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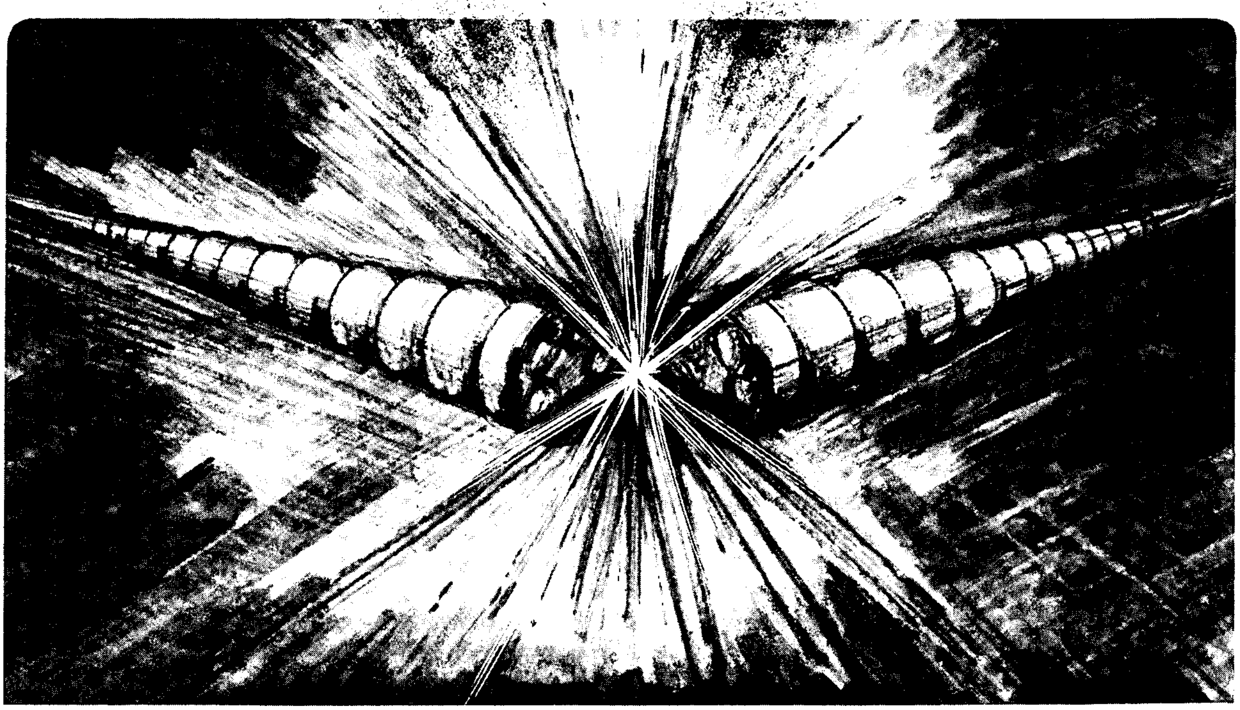

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# Generalized Ohm's Law for Plasma Including Neutral Particles

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## Abstract

A simplified derivation is presented for the so-called "ion-slip", i.e., for the relative motion of ions with respect to the fluid center-of-mass motion, which gives rise to an extra term in the generalized Ohm's law for partially ionized gases. In the usual formulation, this term is shown to be proportional to $\vec{j}_\perp$, the current density perpendicular to $\vec{B}$. This turns out to be valid when the slip is not strongly affected by a pressure gradient of the neutral component. In the opposite limit, i.e. when $\nabla p_n$ dominates the relative motion perpendicular to $\vec{B}$, the ion-slip term takes on a simple form again, provided inertial effects are small, i.e. under quasistatic conditions:

$$(\Delta\vec{E})_{slip} = \eta\ s_D\ (\vec{j}_\perp - \vec{j}_D) \qquad \text{where}$$

$$s_D \equiv \Omega_e \Omega_i m_n n_n / (m_n n_n + m_i n_i)\ (\nu_{ei} + \nu_{en})\ \nu_{in}$$

is the modified ion-slip coefficient, while

$$\vec{j}_D \equiv c\vec{B} \times \nabla(p_i + p_e)/B^2$$

denotes the diamagnetic current.


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## I. Introduction

In the macroscopic fluid description of plasma dynamics it is often convenient to make use of an equation relating the electric field $\vec{E}$ to the current density $\vec{j}$ in the plasma. For a fully ionized gas this relation is the well-known "generalized Ohm's law", as given by Spitzer[1]. Many if not most plasmas in the laboratory as well as in astrophysical situations are not fully ionized but contain substantial fractions of neutral particles. Since the neutral particles are not affected directly by the electric and magnetic fields and interact with the charged particles only through short-range forces, i.e. via binary collisions, it is obvious that the effect of the neutrals on such an Ohm's law depends on the collisionality as well as on the composition of the mixture.

The first and often most important feature to note is that the mean motion of the ions, $\vec{u}_i$, can not necessarily be taken as essentially equal to the mean motion of the fluid as a whole, $\vec{u}$, as in the case of the fully ionized gas. This is particularly true in the presence of a strong magnetic field $\vec{B}$ and in the direction perpendicular to this field. The relative motion

$$\vec{u}_{is} \equiv (\vec{u}_i - \vec{u})_\perp \equiv (\vec{u}_i - \vec{u}) - [(\vec{u}_i - \vec{u}) \cdot \vec{B}]\vec{B}/B^2 \tag{1}$$

$$= \vec{B} \times (\vec{u}_i - \vec{u}) \times \vec{B}/B^2$$

has been called the (perpendicular) "ion-slip".[2] It is the purpose of this note to reexamine the role of this ion-slip by giving a simplified derivation of a more general "generalized Ohm's law", and to clarify under what conditions relatively transparent and meaningful relations between $\vec{j}$ and $\vec{E}$ can be constructed.

## II. General Considerations

In terms of the mean motion of the charge-carrying species the current density in an ionized gas is given by

$$\vec{j} = e\,(z_i n_i \vec{u}_i - n_e \vec{u}_e) \simeq e n_e (\vec{u}_i - \vec{u}_e) \qquad (2)$$

where quasineutrality is assumed, as usual. If several different ion species are present, $\vec{u}_i$ refers to a charge-weighted mean velocity here. In general, this $\vec{u}_i$ could be quite different from an average velocity of the ion center-of-mass density that corresponds to their mechanical momentum. In order to keep our treatment as simple as possible we restrict ourselves to a single species of ions, in which case the difference vanishes. Extension to a mixture of ions is straight forward but tedious, and adds no further fundamental insight into the nature of our Ohm's law.

In the macroscopic description, the velocities $\vec{u}_i$ and $\vec{u}_e$ are obtained from the equations of motion of the species. For the electrons this equation can be written in the form[3]

$$m_e n_e \frac{D_e \vec{u}_e}{Dt} + \nabla p_e = -e n_e (\vec{E} + \frac{\vec{u}_e}{c} \times \vec{B})$$

$$+ m_e n_e \nu_{en} (\vec{u}_n - \vec{u}_e) + \frac{m_e \nu_{ei}}{e} \vec{j} + \vec{F}_e^{\,th} \qquad (3)$$

where (2) has been used to eliminate $n_e(\vec{u}_i - \vec{u}_e)$. The symbols $\nu_{ei}$ and $\nu_{en}$ represent momentum-transfer binary collision frequencies to express the effective friction force density accompanying relative drift motion of the species. The subscript $(\ )_n$ refers to the neutral component where, again, a single species is assumed. The electron stress tensor is approximated by a scalar pressure $p_e \equiv n_e k T_e$. The symbol $\vec{F}_e^{th}$ stands for the thermal force. It is usually proportional to $\nabla T_e$. When neutral gas dominates the collisions it is often negligible. But when $\nu_{ei}$ is important $\vec{F}_e^{th}$ can be of the same order as $\nabla p_e$.[4] In the interest of compactness the co-moving time derivative is used to describe the inertial term: $D_e\vec{u}_e/Dt \equiv (\partial/\partial t + \vec{u}_e \cdot \nabla)\vec{u}_e$.

The ions obey a similar equation of motion, but we will not need it explicitly in the derivation of the Ohm's law. In very weakly ionized collision-dominated gases where the term containing $\nu_{en}$ dominates over $\nabla p_e$ and $\vec{F}_e^{th}$ it is customary to set $\vec{u}_n = 0$, i.e., to describe the charge motion and the electric field in the frame of reference where the neutral component is at rest, because any ion motion would have a negligible effect on the center-of-mass of the total mixture. Solving Eq. (3) for $\vec{u}_e$ and similarly the ion equation for $\vec{u}_i$, then leads to the usual expressions for the mobility drifts[5], so that the Ohm's law is simply constructed by inserting these drifts into Eq. (2). For arbitrary degrees of ionization, and for a wider range of collisionality it is more practical to proceed in a different manner. We restrict our discussion to relatively low frequencies so that the electron inertia can be neglected in comparison with the other terms in Eq. (3). Elimination of $\vec{u}_e$ between (2) and (3), setting $m_e D_e\vec{u}_e/Dt = 0$, then yields an Ohm's law of the form

$$\vec{E}'' \equiv \vec{E} + \frac{\vec{u}_i}{c} \times \vec{B} = \eta\vec{j} + (\vec{j} \times \vec{B}/c - \nabla p_e + \vec{F}_e^{th})/en_e + m_e\nu_{en}(\vec{u}_n - \vec{u}_i)/e \qquad (4)$$

i.e., the electric field is expressed in the ion frame of reference, and we have introduced the resistivity $\eta$

$$\eta \equiv \frac{m_e}{e^2 n_e} (\nu_{ei} + \nu_{en}) = 4\pi \frac{\nu_{ei} + \nu_{en}}{\omega_p^2} \tag{5}$$

Note: for $\nu_{en} \to 0$ this reduces to the usual generalized Ohm's law for fully ionized gases.

In the general case we must refer to the fluid flow velocity $\vec{u} \neq \vec{u}_i$, and $\vec{u}$ is given by

$$\vec{u} = \sum_s m_s n_s \vec{u}_s / \sum_s m_s n_s \qquad s = e, i, n$$

$$\approx (1 - X)\, u_n + X\, u_i \tag{6}$$

where we have introduced the ion mass fraction

$$X \equiv m_i n_i / \sum_s m_s n_s \approx m_i n_i / (m_i n_i + m_n n_n) \tag{7}$$

Note that for $m_i = m_n$, X represents the degree of ionization. But in general we cannot expect $m_i = m_n$, and the difference may be considerable; for instance when we have $C_s^+$ ions in a background of helium gas.

Transformed to the proper fluid frame the Ohm's law now displays the ion-slip introduced in Eq. (1):

$$\vec{E}' \equiv \vec{E} + \frac{\vec{u}}{c} \times \vec{B} = \vec{E}'' - \vec{u}_{is} \times \vec{B}/c$$

$$= \vec{E}'' - (1 - X)(\vec{u}_i - \vec{u}_n) \times \vec{B}/c \tag{8}$$

Note that the component of $\vec{u}_i - \vec{u}$ parallel to $\vec{B}$ is usually unimportant since it contributes only a small correction to $\eta\vec{j}$ via the last term in $\vec{E}''$, Eq. (4). But the slip $\vec{u}_{is}$ always affects $\vec{E}_\perp$, sometimes significantly. To obtain an Ohm's law in the form $\vec{E}' = F(\vec{j})$ where $F$ does not contain $\vec{u}_i - \vec{u}_n$

explicitly we must find a relation between $\vec{j}$ and $(\vec{u}_i - \vec{u}_n)$ so that we can eliminate the individual species velocities from (8). The most direct way is via the equation of motion for the neutral component:

$$m_n n_n \frac{D_n \vec{u}_n}{Dt} + \nabla p_n = m_n n_n [\nu_{ni}(\vec{u}_i - \vec{u}_n) + \nu_{ne} (\vec{u}_e - \vec{u}_n)]$$

$$= m_i n_i \nu_{in}(\vec{u}_i - \vec{u}_n) + m_e n_e \nu_{en}(\vec{u}_e - \vec{u}_n) \quad (9)$$

The last step follows from conservation of momentum in the interspecies friction, which implies that always

$$m_n n_n \nu_{ns} = m_s n_s \nu_{sn}$$

Finally, using (2) to eliminate $\vec{u}_e$, and rearranging we have

$$\vec{u}_i - \vec{u}_n = \left(m_n n_n \frac{D_n \vec{u}_n}{Dt} + \nabla p_n + \frac{m_e \nu_{en}}{e} \vec{j}\right) \Big/ \left(m_i n_i \nu_{in} + m_e n_e \nu_{en}\right) \quad (10)$$

where we can usually neglect the electron term in the denominator.

We are interested in the perpendicular component of (10), in which case the electron term in the numerator tends to be negligible also as soon as the electron gyrofrequency becomes comparable or exceeds $\nu_{en}$. Thus, when

$$\Omega_e \equiv \frac{e B}{m_e c} > \nu_{en} \qquad \text{we can approximate}$$

$$(\vec{u}_i - \vec{u}_n)_\perp \approx \left(m_n n_n \frac{D_n \vec{u}_n}{Dt} + \nabla p_n\right)_\perp / m_i n_i \nu_{in} \quad (11)$$

and the problem is reduced to relating the r.h.s. of (11) to $\vec{j}$.

In limiting cases this can be done with the help of the equation of motion for the entire fluid as a whole:

$$\left(m_n n_n + m_i n_i\right) \frac{D\vec{u}}{Dt} + \nabla \left(p_n + p_i + p_e\right) = \vec{j} \times \vec{B}/c \tag{12}$$

III. Case A: Small neutral slip

When
$$|\vec{u}_n - \vec{u}| = X\,|\vec{u}_n - \vec{u}_i| \ll |\vec{u}| \tag{13}$$

we can approximate $D_n\vec{u}_n/Dt \simeq D\vec{u}/Dt$ and substitute from (12) into (11) and obtain

$$(\vec{u}_i - \vec{u}_n)_\perp \cong [(1 - X)\,\vec{j}\times\vec{B}/c + X\nabla_\perp p_n - (1 - X)\nabla_\perp(p_i + p_e)]/m_i n_i \nu_{in} \tag{14}$$

If the first term on the right dominates over the others in this expression, i.e., if the ion-slip is driven by the Lorentz force rather than by pressure gradients, combination of (14) with (6) yields an ion-slip velocity given by

$$\vec{u}_{is} \equiv (\vec{u}_i - \vec{u})_\perp = (1 - X)^2\,\vec{j}\times\vec{B}/m_i n_i \nu_{in} c \tag{15}$$

Substitution into (8) finally gives us the desired result

$$\vec{E}' \equiv \vec{E} + \vec{u} \times \vec{B}/c = \eta[\vec{j}_{\|} + (1 + s)\vec{j}_\perp] + [\vec{j} \times \vec{B}/c - \nabla p_e + \vec{F}_e^{\,th}]/en_e \tag{16}$$

where we have introduced the ion-slip coefficient

$$s \equiv (1 - X)^2\,B^2/\eta m_i n_i \nu_{in} c^2 = (1 - X)^2\,\Omega_e\Omega_i/(\nu_{ei} + \nu_{en})\nu_{in} \tag{17}$$

The last term of Eq. (4) has been dropped because it adds a correction to the Hall term $\vec{j} \times \vec{B}/en_e c$ that is at most of the order $m_e\nu_{en}/m_i\nu_{in}$, i.e. very small.

Equation (16), possibly expressed in a somewhat different form, or without inclusion of the thermoelectric term, has appeared in the literature before. A longish and very explicit derivation is given by Mitchner and Kruger(6) as well as by Sutton and Sherman(2). The result is identical so that the physical assumptions and restrictions of validity are presumably equivalent. When the ion fraction $X \to 0$, Eq. (16) agrees with the Ohm's law based on mobilities in very weakly ionized gases, whereas for $X \to 1$ we recover the relation for fully ionized plasma, as expected. This form is applicable to weakly up to moderately ionized gas flows across magnetic fields such as occur in the ionosphere and in certain astrophysical situations as well as in MHD generators.

## IV. Case B: Gradient-Driven Slip

Where $D/_{Dt}$ can be neglected in both (10) and (12), but when pressure gradients are important, elimination of $\nabla p_n$ between (10) and (12) yields

$$\vec{u}_i - \vec{u}_n = [\vec{j} \times \vec{B}/c - \nabla(p_i + p_e) + m_e \nu_{en} \vec{j}/e]/m_i n_i \nu_{in} \tag{18}$$

Note that for $\vec{j} = 0$ this reduces to the statement for the familiar ambipolar diffusion, as expected. The perpendicular ion-slip velocity thus becomes

$$\vec{u}_{is} = (1 - X)[\vec{j}x\vec{B}/c - \nabla_\perp(p_i + p_e)]/m_i n_i \nu_{in} \tag{19}$$

leading to the generalized Ohm's law for Case B

$$\vec{E}' = \vec{E}'' + \frac{(1 - X)B^2}{m_i n_i \nu_{in} c^2} \left[ \vec{j}_\perp - \frac{c\vec{B} \times \nabla(p_i + p_e)}{B^2} \right]$$

$$= E'' + \eta s_D(\vec{j} - \vec{j}_D) \qquad (20)$$

In the last expression we have identified the "diamagnetic current density"

$$\vec{j}_D \equiv c\vec{B} \times \nabla(p_i + p_e)/B^2 \qquad (21)$$

and the ion-slip coefficient

$$s_D = (1 - X)\, \Omega_e\, \Omega_i/(\nu_{ei} + \nu_{en})\, \nu_{in} = s/(1 - X) \qquad (22)$$

This formulation for Case B is rather transparent. The definition (21) states

$$\vec{j}_D \times \vec{B}/c = \nabla_\perp(p_i + p_e)$$

so that (12) without inertial term can be written as

$$\nabla_\perp p_n = (\vec{j}_\perp - \vec{j}_D) \times \vec{B}/c$$

In words, this says that the collisional friction which balances $\nabla_\perp p_n$ in Eq. (9) to maintain the steady state is accompanied by a cross-field current that is additional to the diamagnetic current. An analysis of particle motion readily shows that this is indeed the guiding center current produced by the inter-species friction force.

## V. Discussion

It should be clear from both the derivation and the discussion presented above that the relation (9) and the entire development following it are valid only when the ion-neutral momentum transfer rate is sufficiently high.

Quantitatively we can express this criterion by requiring that the momentum transfer mean free paths must be much shorter than the scale length of the gradient involved; for example:

$$\nabla n_n << n_n n_i \sigma_{in}^m$$

or equivalently (23)

$$(kT_n/m_n)^{\frac{1}{2}} \, |\nabla \ell n \; n_n| << \nu_{ni}$$

A quantitative statement is possible because for practically all species $\sigma_{in}^m \simeq 10^{-15}\ \mathrm{cm}^2$ within one order of magnitude. If we identify the scale length by L (e.g. $L_s \equiv |\nabla \ell n\; n_s|^{-1}$), condition (23) combined with the requirement of nearly uniform degree of ionization, $L_s \nabla X_A << 1$, says that

$$n_i L_n \simeq n_i L_i >> 10^{15}\ \mathrm{cm}^{-2} \tag{24}$$

must be satisfied for close coupling between plasma and neutrals as far as mass motion is concerned at arbitrary degrees of ionization. The inequality (24) tends to be satisfied, for example, in MHD converters and in some high-pressure arc discharges, because of their high densities, and under certain circumstances, in the ionosphere and in astrophysical situations because of the large scale. When the requirement (23) is combined with (10) under conditions where $\nabla p_n$ dominates on the right we arrive at the conclusion

$$|\vec{u}_i - \vec{u}_n| << (kT_n/m_n)^{\frac{1}{2}} \tag{25}$$

In other words, when the plasma and the neutral gas are strongly coupled, the relative motion (the ion drift speed) must be much slower than thermal speeds.

Up to this point we have restricted our generalized Ohm's law for partially ionized gases to one of two simplifying extreme conditions: either $\nabla p_n$ or $m_n\, n_n\, (D_n\vec{u}_n/Dt)$ can be neglected in Eq. (10), so that $(\vec{u}_i - \vec{u}_n)_\perp$ can be related to $\vec{j} \times \vec{B}$ or to $\nabla_\perp p_n$. When the inertial and the pressure gradient terms are comparable, as for example in a gas dynamic shock wave, the approximations made break down. In that case the generalized Ohm's law can only be used in the limit $X \to 1$, where the effect of the neutral gas is insignificant, or when condition (23) is reversed (the long mean free path limit), where the plasma is not coupled to the neutral gas. We call the latter Case C, for short, or "uncoupled plasma."

In this limit, the right-hand side of (9) is either negligible, or $(\vec{u}_i - \vec{u}_n)$ can become arbitrarily large. This means that the neutral component and "the plasma" are not coupled mechanically and the transformation (8) is meaningless. In that case it is justified to treat the ion-electron plasma as a separate gas for which Eq. (4) can be used as the appropriate generalized Ohm's law, while Eq. (12) for $n_n = 0$ determines $\vec{u}_i$, with $\vec{u}_i = \vec{u}$. If $(\vec{u}_i - \vec{u}_n)$ can be found from other considerations, the last term in (4) can be included as a perturbation. However, none of the equations (14) to (20) are applicable in that case. The ions and electrons are then to be treated as components of a fully ionized plasma, and the neutral component obeys the laws of ordinary gas dynamics. Any existing friction between this gas and the plasma must be added as a correction in the spirit of weak coupling between the two fluids. This can presumably still be done analytically, but the matter is not pursued here, because it does not lead to an Ohm's Law.

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