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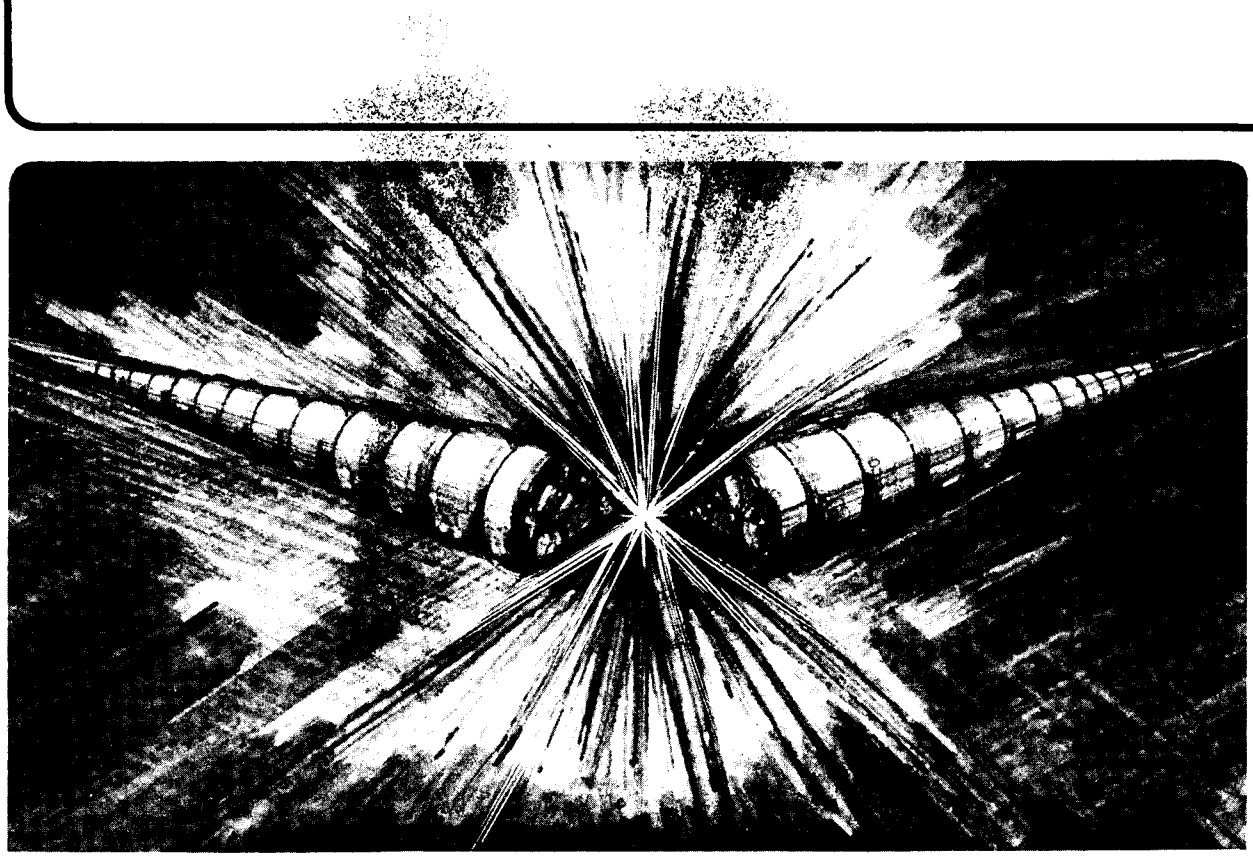

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RELATIVISTIC PONDEROMOTIVE HAMILTONIAN*

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ABSTRACT

The relativistic ponderomotive hamiltonian is derived under as general conditions as possible. Arbitrary $k\rho$, $E_{\parallel}/E_{\perp}$, ω/Ω, $v/c < 1$, wave polarization, spatial modulation of the wave, and nonuniformities in the background electric and magnetic fields are introduced in a systematic way. This calculation is a modification of guiding center theory, because in addition to averaging over gyration, there is also an averaging over rapid oscillations of the wave. Therefore, as a by-product of our objective, we derive a new formulation of the relativistic guiding center motion.

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I. INTRODUCTION

The study of plasma stabilization by a population of energetic electrons is increasing in theoretical interest[1] due to the experimental advances in both open and closed magnetic field configurations. In present fusion devices, such as the ELMO bumpy torus and tandem mirrors, hot electrons are produced mainly by electron cyclotron heating, for which the electron energy is already reaching 1 MeV ($\gamma > 2$). Thus, heating studies must be done relativistically.

There are many processes in plasmas that involve the concept of a ponderomotive force or potential, which represents the effect of high frequency fields on the slow plasma motion. Our derivation produces general expressions for the ponderomotive Hamiltonian that applies to relativistic processes such as found in ELMO bumpy torus rings, thermal barriers in tandem mirror, CO_2 laser fusion plasmas in Helios, free electron lasers, and so on. We find that relativistic effects introduce new terms in the expression for the ponderomotive Hamiltonian (see Ref. 2 for the nonrelativistic expression) and, at the same time, reduce the magnitude of terms already existent in the nonrelativistic case.

In the following analysis we attempt to be as general as possible. We allow for arbitrary $k\rho$, $E_{\parallel}/E_{\perp}$, ω/Ω, $v/c < 1$, polarization, and spatially modulated wave. Nonuniformities in the background fields are introduced in a systematic way. To solve this problem, we use a Hamiltonian approach, because it would be virtually impossible by any other. Furthermore, the Hamiltonian formulation has the advantage of expressing the vector evolution equations in terms of a single scalar function on phase space. As plasma problems become more complex, this advantage is all the greater.

This calculation is a modification of guiding center theory, because in addition to averaging over gyration, there is also an averaging over rapid oscillations[2-4] due to the wave. For example, (i) the usual parallel acceleration $\dot{V}_{\parallel}$ of the guiding center is modified by the wave (this is the usual idea of a ponderomotive force); (ii) the usual magnetic moment μ is no longer constant, it must be replaced by $\bar{\mu} = \mu +$ (wave terms); (iii) the usual drifts are modified and now include wave terms.

In Sec. II, before studying the nonlinear response of a relativistic particle to an electromagnetic wave, we derive the relativistic guiding center motion. The Hamiltonian formalism of the guiding center motion provides us with an unperturbed problem to which a time dependent electromagnetic wave is then added as a perturbation. In Sec. III, we introduce the electromagnetic wave and express the Hamiltonian in terms of the relativistic guiding center variables. In Sec. IV, we obtain the ponderomotive Hamiltonian by subjecting the Hamiltonian system to an averaging tranformation using Lie transforms. We show that the resulting Hamiltonian is indeed gauge invariant. In Sec. V we give a discussion of our results which is independent of the derivation in preceding sections, and which may be read by those who are not interested in the details of the derivation. In this section we apply our results to a calculation of the shift in the turning point of mirroring particles caused by the ponderomotive effects. Finally, in Sec. VI, we present some simplifying limits which might be useful in specific applications.

II. RELATIVISTIC GUIDING CENTER MOTION

Before we analyze the ponderomotive effects of a wave, we must first study the motion of a particle when there is no wave. This simpler system will form our "unperturbed system". In this section we will use unsubscripted variables to describe the unperturbed system, such as $\underset{\sim}{E}$, $\underset{\sim}{B}$; later, when we need to distinguish the background from the perturbation, we will append the subscript 0.

The unperturbed system is merely a matter of guiding center motion in the background fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$, but here we require both a relativistic and a Hamiltonian description of that motion. Relativistic guiding center motion has previously been studied by Vandervoort[5] and reviewed by Northrop,[6] but in a non-Hamiltonian context. Thus, Vandervoort failed to present conservation laws for energy or phase volume.

On the other hand, a Hamiltonian description of nonrelativistic guiding center motion has been given by one of us (RGL).[7] It turns out that it is quite easy to generalize this work to relativistic motion, and that the algebra (to lowest order) is almost identical to the nonrelativistic case. The only differences are that one must use the relativistic definition of the momentum (with the factor of γ), and that one must use the relativistic expression for the energy as the Hamiltonian.

In a systematic theory such as the one we will present here, it helps to have an explicit ordering scheme. We let ε be the dimensionless parameter which indicates the order of various terms in the guiding center expansion. We attempt to use, as much as possible, only symbols which are $O(1)$, so that the order of a term will be indicated by the factor of ε^n which explicitly appears with it. Physical formulas result by setting $\varepsilon = 1$.

The parameter ε is introduced into physical formulas in several steps. In the first step, we understand the guiding center approximation to mean physically that the particle motion is dominated by the effects of the magnetic field, i.e., that B is "large" (in a sense that can be made more precise). In accordance with this, we replace $\underset{\sim}{B}$ whenever it occurs by $\underset{\sim}{B}/\varepsilon$. (Actually, we will replace $\underset{\sim}{A}$ by $\underset{\sim}{A}/\varepsilon$, since in a Hamiltonian theory we must deal with potentials.) As a result of this step, the gyroradius $\rho = \gamma mcv_{\perp}/eB$ is replaced by $\varepsilon\rho$, showing that the gyroradius is $O(\varepsilon)$. Similarly, the gyrofrequency becomes $O(1/\varepsilon)$, representing a fast time scale.

Unlike $\underset{\sim}{B}$, we do nothing with $\underset{\sim}{E}$ (or ϕ). Physically, this means that $E/B = O(\varepsilon)$, or that $\underset{\sim}{E}\times\underset{\sim}{B}$ drifts are of the same order as ∇B and curvature drifts. It would be possible to analyze stronger electric fields, but the case we are considering is most common in practice.

Finally, we allow $\underset{\sim}{E}$ and $\underset{\sim}{B}$ to be slow functions of time, i.e., functions of $\tau = \varepsilon t$. Physically, this means that $\underset{\sim}{E}$ and $\underset{\sim}{B}$ can change appreciably on a drift time scale.

We begin with the relativistic Lagrangian of a particle in configuration space,

$$L = -\frac{mc^2}{\gamma} + \frac{e}{\varepsilon c}\,\underset{\sim}{v}\cdot\underset{\sim}{A} - e\phi \ , \tag{1}$$

in which ε has been appropriately introduced. Here γ is the usual relativistic quantity,

$$\gamma = (1 - v^2/c^2)^{-1/2} \ . \tag{2}$$

The canonical momentum is given by

$$\underset{\sim}{p} = \frac{e}{\varepsilon c} \underset{\sim}{A} + m\underset{\sim}{u} \ , \tag{3}$$

in which $\underset{\sim}{u} = \gamma \underset{\sim}{v}$ is the world velocity (velocity with respect to proper time). In the usual way in mechanics, the Hamiltonian H may be derived from L. The result is

$$H = mc^2\gamma + e\phi = [m^2c^4 + (c\underset{\sim}{p} - \frac{e}{\varepsilon} \underset{\sim}{A})^2]^{1/2} + e\phi \ . \tag{4}$$

The theory of phase space Lagrangians is presented in Ref. 8 and applied to nonrelativistic guiding center motion in Ref. 7. Applying the same kind of analysis here, we find the phase space Lagrangian, $L = \underset{\sim}{p} \cdot \dot{\underset{\sim}{x}} - H$, in the following form:

$$L(\underset{\sim}{x}, \underset{\sim}{u}, \dot{\underset{\sim}{x}}) = (m\underset{\sim}{u} + \frac{e}{\varepsilon c} \underset{\sim}{A}) \cdot \dot{\underset{\sim}{x}} - H \ . \tag{5}$$

This is identical in form to the phase space Lagrangian for a nonrelativistic particle [see Eq. (18) of Ref. 7], except that the world velocity $\underset{\sim}{u}$ has replaced the ordinary velocity $\underset{\sim}{v}$, and the Hamiltonian H is the relativistic version of the energy. Therefore the transformation to guiding center variables is algebraically identical to the corresponding transformation in the nonrelativistic case, and only the interpretation of the symbols is different. For this reason, we will skip the algebraic details of the transformation, which may be found in Ref. 7, and merely summarize the results.

We let $\hat{b}$ be the unit vector along $\underset{\sim}{B}$, and we decompose $\underset{\sim}{u}$ into its

components $u_\parallel$ and $u_\perp$. We define perpendicular unit vectors $\hat{a}$ and $\hat{c}$, rotating with the gyration of the particle, by $\underset{\sim}{u} = u_\parallel \hat{b} + u_\perp \hat{c}$, and $\hat{a} = \hat{b} \times \hat{c}$. The gyrophase θ is defined implicitly by

$$\hat{a} = \cos\theta\, \hat{\tau}_1 - \sin\theta\, \hat{\tau}_2 \,,$$
$$\hat{c} = -\sin\theta\, \hat{\tau}_1 - \cos\theta\, \hat{\tau}_2 \,, \tag{6}$$

in which $\hat{\tau}_1$ and $\hat{\tau}_2$ are perpendicular unit vectors which do not rotate with the particle, and which satisfy $\hat{\tau}_1 \times \hat{\tau}_2 = \hat{b}$. $\hat{\tau}_1$ and $\hat{\tau}_2$ may have a slow dependence on time, since $\underset{\sim}{B}$ itself does.

The guiding center variables are $\underset{\sim}{X}$, the guiding center position; $U_\parallel$, the parallel world velocity of the guiding center; μ, the magnetic moment; $U_\perp$, a variable which is essentially the gyroaverage of $u_\perp$, and which we define to all orders in ε by

$$\mu = \frac{mU_\perp^2}{2B(\underset{\sim}{X}, \varepsilon t)} \,; \tag{7}$$

and Θ, the gyroaveraged gyrophase. The guiding center variables are functions of the particle variables $(\underset{\sim}{x}, u_\parallel, u_\perp, \theta)$, where θ is the instantaneous gyrophase. Explicitly, we have

$$\underset{\sim}{X} = \underset{\sim}{x} - \varepsilon \frac{mcu_\perp}{eB} \hat{a} + O(\varepsilon^2) \,,$$

$$U_\parallel = u_\parallel + O(\varepsilon) \,,$$

$$\mu = \frac{mu_\perp^2}{2B} + O(\varepsilon) \,,$$

$$\textcircled{H} = \theta + O(\varepsilon) \ . \tag{8}$$

All fields on the right-hand sides of Eqs. (8) are evaluated at $(\underset{\sim}{x}, \varepsilon t)$. The higher order correction terms may also be computed, but we omit them here. Note that to lowest order the variables $(u_{\parallel}, \theta)$ are identical to $(U_{\parallel}, \textcircled{H})$.

When the Hamiltonian is transformed to guiding center variables, the result is

$$H = mc^2\Gamma + e\phi + O(\varepsilon) \ . \tag{9}$$

The quantity Γ is the first term in an expansion of γ in terms of guiding center variables, so that $\gamma = \Gamma + O(\varepsilon)$. Γ is a function of $(\underset{\sim}{X}, \theta_{\parallel}, \mu)$, given explicitly by

$$\Gamma = \left(1 + \frac{U_{\parallel}^2}{c^2} + \frac{2\mu B}{mc^2}\right)^{1/2} \ . \tag{10}$$

The fields ϕ and B in Eqs. (9) and (10) are evaluated at $(\underset{\sim}{X}, \varepsilon t)$.

As in the nonrelativistic motion, it is convenient to express the phase space Lagrangian and the equations of motion in terms of the modified fields $\underset{\sim}{A}^*$, $\underset{\sim}{B}^*$, and $\underset{\sim}{E}^*$. These are defined by

$$\underset{\sim}{A}^* = \underset{\sim}{A} + \frac{\varepsilon mc}{e} U_{\parallel} \hat{b} \ ,$$

$$\underset{\sim}{B}^* = \underset{\sim}{B} + \frac{\varepsilon mc}{e} U_{\parallel} \nabla \times \hat{b} \ ,$$

$$\underset{\sim}{E}^* = \underset{\sim}{E} - \frac{\varepsilon m}{e} U_{\parallel} \frac{\partial \hat{b}}{\partial \tau} \ . \tag{11}$$

In terms of the modified fields, the guiding center Lagrangian is given by

$$L = \frac{e}{\varepsilon c} \underset{\sim}{A}^{*} \cdot \dot{\underset{\sim}{X}} + \frac{\varepsilon mc}{e} \mu \dot{\textcircled{H}} - (e\phi + mc^2\Gamma) \ . \tag{12}$$

This may be compared to its nonrelativistic counterpart, Eq. (2) of Ref. 7.

The equations of motion (the drift equations) are given implicitly by

$$\delta \int L dt = 0 \ , \tag{13}$$

or explicitly by

$$\dot{\mu} = 0 \ , \tag{14a}$$

$$\dot{\textcircled{H}} = \frac{eB}{\varepsilon \Gamma mc} \ , \tag{14b}$$

$$m\dot{U}_{\parallel} = \frac{1}{B_{\parallel}^{*}} \underset{\sim}{B}^{*} \cdot [e\underset{\sim}{E}^{*} - \frac{\mu \nabla B}{\Gamma}] \ , \tag{14c}$$

$$\dot{\underset{\sim}{X}} = \frac{1}{B_{\parallel}^{*}} [\frac{U_{\parallel} \underset{\sim}{B}^{*}}{\Gamma} + \varepsilon \hat{b} \times (-c\underset{\sim}{E}^{*} + \frac{c}{e} \frac{\mu \nabla B}{\Gamma})] \ , \tag{14d}$$

in which $B_{\parallel}^{*} = \hat{b} \cdot \underset{\sim}{B}^{*}$. Equation (14a) shows that μ is a (formal) constant of the motion, and (14b) shows that the relativistic gyrofrequency is $\Omega = eB/\Gamma mc$. Equation (14c) gives the parallel force on the guiding center, and Eq. (14d) shows the various drifts, as well as the fact that $U_{\parallel} = \Gamma \hat{b} \cdot \dot{\underset{\sim}{X}}$. The denominators $B_{\parallel}^{*}$ in Eqs. (14c) and (14d) are ordered in ε, since $B_{\parallel}^{*} = B + (\varepsilon mc/e)U_{\parallel}(\hat{b} \cdot \nabla \times \hat{b})$. If these denominators are expanded in powers of ε, it is easy to produce the relativistic drift equations as

presented by Northrop.[6]

As they stand, however, Eqs. (14) possess exact conservation laws which their expanded counterparts do not possess. For example, the energy of the particle varies according to the equation,

$$\frac{d}{dt}(e\phi + mc^2\Gamma) = e\frac{\partial\phi}{\partial t} + \frac{\mu}{\Gamma}\frac{\partial B}{\partial t} - \frac{e}{c}\dot{\underset{\sim}{X}} \cdot \frac{\partial \underset{\sim}{A}^*}{\partial t}, \tag{15}$$

which shows that energy is conserved in static fields. Similarly, one can show that angular momentum is exactly conserved in azimuthally symmetric fields, and that phase volume is exactly conserved in all cases.

Finally, we may derive the Poisson brackets of the guiding center variables among themselves, as was done in Ref. 7 for the nonrelativistic motion. There is no work to be done here, because the results are formally identical to the nonrelativistic case [with ordinary velocities replaced by world velocities; see Eq. (43) of Ref. 7]. The results are

$$\{X_i, X_j\} = -\frac{\varepsilon c}{e}\frac{b_{ij}}{B_\parallel^*},$$

$$\{\underset{\sim}{X}, U_\parallel\} = \frac{1}{mB_\parallel^*}\underset{\sim}{B}^*,$$

$$\{\textcircled{H}, \mu\} = \frac{e}{\varepsilon mc}, \tag{16}$$

where b_{ij} is the tensor dual to the unit vector $\hat{b}$, i.e., $b_{ij}V_j = -(\hat{b} \times \underset{\sim}{V})_i$ for any vector $\underset{\sim}{V}$.

These brackets can be combined into a single formula for the Poisson bracket of two functions f and g, which are expressed in terms of the guiding center variables $(\underset{\sim}{X}, U_\parallel, \mu, \textcircled{H})$. The formula is

$$\{f,g\} = -\frac{\varepsilon c}{eB_{\parallel}^{*}}\,\hat{b}\cdot(\nabla f \times \nabla g) + \frac{1}{mB_{\parallel}^{*}}\,\underset{\sim}{B}^{*}\cdot\left(\nabla f\,\frac{\partial g}{\partial U_{\parallel}} - \nabla g\,\frac{\partial f}{\partial U_{\parallel}}\right)$$

$$+\frac{e}{\varepsilon mc}\left(\frac{\partial f}{\partial \textcircled{H}}\frac{\partial g}{\partial \mu} - \frac{\partial f}{\partial \mu}\frac{\partial g}{\partial \textcircled{H}}\right) . \qquad (17)$$

This will prove useful in the perturbation analysis of the next section.

III. TRANSFORMING THE HAMILTONIAN

We now introduce an electromagnetic wave and study its ponderomotive effects. The wave is represented by the potentials ϕ_1, $\underset{\sim}{A}_1$ in arbitrary gauge, which are assumed to have the eikonal form

$$\phi_1(\underset{\sim}{x},t) = \tilde{\phi}_1 e^{i\psi(\underset{\sim}{x},t)/\varepsilon} + \text{c.c.} \ ,$$

$$\underset{\sim}{A}_1(\underset{\sim}{x},t) = \tilde{\underset{\sim}{A}}_1 e^{i\psi(\underset{\sim}{x},t)/\varepsilon} + \text{c.c.} \ , \tag{18}$$

where the over-tilde represents the slowing varying amplitude of the wave packet. We assume that $\tilde{\phi}_1$ and $\tilde{\underset{\sim}{A}}_1$ depend on $(\underset{\sim}{x},\varepsilon t)$, i.e., that the amplitudes obey the same length and time scales as the background fields $\underset{\sim}{E}_0$ and $\underset{\sim}{B}_0$. This is roughly consistent with a bulk mode of an experimental device, for example.

The local wave number $\underset{\sim}{k}$ and frequency ω are given by

$$\underset{\sim}{k}(\underset{\sim}{x},t) = \nabla\psi(\underset{\sim}{x},t) \ ,$$

$$\omega(\underset{\sim}{x},t) = -\frac{\partial\psi}{\partial t}(\underset{\sim}{x},t) \ . \tag{19}$$

For a wave of fixed frequency ω_0, we can set $\psi(\underset{\sim}{x},t) = \psi_0(\underset{\sim}{x}) - \omega_0 t$, for some purely spatial eikonal function $\psi_0(\underset{\sim}{x})$. The factor of $1/\varepsilon$ in the phase indicates the orderings $k\rho \sim O(1)$ and $\omega/\Omega \sim O(1)$. Once we have obtained the final results, however, we will be free to take the simplifying limits of small k and/or ω. Our aim here is to be as general as possible.

Henceforth we will indicate the background fields with the subscript 0, and the wave field with the subscript 1, as in Eq. (18). The remainder of the calculation is really a double perturbation expansion, the major one being in the wave amplitude, and the secondary one being in the guiding center parameter. To formalize this, we will introduce the dimensionless parameter λ, which indicates the order in the wave amplitude. Thus, we have $\phi = \phi_0 + \lambda\phi_1$ and $\underset{\sim}{A} = \underset{\sim}{A}_0 + \lambda\underset{\sim}{A}_1$ for the total electromagnetic potentials. Physical formulas result by setting $\lambda = 1$. At each order in λ, we expand the solutions in ε. We have just done this, in Sec. II, for the order λ^0; the ponderomotive Hamiltonian is obtained at order λ^2.

Let us return to Eqs. (1)-(5), which describe the exact dynamics of the particle, and let us now use the total fields ϕ and $\underset{\sim}{A}$ (background plus wave). The exact phase space Lagrangian of the particle, corresponding to Eq. (5), is

$$L(\underset{\sim}{x},\underset{\sim}{u},\dot{\underset{\sim}{x}}) = [m\underset{\sim}{u} + \frac{e}{c}(\frac{1}{\varepsilon}\underset{\sim}{A}_0 + \lambda\underset{\sim}{A}_1)] \cdot \dot{\underset{\sim}{x}} - [e(\phi_0 + \lambda\phi_1) + (m^2c^4 + c^2u^2)^{1/2}] , \tag{20}$$

with appropriate factors of ε and λ. The exact equations of particle motion are contained in the variational principle of Eq. (13).

It would be natural to write Eq. (20) in the form $L = L_0 + \lambda L_1$, in which L_0 would be precisely the guiding center system analyzed in Sec. II, and L_1 would represent the perturbation due to the wave. In order to carry out a perturbation analysis of the total system, one would require a perturbation theory for phase space Lagrangians. Such a theory exists; it is described in Refs. 8 and 9, and it is applied to guiding center theory

in Ref. 7. Note that the perturbation λL_1 of Eq. (20) includes a term $\underset{\sim}{A}_1 \cdot \dot{\underset{\sim}{x}}$ in the first major term on the right-hand side, as well as a term $e\phi_1$ in the Hamiltonian. The term $\underset{\sim}{A}_1 \cdot \dot{\underset{\sim}{x}}$ contributes to the symplectic structure, i.e., the structure of Poisson brackets (which in canonical variables would be represented by $\underset{\sim}{p} \cdot \dot{\underset{\sim}{q}}$). Thus, not only is the Hamiltonian perturbed, but also the Poisson bracket structure. The perturbation theory of Refs. 8 and 9 is quite capable of handling this, and indeed the present problem provides nearly an ideal application of that theory.

Nevertheless, the most familiar form of Hamiltonian perturbation theory is the canonical theory,[10] in which the Poisson bracket structure remains form-invariant and only the Hamiltonian is allowed to change. It is possible to bring Eq. (20) into a form in which only the Hamiltonian suffers a perturbation, so that a variant of the more standard Hamiltonian perturbation theory can be applied, instead of the less familiar phase space Lagrangian perturbation theory mentioned in the preceding paragraph. The advantage of this approach is its familiarity; the disadvantage is that it involves nonphysical variables, and that the results are not manifestly gauge invariant.

In order to carry out this alternate approach, we define a velocity-like variable $\underset{\sim}{u}_0$ by

$$m\underset{\sim}{u}_0 = m\underset{\sim}{u} + \frac{\lambda e}{c} \underset{\sim}{A}_1 \ . \tag{21}$$

The new quantity $\underset{\sim}{u}_0$ is not physical, in the sense that it changes when a new gauge is chosen for the wave fields $\phi_1, \underset{\sim}{A}_1$. However, in terms of $\underset{\sim}{u}_0$ the Lagrangian of Eq. (20) becomes

$$L = (m\underset{\sim}{u}_0 + \frac{e}{\varepsilon c}\underset{\sim}{A}_0) \cdot \dot{\underset{\sim}{x}} - \{e(\phi_0 + \lambda\phi_1) + [m^2c^4 + c^2(m\underset{\sim}{u}_0 - \frac{\lambda e}{c}\underset{\sim}{A}_1)^2]^{1/2}\} . \tag{22}$$

Notice that now the perturbation appears solely in the Hamiltonian, and that the perturbed part of Eq. (22) is formally identical to the guiding center system of Sec. II, with the symbols $\phi, \underset{\sim}{A}, \underset{\sim}{u}$ replaced by $\phi_0, \underset{\sim}{A}_0, \underset{\sim}{u}_0$. In a similar notation, we will write γ_0 for the quantity $(1 + u_0^2/c^2)^{1/2}$. Because of Eq. (21), this is not the true γ of the particle, nor is it gauge invariant.

Our approach here is something of a hybrid, being a cross between the canonical theory of Ref. 10 and the phase space Lagrangian theory of Refs. 8 and 9. In canonical theory, one must use the canonical momentum $\underset{\sim}{p}$, which changes under a redefinition of gauge for either the background fields ϕ_0 and $\underset{\sim}{A}_0$ or the wave fields ϕ_1 and $\underset{\sim}{A}_1$. In the phase space Lagrangian theory, one can, if one wishes, use purely physical variables, which are gauge invariant. In the present approach, however, the quantity $\underset{\sim}{u}_0$ is invariant under a change of gauge for the background fields ϕ_0 and $\underset{\sim}{A}_0$, but not for the wave fields ϕ_1 and $\underset{\sim}{A}_1$.

To proceed, we first expand the Hamiltonian of Eq. (22) to second order in λ. The results are

$$H_0 = e\phi_0 + mc^2\gamma_0 , \tag{23a}$$

$$H_1 = e[\phi_1 - \frac{1}{\gamma_0 c}(\underset{\sim}{u}_0 \cdot \underset{\sim}{A}_1)] , \tag{23b}$$

$$H_2 = \frac{e^2}{2\gamma_0 mc^2}\left[\underset{\sim}{A}_1^2 - \frac{1}{\gamma_0^2 c^2}(\underset{\sim}{u}_0 \cdot \underset{\sim}{A}_1)^2\right]$$

$$= \frac{e^2}{2\gamma_0^3\, mc^2}\left[\underset{\sim}{A}_1^2 + \frac{1}{c^2}(\underset{\sim}{u}_0 \times \underset{\sim}{A}_1)^2\right] . \qquad (23c)$$

Next, we transform this Hamiltonian to guiding center variables. The guiding center variables are defined by Eq. (8), except that now we use $\underset{\sim}{u}_0$ on the right-hand side, instead of $\underset{\sim}{u}$. Thus, we are subjecting Eq. (23) to the transformation $(\underset{\sim}{x},\underset{\sim}{u}_0) \rightarrow (\underset{\sim}{X},U_{\parallel},\mu,\textcircled{H})$. The guiding center variables have all the algebraic properties developed in Sec. II, most notably the Poisson bracket relations of Eq. (17). However, their time evolution is now modified by the wave terms of the Hamiltonian. For example, μ is no longer a constant of motion, since it possesses rapid oscillations at the wave frequency ω. In addition, the guiding center variables are now gauge dependent, since they are defined in terms of $\underset{\sim}{u}_0$.

Transforming H_1 and H_2 to guiding center variables is straightforward, although it is here that most of the work of the present calculation lies. The Hamiltonian transforms as a scalar, as Hamiltonians always do under time-independent transformations. This means that we must simply eliminate the particle variables in favor of guiding center variables in Eq. (23). [Actually, the transformation of Eq. (8) does have a slow time dependence, since the fields $\underset{\sim}{E}_0$ and $\underset{\sim}{B}_0$ depend on εt. But this only affects terms in the Hamiltonian which are higher order in ε, which we neglect here.]

We begin by transforming the phase factor of Eq. (18) to guiding center variables. The transformation equation is $\underset{\sim}{x} = \underset{\sim}{X} + \varepsilon\underset{\sim}{\rho} + O(\varepsilon^2)$, with $\underset{\sim}{\rho} = mc\, U_{\perp}\, \hat{a}/e\, B_0$, which gives

$$\exp[i\psi(\underset{\sim}{x},t)/\varepsilon] = \exp[i\psi(\underset{\sim}{X},t)/\varepsilon + i\underset{\sim}{k}\cdot\underset{\sim}{\rho}][1 + O(\varepsilon)] \ . \tag{24}$$

We let α be the angle between $\underset{\sim}{k}_\perp$ (which depends on $\underset{\sim}{x}$) and $\hat{\tau}_1$ (which also depends on $\underset{\sim}{x}$), so that we have

$$\underset{\sim}{k} = k_\parallel \hat{b} + k_\perp(\cos\alpha\ \hat{\tau}_1 + \sin\alpha\ \hat{\tau}_2) \ . \tag{25}$$

Combining these, we obtain the Bessel sum,

$$e^{i\underset{\sim}{k}\cdot\underset{\sim}{\rho}} = \sum_{\ell=-\infty}^{+\infty} e^{i\ell(\alpha+\Theta+\pi/2)} J_\ell(k_\perp\rho) \ , \tag{26}$$

where all appropriate quantities are evaluated at $(\underset{\sim}{X},\varepsilon t)$ for use in Eq. (24).

There is no work to be done to transform H_0, since we have already done this in Sec. II. The result is

$$H_0 = e\phi_0 + mc^2\Gamma \ , \tag{27}$$

with Γ defined in Eq. (10).

As for H_1, it is convenient to adopt the Fourier expansion,

$$H_1(\underset{\sim}{X},U_\parallel,\mu,\Theta,t) = \sum_\ell H_{1\ell}(\underset{\sim}{X},U_\parallel,\mu,\varepsilon t)e^{i\ell(\Theta+\alpha+\pi/2)}\, e^{i\psi(\underset{\sim}{X},t)/\varepsilon} + \text{c.c.} \tag{28}$$

We introduce a new triad of unit vectors, $(\hat{b},\ \hat{k}_\perp,\ \hat{b}\times\hat{k}_\perp)$, in which $\hat{k}_\perp = \cos\alpha\ \hat{\tau}_1 + \sin\alpha\ \hat{\tau}_2$. Using $\underset{\sim}{u}_0 = u_{0\parallel}\hat{b} + u_{0\perp}\hat{c}$ in Eq. (23b) and transforming to guiding center variables, we find the Fourier coefficients in the form

$$H_{1\ell}(\underset{\sim}{X},U_{\parallel},\mu,\varepsilon t) = eJ_{\ell}\tilde{\phi}_1 - \frac{e}{c}\,\underset{\sim}{\tilde{A}}_1\cdot\left[J_{\ell}V_{\parallel}\hat{b} + \frac{\Omega}{k_{\perp}}\left(\ell J_{\ell}\hat{k}_{\perp} + 2i\mu\,\frac{\partial J_{\ell}}{\partial\mu}\,\hat{b}\times\hat{k}_{\perp}\right)\right] , \tag{29}$$

in which $J_{\ell} = J_{\ell}(k_{\perp}\rho)$, $V_{\parallel} = U_{\parallel}/\Gamma$, $\Omega = eB_0/\Gamma mc$, and everything is evaluated at $(\underset{\sim}{X},\varepsilon t)$. There are $O(\varepsilon)$ corrections to Eq. (28) which we neglect.

As for the term H_2, it turns out that we will only need the part of it which is averaged over the rapid gyration and oscillation of the wave. After some algebra, we find the result in the form

$$\bar{H}_2 = \frac{e^2}{\Gamma mc^2}\left[|\underset{\sim}{\tilde{A}}_1|^2 - \frac{1}{\Gamma^2 mc^2}\left(\mu B_0|\underset{\sim}{\tilde{A}}_{1\perp}|^2 + mU_{\parallel}^2|\tilde{A}_{1\parallel}|^2\right)\right]$$

$$= \frac{e^2}{\Gamma^3 mc^2}\left\{|\underset{\sim}{\tilde{A}}_1|^2 + \frac{1}{mc^2}\left[(\mu B_0 + mU_{\parallel}^2)|\underset{\sim}{\tilde{A}}_{1\perp}|^2 + 2\mu B_0|\underset{\sim}{\tilde{A}}_{1\parallel}|^2\right]\right\} , \tag{30}$$

in which the overbar on H_2 indicates the averaged part.

We are now ready for the perturbation analysis.

IV. THE PONDEROMOTIVE HAMILTONIAN

We now subject the Hamiltonian of Eq. (23) to an averaging transformation, based on the Lie transform perturbation theory of Dragt and Finn,[11] as reviewed by Cary.[10] We will denote the averaged variables with an overbar, so that the averaging transformation has the form $(\underset{\sim}{X}, U_{\parallel}, \mu, \textcircled{H}) \rightarrow (\bar{\underset{\sim}{X}}, \bar{U}_{\parallel}, \bar{\mu}, \bar{\textcircled{H}})$. The purpose of the averaging transformation is to remove the rapid oscillations from the Hamiltonian H and to produce thereby the ponderomotive Hamiltonian K.

We denote the Lie generators by W_1, W_2, ..., and the Poisson bracket operators by $L_n = \{W_n, \}$. The averaging transformation T is given in terms of the L_n by

$$T = \ldots \exp(-\lambda^2 L_2)\exp(-\lambda L_1) \ ,$$

$$T^{-1} = \exp(\lambda L_1)\exp(\lambda^2 L_2) \ldots \ , \tag{31}$$

which, by expanding the series, becomes

$$T = I - \lambda L_1 + \lambda^2(-L_2 + \frac{1}{2} L_1^2) + \ldots \ ,$$

$$T^{-1} = I + \lambda L_1 + \lambda^2(L_2 + \frac{1}{2} L_1^2) + \ldots \ . \tag{32}$$

The averaging transformation is $\bar{\underset{\sim}{Z}} = T\underset{\sim}{Z}$ or $\underset{\sim}{Z} = T^{-1}\bar{\underset{\sim}{Z}}$, where $\underset{\sim}{Z}$ and $\bar{\underset{\sim}{Z}}$ represent the old guiding center variables and their averaged counterparts, respectively.

We write $K = K_0 + \lambda K_1 + \lambda^2 K_2 + \ldots$ for the new Hamiltonian. Then the transformation equations, as presented by Cary,[10] are

$$0 = K_0 - H_0 \ , \tag{33a}$$

$$\left(\frac{dW_1}{dt}\right)_0 = K_1 - H_1 \ , \tag{33b}$$

$$\left(\frac{dW_2}{dt}\right)_0 = K_2 - H_2 - \frac{1}{2} L_1(H_1 + K_1) \ . \tag{33c}$$

The notation on the left-hand side indicates the total time derivative along unperturbed orbits.

Traditional applications of Hamiltonian perturbation theory have required canonical variables, and so also has the presentation of Cary. Our variables, however, are noncanonical, as shown by Eqs. (16). Nevertheless, the entire formalism developed for canonical variables also works for noncanonical variables, with only two minor changes.

The first is that one must use the appropriate noncanonical expression for the Poisson bracket, which in our case is given by Eq. (17). This expression is derived from the phase-space Lagrangian, as described in Refs. 7 and 8. The basic reason why it is possible to use the noncanonical bracket is that the Poisson bracket is an object with an invariant geometrical meaning in phase space, independent of the coordinate system employed. Noncanonical variables have now been used in this way for several applications.[2, 12-14]

The second change concerns the use of time-dependent transformations and time-dependent Hamiltonians, such as we have here. It is possible to deal with an explicit time-dependence by introducing time and energy as conjugate variables in an extended phase space with one extra degree of freedom, as has been done in several applications.[2 13 14] However, it

turns out that Eqs. (33) are valid as they stand, even for time-dependent, noncanonical transformations, so that the artifice of adding extra variables is not necessary. This is proved in Refs. 7 and 8, but it is plausible in any case, since the total time derivative, like the Poisson bracket, has an invariant geometrical meaning. Nevertheless, it is important to use the correct formula for the total time derivative. In our case, the total time derivative of a scalar W_n along unperturbed orbits is given by

$$\left(\frac{dW_n}{dt}\right)_0 = \frac{\partial W_n}{\partial t} + \dot{\underset{\sim}{X}}\cdot\nabla W_n + \dot{U}_{\parallel}\frac{\partial W_n}{\partial U_{\parallel}} + \dot{\textcircled{H}}\,\frac{\partial W_n}{\partial \textcircled{H}}, \tag{34}$$

where $\dot{\underset{\sim}{X}}$, $\dot{U}_{\parallel}$, and $\dot{\textcircled{H}}$ are given by Eqs. (14). The point is that this is not the same as

$$\frac{\partial W_n}{\partial t} + \{W_n, H_0\}\ , \tag{35}$$

which is the correct expression for $(dW_n/dt)_0$ only when the Poisson bracket structure is time-independent. This distinction is explained in Refs. 6-9. In our case, the time dependence of the Poisson brackets is slow anyway, so the difference between expressions (34) and (35) is of higher order in ε than we shall need. Nevertheless, we are explaining the distinction, both for future reference and in order to be careful about our presentation.

We begin by analyzing Eq. (33b). The term H_1 is given by Eqs. (28)-(29). As usual, we demand that the generators W_n be purely oscillatory, in order to avoid secular terms. Then on taking the average of Eq. (33b) we find $K_1 = 0$, since $\bar{H}_1 = 0$. The physical meaning of this is that the

wave causes only rapid oscillations at first order, with average of zero. A nonzero average appears only at second order.

On taking the oscillatory part of Eq. (33b), we obtain an equation for W_1, namely,

$$\left(\frac{dW_1}{dt}\right)_0 = - \sum_{\ell} H_{1\ell}\, e^{i\ell(\Theta+\alpha+\pi/2)}\, e^{i\psi(\underset{\sim}{X},t)/\varepsilon} + \text{c.c.} \tag{36}$$

We solve this by positing the ansatz,

$$W_1 = \varepsilon \sum_{\ell} W_{1\ell}\, e^{i\ell(\Theta+\alpha+\pi/2)}\, e^{i\psi(\underset{\sim}{X},t)/\varepsilon} + \text{c.c.} \tag{37}$$

The factor of ε is only a matter of convenience. We use Eqs. (34) and (14), and we keep the leading order term in ε of Eq. (36). The result is

$$W_1 = \varepsilon \sum_{\ell} \frac{-i\, H_{1\ell}\, e^{i\ell(\Theta+\alpha+\pi/2)}\, e^{i\psi(\underset{\sim}{X},t)/\varepsilon}}{\omega - k_{\parallel}V_{\parallel} - \ell\Omega} + \text{c.c.} \tag{38}$$

We move on to Eq. (33c), and again take the averaged part. This gives the ponderomotive Hamiltonian in the form

$$K_2 = \bar{H}_2 + \frac{1}{2}\,\overline{\{W_1,H_1\}}\ . \tag{39}$$

The first term, which we evaluated and displayed in Eq. (30), represents a simple average over rapid oscillations, whereas the second term represents the beating of two first order terms. Both contribute to the ponderomotive Hamiltonian.

It is straightforward to evaluate the second term, using the Poisson bracket formula of Eq. (17). To lowest order in ε, we find

$$K_2 = \bar{H}_2 + \sum_{\ell} \left(\frac{k_{\parallel}}{m}\frac{\partial}{\partial \bar{U}_{\parallel}} + \frac{e\ell}{mc}\frac{\partial}{\partial \bar{\mu}}\right)\left(\frac{|H_{1\ell}|^2}{\omega - k_{\parallel}\bar{V}_{\parallel} - \ell\Omega}\right)\ . \tag{40}$$

The ponderomotive Hamiltonian is a function of the averaged (overbarred) variables, as we have indicated. (The terms $\bar{H}_2$, $H_{1\ell}$, Ω, etc. are all now evaluated at the overbarred variables.) In the nonrelativistic limit, Eq. (40) reduces to Eq. (5) of Ref. 2. If in addition we assume a uniform magnetic field, it reduces to Eq. (91) of Ref. 4.

We will discuss this result presently, but first let us consider the averaged variables. As is usual in Lie transform theory, one need not derive explicit expressions for these if they are not needed; one can obtain considerable information from the Hamiltonian alone, and often this is sufficient. (In non-Hamiltonian approaches, however, one must carry around the averaging transformation in explicit form, whether it is needed or not.) For all six phase space coordinates, the averaging transformation is contained implicitly in the Lie generators W_n, through Eq. (32) and the formula $\bar{\underset{\sim}{Z}} = T\underset{\sim}{Z}$.

For example, for the magnetic moment we have $\bar{\mu} = \mu - \lambda\{W_1, \mu\} + O(\lambda^2)$, or, explicitly,

$$\bar{\mu} = \mu - \frac{\lambda e}{mc}\left[\sum_{\ell} \frac{\ell\, H_{1\ell}\,(\underset{\sim}{X}, U_{\parallel}, \mu, t)\, \exp(i\,\Phi_{\ell})}{\omega - k_{\parallel}V_{\parallel} - \ell\Omega} + \text{c.c.}\right] + O(\lambda^2)\ , \tag{41}$$

where we make the abbreviation

$$\Phi_{\ell} = \ell[\,\textcircled{H} + \alpha(\underset{\sim}{X}, t) + \pi/2] + \psi(\underset{\sim}{X}, t)/\varepsilon\ . \tag{42}$$

The $O(\lambda)$ term of Eq. (41) removes the rapid oscillations in μ due to the wave and produces the variable $\bar{\mu}$, which is a constant of the motion. ($\bar{\mu}$ is constant because the averaged Hamiltonian is independent of $\bar{\textcircled{H}}$.)

Similarly, we find the following formulas for the averaged variables $\bar{U}_\parallel$ and $\bar{\underset{\sim}{X}}$:

$$\bar{U}_\parallel = U_\parallel - \frac{\lambda k_\parallel}{m}\left[\sum_\ell \frac{H_{1\ell}\exp(i\Phi_\ell)}{\omega - k_\parallel V_\parallel - \ell\Omega} + \text{c.c.}\right] + O(\lambda^2) \ , \tag{43}$$

$$\bar{\underset{\sim}{X}} = \underset{\sim}{X} + \varepsilon\lambda\left[\sum_\ell \left(\frac{c}{e\,B_0}\hat{b} \times \underset{\sim}{k} - \frac{i}{m}\frac{\partial}{\partial U_\parallel}\right)\left(\frac{H_{1\ell}\exp(i\Phi_\ell)}{\omega - k_\parallel V_\parallel - \ell\Omega}\right) + \text{c.c.}\right] + O(\lambda^2) \ . \tag{44}$$

A similar formula exists for $\bar{\textcircled{H}}$. Unlike $\bar{\mu}$, the averaged variables $\bar{\underset{\sim}{X}}$ and $\bar{U}_\parallel$ are not constants of the motion. But they are free of rapid oscillation due to the wave (as well as due to the gyromotion). Thus, $\bar{\underset{\sim}{X}}$ is a kind of combined guiding center and oscillation center coordinate. The variables $\bar{\underset{\sim}{X}}$ and $\bar{U}_\parallel$ evolve according to a set of drift equations, which are like the ordinary drift equations, but with ponderomotive effects included.

We have mentioned that the variables $(\underset{\sim}{X}, U_\parallel, \mu, \textcircled{H})$ are not gauge invariant. Nevertheless, the averaged variable $\bar{\mu}$ certainly must be gauge invariant, because it is constant. Actually, all the averaged variables $(\bar{X}, \bar{U}_\parallel, \bar{\mu}, \bar{\textcircled{H}})$ are gauge invariant, because otherwise, a gauge transformation could introduce rapid oscillations at the wave frequency. In the appendix we show how this gauge invariance can be demonstrated explicitly.

Since K_2 is gauge invariant, we can express it in terms of the physical wave fields $\underset{\sim}{E}_1$ and $\underset{\sim}{B}_1$. To lowest order in ε, these have amplitudes $\tilde{\underset{\sim}{E}}_1 = (i\omega/c)\tilde{\underset{\sim}{A}}_1 - i\underset{\sim}{k}\tilde{\phi}_1$ and $\tilde{\underset{\sim}{B}}_1 = i\underset{\sim}{k} \times \tilde{\underset{\sim}{A}}_1$. An easy way to express K_2 in terms of physical fields is to use radiation gauge, in which $\tilde{\phi}_1 = 0$. The result is

$$K_2 = \frac{e^2}{\Gamma m\omega^2}\left[|\tilde{\underset{\sim}{E}}_1|^2 - \frac{1}{\Gamma^2 mc^2}\left(\bar{\mu}B_0|\tilde{\underset{\sim}{E}}_{1\perp}|^2 + m\bar{U}_\parallel^2|\tilde{E}_{1\parallel}|^2\right)\right]$$

$$+ \frac{e^2}{\omega^2}\sum_\ell \left(\frac{k_\parallel}{m}\frac{\partial}{\partial\bar{U}_\parallel} + \frac{e\ell}{mc}\frac{\partial}{\partial\bar{\mu}}\right)\left(\frac{|Q_\ell|^2}{\omega - k_\parallel\bar{V}_\parallel - \ell\Omega}\right) , \tag{45}$$

where

$$Q_\ell = \tilde{\underset{\sim}{E}}_1 \cdot \left[J_\ell\bar{V}_\parallel\hat{b} + \frac{\Omega}{k_\perp}\left(\ell J_\ell\hat{k}_\perp + 2i\bar{\mu}\frac{\partial J_\ell}{\partial\bar{\mu}}\hat{b}\times\hat{k}_\perp\right)\right] , \tag{46}$$

and where everything is evaluated at $(\bar{\underset{\sim}{X}},\bar{U}_\parallel,\bar{\mu})$.

Another form of this is useful. We use partial fractions to clear the term in Q_ℓ which is linear in ℓ, and which degrades the convergence of the series. We also use Faraday's law to simplify the result. We find

$$K_2 = \frac{e^2}{\Gamma m\omega^2}\left[|\hat{b}\times\hat{k}_\perp\cdot\tilde{\underset{\sim}{E}}_1|^2\left(1 - \frac{\bar{\mu}B_0}{\Gamma^2 mc^2}\right)\right.$$

$$\left. + \frac{\omega^2}{k_\perp^2c^2}|\hat{b}\times\hat{k}_\perp\cdot\tilde{\underset{\sim}{B}}_1|^2\left(1 - \frac{\bar{U}_\parallel^2}{\Gamma^2c^2}\right)\right] +$$

$$+ \frac{e^2}{k_\perp^2}\sum_\ell\left(\frac{k_\parallel}{m}\frac{\partial}{\partial\bar{U}_\parallel} + \frac{e\ell}{mc}\frac{\partial}{\partial\bar{\mu}}\right)\left(\frac{|R_\ell|^2}{\omega - k_\parallel\bar{V}_\parallel - \ell\Omega}\right) , \tag{47}$$

where

$$R_\ell = (\hat{k}_\perp\cdot\tilde{\underset{\sim}{E}}_1)J_\ell + (\hat{b}\times\hat{k}_\perp)\cdot\left(-\frac{\bar{V}_\parallel}{c}J_\ell\tilde{\underset{\sim}{B}}_1 + \frac{2i\Omega\bar{\mu}}{\omega}\frac{\partial J_\ell}{\partial\bar{\mu}}\tilde{\underset{\sim}{E}}_1\right) . \tag{48}$$

In Eqs. (45)-(48), $J_\ell = J_\ell(k_\perp\rho)$, and $\rho = (2mc^2\bar{\mu}/e^2B_0)^{1/2}$.

V. HOW TO USE THE PONDEROMOTIVE HAMILTONIAN

This section may be read independently of the derivation of the ponderomotive Hamiltonian in the previous sections. Here we will describe how to interpret and use the results we have obtained. For an alternative discussion of this subject, see the comments preceding Eq. (92) of Ref. 4.

To fix ideas, let us consider for a moment a one-dimensional problem with no magnetic field. The basic physical notion of ponderomotive phenomena is that of a force. In simple cases, one can write the ponderomotive force as the gradient of a potential which depends on the position x of the particle (more precisely, the oscillation center) so that one then speaks of a ponderomotive potential. However, sometimes this potential depends as well on the velocity of the particle, and this circumstance has led to a certain amount of confusion regarding velocity dependent forces.

The correct way to interpret the velocity dependent ponderomotive potentials which arise in various contexts is through Hamilton's equations. Thus, it is better to think in terms of the momentum rather than the velocity of the particle, so that Hamilton's equations, $\dot{x} = \partial H/\partial p$, $\dot{p} = -\partial H/\partial x$, can be used. Then the ponderomotive potential is seen to be a correction to the Hamiltonian describing the particle motion in the background fields (if any), which is due to the nonlinear effects of the wave. Furthermore, it is recognized that the ponderomotive potential (or Hamiltonian, as we will say), affects not only the force equation, $\dot{p} = - \partial H/\nabla p$, but also the equation which relates the velocity of the oscillation center to the momentum, $\dot{x} = \partial H/\partial p$. That is, the definition of the momentum is altered by the introduction of a ponderomotive term, so that p no longer has its original form (e.g., p = mv in

simple problems), but rather has an additional term.

One of the advantages of the Hamiltonian formulation is that often the equations of motion, found by applying Hamilton's equations and taking the required derivatives, are vastly more complicated than the Hamiltonian itself. Thus, when consideration of the Hamiltonian alone will suffice, as in finding turning points, one need not deal with all the complexity of the equations of motion. It will be appreciated presently that this advantage is dramatic for the relativistic ponderomotive Hamiltonian, for which the equations of motion contain many terms.

For the relativistic problem we are considering, Hamilton's equations in the usual sense cannot be used, because we have expressed things in terms of noncanonical variables. There is a great advantage in doing this for guiding center problems. Instead, one must use Hamilton's equations in Poisson bracket form. That is, if A is any quantity whose time derivative we desire, then

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A,K\} \ , \tag{49}$$

where $A = A(X,U_{\parallel},\mu,\textcircled{H},t)$ and where $K = H_0 + K_2$ is the total ponderomotive Hamiltonian, given by Eqs. (9) and (45)-(48). Here we have dropped the overbars used previously, but the variables $(\underset{\sim}{X},U_{\parallel},\mu,\textcircled{H})$ are to be interpreted physically as guiding center variables, averaged over the wave oscillations. Usually we will want to take A as one of the guiding center variables themselves, so that the first term of Eq. (49) will vanish.

The second term is the Poisson bracket term, which is computed according to Eq. (17). Lest this formula seem strange, we point out that this is merely an application of the chain rule formula for Poisson

brackets. This rule is the following. If f and g are any two dynamical variables, and g depends on the collection of further dynamical variables $(h_1, \ldots, h_n)$, then

$$\{f,g\} = \sum_{k=1}^{n} \{f,h_k\} \frac{\partial g}{\partial h_k} . \tag{50}$$

A similar formula applies if f depends on further variables. This formula is easily proved using the usual definition of the Poisson bracket found in mechanics texts. In our case the further variables (h_k) are represented by the guiding center variables themselves, and Eq. (16) is used.

When these rules are applied to the background Hamiltonian H_0, the usual relativistic drift equations are produced, as shown in Eqs. (14). But the term K_2 causes ponderomotive corrections. Consider, for example, the parallel force equation, for which the usual notion of a ponderomotive force is most clear. (Note that in relativity, the force is the time rate of change of the relativistic momentum, which in our variables is represented by $mU_{\parallel}$.) Then we have

$$m\dot{U}_{\parallel} = \{U_{\parallel},H_0\} + \{U_{\parallel},K_2\}$$

$$= \hat{b} \cdot \left(e\underset{\sim}{E}_0 - \frac{\mu\nabla B_0}{\Gamma}\right) - \hat{b}\cdot\nabla K_2 . \tag{51}$$

As for the background term, we have used Eq. (14c), but we have expanded things out to the dominant order in the guiding center parameter ε, and we have indicated the background fields. The result is recognized as the usual parallel force on a guiding center. As for the ponderomotive term,

a glance at Eqs. (45)-(48) shows that it is complicated indeed. [Note that Γ depends on $\underset{\sim}{X}$ through the quantity $B(\underset{\sim}{X})$ in Eq. (10), and that even $V_{\parallel}$ depends on $\underset{\sim}{X}$, because $V_{\parallel} = U_{\parallel}/\Gamma$. Nonrelativistically, both these dependencies vanish, although even there we have $\Omega = \Omega(\underset{\sim}{X})$. We also note that there are efforts underway at present to derive a covariant form of the relativistic ponderomotive force in the limit of zero gyroradius.[15] A covariant formulation of relativistic guiding center theory has previously been developed by Fradkin[16] and Littlejohn.[13]] We see here the advantage of having the result in Hamiltonian form. We will presently discuss this ponderomotive term for some special cases.

Next let us consider the guiding center drifts, which are also modified by ponderomotive effects. Proceeding as before, we find

$$\dot{\underset{\sim}{X}} = \{\underset{\sim}{X}, H_0\} + \{\underset{\sim}{X}, K_2\}$$

$$= \hat{b}\left(\frac{U_{\parallel}}{\Gamma} + \frac{1}{m}\frac{\partial K_2}{\partial U_{\parallel}}\right)$$

$$+ \frac{\hat{b}}{B_0} \times \left(\frac{c}{\Gamma e}\,\mu\nabla B_0 + \frac{mc}{\Gamma e}\,U_{\parallel}^2\,\hat{b}\cdot\nabla\hat{b} - c\underset{\sim}{E}_0 + \nabla K_2\right) . \tag{52}$$

The first term on the right-hand side is the parallel component of the guiding center velocity, which contains a ponderomotive correction. This correction means that the quantity $V_{\parallel}$, which is best interpreted in terms of the parallel momentum, is no longer $U_{\parallel}/\Gamma$. The ponderomotive correction terms which occur in the definition of the momentum are mathematically analogous to the term $(e/c)\underset{\sim}{A}$, which is added to $m\underset{\sim}{v}$ to get the momentum in ordinary particle mechanics.

The second major term shows the drifts, which consist of the usual

relativistic drifts plus a ponderomotive term. Note that the ponderomotive force affects the drifts in the same way as any external force (e.g., gravitational) would do, as one might expect.

For another application of the ponderomotive Hamiltonian, consider the effect of the ponderomotive forces on the turning points of a mirroring particle. For this problem, let us suppose that the frequency ω is constant in time, so that the ponderomotive Hamiltonian is time-independent. Then there exists a form of energy conservation for the guiding center, namely, $E = H_0 + K_2 = \text{const}$. The turning points are specified by $V_\parallel = 0$. This is equivalent to $U_\parallel = 0$, since the ponderomotive correction to $U_\parallel$ does not enter to the order we require. Then the turning point is found by finding a root in $s = X_\parallel$ of the equation,

$$E = \left(1 + \frac{2\mu B_0}{mc^2}\right)^{1/2} + K_2 \,, \tag{53}$$

where K_2 is evaluated at $U_\parallel = 0$. If the turning point in the absence of the wave is s_0, then the turning point in the presence of the wave (for the same value of E) is shifted by

$$\Delta s = -\frac{\Gamma K_2}{\mu(\partial B/\partial s)} \,, \tag{54}$$

where everything is evaluated at $s = s_0$, $U_\parallel = 0$. We see here the usefulness of the Hamiltonian, apart from the equations of motion. In spite of the complicated nature of K_2, it would be straightforward to evaluate Eq. (54) numerically.

VI. PONDEROMOTIVE HAMILTONIAN IN SOME SIMPLIFYING LIMITS

Next let us consider the ponderomotive Hamiltonian in some simplifying limits. It turns out that our results contain the case of ponderomotive forces in a one-dimensional electrostatic wave (but still relativistic), and it is useful to consider this case for purposes of illustration. This is so because if we set $\mu = 0$, $\underset{\sim}{B}_1 = 0$, and $\underset{\sim}{E}_{1\perp} = 0$, we get essentially one-dimensional motion along the field line. It is easiest to use Eqs. (45)-(46) when taking this limit; the result can be written in the forms

$$K_2 = \frac{e^2}{mk_\parallel} \frac{\partial}{\partial U_\parallel} \left(\frac{|\underset{\sim}{\tilde{E}}_1|^2}{\omega - k_\parallel V_\parallel}\right)$$
$$= \frac{e^2 |\underset{\sim}{\tilde{E}}_1|^2}{m\Gamma^3} \frac{1}{(\omega - k_\parallel V_\parallel)^2} . \tag{55}$$

This result has been derived previously, in the context of a three-dimensional, unmagnetized plasma [c.f., Ref. 4, Eq. (59)]. Apart from the factor Γ^3, this is recognized as the nonrelativistic ponderomotive potential in electrostatic fields. The parallel ponderomotive force, $F_\parallel = m\dot{U}_\parallel$, is then found in the same way as shown in Eq. (51). It is given by

$$F_\parallel = -\frac{e^2}{m\Gamma^3} \left[\frac{1}{(\omega - k_\parallel V_\parallel)^2} \frac{\partial}{\partial s} |\underset{\sim}{\tilde{E}}_1|^2 + \frac{2V_\parallel |\underset{\sim}{\tilde{E}}_1|^2}{(\omega - k_\parallel V_\parallel)^3} \frac{\partial k_\parallel}{\partial s}\right] , \tag{56}$$

where we have dropped the background terms and where $s = X_\parallel$. Note that the two terms may have opposite signs.

Another limit which is often useful is $\omega/\Omega \sim k_\parallel/k_\perp \ll 1$. Here we must be careful, however; if ω is so much less than Ω that it becomes comparable to the bounce frequency, then it makes no sense to average simultaneously over the gyration and the wave oscillations, while at the same time to leave the bounce oscillations out. When $\omega \sim \omega_b$, a correct analysis will show bounce resonances in the denominators. Let us assume, therefore, that ω is intermediate between ω_b and Ω. [If we wish to be formal, we can treat ω/Ω and $k_\parallel/k_\perp$ as $O(\varepsilon^{1/2})$, since $\omega_b/\Omega = O(\varepsilon)$.]

In taking this limit it is best to use Eqs. (47)-(48), since the term in Eqs. (45)-(46) which is apparently dominant in this limit actually vanishes. The result can be written in the form

$$K_2 = \frac{e^2}{\Gamma m\omega^2} |\hat{b} \times \hat{k}_\perp \cdot \tilde{\underset{\sim}{E}}_1|^2 \left(1 - \frac{\mu B_0}{\Gamma^2 mc^2}\right) + \frac{e^2}{\Gamma m k_\perp^2 c^2} |\hat{b} \times \hat{k}_\perp \cdot \tilde{\underset{\sim}{B}}_1|^2 \left(1 - \frac{U_\parallel^2}{\Gamma^2 c^2}\right)$$
$$+ \frac{e^2 k_\parallel}{m k_\perp^2} J_0^2(k_\perp \rho) \frac{\partial}{\partial U_\parallel} \left(\frac{|R|^2}{\omega - k_\parallel V_\parallel}\right) - \frac{e^2}{k_\perp^2 B_0} \frac{\partial}{\partial \mu} (\Gamma |R|^2) , \qquad (57)$$

where

$$R = \hat{k}_\perp \cdot \left(\tilde{\underset{\sim}{E}}_1 + \frac{V_\parallel}{c} \hat{b} \times \tilde{\underset{\sim}{B}}_1\right) . \qquad (58)$$

Evidently the third term of Eq. (57) is analogous to Eq. (55), but an exact correspondence cannot be established because we have assumed $k_\parallel \ll k_\perp$ in deriving Eq. (57), whereas Eq. (55) assumes $k_\perp = 0$. The fourth major term of Eq. (57) is purely relativistic, as are several subterms.

The limit $\tilde{E}_{1\parallel} = \tilde{B}_{1\parallel} = 0$ and $k_\perp = 0$ is often found in applications, like in free electron lasers. In this case, we simplify Eqs. (45)-(46)

and obtain

$$K_2 = \frac{e^2}{\Gamma m\omega^2}\left(1 - \frac{\mu B_0}{\Gamma^2 mc^2}\right)|\tilde{\underset{\sim}{E}}_{1\perp}|^2$$
$$+ \frac{e^2 k_\parallel |\tilde{\underset{\sim}{E}}_{1\perp}|^2}{2m\omega^2}\frac{\partial}{\partial U_\parallel}\left[\frac{(\omega - k_\parallel V_\parallel)V_\perp^2}{(\omega - k_\parallel V_\parallel)^2 - \Omega^2}\right]$$
$$+ \frac{e^3 |\tilde{\underset{\sim}{E}}_{1\perp}|^2}{2mc\omega^2}\frac{\partial}{\partial \mu}\left[\frac{\Omega V_\perp^2}{(\omega - k_\parallel V_\parallel)^2 - \Omega^2}\right] . \tag{59}$$

This is the relativistic expression of the ponderomotive Hamiltonian of an electromagnetic wave propagating along the static magnetic field. Except for the Γ factors and for the term $\mu B_0/\Gamma^2 mc^2$, this expression is in perfect agreement with the nonrelativistic ponderomotive Hamiltonian.

VI. CONCLUSIONS

In spite of the fact that this paper has dealt exclusively with single particle motion, we have completed the large majority of the work for a derivation of the nonlinear, relativistic gyrokinetic equation. This is because the Vlasov equation can be written in Poisson bracket form,

$$\frac{\partial F}{\partial t} + \{F,K\} = 0 \quad , \tag{60}$$

and when this is expressed in terms of our doubly averaged variables, the result is the gyrokinetic equation. The only additional work to be done is to calculate the linear and nonlinear currents and charges, to complete the self-consistency. We will report on this application of our results in the future.

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APPENDIX

Let the gauge scalar be S_1, so that the wave fields ϕ_1, A_1 transform according to $\phi_1 \to \phi_1 - (1/c)\partial S_1/\partial t$, $\underset{\sim}{A}_1 \to \underset{\sim}{A}_1 + \nabla S_1$. In order to preserve the eikonal form of the wave packet, S_1 must have the eikonal form. Here we take

$$S_1(\underset{\sim}{x},t) = \varepsilon\tilde{S}_1(\underset{\sim}{x},\varepsilon t)e^{i\psi(\underset{\sim}{x},t)/\varepsilon} + \text{c.c.} \tag{A1}$$

Thus, to lowest order in ε, the amplitudes transform according to $\tilde{\phi}_1 \to \tilde{\phi}_1 + (i\omega/c)\,\tilde{S}_1$, $\tilde{A}_1 \to \tilde{A}_1 + i\underset{\sim}{k}\tilde{S}_1$. Next, because of Eq. (21), $\underset{\sim}{u}_0$ transforms according to $\underset{\sim}{u}_0 \to \underset{\sim}{u}_0 + (\lambda e/mc)\underset{\sim}{k}\sigma(\underset{\sim}{x},t)$, where

$$\sigma(\underset{\sim}{x},t) = i\tilde{S}_1 e^{i\psi(\underset{\sim}{x},t)/\varepsilon} + \text{c.c.} \tag{A2}$$

Combining this with Eq. (8), we find the gauge dependence of the guiding center variables:

$$\underset{\sim}{X} \to \underset{\sim}{X} - \frac{\lambda\varepsilon}{B_0}(\hat{b} \times \underset{\sim}{k})\sigma(\underset{\sim}{x},t) \ ,$$

$$U_{\parallel} \to U_{\parallel} + \frac{\lambda e k_{\parallel}}{mc}\,\sigma(\underset{\sim}{x},t) \ ,$$

$$\mu \to \mu + \frac{\lambda e}{B_0 c}(\underset{\sim}{u}_0 \cdot \underset{\sim}{k})\sigma(\underset{\sim}{x},t) + O(\lambda^2) \ . \tag{A3}$$

On the other hand, the $O(\lambda)$ terms of Eqs. (41)-(44) also have a gauge dependence, which is contained in the transformation rule

$$H_{1\ell} \to H_{1\ell} + \frac{ie}{c}\,J_\ell\tilde{S}_1(\omega - k_{\parallel}V_{\parallel} - \ell\Omega) \ . \tag{A4}$$

When we apply this to Eqs. (41)-(44), we find that the gauge dependence overall exactly cancels, leaving us with the gauge invariant quantities $(\bar{\underset{\sim}{X}}, \bar{U}_{\parallel}, \bar{\mu})$.

Similarly, we may consider the gauge dependence of the averaged Hamiltonian, $K = K_0 + \lambda^2 K_2$. Since this is a function of the averaged variables, which are themselves gauge invariant, K has no gauge dependence from this source. Furthermore, when the fields $\tilde{\underset{\sim}{A}}_1, \tilde{\phi}_1$ appearing in K_2 are subjected to a gauge transformation, we find after some algebra that K_2 is overall gauge invariant. We might have taken this for granted, but the verification of gauge invariance provides a useful check on the derivation of K_2. We note, however, that individually the two major terms of K_2 [$\bar{H}_2$ and the beat term, in Eq. (40)] are not gauge invariant; only their sum is. Thus, the division of K_2 into these two terms has no invariant meaning, in spite of their rather different appearance.

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