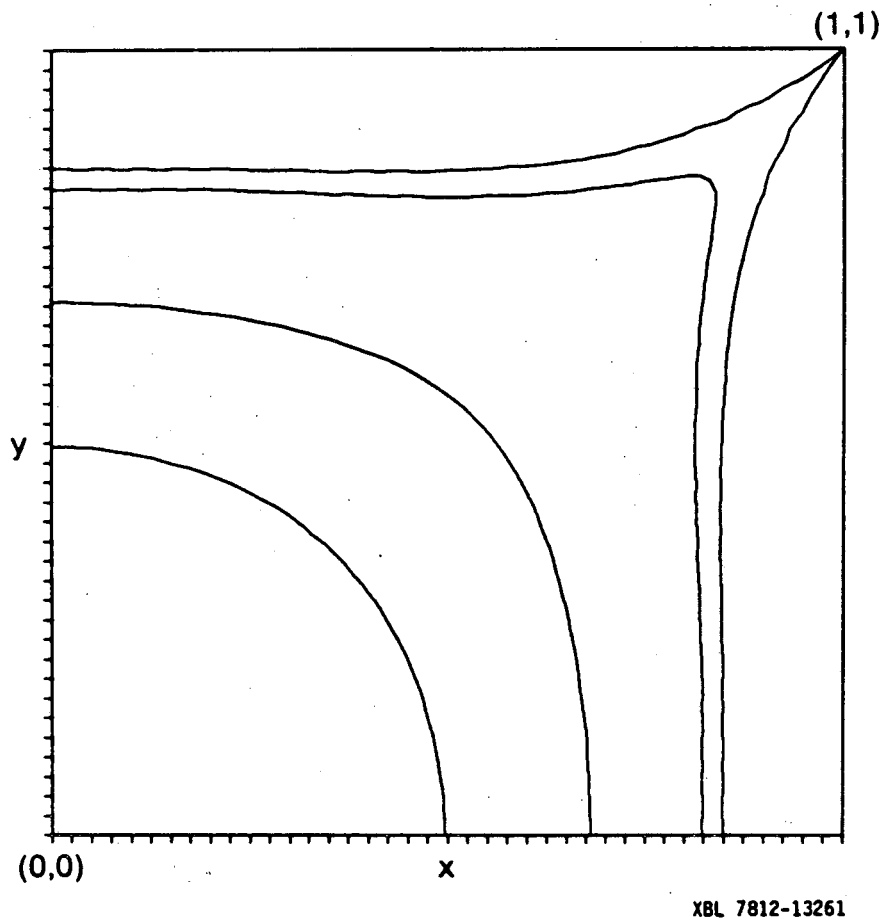

Figure 3e

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