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NICOLAI MAPS FOR QCD_4

M. Claudson and M.B. Halpern

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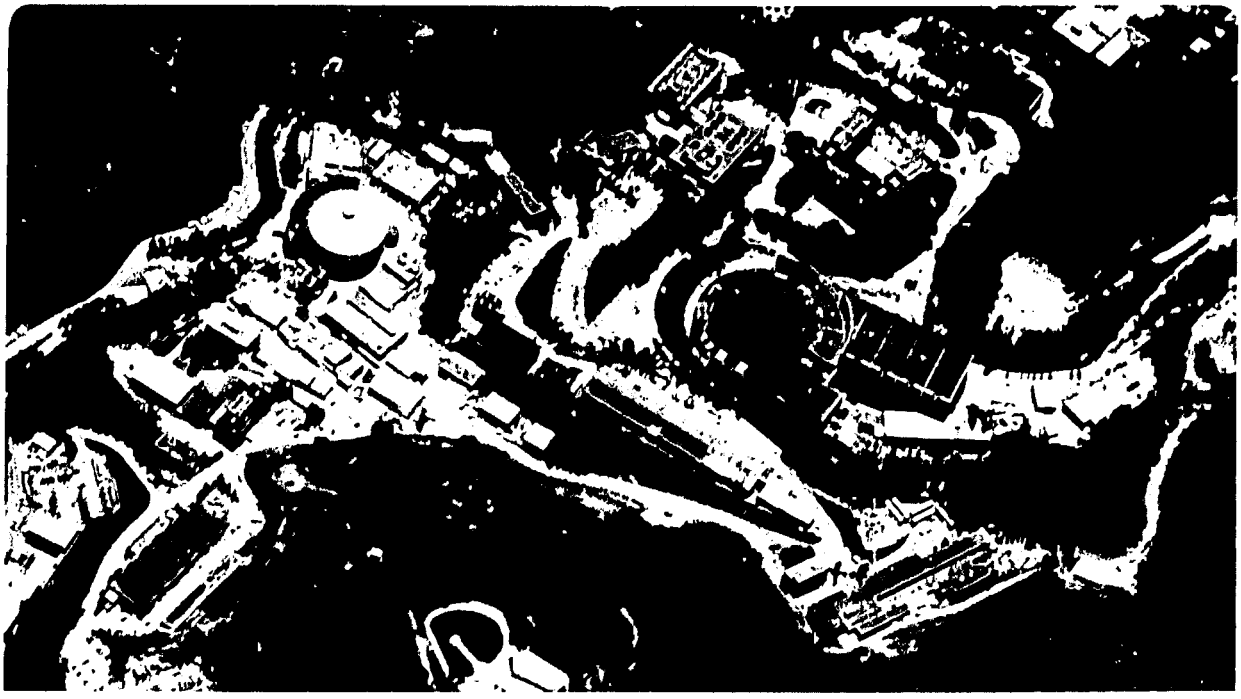

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NICOLAI MAPS FOR QCD_4

M. Claudson

and

M. B. Halpern

Lawrence Berkeley Laboratory
and
Department of Physics
University of California
Berkeley, California 94720

Abstract

We find two Nicolai-Langevin maps in ordinary non-supersymmetric quantum chromodynamics in four dimensions. The maps are constant Jacobian transformations from the gluon field to a Gaussian noise, and run as stochastic equations in real (Euclidean) time.

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The purpose of this letter is to point out the existence of a class of purely bosonic, non-supersymmetric theories which possess Nicolai maps[1,2] by studying an important member of this class, quantum chromodynamics in four space-time dimensions. In the case of QCD_4, we have found two maps, each of which expresses the theory as a local Langevin equation with a Gaussian noise. Each Langevin equation runs in real (Euclidean) time, not fifth time.[3] The Fokker-Planck Hamiltonia, which control the time development of the Nicolai-Langevin maps, are, not surprisingly, the Hamiltonian of QCD itself.

Nicolai maps generally involve the cancellation of a bosonic Jacobian by a fermionic determinant. The crucial feature of our maps is that they each possess a constant Jacobian as a change of variable from the gauge fields to the Gaussian noise, and therefore no fermions are needed for any cancellation. Non-linear transformations with constant Jacobians are remarkable, and it is worth noting how such a circumstance can arise.

It is well known that Jacobians from Langevin systems tend to be very simple because of the retarded boundary conditions of the Langevin equation. In fact, with retarded boundary conditions, $\partial_t^{-1}\delta(t-t') = \theta(t-t')$ so that[4]

$$\det_R(\partial_t + B) = [\det_R \partial_t] e^{\frac{1}{2}\int dt\, \mathrm{Tr} B} \tag{1}$$

for any matrix B(t). A constant determinant is always obtained if $\mathrm{Tr}\, B = 0$.

We will first state and then prove the following two identities in $A_0 = 0$ gauge QCD_4:

$$\begin{aligned} &\langle F_i^a(\vec{x})|e^{-\hat{H}T}|I_i^a(\vec{x})\rangle \\ &\quad = e^{\pm(W[F]-W[I])}\langle e^{\pm\xi[A^\pm]}\delta[A_i^{\pm a}(\vec{x}, t_2) - F_i^a(\vec{x})]\rangle_\eta \end{aligned} \tag{2}$$

On the left is the Euclidean finite time operator transition amplitude for QCD_4 in this gauge. The transition is from an initial configuration $I_i^a(\vec{x})$ at t_1 to a final configuration $F_i^a(\vec{x})$ at t_2, $\hat{H}$ is the QCD Hamiltonian and $T = t_2 - t_1$. Also, i, j, ... = 1, 2, 3 are spatial indices, while a, b, ... run over colors. The quantity on the right involves an average over a Langevin noise $\eta_i^a(\vec{x}, t)$ which is computed with the Gaussian Boltzmann factor

$$e^{-\frac{1}{2}\int_{t_1}^{t_2} dt \int d^3x\, \eta_i^a(\vec{x}, t)\eta_i^a(\vec{x}, t)} \tag{3}$$

using either of a pair of Langevin-Nicolai maps[F1]

$$E_i^{\pm a} = \mp B_i^{\pm a} + \eta_i^a. \tag{4}$$

In these maps, $E_i^{\pm a} = E_i^a(A^\pm)$, etc. where

$$\begin{aligned} &E_i^a = \dot{A}_i^a \qquad\qquad B_i^a = \tfrac{1}{2}\,\varepsilon_{ijk}F_{jk}^a \\ &F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a - g\, f^{abc}A_i^b A_j^c. \end{aligned} \tag{5}$$

The fields $A_i^{\pm a}$ are the non-linear functionals of η which solve the Langevin equations with the initial condition

$$A_i^{\pm a}(\vec{x}, t_1) = I_i^a(\vec{x}). \tag{6}$$

The final configuration enters on the right through the δ-function to which we shall refer as the final state constraint. The prefactors on the right involve the winding number current:

$$\begin{aligned} &W[A] = \int d^3x\, J_0(A) \qquad\qquad \xi[A] = \int_{t_1}^{t_2} dt \int dS_i J_i(A) \\ &J_0(A) = \tfrac{1}{2}\,\varepsilon_{ijk}A_i^a(\partial_j A_k^a - \tfrac{1}{3} f^{abc}A_j^b A_k^c) \\ &J_i(A) = \tfrac{1}{2}\,\varepsilon_{ijk}A_j^a E_k^a \end{aligned} \tag{7}$$

so that

$$\int_{t_1}^{t_2} dt \int d^3x\, E_i^a B_i^a = W[F] - W[I] + \xi[A]. \tag{8}$$

The identities Eq. 2 may be simplified by choosing spatially periodic boundary conditions so that $\xi = 0$. This leaves only the Greensite[5] wave functions $e^{\mp W}$ as prefactors. Finally, we note that the final state constraint and initial condition Eq. 6 ensure that the identities are correctly normalized since both sides equal $\delta[F\text{-}I]$ for $T = 0$.

The proof of Eq. 2 is simply a change of variables, via either map, from η back to the gauge fields $A^\pm = A$. The QCD action emerges from the Gaussian Boltzmann factor since, using Eq. 8,

$$\begin{aligned} &\tfrac{1}{2}\int_{t_1}^{t_2} dt \int d^3x\, \eta_i^a\eta_i^a = S_{QCD} \pm \{W[F] - W[I] + \xi[A]\} \\ &\quad S_{QCD} = \tfrac{1}{2}\int_{t_1}^{t_2} dt \int d^3x\, [E_i^a E_i^a + B_i^a B_i^a] \end{aligned} \tag{9}$$

We must also compute the Jacobian of the transformation. First, note that

$$\frac{\delta\eta_i^a(\vec{x}, t)}{\delta A_j^{\pm b}(\vec{x}', t')} = [\delta^{ab}\delta_{ij}\partial_t \pm \varepsilon_{ijk}D_k^{ab}(\vec{x}, t)]\,\delta^4(x - x') \tag{10}$$

where

$$D_i^{ab}(\vec{x}, t) = \delta^{ab}\partial_i^x + gf^{abc}A_i^c(\vec{x}, t) \tag{11}$$

is the covariant derivative. Then, using the retarded boundary conditions of the Langevin equations,

$$\det\left(\frac{\delta\eta}{\delta A^{\pm}}\right) = [\det{}_R\partial_t]e^{\Gamma_{\pm}} \tag{12}$$

where

$$\begin{aligned}\Gamma_{\pm} &= \mathrm{Tr}\,\ln\{[\delta^{ab}\delta_{ij}\delta(t-t') \pm \varepsilon_{ijk}D_k^{ab}(\vec{x}, t')\theta(t-t')]\delta^3(\vec{x}-\vec{x}')\} \\ &= \pm\,\mathrm{tr}[\varepsilon_{ijk}D_k^{ab}(\vec{x}, t')\,\theta(t-t')\delta^3(\vec{x}-\vec{x}')] \\ &= 0\end{aligned} \tag{13}$$

since ε_{ijk} is antisymmetric. As promised, the maps have constant Jacobians. Ignoring overall constants, we have therefore reduced the right-hand side of Eq. 2 to

$$\int_{A_i^a(\vec{x}, t_1) = I_i^a(\vec{x})}^{A_i^a(\vec{x}, t_2) = F_i^a(\vec{x})} D\,A_i^a\, e^{-S_{QCD}} \tag{14}$$

which establishes the identities Eq. 2.

Similarly, for the finite time expectation values of an arbitrary functional G[A], the correspondence is

$$\langle G[A]\rangle_{QCD} = \frac{\langle e^{\pm\xi[A^{\pm}]}\delta[A_i^{\pm a}(\vec{x}, t_2) - F_i^a(\vec{x})]G[A^{\pm}]\rangle_\eta}{\langle e^{\pm\xi[A^{\pm}]}\delta[A_i^{\pm a}(\vec{x}, t_2) - F_i^a(\vec{x})]\rangle_\eta} \tag{15}$$

These averages approach the usual Euclidean ground state expectation values at large times.

We turn now to the Fokker-Planck formulation of the maps. The normalized Fokker-Planck probability densities are

$$\rho_{\pm}[A_i^a(\vec{x}), t|I_i^a(\vec{x}), t_1] = \langle\, \delta[A_i^{\pm a}(\vec{x}, t) - A_i^a(\vec{x})]\rangle_\eta. \tag{16}$$

These can be used to compute the equal time Langevin averages

$$\int D\,A_i^a(\vec{x})\,\rho_{\pm}[A_i^a(\vec{x}), t\,|\,I_i^a(\vec{x}), t_1]G[A_i^a(\vec{x})] = \langle G[A_i^{\pm a}(\vec{x}, t)]\rangle_\eta \tag{17}$$

of functionals which depend on only A_i^a at time t. Note that these are not the same as the QCD averages Eq. 15 which depend on the final configuration. We can interpret $\rho_{\pm}$ in QCD as follows: Using Eq. 2 and Eq. 16 with spatially periodic boundary conditions,

$$\langle F|e^{-HT}|I\rangle = e^{\pm(W[F]-W[I])}\rho_{\pm}[F, t_2|I, t_1]. \tag{18}$$

The QCD transition probability <u>amplitude</u> is linear in the Fokker-Planck probability <u>densities.</u>

The Fokker-Planck equations themselves may be derived from the maps by standard methods,

$$\dot{\rho}_{\pm} = \int d^3x\{\pm\, B_i^a(\vec{x})\,\delta/\delta A_i^a(\vec{x}) + \tfrac{1}{2}\,\delta^2/\delta A_i^a(\vec{x})\delta A_i^a(\vec{x})\}\,\rho_{\pm}. \tag{19}$$

It is particularly interesting to find the hermitean Fokker-Planck Hamiltonia $\hat{H}_{\pm}$ which generate the time development of the Langevin-Fokker-Planck systems. To eliminate the first derivative term of (19), we make the similarity transformation

$$\rho_{\pm}[A, t|I, t_1] = e^{\mp W[A]}\tilde{\rho}_{\pm}[A, t|I, t_1]e^{\pm W[I]}$$
$$\tilde{\rho}_{\pm}[A, t_1|I, t_1] = \delta[A-I]. \tag{20}$$

Since, with spatially periodic boundary conditions,

$$\frac{\delta W[A]}{\delta A_i^a(\vec{x})} = B_i^a(\vec{x}), \tag{21}$$

we find that $\tilde{\rho}_{\pm}$ evolves according to the Euclidean Schrödinger equations

$$\dot{\tilde{\rho}}_{\pm} = -\hat{H}_{\pm}\tilde{\rho}_{\pm}$$
$$\hat{H}_{\pm} = \hat{H} = \tfrac{1}{2}\int d^3x\,[-\delta^2/\delta A_i^a(\vec{x})\delta A_i^a(\vec{x}) + B_i^a(\vec{x})B_i^a(\vec{x})]. \tag{22}$$

The time development of both systems is controlled by the QCD Hamiltonian itself. Of course, $\tilde{\rho} = \tilde{\rho}_{\pm}$ is precisely the original QCD operator transition amplitude (with spatially periodic boundary conditions) as seen in Eq. 18, or Eq. 20 and Eq. 22.

If ρ relaxes to a time-independent distribution at large times, we say that the Langevin-Fokker-Planck system equilibrates. The condition for this to happen is that the Fokker-Planck Hamiltonian has a normalizable zero energy ground state. Since the QCD ground state energy $E_0 \neq 0$, our systems do not equilibrate, and the solutions to the maps tend to run away from the initial condition. The final state constraint filters out such runaway behavior in the sense that this constraint selects out the noise for which $A^{\pm}$ do not run away, but instead approach the final configuration at t_2.[F2] However, this non-

equilibration may present technical difficulties in an attempt to study the maps numerically. Further properties of non-equilibrating maps are discussed in Reference [6].

As an example, we will solve the maps for the free abelian case (zeroth order perturbation theory for the non-abelian case). The free maps

$$\dot{A}^{\pm} = \mp DA^{\pm} + \eta, \qquad D_{ij} = -\varepsilon_{ijk}\partial_k \tag{23}$$

are immediately solved by

$$A^{\pm}(\vec{x}, t) = e^{\mp(t-t_1)D}\{I(\vec{x}) + \int_{t_1}^{t} d\tau\, e^{\pm(\tau-t_1)D}\eta(\vec{x}, \tau)\}$$
$$e^{\lambda D} = L + e^{+\lambda\sqrt{-\nabla^2}}T^{+} + e^{-\lambda\sqrt{-\nabla^2}}T^{-}$$
$$T^{\pm} = \tfrac{1}{2}(T \pm D/\sqrt{-\nabla^2}). \tag{24}$$

Here L and T are standard longitudinal and transverse projection operators, while $T^{\pm}$ are transverse helicity projection operators. Note that the longitudinal mode undergoes a random walk and that one of the helicities always runs away. However, both of these behaviors are filtered out by the final state constraint, and the usual $A_0 = 0$ gauge results are obtained for the finite time propagator.

As a check on the gauge invariance of the results, we will discuss the implementation of the operator Gauss' Law $D\cdot\hat{E}^a = 0$. In the usual functional formalism, it is not hard to show that the correct generating functional in $A_0^a = 0$ gauge with a source ρ^a for $D\cdot\hat{E}^a$ is just the $A_0^a = \rho^a$ gauge transition amplitude $Z[F, I; \rho^a]$. That is, the transition matrix elements of powers of $D\cdot\hat{E}^a$ in $A_0^a = 0$ gauge are the derivatives of

$Z[F, I; \rho^a]$ at $\rho^a = 0$. By smearing $Z[F, I; \rho^a]$ with a gauge invariant initial state $\Psi_i[I]$ and by transforming from $A_0{}^a = \rho^a$ gauge to $A_0{}^a = 0$ gauge (with a gauge transformation which equals the identity at t_2), we see that the transition matrix elements of powers of $(D\cdot\hat{E}^a)$ vanish when the initial state is gauge invariant. The same result may be obtained using the maps

$$A_i{}^a = D_i{}^{ab}\rho^b \mp B_i{}^a + \eta_i{}^a \tag{25}$$

which also have constant Jacobians. Indeed, these maps convert $Z[F, I; \rho^a]$ into the right hand side of (2) with a gauge transformed initial configuration.

The discussion here has been formal in that it lacks regularization. An important next step would be a regularization scheme which preserves the residual spatial gauge invariance. This could be investigated with a spatial lattice or possibly with a stochastic regularization in which the noise is smeared in time only. The latter scheme corresponds to the regulated actions

$$\begin{aligned} S_{QCD}{}^{\Lambda\pm} = \tfrac{1}{2}\int_{t_1}^{t_2} d\tau dt\, d^3x\, (E\pm B)_i{}^a(\vec{x}, t)\alpha_\Lambda(t-\tau)(E\pm B)_i{}^a(\vec{x},\tau) \\ \mp \int_{t_1}^{t_2} dt\, d^3x\, E_i{}^a B_i{}^a \end{aligned} \tag{26}$$

in which $\alpha_\Lambda(t-\tau)$ is the smearing function.

Originally, we discovered the existence of maps in certain purely bosonic theories while studying Nicolai maps in fermion sectors of various supersymmetric theories.[6] We suspect that any bosonic theory with a constant Jacobian Nicolai map is the effective bosonic theory in a

"null-fermion" sector of a supersymmetric theory. For example, in $N=1$ supersymmetric QCD, the entire fermionic part of the Hamiltonian vanishes in each of the sectors defined by $\Lambda_\alpha{}^a|+> = 0$ or $\bar{\Lambda}_{\dot\alpha}{}^a|->= 0$ where $\Lambda_\alpha{}^a$ and $\bar{\Lambda}_{\dot\alpha}{}^a$ are the Weyl fermions of the theory. The effective bosonic theory in both sectors is then just ordinary QCD. In Reference [6], we will discuss Nicolai maps in the null fermion sectors of other models, including dimensionally reduced Wess-Zumino models and the sigma model.

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Footnotes

F1. In Reference [2], a similar map with different boundary conditions and a non-constant Jacobian was proposed by Nicolai in an unsuccessful attempt to find a map for the $N = 1$ supersymmetric Yang-Mills theory.

F2. The large T behavior e^{-E_0T} of the transition amplitude is a direct consequence of the runaway behavior of the maps: For a non-equilibrating map, the return to F is less likely, and still less at larger T.

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