Submitted to American Journal of Physics

LBL-1783
Preprint c.1

HIGHER MODES OF A BOTTLE

Frank S. Crawford

May 1973

For Reference

Not to be taken from this room

Prepared for the U. S. Atomic Energy Commission
under Contract W-7405-ENG-48

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

HIGHER MODES OF A BOTTLE

Frank S. Crawford

Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

ABSTRACT

For nearly a century we have known how to predict the frequency of the lowest mode of a bottle with a long cylindrical neck, by means of a well-known model due to Helmholtz wherein the main volume acts like a massless spring and the mass of air in the neck moves as a whole. It has also been widely believed that to predict higher modes one must take into account the inertia in the main volume, and that therefore the calculations are difficult. However, I have found a simple model that continues to neglect the inertia in the main volume (à la Helmholtz) but does not assume the air in the neck moves as a whole. Instead, I treat the neck as a distributed massive spring. With this model not only can I obtain the Helmholtz result for the lowest mode but I can also predict the frequencies of higher modes. The predictions of the model are beautifully confirmed by experiment for the lowest two modes, where I have tested them. Details of the calculations are given so that the theory can be understood by an undergraduate physics student. Further experiments are suggested that could easily be done in an undergraduate laboratory to find out whether the model breaks down at higher modes, where I have not tested it.

1. INTRODUCTION

If you blow gently across the mouth of a jug or bottle you may produce a tone by exciting the lowest mode. The frequency of this tone can be calculated by a well-known approximation due to Helmholtz.[1] In that approximation the air in the main volume of the bottle acts like a massless spring. The air in the neck is assumed to move as a whole and acts as a mass attached to the spring. The approximation consists in assuming all the moving mass is in the neck, and all the return force comes from pressure changes in the main volume.

If you blow very hard across the mouth of the bottle you can excite higher modes. Once you have heard them by blowing hard, you can usually hear their faint presence even when blowing softly so as to excite mainly the lowest mode.

I had always assumed that these higher modes were difficult to calculate.[2] However, while teaching Physics 4C at Berkeley I have recently found a simple model that is easy to work with and that gives predictions that agree with experiment to the extent that I have tested them. For a bottle whose main volume V_0 is large compared with the volume V of the neck, my model gives the Helmholtz formula for the lowest mode. But my result also correctly predicts the lowest mode frequency even for small ratios of V_0 to V, where the Helmholtz formula fails. My model also gives the higher mode frequencies which cannot be calculated at all by the Helmholtz technique.[3]

My model applies to bottles shaped like that in Fig. 1. The neck is a uniform cylinder of length L, cross section A, and volume V. The main body of the bottle has volume V_0, and can have any reasonable shape. Like Helmholtz, I neglect the inertia of the air in volume V_0, because it does not move very much. Thus I treat V_0

as a massless spring.[4] Unlike Helmholtz, I do not assume that the air in the neck moves as a whole. Instead I treat the neck as a distributed massive spring.

We shall show later that each of these two springs has an effective spring constant K that is inversely proportional to the volume of air in the spring:

$$K/K_0 = V_0/V. \tag{1}$$

Thus, for example, for an ordinary bottle with neck volume V small compared to the main volume V_0, the neck acts like a comparatively stiff spring and the body a comparatively weak spring.

2. QUALITATIVE PREDICTIONS OF THE MODEL

Let us guess in advance of the detailed theory the results expected from the model for the two limiting cases: (a) main volume large compared with neck volume and (b) main volume small compared with neck volume.

a. Large main volume: $V_0/V >> 1$

Then according to Eq. (1) the neck is a stiff spring and the body is a weak spring. In the lowest mode the stiff massive spring (neck) moves back and forth as a whole, barely expanding and contracting, with the return force nearly all provided by the weak massless spring. This is nearly Helmholtz's approximation, and our result will be nearly Helmholtz's, with a small correction for the fact that the stiff spring does expand and contract slightly as it moves back and forth.

In the second and higher modes the massive spring (neck) expands and contracts. Since the neck is relatively stiff, it hardly feels the weak spring (the main volume). Therefore the center of mass of the neck remains nearly at rest. Thus in the higher modes the neck oscillates like a tube with both ends open. Our formula will therefore

predict modes like that of an open-ended tube, with corrections for the fact that the weak spring (volume V_0) does have some influence, and therefore the center of mass of the neck does not quite remain at rest.

We might have guessed these results without the model. From Fig. 1 we see that if V_0 is huge, the inner end of the neck opens into a huge volume and the neck should behave like a tube open at both ends.[5]

b. Small main volume: $V_0/V << 1$

Then according to Eq. (1) the neck is a relatively weak spring and the body is a stiff spring. In the limit that V_0 goes to zero the neck is a massive spring with the inner end fixed and the outer end free. In that limit the system is simply a tube with one end closed and one end open. Its modes are well known. For V_0 not quite zero, our formula should give a result not exactly that of a tube with one end closed, since the stiff spring V_0 does expand and contract slightly.

Again, we can guess these limiting results without the model. If we fill the flask with water until V_0 is zero we obviously have simply a tube (the neck) with a closed bottom end.

These qualitative predictions agree with experiment. Using a 500-ml volumetric flask I can, after some practice, blow the lowest three modes. The second and third modes have the same pitches as the lowest two modes of an open-ended glass tube having the same diameter and length as the neck. When I fill the main volume with water I find, as expected, that the first three modes have the same pitches as the first three modes of a test tube having the same diameter and same length as the neck.

c. Transition between large V_0 and small V_0

Now that we have considered the two limiting cases, $V_0/V >> 1$ and $V_0/V << 1$, we can make a good guess as to the predictions of the model if we start with a bottle with V_0 large compared to V and then gradually decrease V_0 to zero by filling the bottle with water. The lowest mode (mode zero) will start out with a frequency f_0 that agrees with Helmholtz's prediction. As we fill the bottle my model will start to disagree with that of Helmholtz when V_0 is no longer large compared with V. The pitch f_0 of the lowest mode will gradually increase until finally, when V_0 goes to zero, the frequency f_0 will be $f_0 = f_{00}$, where f_{00} is the frequency of the lowest mode of a tube of length L with one end closed; then the tube length L is one quarter wavelength. The corresponding frequency is

$$f_{00} = c/4L,$$

where c is the velocity of sound in air.

The second mode of the bottle will start out, when V_0/V is large, as the lowest mode of a tube open at both ends and having the length of the neck. The neck length will be one half wavelength. Thus the frequency f_1 of this second mode (first overtone) will be $2f_{00}$ for large V_0. As V_0 is decreased this mode will gradually increase its frequency until, when V_0 goes to zero, it has the frequency of the second mode of a tube with one end closed. The neck length will then be three quarter wavelengths, and the frequency will be $3\ f_{00}$.[5]

3. ADVANCE SUMMARY OF THEORETICAL AND EXPERIMENTAL RESULTS

The reader who wishes to skip (or temporarily postpone) details of the theory can now turn to Fig. 3 where the detailed predictions of my model are plotted for the lowest mode, with frequency f_0, and the first overtone, of frequency f_1, as a function of V_0, for a 500-ml

flask. The three limiting frequencies f_{00}, $2 f_{00}$, and $3 f_{00}$ discussed above are also shown in Fig. 3. Helmholtz's prediction for f_0 is also shown, as are my experimental results.

4. THEORY

a. Helmholtz's approximation for the lowest mode

First I shall review Helmholtz's model. The mass in the neck is

$$M = \rho_0 V, \tag{2}$$

where ρ_0 is the density of the air at equilibrium. We need to calculate the effective spring constant K_0 of the volume V_0. When we have found K_0 we will have found the angular frequency,

$$2\pi f_0 = \omega_0 = \sqrt{K_0/M}, \tag{3}$$

for the simple harmonic oscillator of Helmholtz's model. The air in volume V_0 satisfies the adiabatic gas law,

$$p_0 V_0^{\gamma} = \text{constant}. \tag{4}$$

Here p_0 is the pressure in the volume V_0, and γ is the ratio of specific heats. Differentiating Eq. (4) gives that an increase in volume, ΔV_0, yields a decrease in pressure

$$\Delta p_0 = - (\gamma p_o/V_0)\Delta V_0. \tag{5}$$

Now consider the motion of the air in the neck. If the air moves as a whole, a distance x to the right, the volume V_0 increases by an amount

$$\Delta V_0 = Ax. \tag{6}$$

The return force F on the air in the neck is due to the pressure differential between the air in volume V_0 and the air outside the end of the neck:

$$F = \Delta p_0 A. \tag{7}$$

If we think of this force as due to an equivalent spring of spring constant K_0 we must have

$$F = - K_0 x. \tag{8}$$

Conbining Eqs. (5) to (8) gives

$$K_0 = (\gamma p_0 A^2)/V_0. \tag{9}$$

Then (3), (2), and (9) give

$$\omega_0^2 = (\gamma p_0/\rho_0)\,(A^2/V_0 V). \tag{10}$$

But, as is well known (and as we shall later review) the sound velocity in air, c, is given by

$$c^2 = (\gamma p_0)/\rho_0. \tag{11}$$

The volume V of the neck is given by

$$V = AL\,. \tag{12}$$

Combining (11), (12) and (10) gives

$$\boxed{\omega_0 = (c/L)\sqrt{V/V_0}} \tag{13}$$

which is Helmholtz's formula.[6]

b. Spring constant for air in the neck

In my model we shall need the spring constant for the air in the neck. The derivation follows the same steps as in Eqs. (4) to (9). We find, therefore, that

$$K = (\gamma p_0 A^2)/V. \tag{14}$$

Comparing (14) and (9) we see that Eq. (1) holds: The effective spring constants are inversely proportional to the volumes.

c. Lumped-parameter model

For conceptual clarity we start our calculation with a model where the massive spring (the neck) of spring constant K, length L, and mass M is thought of as a string of N masses and N springs in series. See Fig. 2. Each mass has value $M_1 = M/N$. Neighboring masses are separated by distance $L_1 = L/N$. The tiny springs connecting neighboring masses each have spring constant K_1 that is related to K by

$$K_1 L_1 = KL. \tag{15}$$

To understand (15) we can use dimensional analysis: K has dimensions dyne/cm, so KL has dimensions dyne, which sould be independent of the length of the spring. More directly: If N identical springs are connected in series and the combined spring is extended an amount x, then each component will be extended an amount x/N and will exert a force $K_1 x/N$. But that is also the force exerted by the combined spring. Therefore the combined spring has spring constant $K = K_1/N$. Since $N = L/L_1$, that gives (15).

d. Equation of motion for the model

Consider one of the masses in Fig. 2. Call it mass n, where n is neither 1 nor N. (We will later consider the end masses, 1 and N, to obtain the boundary conditions.) Apply Newton's second law to this mass. Let ψ_n be the displacement in centimeters of mass n to the right, away from its equilibrium position. Then the spring to the right of mass n is stretched by an amount $\psi_{n+1} - \psi_n$ and pulls to the right with a force $K_1(\psi_{n+1} - \psi_n)$. The spring to the left pulls to the left with a force $K_1(\psi_n - \psi_{n-1})$. The net force gives mass times acceleration:

$$M_1(d^2\psi_n/dt^2) = K_1(\psi_{n+1} - \psi_n) - K_1(\psi_n - \psi_{n-1}). \tag{16}$$

Now we go to the continuous limit. Instead of the label n we use the label x which gives the equilibrium position of a mass. We also assume that neighboring masses suffer nearly the same displacements, so that $\psi(x,t)$, which replaces $\psi_n(t)$, is a smooth function of x. Then we set

$$(\psi_{n+1} - \psi_n)/L_1 = (\partial\psi/\partial x)_{n+\frac{1}{2}}, \tag{17}$$

$$(\psi_n - \psi_{n-1})/L_1 = (\partial\psi/\partial x)_{n-\frac{1}{2}}, \tag{18}$$

where the subscript $(n+\frac{1}{2})$ refers to the point midway between the equilibrium positions of masses n and n + 1. Then (16) becomes

$$M_1(\partial^2\psi/\partial t^2) = K_1 L_1[(\partial\psi/\partial x)_{n+\frac{1}{2}} - (\partial\psi/\partial x)_{n-\frac{1}{2}}]$$

$$= K_1 L_1^2(\partial^2\psi/\partial x^2)_n, \tag{19}$$

where the last step was analogous to Eqs. (17) or (18).

e. Classical wave equation. Velocity of sound

Equation (19) is the classical wave equation,

$$\partial^2\psi/\partial t^2 = c^2(\partial^2\psi/\partial x^2), \tag{20}$$

with wave velocity c given by

$$c^2 = (K_1 L_1)\, L_1/M_1 = (KL)\, L/M$$

$$= \frac{\gamma p_0 A^2}{V} L^2/M = \gamma p_0 \frac{V}{M} = \frac{\gamma p_0}{\rho_0}, \tag{21}$$

where we used (15), and (14), (2), and (12). Thus we have derived Eq. (11) as promised.

f. Standing wave solution. Relation between frequency and wave number

We are looking for modes that produce tones. These are standing waves of the form

$$\psi(x,t) = \cos(kx + \alpha)\cos\omega t. \tag{22}$$

Substitution of (22) into (20) shows that (22) is a solution, provided

$$\omega^2 = c^2 k^2. \tag{23}$$

In order to find the spatial phase constant α and the allowed values of the angular wave number k we must examine the boundary conditions. Once we have found the possible values of k (one for each mode), (23) will give us the corresponding frequencies.

g. Boundary condition at x = L (the open end)

Go back temporarily to the lumped parameter model and consider the equation of motion of the last mass, number N. It has a spring to its left but none to its right. Newton's law gives

$$M_1(d^2\psi_N/dt^2) = -K_1(\psi_N - \psi_{N-1}). \tag{24}$$

Now go to the continuous limit:

$$(\psi_N - \psi_{N-1})/L_1 = (\partial\psi/\partial x)_{x=L}\,. \tag{25}$$

Then (24) becomes

$$M_1(\partial^2\psi/\partial t^2) = -K_1L_1\,(\partial\psi/\partial x)_{x=L} = -KL\,(\partial\psi/\partial x)_{x=L}\,. \tag{26}$$

But in the continuous limit where N goes to infinity, each tiny mass M_1 goes to zero, whereas KL remains constant. Thus the left side of (26) goes to zero, and therefore we need

$$(\partial\psi/\partial x)_{x=L} = 0\,. \tag{27}$$

We can satisfy (27) by choosing α such that (22) becomes

$$\psi(x,t) = \cos k\,(x-L)\,\cos\omega t\,. \tag{28}$$

We still need to find the allowed values of k. Those are determined by the remaining boundary condition.

h. Boundary condition at x = 0 (connection to spring K_0)

Consider the first mass, n = 1. Its equation of motion (see Fig. 2) is

$$M_1(d^2\psi_1/dt^2) = K_1(\psi_2 - \psi_1) - K_0\psi_1\,. \tag{29}$$

In the continuous limit we have

$$\psi_2 - \psi_1 = L_1\,(\partial\psi/\partial x)_{x=0}\,;$$

also, $K_1L_1 = KL$, and $M_1 = 0$.

Thus (29) becomes

$$0 = KL\,(\partial\psi/\partial x)_{x=0} - K_0\,\psi\,. \tag{30}$$

Substituting the solution (28) into the condition (30) gives

$$KL\,(\partial/\partial x)\,[\cos k(x-L)] - K_0\cos k(x-L) = 0$$

at x = 0, i.e.,

$$K\,(kL)\sin kL - K_0\cos kL = 0$$

i.e.,

$$\boxed{kL\tan kL = K_0/K = V/V_0}\,. \tag{31}$$

Equation (31) is our desired result. It gives the allowed values of kL. The corresponding mode frequencies are given by Eq. (23):

$$\boxed{f \equiv \omega/2\pi = (c/2\pi L)\,(kL)}\,, \tag{32}$$

where the kL values to insert in (32) are the solutions of Eq. (31).

5. THEORETICAL RESULTS FOR LIMITING CASES

Equations (32) and (31) give the general results for the model. Since (31) has an infinite number of solutions there are an infinite number of modes. We shall only consider the lowest two modes, which we call mode zero and mode 1. In order to compare the model with the qualitative discussion in Sec. 2 we shall here reduce the general formulas (32) and (31) to simpler forms, for the two limiting cases of comparatively large main volume, and comparatively small main volume.

a. Large main volume: $V_0/V >> 1$

(1) Lowest mode.

For $V/V_0 << 1$, the smallest value of kL that will satisfy (31) has $k = k_0$ such that $k_0L << 1$. Then we can approximate

$$\tan k_0L \approx k_0L.$$

Then (31) gives

$$k_0L \approx (V/V_0)^{\frac{1}{2}}\,.$$

Then (32) gives

$$f_0 = (c/2\pi L)\,(k_0L) \approx (c/2\pi L)\,\sqrt{V/V_0}\,. \tag{33}$$

Comparing this with (13) we see that we have recovered Helmholtz's formula. Conceptually the derivations are very different. Helmholtz's derivation does not involve the concept of a wavelength at all, in the neck. Since in his derivation all the material of the neck moves

together, you might think that that would correspond to infinite wavelength, i.e. to $k_0 = 0$, in my model. But that is not exactly so; k_0 is only zero for $V/V_0 = 0$, which would give $f_0 = 0$. Rather than $k_0 = 0$ we should say "k_0 small compared with $1/L$" corresponds to Helmholtz's result.

We can obtain a small correction to Helmholtz's formula by taking the next approximation to $\tan k_0L$, for $k_0L << 1$:

$$\tan k_0L = k_0L[1 + (1/3)(k_0L)^2] .$$

That gives

$$f_0 = (c/2\pi L)\sqrt{V/V_0}\,[1 - (1/6)(V/V_0)] . \qquad (34)$$

(2) Next-to-lowest mode

For $V/V_0 << 1$, Eq. (31) is satisfied for infinitely many values of kL. For all of them, $\tan kL$ is small and positive. The smallest such value gave k_0L, as discussed above. The next-to-smallest value of kL we call k_1L, with $\tan k_1L$ small and positive. That occurs for k_1L slightly larger than π:

$$k_1L = \pi + \epsilon, \quad 0 < \epsilon << 1 . \qquad (35)$$

Then

$$\tan k_1L \approx \epsilon$$

and

$$k_1L \tan k_1L \approx \pi\epsilon .$$

Then (31) gives

$$\pi\epsilon = V/V_0 .$$

Then (35) gives

$$k_1L = \pi + \epsilon = \pi[1 + (1/\pi^2)(V/V_0)] .$$

Then (32) gives

$$f_1 = (c/\pi L)\,k_1L = (c/2L)\,[1 + (1/\pi^2)(V/V_0)] . \qquad (36)$$

But we recognize $c/2L$ as the frequency of the lowest mode of a tube

of length L open at both ends. Equation (36) thus agrees with our earlier qualitative prediction in Sec. 2, where we gave the name $2f_{00}$ to $c/2L$. Equation (36) also gives a small correction term that takes into account that V_0 is not infinite.

b. Small main volume: $V_0/V << 1$

(1) Lowest mode

For $V/V_0 >> 1$, Eq. (31) will have solutions where $\tan kL$ is large and positive. The lowest such value is for $k = k_0$ such that k_0L is slightly less than $\frac{1}{2}\pi$:

$$k_0L = (\pi/2) - \epsilon, \quad 0 < \epsilon << 1. \qquad (37)$$

Then

$$\tan k_0L = \tan[(\pi/2) - \epsilon] \approx 1/\epsilon ,$$

and

$$k_0L \tan k_0L \approx \pi/(2\epsilon) .$$

Then (31) gives

$$\pi/(2\epsilon) \approx V/V_0 .$$

Then (37) gives

$$k_0L = (\pi/2) - \epsilon = (\pi/2)[1 - (V_0/V)] .$$

Then (32) gives

$$f_0 = (c/2\pi L)(k_0L) = [c/(4L)]\,[1 - (V_0/V)] . \qquad (38)$$

We recognize $c/4L$ as the frequency of the lowest mode of a tube of length L closed at one end and open at the other. This verifies our earlier qualitative conclusion in Sec. 2, where we gave the name f_{00} to $c/4L$.

(2) Next-to-lowest mode

The next solution of Eq. (31) for $V/V_0 >> 1$ comes when $\tan kL$ is

again large and positive, namely at $k = k_1$ such that $k_1 L$ is slightly less than $(3/2)\pi$:

$$k_1 L = (3/2)\pi - \epsilon, \quad 0 < \epsilon << 1. \tag{39}$$

Then

$$\tan k_1 L = \tan [(3/2)\pi - \epsilon] \approx 1/\epsilon ,$$

and

$$k_1 L \tan k_1 L \approx 3\pi/2\epsilon .$$

Then Eq. (31) gives

$$3\pi/2\epsilon = V/V_0 .$$

Then (39) gives

$$k_1 L = (3/2)\pi - \epsilon = 3/2 \ \pi [1 - (V_0/V)] .$$

Then (32) gives

$$f_1 = (c/2\pi L)(k_1 L) = (3c/4L)[1 - (V_0/V)] . \tag{40}$$

We recognize $3c/4L$ as being $3f_{00}$, the frequency of the next-to-lowest mode of a tube of length L open at one end and closed at the other. This bears out our qualitative guess in Sec. 2.

It is interesting to compare (40) and (38). We see that for small V_0/V the same correction factor, $1 - (V_0/V)$, occurs in both the lowest mode and the first overtone, and so these have a frequency ratio of 1 to 3, as in an ideal tube with one end closed. Therefore the factor $1 - (V_0/V)$ may be regarded as a common correction factor for the effective length.

6. COMPARISON OF THEORY AND EXPERIMENT FOR ARBITRARY V_0/V

Equation (31) is exact, for the model. But does the model agree with experiment for arbitrary V_0/V? I have compared the predictions of the model with experiment for the two lowest modes, over the range $V_0/V = 10$ to zero. My bottle was a 500 milliliter volumetric

flask[7] with a long cylindrical neck of length 16.5 cm, diameter 1.8 cm. For the length L, I added an end correction of 0.6 times the radius, at each end. (The inner end should perhaps instead have had a correction of 0.8 R, since it is close to being a "flanged" end.[5]) Thus, for L, I took 17.5 cm, and for V, I took 46.0 cm^3. The volume V_0 was varied from 486 cm^3, with the bottle empty, to zero, by adding measured amounts of water to the flask. For each value of V_0, I determined the pitch of the two lowest modes as follows: I tuned my flute to an A-440 tuning fork. I blew on the bottle to produce one or the other mode, then played the closest note to that, on my flute, that I could, using my ear as judge. I believe I can estimate to the nearest quarter tone, with an error of about plus or minus an eighth tone. (When the flask was nearly empty the lowest mode was below the flute range, so I played the corresponding note an octave higher on the flute.) After determining the note by ear with my flute (interpolating between half tones where necessary) I looked up the frequency in the Handbook of Chemistry and Physics. Then I plotted my experimental values of frequency versus V_0/V for the two modes. The experimental points are shown on Fig. 3. It is important that I committed myself to my experimental values before plotting the theoretical curve, and did not repeat the experiment after plotting the theoretical curve, because I could easily be prejudiced by perhaps a quarter tone if I knew the "expected" pitch.

The theoretical curves were plotted as follows. The factor $c/(2\pi L)$ appears in both the Helmholtz result, Eq. (33), and in my result, Eq. (32). For c I took 3.32×10^4 cm/sec, for L, 17.5 cm. That gives $c/(2\pi L) = 301.9$ Hz. For the Helmholtz formula I then simply plot Eq. (33) versus V_0/V. For my model one would expect

to choose a given value of V_0/V, then solve (31) for kL, then use (32) to calculate the frequency f, then plot f. Actually, it is easier to go the other way: Choose f, find kL from (32), and calculate V_0/V from (31).

Figure 3 shows the comparison between theory and experiment. It is apparent that theory and experiment are in excellent agreement. Thus the model works, at least for the two lowest modes of this bottle.

Of course, there must be modes that my model cannot encompass, where the inertia in the main volume cannot be neglected.[4]

With my crude experimental technique I was only able to investigate the lowest two modes. It would be interesting to investigate higher modes using a more powerful technique, say a tone generator and a microphone to detect resonances. How far can this model be pushed? Where does it break down? Does it break down for the reasons Rayleigh suggested?

Footnotes and References

1. Lord Rayleigh, The Theory of Sound, (Dover Publications, 1945), Volume 2, p. 187.

2. For years I had been discouraged by a statement in Ref. 1, p. 189, where Rayleigh states, "The simple method of calculating the pitch of resonators with which we have been occupied is applicable to the gravest mode of vibration only, the character of which is quite distinct. The overtones of resonators with contracted necks are relatively very high, and the corresponding modes of vibration are by no means independent of the inertia of the air in the interior of the reservoir. The character of the modes will be more evident, when we come to consider the vibrations of air within a completely closed vessel, such as a sphere, but it will rarely happen that the pitch can be calculated theoretically." (The italics are mine.)

3. I have not seen any predictions or calculations on higher modes of a bottle. Besides Ref. 1, I searched in the following sources: Philip M. Morse and K. U. Ingard, Theoretical Acoustics (McGraw-Hill, 1968); P. Morse, Vibrations and Sound (McGraw-Hill, 1948); H. Olson and F. Massa, Applied Acoustics, (P. Blakiston and Son, 1939); Robert Randall: An Introduction to Acoustics (Addison-Wesley, 1951).

4. Rayleigh would not expect my model to work at all for overtones, since I neglect the inertia in the main volume. See Ref. 2.

5. There will be small "end corrections" to the effective length of the tube. The outer end of the neck is a "free" end. The inner end, being flush with the wall of the main volume, is closer to being a "flanged" end. End corrections are discussed in Ref. 1, pp. 188, 202. At an open end, one should add 0.6 R to the effective length, where R is the tube radius; at a flanged end, one should add 0.8 R.

6. My Eq. (13) is equivalent to Ref. 1, p. 187, Eq. (4). The correspondence between my notation and Rayleigh's is:

$\omega_0 = 2\pi N$, $c=a$, $A = \sigma$, $L = L$, $V_0 = S$.

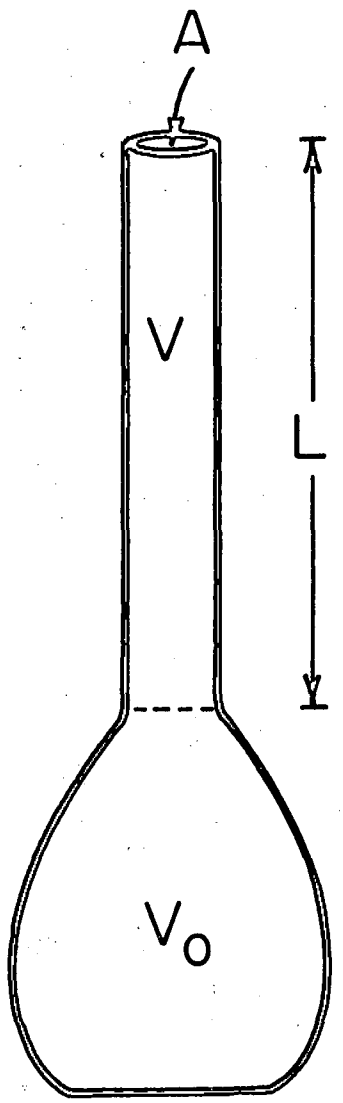
7. Pyrex No. 5581 unstoppered volumetric flask, 500 ml.

FIGURE CAPTIONS

Fig. 1. Bottle with main volume V_0, and a cylindrical neck of length L, volume V, and cross section A. The volume V_0 can be varied from its maximum value to zero by filling the bottle with water.

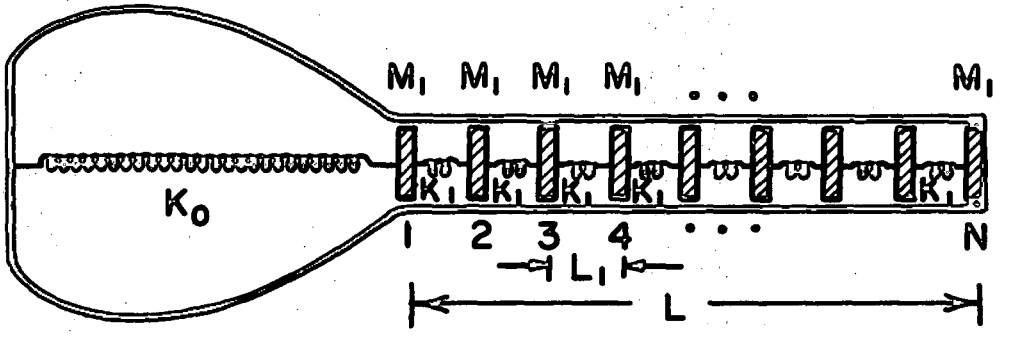
Fig. 2. Lumped parameter model. The main body is a massless spring of spring constant K_0. The neck is a distributed massive spring consisting of N identical masses M_1 separated by N identical springs each of length L_1 and spring constant K_1.

Fig. 3. Theory and experiment. The smooth curves labeled f_0 and f_1 give the predicted frequency for the lowest two modes of a bottle with effective neck length 17.5 cm, as a function of the ratio of main volume to neck volume, for the model discussed. The prediction of Helmholtz for the lowest mode is also plotted. The plotted points are experimental values. The frequency labeled f_{00} has value $c/4L$ and is the lowest mode of a tube of length L closed at one end and open at the other. The frequency labeled $2f_{00}$ is the lowest mode of a tube of length L open at both ends, and is the first overtone of the bottle, when V_0/V is very large. The frequency labeled $3f_{00}$ is the first overtone of a tube of length L closed at one end and open at the other, and is the first overtone of the bottle, when V_0/V is very small.



XBL734-2745

Fig. 1



XBL734-2746

Fig. 2

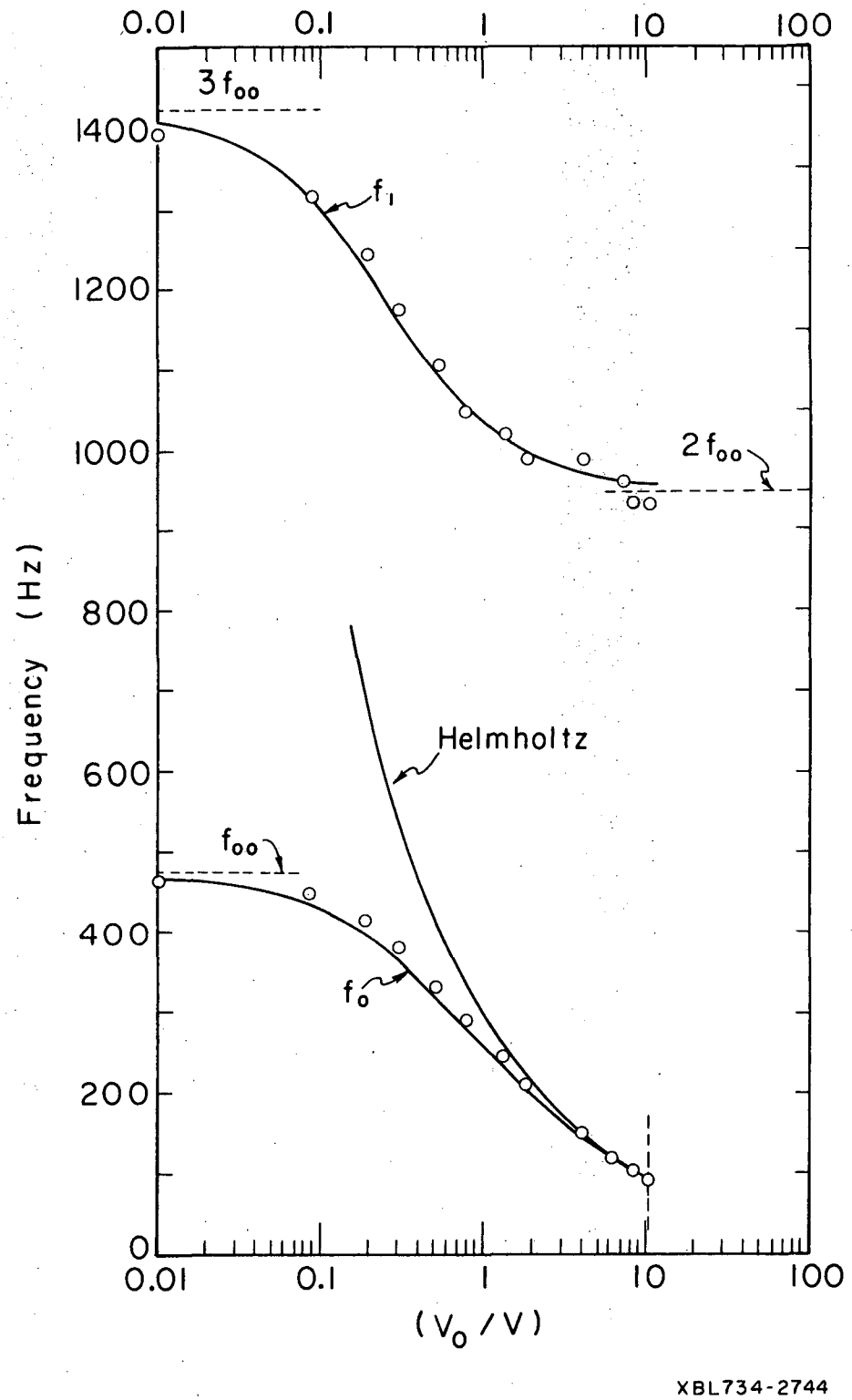

0.01
0.1
1
10
100
3 f00
1400
f1
1200
1000
2 f00
800
Frequency (Hz)
Helmholtz
600
f00
400
f0
200
0
0.01
0.1
1
10
100
(V0 / V)
XBL734-2744

Fig. 3

LEGAL NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

TECHNICAL INFORMATION DIVISION
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720