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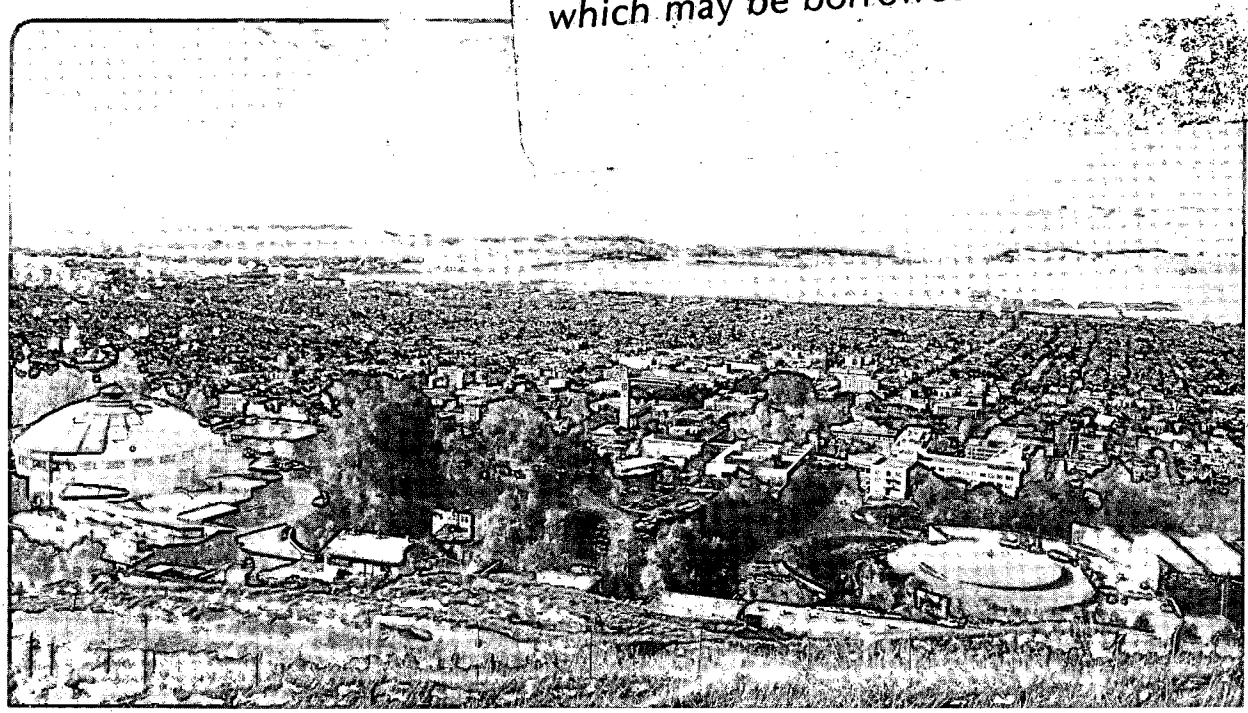

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A Solution for TM-Mode Plane Waves Incident on a Two-Dimensional Inhomogeneity

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Abstract

A solution for the electromagnetic fields scattered from a two-dimensional inhomogeneity in a conducting half space has been obtained for an incident TM mode plane wave; the magnetic field is polarized parallel to the strike of the inhomogeneity. The approach has been to determine the scattering currents within the inhomogeneity using an integral equation for the electric fields. This solution is similar in concept to earlier studies of TE mode scattering from two-dimensional inhomogeneities, and it completes the analysis of the scattering of arbitrary plane waves using the integral equation approach. For simple bodies in the earth integral equation solution offers significant computational advantages over alternate finite element or finite difference methods of solution.

Introduction

Quantitative interpretation of magnetotelluric surveys depends at present on the availability of efficient forward modeling algorithm. To date two major numerical techniques have been used to obtain the scattered fields from buried inhomogeneities in plane wave fields; methods solving the governing differential equation which generally uses a finite element or finite difference approach and methods which solve an integral equation formulation of the problem.

For two-dimensional inhomogeneities a solution for incident fields with the electric field parallel to the strike of the inhomogeneity (TE mode solution) has been developed by Hohmann (1971) using the integral equation approach. For a perfect conductor an integral formulation, for surface scattering currents, for the TM mode (magnetic field parallel to the strike of the inhomogeneity) has been developed by Parry (1969). General two-dimensional solutions in the presence of an arbitrary mode plane wave (mixed TE-TM) have been obtained by Ryu (1971), Swift (1971), and Rijo (1977) using either a finite element or finite difference technique.

To our knowledge the TM integral equation solution for the general case has not been presented. The solution presented here thus completes the analysis for the scattering of arbitrary mode plane waves from two-dimensional inhomogeneities using the integral equation approach. Apart from significant computational advantages in forward modeling of simple geologic bodies for magnetotelluric analysis, this solution is important for evaluating the results of alternate numerical methods used for more complicated geologic models. It is becoming evident that for many of the current numerical modeling schemes, there are no convincing checks on the accuracy of the solution. It is imperative therefore that several solutions be obtained by differing methods and be compared until confidence is attained in these solutions.

Formulation of two-dimensional integral equations

In their general form, Maxwell's equations are written as follows:

$$- \nabla \times \bar{E} = j\omega\mu\bar{H} + \bar{M}^i \tag{1}$$

$$\nabla \times \bar{H} = (\sigma + j\omega\varepsilon)\ \bar{E} + \bar{J}^i \tag{2}$$

where $\bar{\mathbf{M}}^i$ and $\bar{\mathbf{J}}^i$ are impressed magnetic and electric currents. Throughout the paper an e^{jwt} time dependence is assumed. Harrington (1961) has shown that a generalized integral equation solution can be obtained by rewriting (1) and (2) as

$$- \nabla \times \bar{E} = j\omega\mu_1 \bar{H} + \bar{M}^s + \bar{M}^i \tag{3}$$

$$\nabla \times \bar{H} = (\sigma_1 + j\omega\varepsilon_1)\ \bar{E} + \bar{J}^s + \bar{J}^i \tag{4}$$

where $\bar{\mathbf{M}}^s$ and $\bar{\mathbf{J}}^s$ are scattering magnetic and electric currents representing the inhomogeneities in the half space of electric and magnetic constants $\sigma_1, \mu_1,$ and ε_1 (Figure 1). Equating the right hand sides of (1) and (2) to those of (3) and (4) respectively, we find that the scattering currents are zero everywhere except for the inhomogeneous region, in which

$$\bar{M}^s = j\omega(\mu_2 - \mu_1)\ \bar{H} \tag{5.1}$$

$$\bar{J}^s = \Delta\sigma\bar{E} \tag{5.2}$$

where

$$\Delta\sigma = (\sigma_2 - \sigma_1) + j\omega(\varepsilon_2 - \varepsilon_1)\ .$$

In the absence of active sources and magnetic inhomogeneities, equations (3) and (4) become

$$- \nabla \times \bar{E} = j\omega\mu_1 \bar{H} \tag{6}$$

$$\nabla \times \bar{H} = (\sigma_1 + j\omega\varepsilon_1)\ \bar{E} + \bar{J}^s \ . \tag{7}$$

The total field can always be written as the sum of the incident (i) field, the field that would exist in the absence of the inhomogeneity, and the scattered field (s), such that

$$\bar{E} = \bar{E}^i + \bar{E}^s \tag{8}$$

$$\bar{H} = \bar{H}^i + \bar{H}^s \ , \tag{9}$$

and consequently Maxwell's equations for the scattered fields are written as

$$-\nabla \times \bar{E}^s = j\omega\mu_1 \bar{H}^s \tag{10}$$

$$\nabla \times \bar{H}^s = (\sigma_1 + j\omega\varepsilon_1)\ \bar{E}^s + \bar{J}^s \ . \tag{11}$$

Equations (10) and (11) lead to a vector wave equation for the scattered field $\bar{\mathbf{E}}^s$ in the inhomogeneous region

$$\nabla \times \nabla \times \mathbf{E}^s - k_1^2 \mathbf{E}^s = -j\omega\mu_1 \mathbf{J}^s \tag{12}$$

The solution for (12) can generally be written in the form of an integral as

$$\bar{E}^s(\bar{r}) = \iint_{S'} \bar{\bar{G}}^E(\bar{r};\bar{r}') \cdot \bar{J}^s(\bar{r}')\ ds \tag{13}$$

and corresponding magnetic field can be written as

$$\bar{H}^s(\bar{r}) = \iint_{s'} \bar{\bar{G}}^H(\bar{r};\bar{r}') \cdot \bar{J}^s(\bar{r}')\, ds \tag{14}$$

where $\bar{r}$ and $\bar{r}'$ are the vectors describing the positions of observation and source respectively. $\bar{\bar{G}}(\bar{r};\bar{r}')$ is a two-dimensional Green's tensor defined as the scattered field at $\bar{r}$ caused by a unit current density located at $\bar{r}'$.

One way of obtaining $\bar{\bar{G}}(\bar{r};\bar{r}')$ is to find a single vector potential $\bar{\pi}$ and relate it to the scattered fields defined by

$$\bar{E}^s = k^2\bar{\pi} + \nabla(\nabla \cdot \bar{\pi}) \tag{15}$$

$$\bar{H}^s = (\sigma + j\omega\varepsilon)\ \nabla \times \bar{\pi} \tag{16}$$

where $\bar{\pi}$ is the electric Hertz vector satisfying the inhomogeneous Helmholtz equation.

$$(\nabla^2 + k^2)\ \bar{\pi} = -\frac{\bar{J}^s}{\sigma + j\omega\varepsilon}\ \delta(x - x')\ \delta(z - z') \tag{17}$$

in Cartesian coordinates. The particular solution for (17) is shown in Appendix A as

$$\bar{\pi}^p = -\frac{j\omega\mu}{2\pi}\frac{\bar{J}^s}{k^2}\int_0^\infty \frac{1}{u}\, e^{-u|z-z'|} \cos kx(x-x')\ dk_x \tag{18}$$

where

$$u = \left(k_x{}^2 - k^2\right)^{1/2} .$$

The homogeneous solution $\bar{\pi}^s$ for (17) is subject to the boundary

conditions and it satisfies

$$(\nabla^2 + k^2)\ \bar{\pi}^s = 0\ . \tag{19}$$

The total vector potentials in the homogeneous half space generated by current elements in it are given by Appendix A.

$$\pi_{1x} = -\frac{j\omega\mu}{2}\frac{J_x}{k_1^2}\int_0^\infty \frac{1}{u_1}\left[e^{-u_1|z-z'|} + R_{TM}e^{-u_1(z+z')}\right]\cos k_x(x-x')\ dk_x \tag{20}$$

$$\pi_{1z} = -\frac{j\omega\mu}{2\pi}\frac{J_z}{k_1^2}\int_0^\infty \frac{1}{u_1}\left[e^{-u_1|z-z'|} - R_{TM}e^{-u_1(z+z')}\right]\cos k_x(x-x')\ dk_x \tag{21}$$

where

$$R_{TM} = -\frac{k_0^2u_1 - k_1^2u_0}{k_0^2u_1 + k_1^2u_0}$$

and $\mathbf{J_x}$ and $\mathbf{J_z}$ are the scattering current densities directed in x and z respectively.

From equation (15) and (16) and by the definition of Green's function, the elements of Green's tensors $\bar{\bar{\mathbf{G}}}^{\mathbf{E}}(\bar{r};\bar{r}')$ and $\bar{\bar{\mathbf{G}}}^{\mathbf{H}}(\bar{r};\bar{r}')$ can be written as

$$G^E_{xx} = \left(k_1^2 + \frac{\partial^2}{\partial x^2}\right)\pi'_{1x} \tag{22}$$

$$G^E_{zx} = \frac{\partial^2}{\partial z\partial x}\pi'_{1x} \tag{23}$$

$$G^H_{yx} = -\frac{k_1^2}{j\omega\mu}\frac{\partial}{\partial z}\pi'_{1x} \tag{24}$$

$$G^E_{xz} = \frac{\partial^2}{\partial x \partial z}\pi'_{1z} \tag{25}$$

$$G^E_{zz} = \left(k_1^2 + \frac{\partial^2}{\partial z^2}\right)\pi'_{1z} \tag{26}$$

$$G^H_{yz} = \frac{k_1^2}{j\omega\mu}\frac{\partial}{\partial x}\pi'_{1z} \tag{27}$$

where the primed potentials π'_{1x} and π'_{1z} are just the potentials due to a unit current densities of $\mathbf{J_x}$ and $\mathbf{J_z}$ respectively. One can easily verify the reciprocity principle by substituting (20) and (21) to (22) through (27).

With $\bar{\bar{\mathbf{G}}}^{\mathbf{E}}(\bar{r};\bar{r}')$ provided the total field at $\bar{r}$ satisfies the following integral equation

$$\bar{E}(\bar{r}) = \bar{E}^i(\bar{r}) + \iint_{S'} \bar{\bar{G}}^E(\bar{r};\bar{r}') \cdot \bar{J}^s(\bar{r}')\, ds \tag{28}$$

and with $\bar{\bar{\mathbf{G}}}^{\mathbf{H}}(\bar{r};\bar{r}')$, the magnetic field is written as

$$\bar{H}(\bar{r}) = \bar{H}^i(\bar{r}) + \iint_{S'} \bar{\bar{G}}^H(\bar{r};\bar{r}') \cdot \bar{J}^s(\bar{r}')\, ds \quad . \tag{29}$$

Numerical computation of the scattering current $\bar{\mathbf{J}}^{\mathbf{s}}$ in equation (28) is initiated by dividing the inhomogeneous region, $\mathbf{S}'$, into a finite number of rectangular cells, Figure (1), such that a constant current density over each cell can be assumed. With this assumption the integral equation (28) can be

rewritten as

$$E_i = E_i^i + \sum_{j=1}^{N} \Delta\sigma_j \left(\Gamma_{ij}^E \cdot E_j \right), \quad i = 1,N \tag{30}$$

where

$$\Gamma_{ij}^E = \begin{pmatrix} \Gamma_{xxij}^E & \Gamma_{xzij}^E \\ \Gamma_{zxij}^E & \Gamma_{zzij}^E \end{pmatrix} . \tag{31}$$

The elements composing Γ_{ij}^E are integrated Green's functions, for example,

$$\Gamma_{zxij}^E = \iint_{S_j'} G_{zx}^E(\bar{r}_i;\bar{r}')\, ds \tag{32}$$

where the integration is carried out over the jth cell of S'.

Using the quasistatic approximation, $k_0 \approx 0$, we can reduce the factor R_{TM} to

$$R_{TM} = 1, \qquad \text{for all} \quad k_x \ . \tag{33}$$

It can be shown that the integrations in k_x leads to the following analytic results for π_{1x}' and π_{1z}'

$$\pi_{1x}' = - \frac{j\omega\mu}{2\pi} \frac{1}{k_1^2} \left[K_0(jk_1r_1) + K_0(jk_1r_2) \right] \tag{34}$$

$$\pi_{1z}' = - \frac{j\omega\mu}{2\pi} \frac{1}{k_1^2} \left[K_0(jk_1r_1) - K_0(jk_1r_2) \right] \tag{35}$$

where K_0 is the modified Bessel function of the second kind of order zero, and

$$r_1 = \left[(x - x')^2 + (z - z')^2\right]^{1/2}.$$

$$r_2 = \left[(x - x')^2 + (z + z')^2\right]^{1/2}$$

The evaluation of Γ is given by Appendix B. Special care is taken for the situation where the Green's function becomes singular. Rewriting the results for the electric fields,

$$\Gamma^E_{xxij} = \frac{j\omega\mu}{k_1^2} S_{ij} - \frac{\omega\mu}{2\pi k_1} \int_{x_\ell}^{x_r} \left[\frac{z_i - z'}{r_1} K_1(jk_1 r_1) - \frac{z_i + z'}{r_2} K_1(jk_1 r_2)\right] \Bigg|_{z_t}^{z_b} dx' \tag{36}$$

$$\Gamma^E_{zxij} = -\frac{j\omega\mu}{2\pi} \frac{1}{k_1^2} \left[K_0(jk_1 r_1) - K_0(jk_1 r_2)\right] \Bigg|_{x_\ell}^{x_r} \Bigg|_{z_t}^{z_b} \tag{37}$$

$$\Gamma^E_{xzij} = -\frac{j\omega\mu}{2\pi} \frac{1}{k_1^2} \left[K_0(jk_1 r_1) + K_0(jk_1 r_2)\right] \Bigg|_{x_\ell}^{x_r} \Bigg|_{z_t}^{z_b} \tag{38}$$

$$\Gamma^E_{zzij} = \frac{j\omega\mu}{k_1^2} S_{ij} - \frac{\omega\mu}{2\pi k_1} \int_{z_t}^{z_b} \left[\frac{x_i - x'}{r_1} K_1(jk_1 r_1) - \frac{x_i - x'}{r_2} K_1(jk_1 r_2)\right] \Bigg|_{x_\ell}^{x_r} dz' . \tag{39}$$

where

$S_{ij} = 1$, for $x_\ell < x_i < x_r$ and $z_t < z_i < z_b$

$S_{ij} = 0$, for (x_i, z_i) outside of s'_j

and

$$f(t)\Big|_{t_1}^{t_2} = f(t_2) - f(t_1) \ . \tag{40}$$

The notations x_l, x_r, z_t, and z_b are shown in Figure 1, and they are consistent with those used in Appendix B.

Equation (30) can now be rewritten in a numerically equivalent matrix form as

$$K \cdot E = - E^i \tag{41}$$

where the elements of K are given by

$$K_{ij} = \Delta\sigma_j \, \Gamma^E_{ij} - \delta_{ij}, \qquad i,j = 1,N$$

with

$$\delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{for} \quad i = j$$

and

$$\delta_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{for} \quad i \neq j \ .$$

The electric field in the body can now be formally written

$$E = - K^{-1} \cdot E^i \tag{42}$$

where the incident field $\mathbf{E}^i$ can be easily computed in the absence of the inhomogeneity. The scattering current $\mathbf{J}^s$ is then simply given by equation (5.2).

The electric and magnetic fields outside of the body are obtained using the same equations (28) and (29) by substituting the scattering current obtained in the foregoing section. The Green's function is evaluated at the earth's surface. Equation (30) is slightly modified to

$$E_k\Big|_{z_k=0} = E^i\Big|_{z_{k=0}} + \sum_{j=1}^{N} \Gamma^E_{kj}\Big|_{z_k=0} \cdot J^s \tag{43}$$

and for the magnetic field

$$H_k\Big|_{z_k=0} = H^i\Big|_{z_k=0} + \sum_{j=1}^{N} \Gamma^H_{kj}\Big|_{z_k=0} \cdot J^s, \qquad k = 1,M \tag{44}$$

where

$$\Gamma^E_{kj}\Big|_{z_k=0} = \begin{pmatrix} \Gamma^E_{xxkj} & \Gamma^E_{xzkj} \\ \Gamma^E_{zxkj} & \Gamma^E_{zzkj} \end{pmatrix}_{z_k=0} \tag{45}$$

$$\Gamma^H_{kj}\Big|_{z_k=0} = \begin{pmatrix} \Gamma^H_{yxkj} & \Gamma^H_{yzkj} \end{pmatrix}_{z_k=0} \tag{46}$$

and M is the number of field points on the surface.

It can be shown that the integrated Green's functions evaluated at the surface are as follows;

$$\Gamma^E_{xxkj}\Big|_{z_k=0} = \frac{\omega\mu}{\pi k_1} \int_{x_\ell}^{x_r} \left[\frac{z'}{r_0} K_1(jk_1 r_0)\right]\Bigg|_{z_t}^{z_b} dx' \tag{47.1}$$

for $z_\ell > 0$, and

$$\Gamma^E_{xxkj}\Big|_{z_k=0} = \frac{j\omega\mu}{k_1^2} + \frac{\omega\mu}{\pi k_1} \int_{x_\ell}^{x_r} \left[\frac{z'}{r_0} K_1(jk_1 r_0)\right]_{z_b} dx' \tag{47.2}$$

for $z_t = 0$ and $x_r < x_k < x_l$.

$$\Gamma^E_{xzkj}\Big|_{z_k=0} = -\frac{j\omega\mu}{\pi}\frac{1}{k_1^2}K_0(jk_1r_0)\Bigg|_{x_\ell}^{x_r}\Bigg|_{z_t}^{z_b} \tag{48}$$

with r_0 in (47.1), (47.2), and (48) defined as

$$r_0 = \left[(x_k - x')^2 + z'^2\right]^{1/2}$$

It should be noticed that

$$\Gamma^E_{zxkj}\Big|_{z_k=0} = \Gamma^E_{zzkj}\Big|_{z_j=0} = \Gamma^H_{yxkj}\Big|_{z_k=0} = \Gamma^H_{yzkj}\Big|_{z_k=0} = 0\ . \tag{49}$$

Scattering in a two-layered earth

Consider the case of a conductive body buried in the lower half space of a two-layered earth. The vector potentials in the first and second layers are given in Appendix A. Rewriting the results by making use of the quasistatic approximation, $k_0 \approx 0$,

$$\pi'_{1x} = -\frac{j\omega\mu}{2\pi}\frac{1}{k_1^2}\int_0^\infty \frac{u_2}{u_1^2}\,{}^1R_{TM}\cdot\left(e^{u_1z} + e^{-u_1z}\right)e^{-u_2z'}\cos k_x(x-x')dk_x \tag{50}$$

$$\pi'_{1z} = -\frac{j\omega\mu}{2\pi}\frac{1}{k_1^2}\int_0^\infty \frac{1}{u_1}\,{}^1R_{TM}\cdot\left(e^{u_1z} - e^{-u_1z}\right)e^{-u_2z'}\cos k_x(x-x')dk_x \tag{51}$$

$$\pi'_{2x} = -\frac{j\omega\mu}{2\pi}\frac{1}{k_2^2}\left[K_0(jk_2r) + \int_0^\infty \frac{1}{u_2}\,{}^2R_{TM}\cdot e^{-u_2(z+z'-2d)}\cos k_x(x-x')\,dk_x\right] \tag{52}$$

$$\pi'_{2z} = -\frac{j\omega\mu}{2\pi}\frac{1}{k_2^2}\left[K_0(jk_2r) - \int_0^\infty \frac{1}{u_2}\,{}^2R_{TM}\cdot e^{-u_2(z+z'-2d)}\cos kx(x-x')\,dx_x\right] \tag{53}$$

where

$$^1R_{TM} = \frac{e^{u_2 d}\ \mathrm{sech}\ u_1 d}{\left(\frac{k_2}{k_1}\right)^2 + \frac{u_2}{u_1}\tanh u_1 d}$$

$$^2R_{TM} = \frac{\left(\frac{k_2}{k_1}\right)^2 - \frac{u_2}{u_1}\tanh u_1 d}{\left(\frac{k_2}{k_1}\right)^2 + \frac{u_2}{u_1}\tanh u_1 d}$$

and d is the thickness of the first layer. The Green's function in each layer is given by equations (22) through (27). The scattering currents are computed by using π'_{2x} and π'_{2z} in the lower half space in the usual manner discussed in the homogeneous half space case.

The integrated Green's functions to be used for the formulation of equation (30) are given in Appendix B as

$$\Gamma^E_{xxij} = \frac{j\omega\mu}{k_2^2}S_{ij} - \frac{\omega\mu}{2\pi k_2}\int_{x_\ell}^{x_r}\left[\frac{z_i-z'}{r}K_1(jk_2r)\right]\Bigg|_{z_t}^{z_b}dx' \tag{54}$$

$$+\frac{j\omega\mu}{2\pi}\frac{1}{k_2^2}\int_0^\infty \frac{{}^2R_{TM}}{k_x}e^{-u_2z_i}\left[e^{-u_2z'}\sin k_x(x_i-x')\right]\Bigg|_{x_\ell}^{x_r}\Bigg|_{z_t}^{z_b}dk_x$$

$$\Gamma^E_{yxij} = -\frac{j\omega\mu}{2\pi}\frac{1}{k_2^2}K_0(jk_2r)\Big|_{x_\ell}^{x_r}\Big|_{z_t}^{z_b} \tag{55}$$

$$+\frac{j\omega\mu}{2\pi}\frac{1}{k_2^2}\int_0^\infty \frac{{}^2R_{TM}}{u_2}e^{-u_2z_i}\left[e^{-u_2z'}\cos k_x(x_i-x')\right]\Big|_{x_\ell}^{x_r}\Big|_{z_t}^{z_b}dk_x$$

$$\Gamma^E_{xzij} = -\frac{j\omega\mu}{2\pi}\frac{1}{k_2^2}K_0(jk_2r)\Big|_{x_\ell}^{x_r}\Big|_{z_t}^{z_b} - \frac{j\omega\mu}{2\pi}\frac{1}{k_2^2}\int_0^\infty \frac{{}^2R_{TM}}{u_2}e^{-u_2z_i} \tag{56}$$

$$\left[e^{-u_2z'}\cos k_x(x_i-x')\right]\Big|_{x_\ell}^{x_r}\Big|_{z_t}^{z_b}dk_x$$

$$\Gamma^E_{zzij} = \frac{j\omega\mu}{k_2^2}S_{ij} - \frac{\omega\mu}{2\pi k_2}\int_{z_t}^{z_b}\left[\frac{x_i-x'}{r}K_1(jk_2r)\right]\Big|_{x_\ell}^{x_r}dz' \tag{57}$$

$$+\frac{j\omega\mu}{2\pi}\frac{1}{k_2^2}\int_0^\infty \frac{k_x}{u_2^2}{}^2R_{TM}e^{-u_2z_i}\left[e^{-u_2z'}\sin k_x(x_i-x')\right]\Big|_{x_\ell}^{x_r}\Big|_{z_t}^{z_b}dk_x$$

Once the elements of Green's tensor are computed the electric fields in the body are found by (42) and finally the scattering currents are computed by (5.2). These elements can not, in this case, be integrated analytically and consequently time consuming numerical integration in k_x must be carried out.

The electric and magnetic fields on the surface are computed in the usual manner. However, the Green's function to be used for the surface fields differs from that used in the lower half space. The elements of integrated Green's tensors evaluated at the kth field point $(x_k, z_k, = 0)$ are given by Appendix B.

$$\Gamma^E_{xzkj}\Big|_{z_k=0} = -\frac{j\omega\mu}{\pi}\frac{1}{k_1^2}\int_0^\infty \frac{{}^1R_{TM}}{u_2}\left[e^{-u_2z'}\cos k_x(x_k-x')\right]\Big|_{x_\ell}^{x_r}\Big|_{z_t}^{z_b}dk_x \tag{58}$$

$$\Gamma^{H}_{yxkj}\Big|_{z_k=0} = \Gamma^{H}_{yzkj}\Big|_{z_k=0} = \Gamma^{E}_{zxkj}\Big|_{z_k=0} = \Gamma^{E}_{zzkj}\Big|_{z_k=0} = 0. \tag{59}$$

Substituting these elements into equations (43) and (44), we obtain the total electric and magnetic fields on the surface.

Numerical Results and Conclusion

The numerical solution developed in this paper is compared to the one obtained by Ryu (1970) using the finite element method. The model used for this purpose is a half space in which a rectangular conductor of 200m x 50m is buried 50m deep to the top of the conductor. The resistivities used are 1 ohm-m for the conductor and 100 ohm-m for the half space. At the frequency of 8 hertz the finite element solution has been obtained on a grid in which the conductor is simulated by 16 cells of equal size (25m x 25m). At this frequency the cell size used for the finite element solution is considered fine enough, with its skin depth of 177m, to ensure the accurate numerical solution. Compared to the finite element solution, a number of integral equation solutions has been obtained using increasingly more cells for the conductor. The result has been plotted, Figure 2, in terms of the apparent resistivity and the impedance phase. The integral equation solution converges to the finite element solution as the number of cells used increases from 4 cells to 48 cells. The convergence rate is relatively slow compared to the one for the TE-mode solution (Hohmann, 1970), in which a cell size of a quarter of the skin depth would result in a reasonably good numerical solution. The major reason for the slow convergence of the TM-mode numerical solution is because of the assumption we used in which

electric fields are constant in a cell. Unlike the TE-mode situation where there is no charge, the discontinuous normal electric fields at cell boundaries create charges. The Hertz potential of a constant current, by which the electromagnetic field is computed in this paper, does not include potentials due to thus created charges, and eventually affects the quality of the numerical solution. The effect of the cell boundary charges can be somewhat reduced through the approach in which an explicit scalar potential is employed to represent fields due to charges. Hohmann and Ting (1978) used a scheme by which the concentrated boundary charges are uniformly distributed over a volume extending from the center of one cell to the center of the next cell.

The next model compared is a conductive dyke with varying dip angles as is shown in Figure 3. A dipping boundary of the conductor is simulated by stacking small rectangular conductors in such a way that the overall dip angle is preserved. For frequencies of 8 hertz and 100 hertz, the finite element solution and the integral equation solution result in identical apparent resistivity profiles when the conductor (20m x 50m) is vertical. With the dip angle of 45° however, a maximum of 10% difference in peak to peak apparent resistivity can be observed for both frequencies. For the integral equation solutions the conductor has been simulated by 10 cells of equal size; 10m x 10m. Considering the slow convergence rate illustrated by Figure 2, the numerical integral equation solution for the dipping conductor, with its increased boundary surface, may not have reached its full convergence.

Using the integral equation for the modeling of electromagnetic scattering, one is immediately confined to a model finite in extent in an otherwise layered half space. It is unfortunate that the available Green's functions are limited to those for the layered half space. However, this geometrical restriction offers the integral equation technique its major advantage; it is necessary to solve for

the scattering current only in the inhomogeneous region.

Another usefulness of the integral equation approach is that it offers an easy access to the derivatives of the field quantities, a procedure necessary for the inversion of the field data (Weidelt, 1975). Rewriting equation (30) with $\Delta\sigma_j$ substituted by $(\sigma_j - \sigma)$ in the expression:

$$E_i = E_i^i + \sum_{j=1}^{N} (\sigma_j - \sigma) \left(\Gamma_{ij}^E \cdot E_j\right), \quad i = 1,N . \tag{60}$$

The derivatives of $\mathbf{E}_i$ with respect to the conductivity of the lth cell can be written as

$$\frac{\partial E_i}{\partial \sigma_\ell} = \left(\Gamma_{i\ell}^E \cdot E_\ell\right) + \sum_{j=1}^{N} (\sigma_j - \sigma) \left(\Gamma_{ij}^E \cdot \frac{\partial E_j}{\partial \sigma_\ell}\right), \quad \ell = 1,N . \tag{61}$$

Equation (61) is N linear equations for each component from which the derivatives of the electric fields in the body can be obtained. The derivatives of the surface fields with respect to the conductivities of each cell can be found by substituting $\frac{\partial \mathbf{E}_i}{\partial \sigma_l}$ into the similar equation derived from (43) as

$$\frac{\partial E_k|_{z_k=0}}{\partial \sigma_\ell} = \left(\Gamma_{k\ell}^E|_{z_k=0} \cdot E_\ell\right) + \sum_{j=1}^{N} (\sigma_j - \sigma) \left(\Gamma_{kj}^E|_{z_k=0} \cdot \frac{\partial E_j}{\partial \sigma_\ell}\right), \quad \ell = 1,N . \tag{62}$$

Substitutions of $\mathbf{H}$ and $\Gamma^{\mathbf{H}}$ into equations (60), (61), and (62) will produce the derivatives of the surface magnetic fields with respect to the conductivities of each cell. For each iteration in the inversion process the same Green's

functions will be used resulting in a fast iteration time.

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FIGURE CAPTIONS

Figure 1. A simulation of a two-dimensional inhomogeneity by N rectangular cells.

Figure 2. Convergence test for the integral equation numerical solution.

Figure 3. Comparison between solutions for a dipping conductor.

Figure A.1. An N-layered earth with the scattering current $\bar{\mathbf{J}}^s$, in the ith layer.

Figure B.1. A rectangular current cell.

APPENDIX A

Consider an N-layered half space upon which a TM-mode plane wave is incident. A two-dimensional conductive inhomogeneity can be represented by a distribution of current elements over the cross sectional area oriented perpendicular to the strike. Suppose that a point current of density $\bar{J}^s$ is located at (x',z') in the ith layer as shown by Figure A.1. Rewriting equation (17) in the ith layer

$$\left(\nabla^2 + k_i^2\right) \bar{\pi}_i = - \frac{\bar{J}^s}{\sigma_i + j\omega\varepsilon_i} \delta(x - x') \; \delta(z - z') \tag{A.1}$$

Fourier transforming (A-1), we obtain the primary potential as

$$\bar{\pi}_i^P \; (k_x, k_z) = - \frac{j\omega\mu}{k_i^2} \bar{J}^s \frac{e^{-j(k_x x' + k_z z')}}{u_i^2 + k_z^2} \tag{A.2}$$

where

$$k_i^2 = (\omega^2 \mu\varepsilon_i - j\sigma_i \omega\mu)$$

$$u_i^2 = k_x^2 - k_i^2$$

$$\mu = \mu_0 = \mu_1 = \ldots = \mu_N \; .$$

Inversely transforming (A.2) in k_z (Erdelyi, 1954),

$$\bar{\pi}_i^P \; (k_x, z) = - \frac{j\omega\mu}{2k_i^2} \bar{J}^s e^{-jk_x x'} \frac{e^{-u_i |z - z'|}}{u_i} \; . \tag{A.3}$$

Equation (19) can be rewritten in each layer as

$$\left(\nabla^2 + k_j^2\right) \bar{\pi}_j^s = 0, \qquad j = 0,N \ . \tag{A.4}$$

The solutions for (A.4) in k_x space are written as

$$\bar{\pi}_j^s = \left[A_j(k_x)e^{-u_j z} + B_j(k_x)e^{u_j z}\right] \bar{i}_j \tag{A.5}$$

where the directions of unit vectors $\bar{i}_j$ and the coefficients A_j and B_j are all subject to boundary conditions with initial constraints.

$$A_0(k_x) = B_N(k_x) \equiv 0 \ .$$

1. **Homogeneous half space**

a. $\bar{J}^s = J_x \bar{i}_x$

From (A.3) and (A.5), after dropping parantheses

$$\pi_{0x} = B_0 \ e^{u_0 z}, \qquad \text{in the air} \tag{A.6}$$

$$\pi_{1x} = \frac{\alpha}{u_1} e^{u_1 z} + A_1 e^{-u_1 z}, \qquad \text{in the earth} \tag{A.7}$$

where

$$\alpha = - \frac{j\omega\mu}{2k_1^2} J_x \ e^{-jk_x x'} \ e^{-u_1 z'} \ .$$

From equations (15) and (16), we have

$$E_x^s = - u^2 \pi_x \tag{A.8}$$

$$E_z^s = - jk_x \frac{\partial \pi_x}{\partial z} \tag{A.9}$$

$$H_y^s = - \frac{k^2}{j\omega\mu} \frac{\partial \pi_x}{\partial z} . \tag{A.10}$$

Matching boundary conditions at z = 0, we obtain

$$u_0^2 B_0 = u_1^2 \left(\frac{\alpha}{u_1} + A_1\right), \qquad \text{for} \quad E_x^s|_{z=0} \tag{A.11}$$

$$k_0^2 u_0 B_0 = k_1^2 \alpha - k_1^2 u_1 A_1, \qquad \text{for} \quad H_y^s|_{z=0} . \tag{A.12}$$

From (A.11) and A.12),

$$A_1 = \frac{\alpha}{u_1} R_{TM} \tag{A.13}$$

$$B_0 = 2 \frac{\alpha}{u_0} \frac{k_1^2 u_1}{k_0^2 u_1 + k_1^2 u_0} . \tag{A.14}$$

where

$$R_{TM} = - \frac{k_0^2 u_1 - k_1^2 u_0}{k_0^2 u_1 + k_1^2 u_0} .$$

Hence, the vector potential in the half space is

$$\pi_{1x} = \frac{j\omega\mu}{2\pi} \frac{J_x}{k_1^2} \int_0^\infty \frac{1}{u_1} \left[e^{-u_1|z-z'|} + R_{TM} e^{-u_1(z+z')} \right] \cos k_x(x-x') \, dk_x . \tag{A.15}$$

b. $\bar{J}^s = J_z \bar{i}_z$

In much the same way, the z-directed vector potential in the half space can be derived as

$$\pi_{1z} = -\frac{j\omega\mu}{2}\frac{J_z}{k_1^2}\int_0^\infty \frac{1}{u_1}\left[e^{-u_1|z-z'|} - R_{TM}e^{-u_1(z+z')}\right]\cos k_x(x-x')dk_x \quad \text{(A.16)}$$

2. Two-layered half space

Let us suppose that the inhomogeneity is in the second layer.

a. $\bar{J}^s = J_x \bar{i}_x$

The potentials in each layer can be written by using equations (A.3) and (A.5) as

$$\pi_{0x} = B_0\, e^{u_0 z}, \qquad \text{in the air} \quad \text{(A.17)}$$

$$\pi_{1x} = A_1 e^{-u_1 z} + B_1 e^{u_1 z}, \qquad \text{in the first layer} \quad \text{(A.18)}$$

$$\pi_{2x} = \frac{\alpha}{u_2}\, e^{u_2 z} + A_2 e^{-u_2 z}, \qquad \text{in the source region} \quad \text{(A.19)}$$

where

$$\alpha = -\frac{j\omega\mu}{2k_2^2} J_x\, e^{-jk_x x'}\, e^{-u_2 z'} .$$

Matching boundary conditions at $z = z_1$, and $z = z_2$ by substituting π_x into equations (15) and (16), we obtain at $z = z_1 = 0$,

$$B_0^2 u_0 = u_1^2 \, (A_1 + B_1), \qquad \text{for} \quad E_x^s\big|_{z=0} \tag{A.20}$$

$$k_0^2 u_0 B_0 = k_1^2 u_1 \, (-A_1 + B_1), \qquad \text{for} \quad H_y^s\big|_{z=0} \tag{A.21}$$

and at $z = z_2 = d$,

$$u_1^2 \left(A_1 e^{-u_1 d} + B_1 e^{u_1 d}\right) = u_2^2 \left(\frac{\alpha}{u_2} \, e^{u_2 d} + A_2 e^{-u_2 d}\right), \qquad \text{for} \quad E_x^s\big|_{z=d} \tag{A.22}$$

$$k_1^2 u_1 \left(-A_1 e^{-u_1 d} + B_1 e^{u_1 d}\right) = k_2^2 u_2 \left(\frac{\alpha}{u_2} \, e^{u_2 d} - A_2 e^{-u_2 d}\right), \qquad \text{for} \quad H_y^s\big|_{z=d} \tag{A.23}$$

From equations (A.20), (A.21), (A.22), and (A.23)

$$B_0 = 2\alpha \, \frac{k_2^2 u_2}{k_1^2 u_1} \, \frac{1}{u_0} \, {}^1R_{TM} \tag{A.24}$$

$$A_1 = \alpha \, \frac{k_2^2 u_2}{k_1^2 u_1} \, \frac{1}{u_1} \left(\frac{u_0}{u_1} - \frac{k_0^2}{k_1^2}\right) {}^1R_{TM} \tag{A.25}$$

$$B_1 = \alpha \frac{k_2^2 u_2}{k_1^2 u_1} \frac{1}{u_1} \left(\frac{u_0}{u_1} + \frac{k_0^2}{k_1^2} \right) {}^1R_{TM} \tag{A.26}$$

$$A_2 = \alpha \, e^{2u_2 d} \frac{1}{u_2} \, {}^2R_{TM} \tag{A.27}$$

where

$$ {}^1R_{TM} = \frac{e^{u_2 d} \operatorname{sech} u_1 d}{\frac{k_2^2}{k_1^2}\left(\frac{u_0}{u_1} + \frac{k_0^2}{k_1^2} \tanh u_1 d\right) + \frac{u_2}{u_1}\left(\frac{u_0}{u_1} \tanh u_1 d + \frac{k_0^2}{k_1^2}\right)}$$

$$ {}^2R_{TM} = \frac{\frac{k_2^2}{k_1^2}\left(\frac{u_0}{u_1} + \frac{k_0^2}{k_1^2} \tanh u_1 d\right) - \frac{u_2}{u_1}\left(\frac{u_0}{u_1} \tanh u_1 d + \frac{k_0^2}{k_1^2}\right)}{\frac{k_2^2}{k_1^2}\left(\frac{u_0}{u_1} + \frac{k_0^2}{k_1^2} \tanh u_1 d\right) + \frac{u_2}{u_1}\left(\frac{u_0}{u_1} \tanh u_1 d + \frac{k_0^2}{k_1^2}\right)}$$

The vector potentials in the first and second layers are written as

$$\pi_{1x} = -\frac{j\omega\mu}{2\pi} \frac{J_x}{k_1^2} \int_0^\infty \frac{u_2}{u_1} \, {}^1R_{TM} \left[\left(\frac{u_0}{u_1} + \frac{k_0^2}{k_1^2} \right) e^{u_1 z} + \left(\frac{u_0}{u_1} - \frac{k_0^2}{k_1^2} \right) e^{-u_1 z} \right] e^{-u_2 z'} \cos k_x (x-x') \, dk_x \tag{A.28}$$

$$\pi_{2x} = -\frac{j\omega\mu}{2\pi} \frac{J_x}{k_2^2} \int_0^\infty \frac{1}{u_2} \left[e^{-u_2 |z-z'|} + {}^2R_{TM} \, e^{-u_2 (z+z'-2d)} \right] \cdot \cos k_x (x-x') \, dk_x \, . \tag{A.29}$$

b. $\bar{J}^{s} = J_z \bar{i}_z$

By following similar algebraic procedures, we can easily arrive at the following vector potentials in the first and second layers.

$$\pi_{1z} = -\frac{j\omega\mu}{2\pi}\frac{J_z}{k_1^2}\int_0^\infty \frac{1}{u_1}\,{}^1R_{TM} \left[\left(\frac{u_0}{u_1}+\frac{k_0^2}{k_1^2}\right)e^{u_1 z} - \left(\frac{u_0}{u_1}-\frac{k_0^2}{k_1^2}\right)e^{-u_1 z}\right] e^{-u_2 z'} \cos k_x(x-x')\, dk_x \tag{A.30}$$

$$\pi_{2z} = -\frac{j\omega\mu}{2\pi}\frac{J_z}{k_2^2}\int_0^\infty \frac{1}{u_2}\left[e^{-u_2|z-z'|} - {}^2R_{TM}\, e^{-u_2(z+z'-2d)}\right] \cdot \cos k_x(x-x')\, dk_x \tag{A.31}$$

APPENDIX B

The integration of the Green's function over the area occupied by the jth cell of rectangular shape is evaluated at an arbitrary position (x_i, z_i) .

a. Green's functions for the homogeneous half space

Rewriting equation (34) and (35), we have

$$\pi'_{1x} = \alpha \left[K_0(jk_1 r_1) + K_0(jk_1 r_2) \right] \tag{B.1}$$

$$\pi'_{1z} = \alpha \left[K_0(jk_1 r_1) - K_0(jk_1 r_2) \right] \tag{B.2}$$

where

$$\alpha = - \frac{j\omega\mu}{2\pi} \frac{1}{k_1^2} .$$

Throughout the article, the following relationships are used

$$\begin{aligned} \frac{\partial}{\partial x} f(r_1) &= - \frac{\partial}{\partial x'} f(r_1) \\ \frac{\partial}{\partial x} f(r_2) &= - \frac{\partial}{\partial x'} f(r_2) \\ \frac{\partial}{\partial z} f(r_1) &= - \frac{\partial}{\partial z'} f(r_1) \\ \frac{\partial}{\partial z} f(r_2) &= \frac{\partial}{\partial z'} f(r_2) . \end{aligned} \tag{B.3}$$

Figure B.1 shows the geometry of a rectangular cell in the half space. The field point (x_i, z_i) is arbitrary. If the field point is in the cell, an arbitrary circle l_1 of radius R is drawn about the field point. The vectors π_i , i = 1, 2, denote the unit vectors outward normal to the region S_i, i = 1, 2, over which the integration

is carried out.

1. $G^E_{xxij}(\bar{r};\bar{r}')$

From equations (22) and (B.1)

$$\Gamma^E_{xxij} = \alpha \iint_{S'_j} \left(k_1^2 + \frac{\partial^2}{\partial x^2}\right)\left[K_0(jk_1r_1) + K_0(jk_1r_2)\right] dx'dz'. \tag{B.4}$$

If the field point is in the jth cell, S'_j is divided into S_1 and S_2 . The primary potential $K_0(jk_1r_1)$ is singular in S_1. The following equation holds for the vector potential $\bar{\pi}$ in the non-singular region S_2.

$$(\nabla^2 + k^2)\ \bar{\pi} = 0 \tag{B.5}$$

By using (B.5), equation (B.4) may be rewritten

$$\begin{aligned}\Gamma^E_{xxij} &= \alpha \iint_{S_1} \left(k_1^2 + \frac{\partial^2}{\partial x^2}\right) K_0(jk_1r_1)\ ds \\ &\quad - \alpha \iint_{S_2} \frac{\partial^2}{\partial z^2} K_0(jk_1r_1)\ ds \\ &\quad - \alpha \iint_{S'_j} \frac{\partial^2}{\partial z^2} K_0(jk_1r_2)\ ds \\ &= I_1 + I_2 + I_3 .\end{aligned} \tag{B.6}$$

The first part of I_1 may be evaluated in a manner similar to that used by Richmond (1965).

$$\alpha\ k_1^2 \iint_{S_1} K_0(jk_1r_1)\ ds = 2\pi\alpha \left[jk_1RK_1(jk_1R) - 1\right] \tag{B.7}$$

The second part of I_1 may be evaluated in the following way; because of the symmetry,

$$\alpha \iint_{S_1} \left[\frac{\partial^2}{\partial x^2} K_0(jk_1 r_1)\right]_{\bar{r}=\bar{r}_0} ds = \alpha \iint_{S_1} \left[\frac{\partial^2}{\partial z^2} K_0(jk_1 r_1)\right]_{\bar{r}=\bar{r}_0} ds$$

from equation (17)

$$\left(\nabla^2 + k_1^2\right) \alpha k_0(jk_1 r_1) = - 2\pi\alpha\ \delta(\bar{r}-\bar{r}')$$

Integration on both sides in S_1 evaluated at $\bar{r} = \bar{r}_o$ yields,

$$2\alpha \iint_{S_1} \left[\frac{\partial^2}{\partial x^2} K_0(jk_1 r_1)\right]_{\bar{r}=\bar{r}_0} ds = -\alpha k_1^2 \iint_{S_1} \left[K_0(jk_1 r_1)\right]_{\bar{r}=\bar{r}_0} ds$$

$$- 2\pi\alpha \iint_{S_1} \left[\delta(\bar{r}-\bar{r}')\right]_{\bar{r}=\bar{r}_0} ds$$

by (B.7)

$$= 2\pi\alpha - 2\pi\alpha\ jk_1 R\ K_1(jk_1 R) - 2\pi\alpha\ .$$

Hence

$$\alpha \iint_{S_1} \left[\frac{\partial^2}{\partial x^2} K_0(jk_1 r_1)\right]_{\bar{r}=\bar{r}_0} ds = - jk_1 \pi\alpha R\ K_1(jk_1 R)\ . \tag{B.8}$$

$$I_2 = - \alpha \iint_{S_2} \frac{\partial^2}{\partial z^2} K_0(jk_1 r_1)\ ds$$

$$= - \alpha \iint_{S_2} \nabla \cdot \bar{i}_z \frac{\partial}{\partial z} K_0(jk_1 r_1)\ ds$$

$$= - jk_1\alpha \int_{\ell_1} \frac{z_i - z'}{r_1} K_1(jk_1 r_1)\ \bar{i}_z \cdot \bar{n}_2\ d\ell$$

$$- jk_1\alpha \int_{\ell_2} \frac{z_i - z'}{r_1} K_1(jk_1 r_1)\ \bar{i}_z \cdot \bar{n}_2\ d\ell$$

along l_2, we have

$$\bar{i}_z \cdot \bar{n}_2 = 0, \quad \text{for} \quad x' = x_\ell,\ x_r$$

$$\bar{i}_z \cdot \bar{n}_2 = 1, \quad \text{for} \quad z' = z_b$$

$$\bar{i}_z \cdot \bar{n}_2 = -1, \quad \text{for} \quad z' = z_t,$$

hence

$$= - jk_1 \pi\alpha R\ K_1(jk_1 R) - jk_1\alpha \int_{x_\ell}^{x_r} \left[\frac{z_i - z'}{r_1} K_1(jk_1 r_1)\right] \Bigg|_{z_t}^{z_b} dx' \ . \tag{B.9}$$

Similarly,

$$I_3 = - \alpha \iint_{S_j'} \frac{\partial^2}{\partial z^2} K_0(jk_1 r_2)\ ds \tag{B.10}$$

$$= jk_1\alpha \int_{x_\ell}^{x_r} \left[\frac{z_i + z'}{r_2} K_1(jk_1 r_2)\right] \Bigg|_{z_t}^{z_b} dx' \ .$$

In case the field point is outside of the jth cell, it can be easily shown that

$$\alpha \iint_{S_j'} \left(k_1^2 + \frac{\partial^2}{\partial x^2}\right) K_0(jk_1 r_1)\ ds = - jk_1\alpha \int_{x_\ell}^{x_r} \left[\frac{z_i - z'}{r_1} K_1(jk_1 r_1)\right] \Bigg|_{z_t}^{z_b} dx'. \tag{B.11}$$

Finally from equation (B.7) through (B.11)

$$\Gamma^E_{xxij} = -2\pi\alpha\, S_{ij} - jk_1\alpha \int_{x_\ell}^{x_r} \left[\frac{z_i - z'}{r_1} K_1(jk_1 r_1) - \frac{z_i + z'}{r_2} K_1(jk_1 r_2)\right] \Bigg|_{z_t}^{z_b} dx' \tag{B.12}$$

where S_{ij} is defined by

$$S_{ij} \equiv 1, \qquad \text{for } x_\ell < x_i < x_r \text{ and } z_t < z_i < z_b$$

$$S_{ij} \equiv 0, \qquad \text{for } (x_i, z_i) \text{ outside of } S_j' .$$

2. $G^E_{zxij}(\bar{r};\bar{r}')$

From (23) and (B.1)

$$\Gamma^E_{zxij} = \alpha \iint_{S_j'} \frac{\partial^2}{\partial z \partial x} \left[K_0(jk_1 r_1) + K_0(jk_1 r_2)\right] ds . \tag{B.13}$$

By (B.3)

$$\Gamma^E_{zxij} = \alpha \iint_{S_j'} \frac{\partial^2}{\partial z' \partial x'} \left[K_0(jk_1 r_1) - K_0(jk_1 r_2)\right] ds \tag{B.14}$$

$$= \alpha \left[K_0(jk_1 r_1) - K_0(jk_1 r_2)\right] \Bigg|_{x_\ell}^{x_r} \Bigg|_{z_t}^{z_b}$$

3. $G^H_{yxij}(\bar{r};\bar{r}')$

From (24)

$$\Gamma^H_{yxij} = -\frac{\alpha k_1^2}{j\omega\mu} \iint_{S_j'} \frac{\partial}{\partial z} \left[K_0(jk_1 r_1) + K_0(jk_1 r_2)\right] ds. \tag{B.15}$$

By (B.3)

$$\Gamma^{H}_{yxij} = \frac{\alpha k_1^2}{j\omega\mu} \iint_{S_j'} \frac{\partial}{\partial z'} \left[K_0(jk_1r_1) - K_0(jk_1r_2)\right] ds \tag{B.16}$$

$$= \frac{\alpha k_1^2}{j\omega\mu} \int_{x_\ell}^{x_r} \left[K_0(jk_1r_1) - K_0(jk_1r_2)\right] \Big|_{z_t}^{z_b} dx' .$$

4. $G^{E}_{xzij}(\bar{r};\bar{r}')$

From equations (25) and (B.2)

$$\Gamma^{E}_{xzij} = \alpha \iint_{S_j'} \frac{\partial^2}{\partial x \partial z} \left[K_0(jk_1r_1) - K_0(jk_1r_2)\right] ds . \tag{B.17}$$

By (B.3)

$$\Gamma^{E}_{xzij} = \alpha \left[K_0(jk_1r_1) + K_0(jk_1r_2)\right] \Big|_{x_\ell}^{x_r} \Big|_{z_t}^{z_b} . \tag{B.18}$$

5. $G^{E}_{zzij}(\bar{r};\bar{r}')$

From equation (26)

$$\Gamma^{E}_{zzij} = \alpha \iint_{S_j'} \left(k_1^2 + \frac{\partial^2}{\partial z^2}\right) \left[K_0(jk_1r_1) - K_0(jk_1r_2)\right] ds . \tag{B.19}$$

As usual, the area S_j is divided into two regions S_1 and S_2. Then by equation (B.5)

$$\Gamma^{E}_{zzij} = \alpha \iint_{S_1} \left(k_1^2 + \frac{\partial^2}{\partial z^2}\right) K_0(jk_1r_1)\, ds - \alpha \iint_{S_2} \frac{\partial^2}{\partial x^2} K_0(jk_1r_1)\, ds \tag{B.20}$$

$$- \alpha \iint_{S_j'} \frac{\partial^2}{\partial x^2} K_0(jk_1r_2)\, ds .$$

The similar technique used for Γ^{E}_{xxij} yields

$$\Gamma^{E}_{zzij} = -\,2\pi\alpha\, S_{ij} - jk_1\alpha \int_{z_t}^{z_b} \left[\frac{x_i - x'}{r_1} K_1(jk_1 r_1) - \frac{x_i - x'}{r_2} K_1(jk_1 r_2) \right] \Bigg|_{x_\ell}^{x_r} dz' \,. \tag{B.21}$$

6. $G^{H}_{yzij}(\bar{r};\bar{r}')$

From equation (27)

$$\Gamma^{H}_{yzij} = \frac{\alpha k_1^2}{j\omega\mu} \iint_{S_j} \frac{\partial}{\partial x} \left[K_0(jk_1 r_1) - K_0(jk_1 r_2) \right] ds \,. \tag{B.22}$$

Again by (B.3)

$$\Gamma^{H}_{yzij} = -\frac{\alpha k_1^2}{j\omega\mu} \int_{z_t}^{z_b} \left[K_0(jk_1 r_1) - K_0(jk_1 r_2) \right] \Bigg|_{x_\ell}^{x_r} dz' \,. \tag{B.23}$$

b. Green's functions for the two-layered half space

1. Green's functions in the lower half space

Rewriting π'_{2x} and π'_{2z} from equations (52) and (53)

$$\pi'_{2x} = \alpha_2 K_0(jk_2 r) + \alpha_2 \int_0^\infty \frac{1}{u_2}\, {}^2R_{TM}\, e^{-u_2(z+z')} \cos k_x(x-x')\, dk_x \tag{B.24}$$

$$\pi'_{2z} = \alpha_2 K_0(jk_2 r) - \alpha_2 \int_0^\infty \frac{1}{u_2}\, {}^2R_{TM}\, e^{-u_2(z+z')} \cos k_x(x-x')\, dk_x \tag{B.25}$$

where
$$\alpha_2 = -\frac{j\omega\mu}{2\pi}\frac{1}{k_2^2}$$

$$^2R_{TM} = \frac{\dfrac{k_2^2}{k_1^2} - \dfrac{u_2}{u_1}\tanh u_1 d}{\dfrac{k_2^2}{k_1^2} + \dfrac{u_2}{u_1}\tanh u_1 d}\, e^{2u_2 d}$$

and d is the thickness of the first layer. The notations used for the geometry of a rectangular cell is given in Figure B.1.

The evaluation of primary potential $\mathbf{K}_0(jk_2r)$ is the same as was discussed in the previous section with k_1 substituted by k_2 in the expression. By using equation (B.5) and the relationships similar to (B.3), we obtain

$$\begin{aligned}\Gamma^E_{xxij} &= \iint_{S_j}\left(k_2^2 + \frac{\partial^2}{\partial x^2}\right)\pi'_{2x}\, ds \\ &= -2\pi\alpha_2 S_{ij} - jk_2\alpha_2\int_{x_\ell}^{x_r}\left[\frac{z_i - z'}{r}K_1(jk_2r)\right]\Bigg|_{z_t}^{z_b} dx' \\ &\quad - \alpha_2\int_0^\infty \frac{^2R_{TM}}{k_x}e^{-u_2 z_i}\left[e^{-u_2 z'}\sin k_x(x_i - x')\right]\Bigg|_{x_\ell}^{x_r}\Bigg|_{z_t}^{z_b} dk_x \end{aligned} \tag{B.26}$$

$$\begin{aligned}\Gamma^E_{zxij} &= \iint_{S_j}\frac{\partial^2}{\partial z\partial x}\pi'_{2x}\, ds \\ &= \alpha_2 K_0(jk_2r)\Bigg|_{x_\ell}^{x_r}\Bigg|_{z_t}^{z_b} \\ &\quad - \alpha_2\int_0^\infty \frac{^2R_{TM}}{u_2}e^{-u_2 z_i}\left[e^{-u_2 z'}\cos k_x(x_i - x')\right]\Bigg|_{x_\ell}^{x_r}\Bigg|_{z_t}^{z_b} dk_x \end{aligned} \tag{B.27}$$

$$\Gamma^E_{xzij} = \iint_{S'_j} \frac{\partial^2}{\partial x \partial z} \pi'_{2z} \, ds$$

$$= \alpha_2 \, K_0(jk_2 r) \Big|_{x_\ell}^{x_r} \Big|_{z_t}^{z_b}$$

$$+ \alpha_2 \int_0^\infty \frac{{}^2R_{TM}}{u_2} e^{-u_2 z_i} \left[e^{-u_2 z'} \cos k_x(x_i - x') \right] \Big|_{x_\ell}^{x_r} \Big|_{z_t}^{z_b} dk_x \qquad \text{(B.28)}$$

$$\Gamma^E_{zzij} = \iint_{S'_j} \left(k_2^2 + \frac{\partial^2}{\partial z^2} \right) \pi'_{2z} \, ds$$

$$= -\, 2\pi\alpha_2 \, S_{ij}$$

$$- \, jk_2\alpha_2 \int_{z_t}^{z_b} \left[\frac{x_i - x'}{r} K_1(jk_2 r) \right] \Big|_{x_\ell}^{x_r} dz' \qquad \text{(B.29)}$$

$$- \, \alpha_2 \int_0^\infty \frac{k_x}{u_u^2} \, {}^2R_{TM} \, e^{-u_2 z_i} \left[e^{-u_2 z'} \sin k_x(x_i - x') \right] \Big|_{x_\ell}^{x_r} \Big|_{z_t}^{z_b} dk_x \, .$$

2. Green's functions on the surface

Rewriting π'_{1x} and π'_{1z} from equations (50) and (51)

$$\pi'_{1x} = \alpha_1 \int_0^\infty {}^1R_{TM} \frac{u_2}{u_1^2} \left(e^{u_1 z} + e^{-u_1 z} \right) e^{-u_2 z'} \cos k_x(x - x') \, dk_x \qquad \text{(B.30)}$$

$$\pi'_{1z} = \alpha_1 \int_0^\infty {}^1R_{TM} \frac{1}{u_1} \left(e^{u_1 z} - e^{-u_1 z} \right) e^{-u_2 z'} \cos k_x(x - x') \, dk_x \qquad \text{(B.31)}$$

where

$$\alpha_1 = -\frac{j\omega\mu}{2\pi}\frac{1}{k_1^2}$$

$$^1R_{TM} = \frac{e^{u_2 d}\ \text{sech}\ u_1 d}{\frac{k_2^2}{k_1^2} + \frac{u_2}{u_1}\tanh u_1 d} .$$

Integrations of Green's functions at the kth field point ($x_k, z_k = 0$) are found to be

$$\Gamma^E_{xxkj}\Big|_{z_k=0} = -\iint_{S_j'} \frac{\partial^2}{\partial z^2}\pi'_{1x}\, ds \tag{B.32}$$

$$= -2\alpha_1 \int_0^\infty \frac{^1R_{TM}}{k_x}\left[e^{-u_2 z'}\sin k_x(x_k - x')\right]\Big|_{x_\ell}^{x_r}\Big|_{z_t}^{z_b} dk_x$$

$$\Gamma^E_{xzkj}\Big|_{z_k=0} = \iint_{S_j'} \frac{\partial^2}{\partial x \partial z}\pi'_{1z}\, ds \tag{B.33}$$

$$= 2\alpha_1 \int_0^\infty \frac{^1R_{TM}}{u_2}\left[e^{-u_2 z'}\cos k_x(x_k - x')\right]\Big|_{x_\ell}^{x_r}\Big|_{z_t}^{z_b} dk_x$$

$$\Gamma^E_{zxkj}\Big|_{z_k=0} = \Gamma^E_{zzkj}\Big|_{z_k=0} = 0 \tag{B.34}$$

$$\Gamma^H_{yxkj}\Big|_{z_k=0} = \Gamma^H_{yzkj}\Big|_{z_k=0} = 0 . \tag{B.35}$$

The last two results for the vertical (z) electric field and the horizontal (y) magnetic field are consistent with those concluded for the homogeneous half space case.

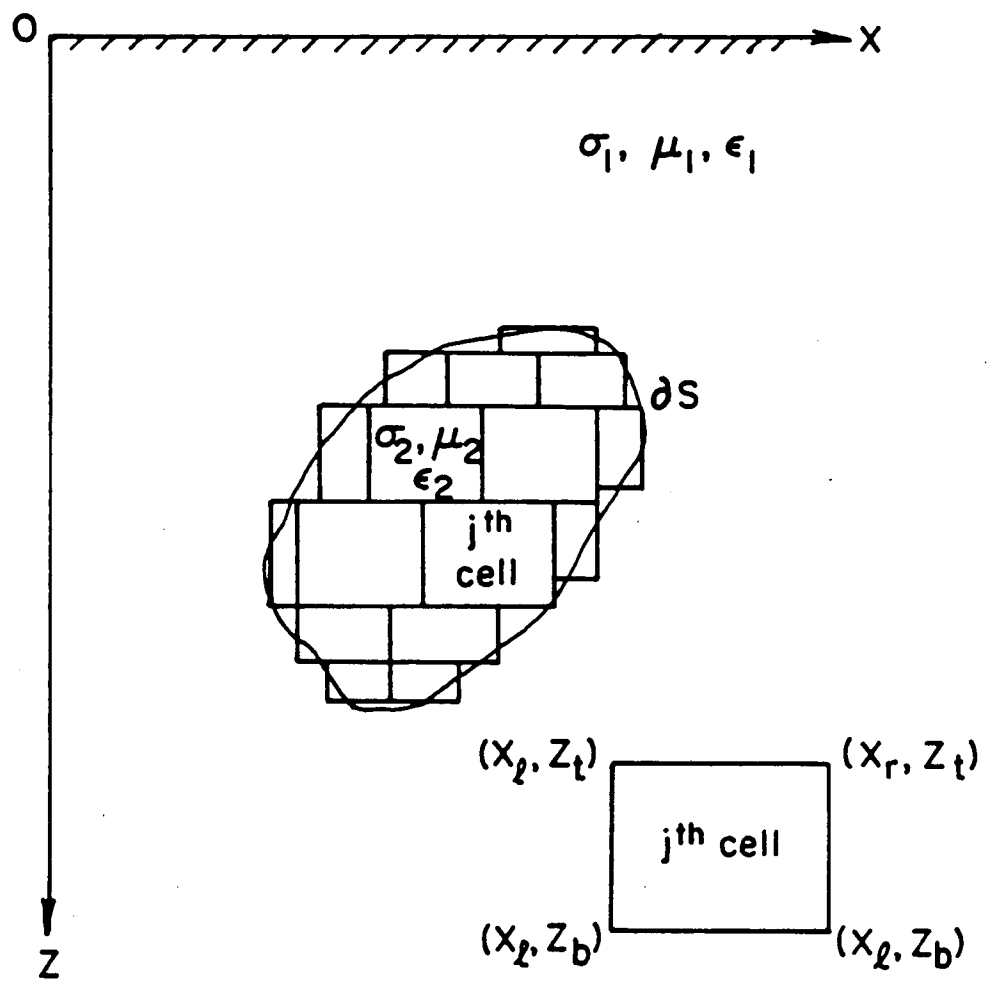

Figure 1. A simulation of a two-dimensional inhomogeneity by N rectangular cells.

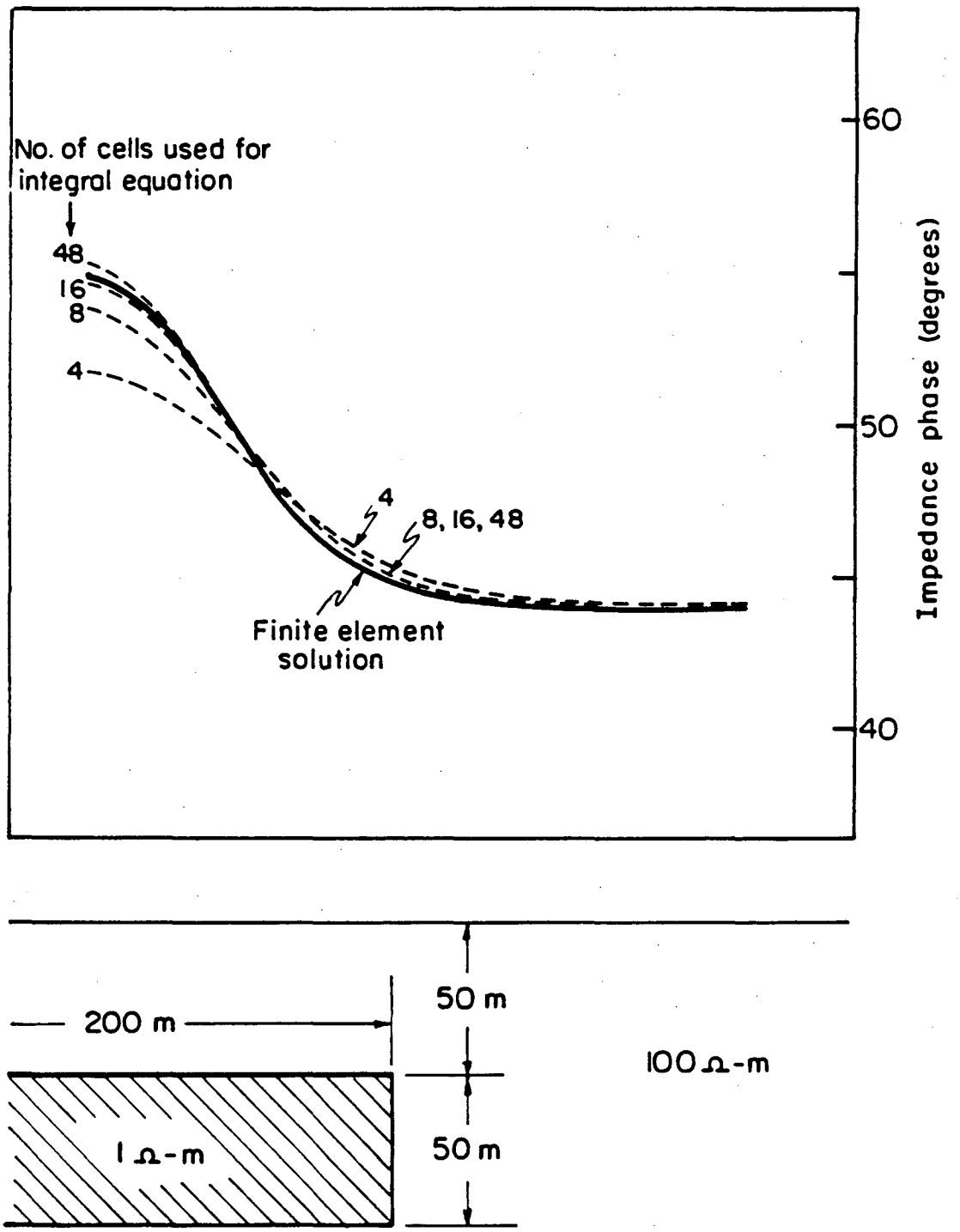

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Figure 2. Convergence test for the integral equation numerical solution.

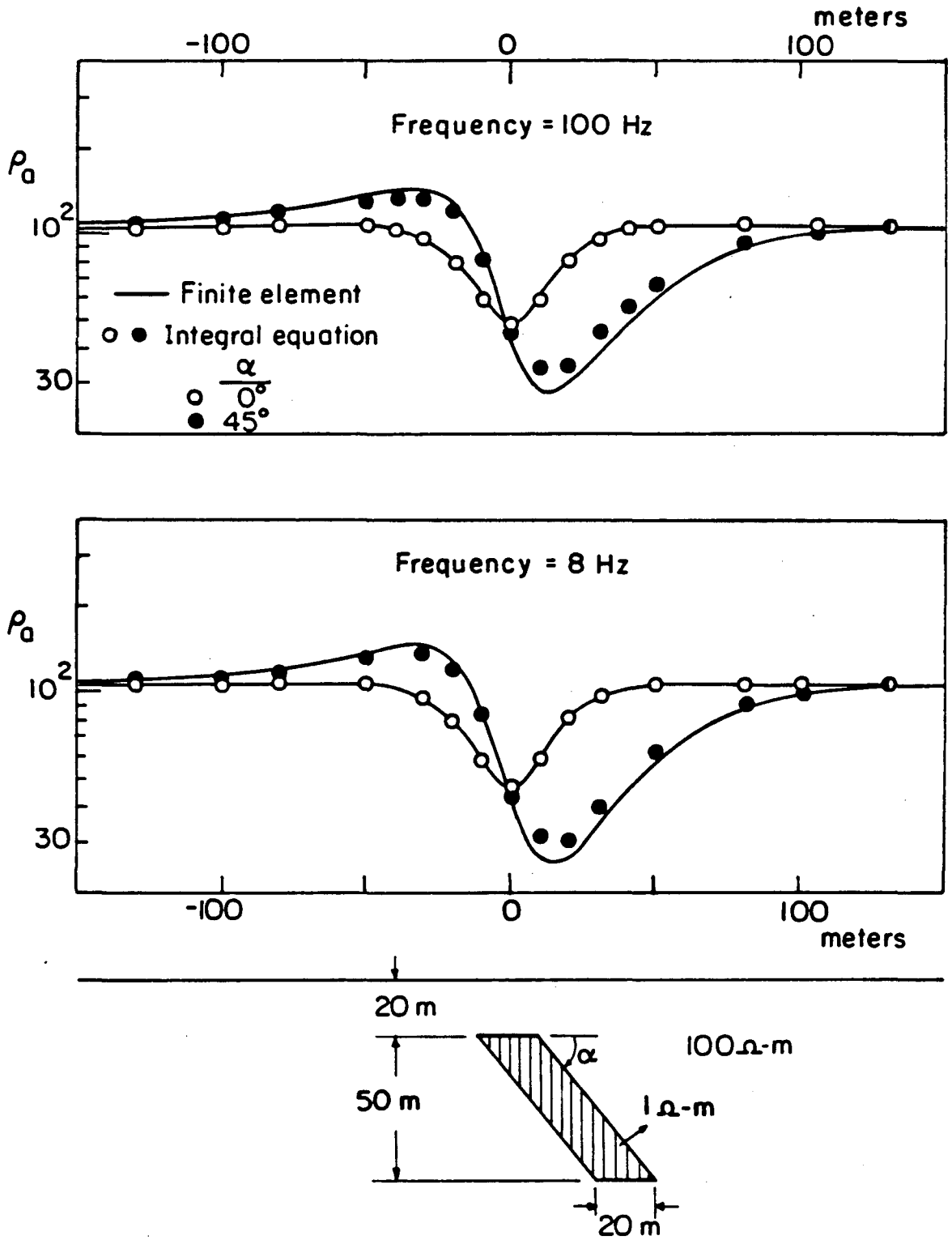

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Figure 3. Comparison between solutions for a dipping conductor.

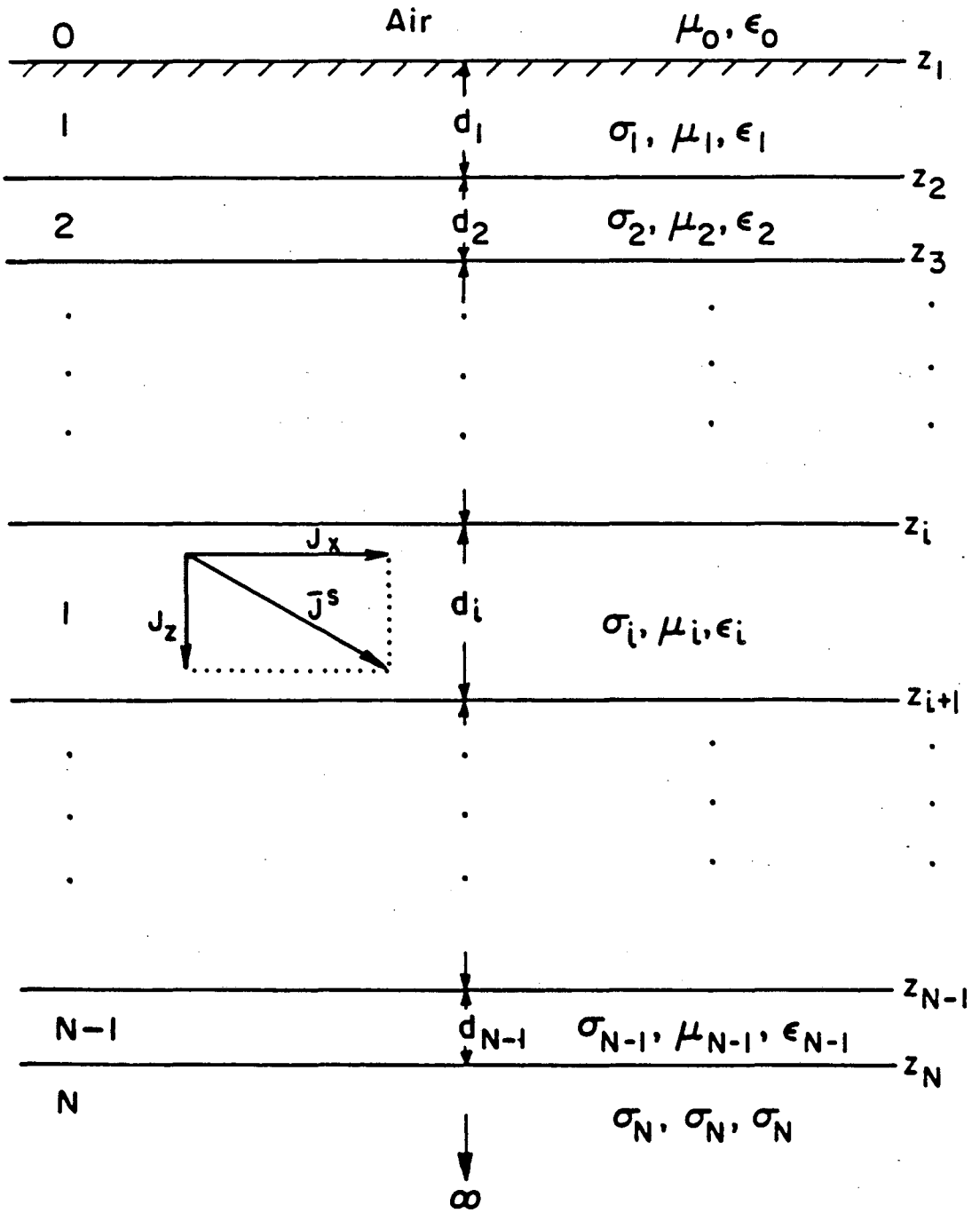

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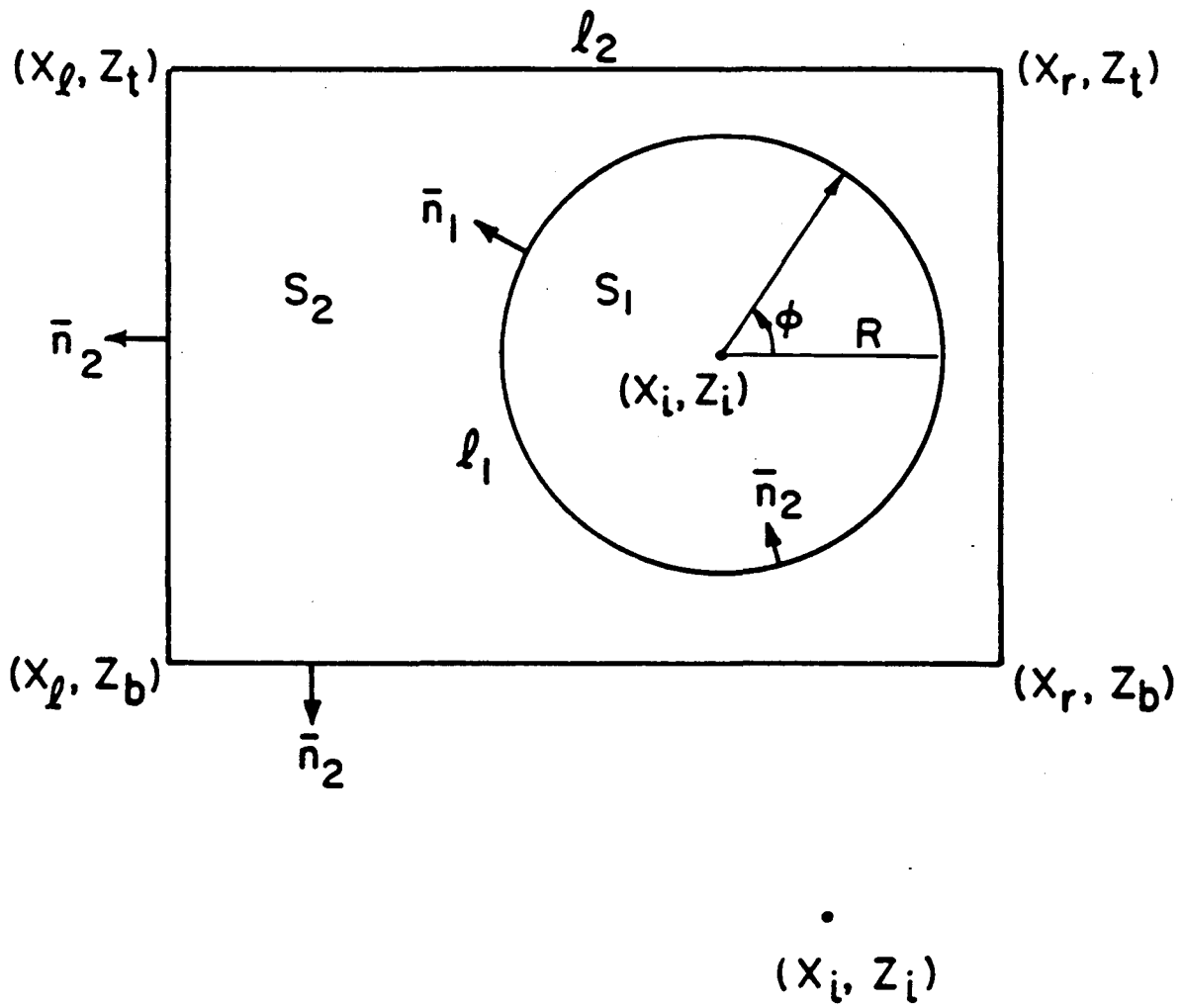
Figure A.1. An N-layered earth with the scattering current $\bar{J}^s$, in the ith layer.



XBL 845-1798

Figure B.1. A rectangular current cell.

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