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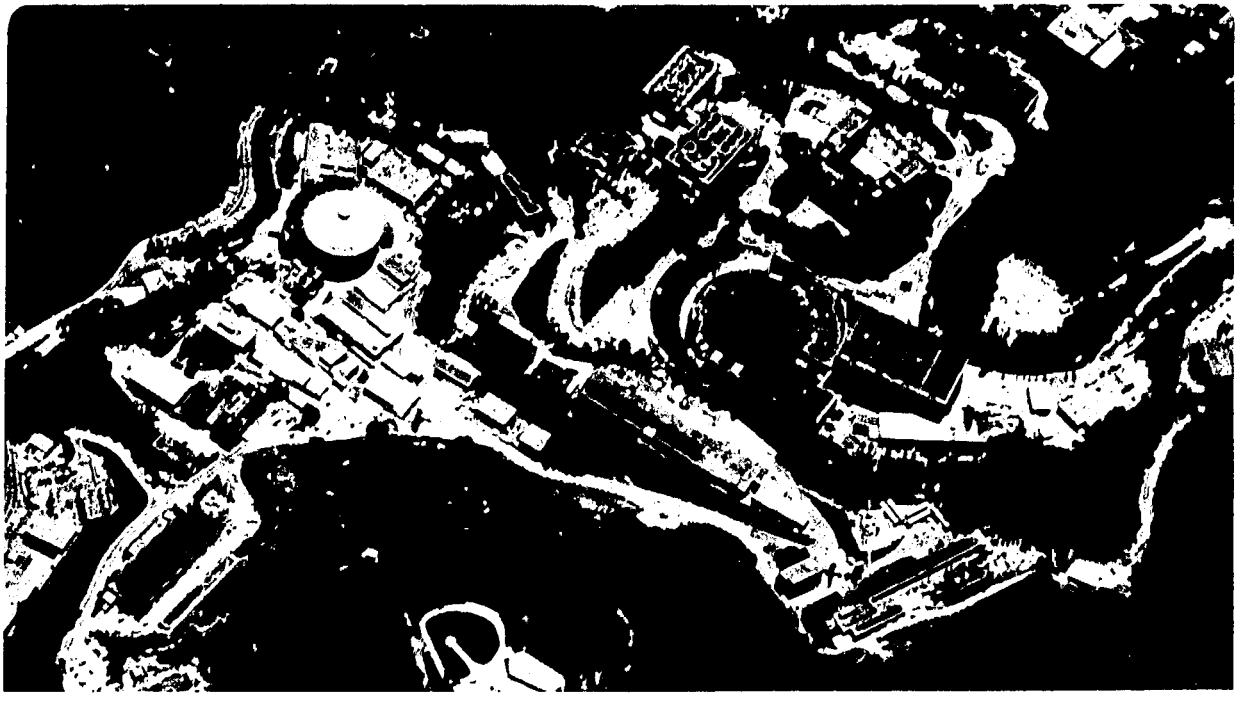

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Ultraviolet Divergences and Supersymmetric Theories*

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ABSTRACT

This article is closely related to the one by Ferrara in these same Proceedings. It deals with what is perhaps the most fascinating property of supersymmetric theories, their improved ultraviolet behavior. My aim here is to present a survey of the state of the art as of August, 1984, and a somewhat more detailed discussion of the breakdown of the superspace power-counting beyond N=2 superfields. I also describe a method for simplifying divergence calculations that uses the locality of subtracted Feynman integrals.

Lecture presented at the International School of Physics "Enrico Fermi", Course XCII, Varenna, Italy, June 1984.

(September, 1984)

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1. Introduction and Survey

Soon after the discovery of supersymmetry [1], it was realized that a symmetry between the bosonic and fermionic degrees of freedom of a field theory would have a beneficial effect on its ultraviolet behavior. For instance, it became clear that supersymmetry, combined with the "minus" sign rule for fermionic loops, would guarantee the absence of corrections to the vacuum energy for an unbroken supersymmetric theory [2]. Moreover, explicit calculations revealed an impressive set of cancellations in specific models. The simplest one of these, known as the Wess-Zumino model [3], is a renormalizable interacting theory of a scalar, a pseudoscalar and a Majorana spinor, all with the same mass and interacting via a single coupling constant. Naively one would expect that, because of supersymmetry, in this case divergences could be absorbed by means of three distinct parameters, one common wave function renormalization for the three fields, one mass renormalization and one coupling constant renormalization. Surprisingly, it was found by explicit calculation that only one wave function renormalization appeared to be needed [4]. Somewhat later it was also shown, again by explicit calculation, that another model, the N=4 Yang-Mills theory [5], possesses no charge renormalization at one and two loops [6,7], and even at three loops [8]. Though it was natural to conjecture that such behavior would persist to all orders, there followed a period of impasse, and for about three years no one succeeded in achieving a satisfactory understanding of the results of Refs. [4-8], and to draw general conclusions from them.

It is undeniable that ultraviolet divergences are a rather bizarre phenomenon to get accustomed to. Nonetheless, the successes of Quantum Electrodynamics first and of Yang-Mills gauge theories later did manage to get physicists accustomed to them, with the result that the improved ultraviolet behavior of supersymmetric theories found in Refs. [4-8] appeared rather remarkable and interesting from a theoretical point of view, but did not add much to the rather limited interest that the majority of particle physicists had in these new ideas. A major wave of interest in these models was only aroused by the observation [9] that tying scalars to spinors by supersymmetry would improve the stability of the parameters of low-energy gauge theories upon renormalization. The stability problem for the parameters is often referred to as the gauge hierarchy problem [10].

The situation for gravitational theories, on the other hand, was quite different. In this case it is obvious on dimensional grounds that their

perturbation expansion is in danger of being nonrenormalizable, because Newton's constant k^2 has the dimension of a negative power of mass. The possible cancellation of ultraviolet divergences does appear a rather crucial phenomenon here, and leads to hope that minor modifications of Einstein's gravity could lead to cure the problems of its perturbation expansion. Indeed, the early achievements of supergravity theories [11] appeared rather spectacular in this context. It was known for some time, after the work of De Witt [12] and 't Hooft and Veltman [13], that *pure* Einstein quantum gravity has finite one-loop corrections to its S matrix, but that coupling it to a single scalar field results in nonrenormalizable ultraviolet divergences. The former result requires little calculation to understand. All one needs is to list the possible counterterms of the right dimensionality, which are

$$\sqrt{-g}\; R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \;,\quad \sqrt{-g}\; R^{\mu\nu} R_{\mu\nu} \;\text{ and }\; \sqrt{-g}\; R^2 \;, \tag{1.1}$$

and notice that they all vanish on-shell, *i.e.* when the classical field equations are used, on account of the Gauss-Bonnet identity for four-dimensional space time. In fact, this guarantees that

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} R_{\mu\nu}{}^{\alpha\beta} R_{\rho\sigma}{}^{\gamma\delta} = -4\, R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + 16\, R^{\mu\nu} R_{\mu\nu} - 4\, R^2 \tag{1.2}$$

is a total derivative, and thus vanishes in perturbation theory. The divergence encountered in Ref. [13] for the case of a single scalar was shown to persist, again by explicit calculations, for a large number of matter couplings [14], and for some time seemed unavoidable. Actually, this is not the case, as the *pure* supergravity theories do share the one-loop finiteness of Einstein's gravity [15]. However, if one adds extra matter, even of the supersymmetric type, divergences appear again [16]. The former result is a direct consequence of what was said above for Einstein's gravity. In fact, irreducible supersymmetry can be used to relate S matrix elements to ones corresponding to external gravitons only. These are finite, as their possible divergences correspond to the harmless invariants in eq. (1). Remarkably, formal arguments are also available that exclude two-loop divergences for supergravity theories. Two-loop finiteness, however, is somewhat more subtle to establish [17], and stems from the impossibility of turning the only candidate two-loop counterterm for pure gravity into a supersymmetric invariant.

Beyond one loop in pure Einstein gravity the situation is pure mystery, and (almost) the same is true for supergravity beyond two loops. So, even assuming

that Einstein gravity does indeed diverge at two loops, we seem very far from having fixed its problems by turning to supergravity. Excluding an infinite number of counterterms of increasing dimensionality on the basis of formal arguments alone seems impossible, and this suggests that trouble is indeed going to show up at the first available opportunity. To be honest, however, the blame is to be put, at present, more on the investigators than on the theories, as our understanding of them is very incomplete. Moreover, there is a subtle point which is easily overlooked. Does it really make sense to write a perturbation series where the expansion parameter is not dimensionless, and actually has the dimensions of a negative power of mass? Just on dimensional grounds, the effective strength of the interaction is bound to increase with momentum, and it is not clear what the small expansion parameter would be in the ultraviolet region. From the mathematical viewpoint, demanding an expansion in integer positive powers of the coupling constant, even of the asymptotic type, is tantamount to demanding analyticity near zero coupling constant of the quantum theory. However, it has been shown that this is actually not the case in toy models. I am referring here to some old work of Parisi [18], where the author infers that the $\frac{1}{N}$ expansion for Φ^4 theory is renormalizable even above four dimensions, the only signature of the nonrenormalizability being the lack of analyticity of the result in the self-coupling of Φ. It goes without saying that an expansion parameter is needed, as there is no hope of solving the quantum theory exactly for complicated four-dimensional models. Unfortunately, it is rather difficult to envisage what a dimensionless expansion parameter could be for supergravity theories, as they are so tightly constrained in their spectra by supersymmetry.

While keeping this in mind, one must admit that the possibility of a theory with a finite perturbation expansion is so attractive that it deserves attention. Moreover, just as renormalizability served as a very useful guiding principle in the search for theories of strong and electroweak interactions, it is conceivable that finiteness can serve as a guiding principle in the search for a truly unified theory of all interactions. Thus, one can proceed and ask the well defined question of what the perturbation expansion in k looks like for (super)gravitational theories. There are two different kinds of approaches to this problem. The first one, obvious in principle but exceedingly difficult in practice, is to proceed to actual calculations, starting from the possibly more accessible case of pure Einstein quantum gravity at two loops. The other one consists in trying to gain some

insight into the problem by indirect means, and possibly attempting to build up formal arguments.

Going back to renormalizable models with (extended) supersymmetry, I already remarked how until 1981 there was a rather impressive set of "experimental" results with apparently no explanation, most remarkable of which the vanishing of the β function for N=4 Yang-Mills up to three loops. The first half of 1982, on the other hand, saw the occurrence of something new. Fairly rigorous formal arguments were proposed leading to an understanding of the results of Refs. [4-8] and to proofs of their persistence to all orders. These arguments all rest on making the supersymmetry more manifest than it is in the usual component formulations, where the balance between the on-shell fermionic and bosonic degrees of freedom is violated by the corresponding off-shell field representations. A crucial observation in this respect came in 1979, and is due to Grisaru, Roček and Siegel [19]. Motivated by their own attempt to calculate the three-loop β function for N=4 Yang-Mills using N=1 superfields, they managed to streamline the method of N=1 superspace in dealing with chiral superfields, and it then became obvious why the Wess-Zumino model had only one renormalization constant. The second (and main) step, is a paper by Grisaru and Siegel [20], where it is shown that combining the familiar properties of N=1 superfields with the background field method and with the *working assumption* that similar manipulations should go through with extended superfields (essentially unknown at the time) leads to a number of rather spectacular conclusions. The N=4 Yang-Mills theory would necessarily be finite to all orders if it were possible to formulate it at least in terms of N=2 extended superfields, and actually all Yang-Mills and matter theories which admitted a formulation in terms of N=2 superfields would be finite beyond one loop. Actually, at the time Ref. [20] appeared, there were some problems left before its conclusions could be made effective for N≥2 gauge and matter multiplets. A major difficulty was the need for a suitable formulation of the N=2 Wess-Zumino multiplet (the so-called hypermultiplet), that would allow quantization along the lines of Ref. [20]. The long known off-shell formulation of this model [21] involved off-shell central charges, *i.e.* extra bosonic generators that vanish on-shell. The corresponding superspace description contained extra bosonic coordinates corresponding to these generators which were not integrated over, with the result that conventional quantization methods ran into difficulties. Howe, Stelle and Townsend [22] succeeded in arriving at a formulation of the hypermultiplet free from this

difficulty, and in fact complete N=2 superfield formulations for gauge and matter multiplets soon became available [23,24]. There followed, in particular, the existence of a whole class of completely finite *renormalizable* theories with N=2 supersymmetry [25].

There were also several independent attempts along different lines, such as the work of Mandelstam *et al.* [26] that led to the construction of a supersymmetric extension of the light-cone formalism. This provided the first formal argument establishing the finiteness to all orders for N=4 Yang-Mills, without the need of any *ad hoc* assumption. An independent argument [27] strongly suggestive of the all-order finiteness of N=4 Yang-Mills had appeared some time before those mentioned above. The observation is quite elegant. It has to do with the long-known result [28] that, already in N=1 supersymmetric theories, anomalies in internal symmetry currents sit in a supermultiplet together with the trace anomaly. As this is well-known to be proportional to the β function [29] for the gauge coupling, it follows that assuming preservation of the chiral SU_4 in N=4 Yang-Mills implies the vanishing of its (only) β function, and thus its finiteness. The (minor) weakness has to do with the assumption about the SU_4 symmetry, which is not manifest in an N=1 superfield formulation. This argument is actually equivalent in its conclusions to assuming the preservation of SU_4 and using the non-renormalization results of Ref. [19].

The arguments of Ref. [20], even if taken seriously and applied to the most interesting case of maximally extended supergravity, run into the ever present problem of the coupling constant with negative mass dimension, and manage at most to stretch the first possible onset of divergences to six loops. This, however, is much better than the well-known result of two-loop finiteness, though obviously very far from the complete solution of the problem. The main difficulty with this approach is that its predictions are obtained at the expense of rather strong assumptions on the unknown off-shell structure of models with extended supersymmetry, and the auxiliary fields that serve to close off-shell the supersymmetry algebra are not known for most supersymmetric theories. In many cases there are also no-go theorems [30] that, under certain hypotheses, exclude the possibility that any be found. These subjects will be discussed in more detail in Section 2, where superspace methods will be described, and the difficulties with off-shell formulations of supersymmetric theories will be reviewed.

Given this rather confused state of affairs, it was found valuable to perform explicit calculations in some controversial case to see how matters would

actually turn out to be, the idea being that an explicit test of the power counting of Ref. [20] would bypass the difficulties connected with *ad hoc* hypotheses. This work was enterprised by Neil Marcus and myself [31-33] at Caltech and was carried out there over the last year and a half. The aim was to probe the power counting of Ref. [20] by using N=4 Yang-Mills in more than four dimensions. As this theory is nonrenormalizable, local invariants of higher order than those simply ruled out by the N=2 power counting can be probed by evaluating higher-loop corrections to the S matrix. A novel feature of this two-loop calculation as compared with similar, and simpler, ones in renormalizable models, was the need to compute divergent parts of Green functions depending on several (in our case three) external momenta. An interesting byproduct of our work is a method [33] to simplify the evaluation of the pole parts of the integrals arising in such calculations. It makes them no more difficult than corresponding ones of two-point functions, and extends to higher orders of the perturbation expansion. The outcome of this work seemed at first very encouraging for supergravity [31], as we found an unexpected cancellation of divergences at two loops in six dimensions. This seemed to indicate that the power counting of Ref. [20] was correct beyond the well established case of N=2 superfields. Clearly, the argument was indirect and at most of suggestive value. However, somewhat later Howe and Stelle [34] succeeded in giving an alternative, and far more conservative, interpretation of the result of Ref. [31] in terms of the available formulation in terms of N=2 superfields [22-24], supplemented by the enforcement of the extra, nonlinearly realized, supersymmetries. This state of affairs undoubtly made the result of Ref. [31] rather empty. However, proceeding with the calculation, after developing the necessary more sophisticated computer techniques [35], we actually obtained a negative, and thus conclusive, result. We found an ***explicit violation*** of the form of the on-shell effective action assumed in Ref. [20]. The conclusion, unfortunately, is rather negative, and we are essentially back to not being able to use the power counting of extended superspace beyond N=2 superfields. As the N=2 power counting carries no useful information for the case of supergravity, it appears likely that (extended) supergravity theories all diverge starting at three loops. This conclusion is also supported by the counterterm analysis of Ref. [36], which appears to exclude further cancellation mechanisms, such as the one noted in Ref. [34]. As I have already remarked, it is not completely clear, at the moment, whether a divergent and nonrenormalizable perturbation expansion for gravitational theories should be regarded as a disease of the theories

themselves, or rather of our way of calculating them. At any rate, the miracles of supergravity at the first two loop orders [15,17] are so impressive that they suggest that a very economic solution to the problems of quantum gravity should involve supersymmetry in some way. In this respect, one should keep in mind that multi-local generalizations of supergravity theories, known as superstring theories, have been developed over the past few years by Green and Schwarz [37].

There are *only* three superstring theories in ten dimensions, usually called Type-I, Type-IIa and Type-IIb. Type-II supesrtring theories contain in their massless sectors the two inequivalent forms of N=2 supergravity [38]. On the other hand, the massless sector of the Type-I theory consists of N=1 Yang-Mills coupled to N=1 supergravity [39]. Naively, this theory would sound less attractive than the others, because of the arbitrariness in the choice of the gauge group. It has been known for some time that tree-level unitarity already restricts the gauge group to be U(n), SO(n) or USp(n) [40]. Remarkably, at the time of writing this article Green and Schwarz [41] have just shown that the cancellation of anomalies restricts the gauge group to be SO(32), thus making this also a unique theory. Whereas the features of the perturbation expansion of these models are very little understood at the moment, and even their formulation is not at a definitive stage, it has already been shown that at one loop and in $d>4$ they do behave better than their field theory limits [42]. What happens at the next orders is completely unknown at the present time, but this seems a very promising way of departing from the usual description of gravity, perhaps more so than do conformal theories [43], even though these are obviously renormalizable and, in the supersymmetric case, even finite [44].

The conclusion is that, at the moment, we seem to have at our disposal a large class of finite renormalizable theories, with (extended) global supersymmetry. However, the motivation for regarding such models as fundamental theories is far from clear. Whereas demanding finiteness clearly restricts the freedom of the model builder, it would seem that, once a place is to be found for supersymmetry in particle physics, one should go all the way and consider locally supersymmetric theories, as these offer the perspective of unifying all interactions. In this respect, one could conceive of finite renormalizable models as low-energy truncations of (possibly) finite locally supersymmetric theories†. This could mean superstring theories, as supergravity theories seem candidate

† I thank Neil Marcus for a discussion of this point.

to suffering to from nonrenormalizable ultraviolet divergences after the first two loop orders.

All the foregoing discussion has been rather formal, as I have avoided on purpose to emphasize the possible effect of quantization on supersymmetry. In other words, I have not stressed the well-known fact that symmetries which hold at the classical level may be incompatible with the ever present hidden parameter of the quantum theory, the regulator that disposes of the ultraviolet divergences. This leads to anomalies in classically conserved currents, such as the familiar chiral anomalies [45]. A sufficient (though not necessary) condition in order that a symmetry that holds good at the classical level be free of quantum anomalies is that there exist a regulator compatible with it. Trivially, however, at times a regulator can be lacking only because not discovered yet, as was the case for Yang-Mills gauge theories before the advent of dimensional regularization [46]. Even though, to be precise, one should admit that really rigorous proofs of the renormalizability of Yang-Mills theories have followed the introduction in Ref. [46] of a gauge invariant regulator, one should recall that highly credited work on the subject [47] has actually preceded it. The work of, *e.g.* Refs. [47], was aimed at proving something which, in essence, is equivalent to the existence of a symmetry respecting regulator. This is the possibility of using an existing symmetry violating regulator (in that case the Pauli-Villars method [48]), while still achieving the preservation of the gauge symmetry (*i.e.* the fulfillment of the corresponding Ward identities) by means of the addition of suitable extra finite counterterms.

The state of the art for supersymmetric theories can be summarized as follows. There does exist a proposed symmetry preserving regulator. It was originally designed by Siegel [49] in order to maintain the balance between fermionic and bosonic degrees of freedom unaltered upon continuation to a non integer number of dimensions. This regularization method can thus be regarded as an extension to a non integer number of dimensions of the technique of dimensional reduction, which had so much success in its application to the construction of supersymmetric models (see, most notably, Ref. [50]). The prescription is to continue in the number of momenta, while keeping the Lorentz indices on the external fields untouched, which ensures that subtraction of pole parts alone does preserve supersymmetry. Unfortunately, this scheme involves manipulations of different kinds of indices, and some ambiguities do arise. This became clear very early to the author himself [51]. Recently, some work of van Damme

and 't Hooft [52] raised the question again. The idea behind it is interesting. It sends us back to the situation for Yang-Mills theories at the beginning of the last decade, and to the work of, *e.g.*, Refs. [47]: compute in dimensional regularization, and see whether the results of dimensional reduction can be reached by means of a suitable non minimal subtraction scheme. This work raised a large interest in the community, especially since it appeared that an incompatibility was arising. Clearly, this would be disastrous for theories like supergravity, for which the supersymmetry current is a gauge current. At the time of this writing the situation has apparently been clarified by Jack and Osborn [53], and the work of Ref. [52] has turned out to contain numerical errors which invalidate its earlier conclusions. Thus, there appear to be no difficulties, at the perturbative level, in quantizing supersymmetric theories, and dimensional reduction is equivalent to a non minimal form of dimensional regularization. Strictly speaking, this has been shown to be the case at the two-loop order for renormalizable models. However, it confirms the findings of several groups that have long used dimensional reduction successfully, even with non supersymmetric theories [54,31,32].

This concludes the historical survey of the problem. The remaining Sections are meant to be more technical. They deal in somewhat more detail with some of the points mentioned above. Hopefully, they should make the discussion here more concrete, while at the same time conveying some basic information to the more unexperienced reader. The plan of the remaining Sections is as follows. In Section 2 superspace methods are reviewed, starting from the very beginning, at least insofar as is needed to discuss the power counting of Ref. [20]. This material is mostly well-known, and two textbooks are now available, with different levels of completeness (and complexity!) [55,56]. Thus, I will try to be concise. Section 2 also contains brief discussions of the finite models with global supersymmetry, and of the difficulties one encounters when attempting to find off-shell formulations for supersymmetric theories. Section 3 addresses in somewhat more detail the issue of the consistency of dimensional reduction and illustrates the results of the work of Refs. [31-33]. The discussion summarizes the main points arrived at here. Finally, the Appendix contains a brief discussion of two-component formalism, at least isofar as is needed in Section 2. Here I have made use of Ref. [58], where more details can be found.

An excellent review of these subject matters was written last year by West [59]. Consequently, as I have tried to make this discussion self-consistent, some

overlapping has been unavoidable.

2. Superspace, Superfields and The Superspace Power Counting

Superspace [60] is a very useful tool that enables one to make supersymmetry into a manifest symmetry. This is achieved by adjoining to the commuting space time four-vector coordinate x^μ a set of anticommuting spinorial coordinates (one of them and its complex conjugate for each supersymmetry). These are actually merely labels, since their third power vanishes as a consequence of anticommutativity. Thus†

$$x^{\alpha\dot\alpha} \to x^{\alpha\dot\alpha}\ ,\ \theta^{\alpha i}\ ,\ \overline{\theta}_i^{\dot\alpha}\ . \tag{2.1}$$

Supersymmetry transformations appear as particular coordinate transformations in this extended manifold, such that

$$\delta x^{\alpha\dot\alpha} = \frac{i}{2}\left(\overline{\epsilon}_i^{\dot\alpha}\,\theta^{\alpha i} + \epsilon^{\alpha i}\,\overline{\theta}_i^{\dot\alpha}\right) \tag{2.2a}$$

$$\delta\theta^{\alpha i} = i\,\epsilon^{\alpha i}\ . \tag{2.2b}$$

Ordinary fields then generalize to superfields, which are functions of all the coordinates of the extended manifold. In practice, the θ-dependence is rather trivial as a consequence of anticommutativity, and superfields are just polynomials of finite degree in θ. They provide a convenient way of grouping together the fields of a supersymmetry multiplet. For example:

$$\Phi[x,\theta] = \phi(x) + \theta^{\alpha i}\,\psi_{\alpha i}(x) + \cdots \tag{2.3}$$

The field components can then be recovered by taking successive derivatives of the superfields at $\theta = 0$. The problem, of course, is to recognize a given supermultiplet inside a superfield and to write down actions in terms of superfields. For simplicity, I will now concentrate on the case of N=1 superfields, for which all is known and writable in a rather accessible form for all models of interest. A superfield without any Lorentz indices, $\Phi[x,\theta]$, transforms as

$$\Phi[x,\theta] \to \Phi[x + \delta x, \theta + \delta\theta] \tag{2.4}$$

under a supersymmetry transformation. The coefficient of the highest power of θ

† For two-component notation see the Appendix.

cannot, so to speak, bear any more θ's. It follows that it must transform as a total divergence under supersymmetry. Inside a space time integral, this produces a (harmless) surface term. Thus, one can write invariant actions as integrals over all superspace of products of superfields

$$\int d^4x\, d^4\theta\, L[\Phi[x,\theta], \partial\Phi[x,\theta]]\ , \tag{2.5}$$

where the familiar Berezin integration, which is tantamount to differentiation, picks out the highest component of L.

The problem is to gain control over the formalism, and to write down the proper action for the models of interest. This requires choosing the right superfields, and often achieving a way to truncate their θ-expansion without imposing any condition on the x-dependence of a minimal number of component fields. This truncation is obtained by imposing *constraints* on the superfields, often in the form of differential equations in θ. The constraints are properly distinguished into *on-shell* ones, which do imply the equations of motion for the component fields, and *off-shell* ones, which do not imply the equations of motion for the component fields, and can thus lead to the construction of off-shell Lagrangians. I will describe here the superspace formulations of Wess-Zumino multiplet [61] and of the Yang-Mills multiplet [62]. This material is well known. Therefore, the description that follows is rather sketchy. I hope, however, that it will suffice to make the discussion presented at the end of this Section more intelligible, at least insofar as the main ideas are concerned.

First of all, in analogy with what done for usual Poincaré invariant theories, one introduces representations of the symmetry generators in terms of differential operators on the manifold. Thus, from eqs. (2), and specializing to the case of N=1 superspace,

$$Q_\alpha = i\,\partial_\alpha + \frac{1}{2}\overline{\theta}^{\dot\alpha}\,\partial_{\alpha\dot\alpha}\ , \tag{2.6a}$$

$$\overline{Q}_{\dot\alpha} = i\,\partial_{\dot\alpha} + \frac{1}{2}\theta^\alpha\,\partial_{\alpha\dot\alpha}\ . \tag{2.6b}$$

The supercharges in eqs. (6) satisfy

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = i\,\partial_{\alpha\dot\alpha}\ ; \tag{2.7a}$$

$$\{Q_\alpha, Q_\beta\} = 0\ . \tag{2.7b}$$

For extended supersymmetry eqs. (7) would generalize into

$$\{ Q_{\alpha i}, \overline{Q}^j_{\dot\alpha} \} = i\, \delta_i^j \partial_{\alpha\dot\alpha} \quad , \tag{2.8a}$$

$$\{ Q_{\alpha i}, Q_{\beta j} \} = i\, \epsilon_{\alpha\beta} Z_{ij} \quad . \tag{2.8b}$$

The first of eqs. (8) is the obvious generalization of the first of eqs. (7), but the second of eqs. (8) contains new generators, Z_{ij}, which generate the so called *central charge* transformations [63].

Going back to eqs. (6), it is clear that the θ and $\overline{\theta}$ derivatives do not anticommute with the supersymmetry generators, and are therefore not very convenient in the construction of invariant actions. It is better to work with the covariant spinorial derivatives

$$D_\alpha = \partial_\alpha + \frac{i}{2} \overline{\theta}^{\dot\alpha} \partial_{\alpha\dot\alpha} \quad , \tag{2.9a}$$

$$\overline{D}_{\dot\alpha} = \partial_{\dot\alpha} + \frac{i}{2} \theta^\alpha \partial_{\alpha\dot\alpha} \quad , \tag{2.9b}$$

which do anticommute with the supersymmetry charges. They satisfy the algebra

$$\{ D_\alpha, \overline{D}_{\dot\alpha} \} = i\, \partial_{\alpha\dot\alpha} \tag{2.10a}$$

$$\{ D_\alpha, D_\beta \} = 0 \quad . \tag{2.10b}$$

Then, one can impose differential constraints, for example by applying covariant spinorial derivatives to superfields. For the Wess-Zumino multiplet one needs a complex dimension 1 (*i.e* propagating) scalar, a Weyl spinor and a complex dimension 2 (*i.e.* auxiliary) scalar. This set of fields is contained, for example, in a superfield shortened by the chirality condition

$$\overline{D}_{\dot\alpha} \Phi = 0 \tag{2.11}$$

will do. Indeed, expanding ϕ in components gives

$$\Phi(x) = A(x) + \theta^\alpha \lambda_\alpha(x) + \theta^2 G(x) + \cdots \quad , \tag{2.12}$$

and the remaining components are all space time derivatives of these. The three fields above are just the propagating scalar, the spinor and the auxiliary scalar of the usual component formulation of the Wess-Zumino model [3]. It should be noted that the constraint in eq. (11) does not imply any x-equation for the independent fields inside Φ. The situation would be quite different if, together

with eq. (11), one had also imposed its complex conjugate. Then, on account of eq. (10a), the superfield would contain only fields constant in x-space. The conclusion is that a complex scalar superfield subject to a chirality constraint is what one needs to describe the Wess-Zumino multiplet. The chirality constraint does not imply any component field equations (*i.e.* it is an off-shell constraint), and therefore one can write invariant actions by writing suitable θ-integrals containing Φ and $\overline{\Phi}$. Renormalizable couplings are then selected by the restriction that the corresponding coupling constants be of nonnegative dimension. For example, the Wess-Zumino model with renormalizable couplings would look like

$$\int d^4x\, d^4\theta\, \overline{\Phi}\, \Phi + \left(m \int d^4x\, d^2\theta\, \Phi^2 + g \int d^4x\, d^2\theta\, \Phi^3 + \text{h.c} \right) , \tag{2.13}$$

and this expression can be "guessed" using just dimensional analysis. It should be noted that the last two integrals above are only over a subspace of the superspace. They are called chiral integrals, and do produce supersymmetric invariants on account of the constraint (11) on Φ. Equivalently, chiral superfields could be regarded as derivatives of unconstrained superfields [64], *i.e.*

$$\Phi = \overline{D}^2\, U \quad , \tag{2.14}$$

where U has the gauge invariance

$$\delta\, U = \overline{D}^{\dot\alpha} \Lambda_{\dot\alpha} \quad . \tag{2.15}$$

As a second, and more complicated, example, consider N=1 Yang-Mills [62]. This four-dimensional model describes the interactions of a multiplet of vectors with one of Weyl spinors, both in the adjoint representation of a gauge group. Thus, in components the action is simply

$$S = \int d^4x \left(-\frac{1}{4} F^2_{\mu\nu} + i\, \overline{\lambda} \not{\nabla} \lambda \right) . \tag{2.16}$$

The terms are written in four-component notation, and have the usual definitions:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \quad , \tag{2.17a}$$

$$\nabla_\mu \lambda^a = \partial_\mu \lambda^a + g\, f^{abc} A^b_\mu \lambda^c \quad . \tag{2.17b}$$

To describe the theory in superspace, one introduces gauge covariant derivatives ∇_α, $\nabla_{\dot\alpha}$ and $\nabla_{\alpha\dot\alpha}$, which is tantamount to introducing superfield

potentials (much too many!) A_α, $A_{\dot\alpha}$ and $A_{\alpha\dot\alpha}$. Then one uses the covariantization of the first of eqs. (10) to express $A_{\alpha\dot\alpha}$ in terms of A_α and $A_{\dot\alpha}$, *i.e* one writes

$$\{\nabla_\alpha, \nabla_{\dot\alpha}\} = i\,\nabla_{\alpha\dot\alpha} \quad . \tag{2.18}$$

This is actually a constraint, though a trivial one. It amounts to merely resolving the ambiguity in the definition of $\nabla_{\alpha\dot\alpha}$. In fact eq. (18) could at most look like

$$\{\nabla_\alpha, \nabla_{\dot\alpha}\} = i\,\nabla_{\alpha\dot\alpha} + F_{\alpha\dot\alpha} \quad , \tag{2.19}$$

with $F_{\alpha\dot\alpha}$ a field strength. However, adding a covariant object to a covariant derivative produces an equally suitable covariant derivative. On the other hand, the second of eqs. (10) and its complex conjugate could generalize to

$$\{\nabla_\alpha, \nabla_\beta\} = F_{\alpha\beta} \quad , \tag{2.20a}$$

$$\{\nabla_{\dot\alpha}, \nabla_{\dot\beta}\} = \overline{F}_{\dot\alpha\dot\beta} \quad . \tag{2.20b}$$

The proper way to interpret these equations is to look at "$\theta = 0$". Then $F_{\alpha\beta}$ starts with a self-dual tensor (see the Appendix) of dimension one. This clearly cannot be identified with $A_\mu \sim A_{\alpha\dot\alpha}$, nor with the Weyl spinor λ_α. Thus, one concludes that the field strengths $F_{\alpha\beta}$ and $\overline{F}_{\dot\alpha\dot\beta}$ must both vanish, which leads to the nontrivial constraint

$$\{\nabla_\alpha, \nabla_\beta\} = 0 \quad , \tag{2.21}$$

together with its complex conjugate. These constraints on the covariant derivatives are conditions on the potentials A_α, $A_{\dot\alpha}$ and $A_{\alpha\dot\alpha}$, and allow one to express them in terms of a more basic object, the *prepotential*.

Before heading for the prepotential, let us recall that there exist algebraic relations between the covariant derivatives known as Bianchi identities. These are trivial if no constraints are imposed, but are very useful when working with constrained objects. They allow one to find all the independent field strengths of a theory, both on-shell and off-shell. These are clearly very useful objects in constructing supersymmetric invariants. One starts by listing all the Bianchi identities in order of increasing dimensionality (∇_α and $\nabla_{\dot\alpha}$ both have dimension $\frac{1}{2}$, whereas $\nabla_{\alpha\dot\alpha}$ has dimension 1) :

$$[\{\nabla_\alpha, \nabla_\beta\}, \nabla_\gamma] + [\{\nabla_\beta, \nabla_\gamma\}, \nabla_\alpha] + [\{\nabla_\gamma, \nabla_\alpha\}, \nabla_\beta] = 0 \quad , \tag{2.22a}$$

$$[\{\nabla_\alpha, \nabla_{\dot\alpha}\}, \nabla_\beta] + [\{\nabla_\alpha, \nabla_\beta\}, \nabla_{\dot\alpha}] + [\{\nabla_\beta, \nabla_{\dot\alpha}\}, \nabla_\alpha] = 0 \quad , \tag{2.22b}$$

$$[\{\nabla_{\alpha\dot\alpha}, \nabla_\alpha\}, \nabla_\beta] + [\{\nabla_\alpha, \nabla_\beta\}, \nabla_{\alpha\dot\alpha}] + [\{\nabla_\beta, \nabla_{\alpha\dot\alpha}\}, \nabla_\alpha] = 0 \quad , \tag{2.22c}$$

$$[\{\nabla_{\alpha\dot\alpha}, \nabla_\alpha\}, \nabla_{\dot\beta}] + [\{\nabla_\alpha, \nabla_{\dot\beta}\}, \nabla_{\alpha\dot\alpha}] + [\{\nabla_{\dot\beta}, \nabla_{\alpha\dot\alpha}\}, \nabla_\alpha] = 0 \quad , \tag{2.22d}$$

$$[\{\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}\}, \nabla_\gamma] + [\{\nabla_{\beta\dot\beta}, \nabla_\gamma\}, \nabla_{\alpha\dot\alpha}] + [\{\nabla_\gamma, \nabla_{\alpha\dot\alpha}\}, \nabla_{\beta\dot\beta}] = 0 \quad , \tag{2.22e}$$

$$[\{\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}\}, \nabla_{\gamma\dot\gamma}] + [\{\nabla_{\beta\dot\beta}, \nabla_{\gamma\dot\gamma}\}, \nabla_{\alpha\dot\alpha}] + [\{\nabla_{\gamma\dot\gamma}, \nabla_{\alpha\dot\alpha}\}, \nabla_{\beta\dot\beta}] = 0 \quad . \tag{2.22f}$$

One then enforces in these otherwise trivial identities the constraints discussed before, and other similar ones involving higher order field strengths. In the list of eqs. (22a)-(22f) I have skipped some which are trivially complex conjugates of those listed. Eq.(22a) is identically satisfied, and eq. (22b) implies that

$$[\{\nabla_\alpha, \nabla_{\dot\alpha}\}, \nabla_\beta] = \epsilon_{\alpha\beta}\, W_{\dot\alpha} \quad . \tag{2.23}$$

In fact, the *l.h.s*, being antisymmetric in α and β, can only be proportional to the SU(2) metric $\epsilon_{\alpha\beta}$ (recall that in this notation $\alpha = 1,2$ and $\dot\alpha = 1,2$). This allows one to solve for $\overline{W}_{\dot\alpha}$, and thus for W_α, as

$$W_\alpha = \frac{1}{2}[\{\nabla_\alpha, \nabla_{\dot\alpha}\}, \nabla^{\dot\alpha}] \quad . \tag{2.24}$$

At this point one can proceed in two ways. The first is simply to keep on inserting eqs. (23) and (24) in the remaining Bianchi identities (eqs. (22c)-(22f)). The other, more handy procedure, consists of taking successive derivatives of W_α. There would seem to be the problem of establishing when to stop. This, however, is clear if one recalls what said above about the constraints. Everything is always interpretable, provided one looks at the $\theta = 0$ part of the superfields. Thus, W_α contains the spinor at $\theta = 0$, and differentiating it will say something about the vector field strength. Subsequent differentiations contain information about the field equations, *i.e.* give a foolproof procedure to establish whether the constraints put the theory on-shell, the field equations being easily recognizable using the "$\theta = 0$ trick" mentioned above. The auxiliary fields appear as a whole superfield that, when set to zero, puts the theory on-shell. They are in a multiplet with the field equations, and the auxiliary field superfield contains all the field equations at successive orders in θ. To see how all this works let us differentiate eq. (24) with respect to $\overline{\theta}^{\dot\beta}$. This gives

$$\{\nabla_{\dot\beta}, W_\alpha\} = \frac{1}{2}\{\nabla_{\dot\beta}, [\{\nabla_\alpha, \nabla_{\dot\alpha}\}, \nabla^{\dot\alpha}]\} \tag{2.25}$$

which, using the lower Bianchi identities, the constraints and eq. (23) can be

written

$$\{\nabla_{\dot\beta}, W_\alpha\} = -\frac{1}{2}\{\nabla_{\dot\beta}, W_\alpha\} \quad . \tag{2.26}$$

Thus, W_α is the gauge-covariant generalization of a chiral superfield. One can also differentiate with respect to θ^β. The result can be trivially decomposed into irreducible pieces as

$$\{\nabla_\beta, W_\alpha\} = i\,\epsilon_{\alpha\beta} D + G_{\alpha\beta} \quad , \tag{2.27}$$

where $G_{\alpha\beta}$ is symmetric in its two indices. It is (see the Appendix) the self-dual part of the vector field strength, the only object that the component theory of eq. (16) contains at dimension 2. Thus, the superfield D is the auxiliary field superfield, and its $\theta = 0$-component is the familiar pseudoscalar auxiliary field for the Yang-Mills multiplet [62]. Setting D to zero puts the theory on-shell. Indeed, in this case contracting α and β eliminates the *r.h.s.*, and then differentiating with $\nabla_{\dot\alpha}$ and using the chirality of W_α and eq. (24) yields

$$\{\nabla_{\alpha\dot\alpha}, W^\alpha\} = 0 \quad , \tag{2.28}$$

This clearly contains the Dirac equation for the spinor at $\theta = 0$. One more differentiation would generate the vector field equation.

We have thus seen how setting the auxiliary field superfield to zero puts the theory on-shell. The example I have discussed is rather elementary. The lesson, however, is that deriving on-shell constraints is always conceptually simple, and the component theory serves as a guide all the way through. All one needs to do is set to zero all the field strengths whose $\theta = 0$ parts are not present in the component theory formulated without auxiliary fields. This, however, is by no means a pointless exercise, and provides a very elegant approach for analyzing complicated theories and deriving their complete field equations and supersymmetry transformations (for the application of these techniques to supergravity see, *e.g.*, Refs. [64]). For our purposes, the interest in such formulations stems from their allowing one to simply classify on-shell invariants, and thus divergences of the S matrix as allowed by the theorem of Ref. [20].

The really difficult problem is to formulate the theories off-shell or, in the corresponding component approach, to find the auxiliary fields that close the supersymmetry algebra off-shell. The complete solution to this problem is unknown at present, and many failed efforts have even led to the no go theorems of Ref. [30], which exclude the possibility that auxiliary fields be found for (almost)

all multiplets of N>2 supersymmetry. Whereas such theorems always rest on working hypotheses, the lesson they convey is clear. If off-shell formulations exist in the cases excluded by the theorems, they must have a rather unfamiliar look. Thus, also their quantum properties may be very different from what one would expect. I will return to this point at the end of this Section.

To complete the discussion of the superspace formulation of N=1 Yang-Mills, I will now show how one goes about finding a prepotential and the superspace form of the action. For this case the solution was originally "guessed" [62]. This was possible because the constraint (21) that one must solve is very simple. It is the statement that the covariant derivative D_α is the one suited for the pure gauge case, *i.e.*

$$\nabla_\alpha = e^{-\Omega} D_\alpha e^{\Omega} \quad , \tag{2.29}$$

where Ω is a general (complex) scalar superfield. The solution for $\nabla_{\dot\alpha}$ is then obtained by complex conjugation. Here Ω has the gauge invariance

$$\delta e^{\Omega} = e^{-i\bar\Lambda} e^{\Omega} e^{iK} \quad , \tag{2.30}$$

with K a real superfield and $\bar\Lambda$ an antichiral superfield. It should be noted that the $\bar\Lambda$ transformation affects only the prepotential, but not the potential A_α. It is usually called a pre-gauge transformation. Actually, there exists an elegant way of arriving at this result [24] that also works for more complicated cases. All one does is notice that the covariant derivatives are functions of the gauge coupling constant. Differentiating eq. (21) with respect to it gives

$$\{\nabla_{(\alpha}, \frac{d\nabla_{\beta)}}{dg}\} = 0 \quad , \tag{2.31}$$

where the parentheses denote symmetrization. On account of eq. (21), eq. (31) is clearly solved by

$$\frac{d\nabla_\alpha}{dg} = [\nabla_\alpha, \Omega] \quad , \tag{2.32}$$

with Ω an arbitrary scalar superfield. Ω is the prepotential. The problem is integrating the matrix differential equation. In this case the solution is very simple, and is eq. (29) (once the coupling constant is set back to one). In general, however, a closed-form solution is not possible, and one must content himself with a power series in g. The superspace action for N=1 Yang-Mills is then obtained by noticing that, purely on dimensional grounds, the only gauge

invariant object of the right dimensionality one can write is

$$\mathrm{Tr} \int d^4x\, d^2\theta \left(W^\alpha W_\alpha + \mathrm{h.c} \right) . \tag{2.33}$$

The reader more familiar with N=1 superspace formulations will recognize that this description has some redundancy built in with respect to the conventional one in terms of a *real* scalar superfield V, with gauge invariance

$$\delta e^V = e^{-i\bar{\Lambda}} e^V e^{i\Lambda} , \tag{2.34}$$

where Λ is a chiral superfield. The formulation of eq. (29), usually called vector representation (see Ref. [56] for more details), stands to the usual one as the vielbein formulation of gravity stands to the metric formulation. The conventional formulation can be recovered by means of a partial gauge-fixing, whereby the parameter K in eq. (30) is used to gauge away the imaginary part of Ω.

I have already remarked that superfields are very convenient objects for deriving power-counting restrictions on the occurrence of particular kinds of counterterms. In this approach there are two steps:

(a). An off-shell superfield analysis, which examines the features of the perturbation expansion, and thus sorts out the kinds of structures allowed for the divergences;

(b). An on-shell superfield analysis, which shows which of the structures of the type in (a) survive on-shell. This step is, in principle at least, straightforward in all cases.

The really difficult problem is to find a suitable set of off-shell constraints for the covariant derivatives. This is the starting point for quantum calculations. The idea behind the power counting of Grisaru and Siegel [20] is then simple, at least in principle. One gains experience from the rather handy case of N=1 superfields by performing calculations "cleverly", *i.e.* by generating the least number of unnecessary terms at intermediate stages (in gauge theories this requires the use of the background field method to enforce gauge covariance at all stages). The result is that some structures are not generated at all, and must thus be absent from the list of all possible divergences. This analysis rests on the work of Refs. [19,20]. The conclusion is the following:

(1) All divergences are local *complete* θ-integrals, *i.e.* no integrals over subspaces of the superspace are allowed.

(2) The integrands are gauge-invariant functions of connections and field strengths only (*i.e.*, in the example of N=1 Yang-Mills discussed above, only A_α, $A_{\dot\alpha}$ and $A_{\alpha\dot\alpha}$ are allowed, not V!).

Since connections and field strengths, as well as θ integration measures, all have positive dimensionality, this puts strong restrictions on the existence of possible counterterms.

In order to have some say about the really interesting case of N>1 supersymmetry, for which the off-shell superfield formulations are mostly unknown (and often believed not to exist [25]), one needs some working hypotheses. In Ref. [20] the implicit working hypothesis is that similar manipulations to those possible for N=1 should go through for N>1 formulations, if these exist. This leads to rather surprising results.

For N-extended Yang-Mills at more than one loop in a background field gauge divergences would take the form

$$(g^2)^{L-1} \int d^dx\, d^{4N}\theta\, f(A,W) , \tag{2.35}$$

where A and W denote generically the connections and field strengths of the theory. The lowest-dimensional suitable f is of the form

$$A\,W + (\text{higher order in A}) , \tag{2.36}$$

which is gauge invariant modulo a total derivative on account of

$$\delta A = D\Lambda + (\text{more}) , \tag{2.37}$$

and of the typical on-shell Bianchi identity

$$D\,W + (\text{more}) = 0 . \tag{2.38}$$

Given that f in eq.(35) has at least dimension 2, the condition that the effective action be dimensionless yields the restriction

$$(4-d)L + 2(N-1) > 0 , \tag{2.39}$$

using the dimensionality of g^2, $(4-d)$ in d dimensions. Similarly, the lowest-dimensional counterterm for supergravity is the superspace analogue of the cosmological term,

$$(k^2)^{L-1} \int d^dx\, d^{4N}\theta\, sdet(E) , \tag{2.40}$$

where E is the superfield generalization of the vielbein, usually called

supervielbein (see Refs. [55,56] for more details). Counting powers again and recalling that k^2 has dimensionality $(2-d)$ in d dimensions then gives

$$(2-d)L + 2(N-1) > 0 \quad . \tag{2.41}$$

Eqs. (39) and (41) are what I referred to as the power-counting of Grisaru and Siegel [20]. It should be stressed that, whereas the derivation above is rather trivial, it rests on a nontrivial analysis of the superspace perturbation theory, which is at the heart of Ref. [20] (see also Ref. [25]), where it is shown that conditions (1) and (2) above do indeed hold for N=1 superfields. Referring to the discussion of N=1 Yang-Mills presented above, these statements are proved by showing that one can actually perform calculations in the background field method without ever expressing the background covariant derivatives in terms of the background prepotential.

Several remarks are in order here:

I. First of all, a technical one. In Ref. [20] it was noticed how a difficulty would arise at one-loop. In fact, manifest background covariance would lead in this case to an infinite tower of ghosts coupling only to the background fields, which thus contribute only at one-loop. Breaking background covariance would terminate the chain of ghosts, but would also alter the one-loop counterterms by making them noncovariant. Thus, barring more detailed analysis in special cases, the power counting only applies to diagrams with more than one-loop. For more details see Ref. [25].

II. The second remark is that the power-counting stems from a set of sufficient conditions, and can be improved by detailed case-by-case analysis, if the corresponding lowest dimensional counterterms happen to vanish. As they stand, however, eqs. (39) and (41) already produce a number of very interesting results:

(a). N≥2 Yang-Mills coupled to N=2 matter is finite beyond one loop in four dimensions, if an N=2 superfield formulation respecting the conditions (1) and (2) given above can be found for it. This is a rather impressive result indeed. Since N=2 superfield formulations have been constructed [21], this provides a very simple finiteness proof for N=4 Yang-Mills [20,25], alternative to the one previously given using light-cone superfields [26]. Moreover, a closer analysis reveals that there is a whole class of globally supersymmetric renormalizable theories with

N=2 supersymmetry which are finite to all orders. These were found in the first of Refs. [25] by demanding the cancellation of the one-loop contribution to the β-function, which forces one to choose suitable gauge groups and suitable group representations for the matter multiplets. This is sufficient, because finiteness beyond one loop is guaranteed by the power-counting of Ref. [20] and by the existence of the relevant superspace formulations [22-24]. The condition one finds for one-loop finiteness is

$$\sum_i m_i \; T(R_i) = C_2(G) \quad . \tag{2.42}$$

where one considers m_i N=2 hypermultiplets in the representations R_i and $\overline{R}_i$, and where $C_2(G)$ denotes the Casimir for the adjoint representation of the gauge group G. Clearly, there are many solutions to this condition. Actually, eq. (42) can also be generalized [66]. In fact, if some of the representations for the matter fields are pseudoreal, they do not need to be "doubled", and one obtains the condition

$$\sum_i m_i \; T(R_i) + \frac{1}{2}\sum_j m_j \; T(R_{pj}) = C_2(G) \quad . \tag{2.43}$$

For more details, see Ref. [66], where a list of solutions can be found. Interestingly, one can also add extra terms that violate N=2 supersymmetry, while still preserving the finiteness. This was noticed by Parkes and West [57]. The interest in this option is, of course, the greater freedom this allows in the process of model building with finite theories. The solution is not as inelegant as one may think, because supergravity couplings, after spontaneous breaking of local supersymmetry, induce terms which, at low energies, look like explicit breakings [67]. For more details, see Ref. [59].

(b) For supergravity, the success of a power-counting approach is doomed to be rather limited, because these theories are power-counting non-renormalizable. Still, eq. (41) tells us that, if N=8 supergravity could be formulated in terms of N=8 superfields, the first possible onset of ultraviolet divergences would be *ipso facto* postponed to seven loops, an encouraging improvement with respect to the obvious three-loop barrier. The state of the art, however, is not so encouraging. The theorems of Ref. [30] do not allow any conventional formulation of N=8

supergravity in terms of N=8 superfields, leaving at most the possibility of a straightforward formulation in terms of N=4 superfields. This could in principle be done using the known off-shell formulation of N=1 supergravity in ten dimensions and allowing for the existence of a set of four off-shell gravitino multiplets [25], a possibility not excluded by the counting arguments of Ref. [30]. At any rate, this would bring the first possible onset of divergences down to three loops again, just as would be the case for a component formulation. The possibility of unconventional formulations is, of course, left open. The very applicability of the power counting beyond N=2 superfields can be examined by direct calculation. I will come back to this point in the next Section.

III. The third remark is meant to emphasize what the heart of the problem is. I have already stressed that the power counting rests on a working assumption, namely that the manipulations carried out in the handy (and well known) case of N=1 superfields do have some bearing on the really interesting case of N>1 superfields. This is a very strong hypothesis, in fact stronger than the hypothesis that the corresponding off-shell superfield formulations exist. It amounts to assuming that N>1 superfield formulations bear a close resemblance to N=1 ones, which appears dubious, in view of the no go theorems of Refs. [30]. The main hypothesis these theorems rest upon is that the spectra of the theories can be read from their quadratic terms, *i.e.* that no cubic or higher order Lagrange multiplier terms are needed to determine them. Within this assumption, it can be shown that there are no auxiliary fields closing the off-shell algebra for N=4 Yang-Mills and for a number of other models [30]. This is done by showing that two equivalent ways of counting the fermionic auxiliary fields needed for the off-shell closure of the supersymmetry algebra give different, and thus inconsistent, results. One of these uses the known dimension of off-shell representations or superfields, and the other follows from the observation that fermionic auxiliary fields must come in pairs, say

$$\overline{\chi}\psi \ , \tag{2.44}$$

simply because one needs two different fields of half-odd-integer dimension to construct an invariant of dimension four. As a word of caution, one should note that the apparently insignificant restriction on the Lagrange multipliers actually fails for as familiar a case as that of nonlinear σ models,

and relaxing it allows off-shell Lorentz covariance [68] for one of the two inequivalent forms of N=2 supergravity in ten dimensions [69], a problem known to be otherwise insoluble [70].

3. The Breakdown of The Power Counting

The discussion presented in Section 2 was meant to emphasize one point. Superspace is a very powerful tool for analyzing divergences in supersymmetric theories, with great effectiveness in the case of renormalizable, and thus globally supersymmetric models, where the hypotheses underlying the power-counting of Ref.[20] have been substantiated by the explicit construction of the corresponding N=2 superfield formulations [22-24]. The reason for this success is twofold. First of all, the models are renormalizable, and thus power counting arguments have a good chance of being effective. Moreover, one only needs N=2 superfield formulations in order to exploit the power counting of Ref. [20] in all its strength. On the other hand, for supergravity theories the nonrenormalizability requires that stronger hypotheses be made on their superspace formulations to arrive at nontrivial, though only partial, results. The same is true for higher-dimensional Yang-Mills theories, again by virtue of their nonrenormalizability. Since N=4 Yang-Mills [5] is one of the models for which no conventional superfield formulation is possible, the study of its ultraviolet behavior in $d>4$ can in principle lead to useful indications for the more interesting, and far more difficult, case of supergravity. This analysis was carried out in Refs. [31-33] by Neil Marcus and the author. There we computed the divergences of the four-spinor S matrix amplitude at two loops for N=4 Yang-Mills in four, five, six, seven and nine dimensions. We used the Wick rotated action in components written in ten-dimensional notation,

$$S = \int d^{10}x \left(-\frac{1}{4}F_{\mu\nu}^2 - \frac{i}{2}\overline{\lambda}\not{\nabla}\lambda \right) , \tag{3.1}$$

where λ is a Majorana-Weyl spinor and the definitions of the terms above are the same as in eqs. (2.17). The corresponding one-loop analysis had already been carried out by Green, Schwarz and Brink [42], by taking the zero slope limit of the corresponding superstring amplitude. Their result, however, has no direct bearing on the validity of the superspace power-counting which, as I have emphasized in the previous Section, does not apply to one loop.

The theory was regularized using the dimensional reduction scheme [49], whereby external field indices are kept of fixed dimensionality upon regularization, which only affects the momenta. These are continued from d dimensions to $(d-\epsilon)$ dimensions, just as in conventional dimensional regularization. Dimensional reduction preserves the equality of Bose and Fermi degrees of freedom upon regularization. Thus, it allows for calculations done in the convenient minimal subtraction scheme, whereby only pole parts are subtracted, which at the same time preserve the Ward identities of supersymmetry.

Dimensional reduction is somewhat tricky to use, as it involves an algebra of formal manipulations of objects of two distinct kinds, d-dimensional indices and $(d-\epsilon)$-dimensional indices. That there are some ambiguities hidden in the corresponding large set of rules has long been noticed by Siegel himself [51], shortly after introducing the new regularization scheme. In fact, the ambiguities all have to do with manipulations of ϵ symbols, objects which are actually not susceptible of a consistent definition in the conventional dimensional regularization scheme [45]. The point is that dimensional reduction can give one the illusion that ϵ's can be manipulated naively, with potentially disastrous consequences. For example, in d dimensions, the product of two ϵ symbols can be turned into a product of δ's . Considering for simplicity the case of two-dimensional Euclidean space, one has

$$\epsilon^{\alpha\beta}\,\epsilon_{\gamma\delta} = \delta^{\alpha}_{\gamma}\,\delta^{\beta}_{\delta} - \delta^{\beta}_{\gamma}\,\delta^{\alpha}_{\delta} \quad . \tag{3.2}$$

This, however, yields zero if one specializes α and β to lie in a $(d-\epsilon)$-dimensional subspace and γ and δ to lie in the orthogonal ϵ-dimensional one, and leads to different results for the two manifestly identical expressions :

$$\epsilon^{\alpha\beta}\,\epsilon_{\gamma\delta}\,\epsilon_{\alpha\beta}\,\epsilon^{\gamma\delta} \quad , \tag{3.3}$$

and

$$\epsilon^{\alpha\beta}\,\epsilon_{\alpha\beta}\,\epsilon^{\gamma\delta}\,\epsilon_{\gamma\delta} \quad . \tag{3.4}$$

On the other hand, dimensional regularization works with $(d-\epsilon)$-dimensional indices, and does not allow for a consistent definition of ϵ altogether. In particular, it does not let one write eq. (2). Thus, it calls for caution whenever such quantities are present. We know that this should be the case, because ϵ symbols call for axial anomalies. These, in turn, imply an incompatibility between the

conservation of two quantities, that as such can be resolved only as a result of a deliberate choice. Whenever anomalies happen to cancel, the ambiguities also resolve because the choice then becomes unique. The conclusion is that ϵ's have to be dealt with carefully anyway, and the remarks of Ref. [51] appear not very significant. This subject matter can actually be investigated in detail, by comparing results obtained by dimensional reduction with corresponding ones obtained by the conventional dimensional regularization scheme, where one allows for nonminimal subtractions. This approach was suggested by van Damme and 't Hooft [52], and also pursued by others [53]. Whereas Ref. [52] contained several numerical errors and correspondingly wrong conclusions, the work of Jack and Osborn [53] is apparently free of errors. Their result is that, at the two-loop level, minimal dimensional reduction is equivalent to a nonminimal form of dimensional regularization. This is tantamount to saying that, to this order, there are no inconsistencies between supersymmetry and quantization, and implies that supersymmetry currents do not suffer from anomalies. This is very relieving, because for supergravity the supersymmetry current is a gauge current, and anomalies in it would destroy such theories completely, in a far more unquestionable way than ultraviolet divergences.

Then, going back to the discussion of Refs. [31-33], one recognizes that, when calculating in components, supersymmetry is only broken by the gauge-fixing procedure. The S-matrix, being gauge independent, is guaranteed to be supersymmetric. Calculating a four-spinor S matrix element thus tells the whole story about all the four-particle amplitudes. The results we found are summarized in the table below.

N=4 Yang-Mills at One and Two Loops							
	Dimensions						
Loops	4	5	6	7	8	9	10
1	F	T	F	T	I_8	T	I_{10}
2	F	F	F	I_7	–	I_9	–

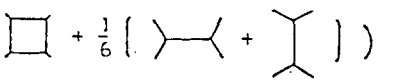
Here T stands for trivially finite, which is the case of all dimensionally regularized amplitudes in odd dimensions at odd numbers of loops, F stands for finite, and I stands for infinite. The subscripts label the form of the divergences encountered. The cases of d=8,10 at two loops were not considered, because the

theory was already known to be one-loop divergent there. A very useful device in analyzing the results was provided by Ref. [42]. There it was shown that all one-loop infinities in the four-particle S matrix of this theory could be recast as products of a common kinematic factor, which can be written

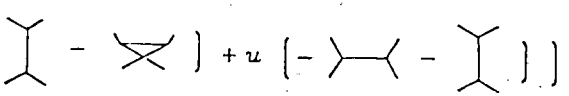
$$F_{\mu\nu}F_{\nu\rho}F_{\rho\sigma}F_{\sigma\mu} - \frac{1}{4}F_{\mu\nu}F_{\mu\nu}F_{\rho\sigma}F_{\rho\sigma} + 2i\bar{\lambda}\gamma_\alpha\partial_\beta\lambda F_{\alpha\gamma}F_{\beta\gamma}$$

$$-i\bar{\lambda}\gamma_{\alpha\beta\gamma}\lambda F_{\alpha\delta}\partial_\gamma F_{\beta\delta} - \frac{1}{3}\bar{\lambda}\gamma_\alpha\partial_\beta\lambda F_{\alpha\gamma}F_{\beta\gamma} \quad , \tag{3.5}$$

and of totally symmetric "group theory" factors, containing at times the Mandelstam-type differential operators $s = 2p_1p_2$, $t = 2p_1p_3$ and $u = 2p_1p_4$. Thus, apart from the common kinematic factor of eq. (5),

$I_8 = \frac{g^4}{(4\pi)^4\epsilon}$  , (3.6)

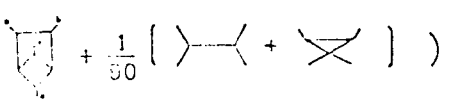
and

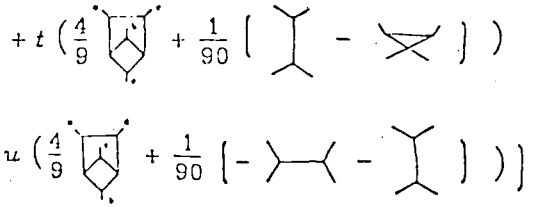
$I_{10} = \frac{1}{360}\frac{g^4}{(4\pi)^5\epsilon}$  . (3.7)

Here the group theory factors are described graphically using a notation first introduced by Cvitanović [71], whereby the structure constants f^{abc} are represented by a trilinear vertex:

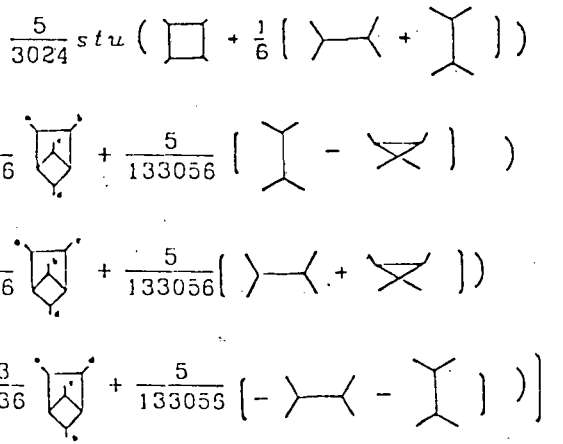
$f^{abc} \rightarrow$ (3.8)

On account of total antisymmetry, the vertex changes sign upon interchange of any two of its three legs. The features of this notation as applied to this case are described in detail in Ref. [32]. At two loops the divergences take similar forms with, however, more complicated group theory factors, as allowed by the larger number of structure constants present in two-loop graphs. Thus

$I_7 = \frac{g^6\pi}{(4\pi)^7\epsilon}$ 

 (3.9)

and

$I_9 = \frac{g^6\pi}{(4\pi)^9\epsilon}$  (3.10)

Actually, in Ref. [32] we have only computed the four-spinor piece of these terms, and we have let supersymmetry determine the other pieces. It should be noted how the two-loop divergences continue the structure of the one-loop ones first found in Ref. [42], where this was actually conjectured to occur. Assuming that this pattern continues to all orders (probably a consequence of supersymmetry, but this has not been proved in general), there follow the power counting rules [42]

$$d < 4 + \frac{2}{L} \tag{3.11}$$

for N=4 supersymmetric Yang-Mills, and

$$d < 2 + \frac{2}{L} \tag{3.12}$$

for N=8 supergravity. It should be noted how eq. (11) fails to predict the finitiness at two loops in six dimensions. The result (12) suggests that N=8 supergravity should diverge at three loops in four dimensions.

The two-loop finiteness in six dimensions would seem to be the most interesting result in the table altogether, because only the far more sophisticated power counting of eq. (2.37) appears to be able to explain it, and only for N=4, *i.e.* if a formulation in terms of N=4 superfields is available. This initially led us to an optimistic interpretation as suggesting that the power counting of Ref. [20] was correct for all N [31]. This, however, turns out not to be the case. A different, and more conservative, explanation, was provided by Howe and Stelle [34], who managed to explain the result of Ref. [31] using the available formulation of six-dimensional N=1 Yang-Mills in terms of six-dimensional N=1 superfields, properly supplemented by the enforcement of the extra, nonlinearly realized, supersymmetries. On the other hand, finiteness in $d=7$ is not guaranteed by the power counting of Ref. [20], even with N=4, and therefore naively the corresponding result in the table above sounds rather uninformative. However, matters turn out to be quite different. To understand all this, one should recall the discussion in the previous Section, where the main points of the analysis of Ref. [20] have been summarized (see remarks (1) and (2)). The point is that, not only do we have a power counting at our disposal, but we also have a very precise statement about the form of the on-shell effective action, namely that its divergent part is a local gauge invariant functional of connections and field strengths only (no prepotentials explicitly!) and containing a full θ-integral. For the case of $d=7$, on dimensional grounds, only one term is allowed:

$$\int d^7x\, d^{16}\theta\, \{ A_A\, w^A + \text{higher order} \} \quad , \tag{3.13}$$

where (see Ref. [32] for more details) A is a ten-dimensional spinor index, and

$$w^A = (\gamma_a)^{AB} (\gamma^a)^{CD} (D_C\, D_D\, A_B - D_B\, D_C\, A_D) \tag{3.14}$$

denotes the linearized form of the dimension-$\frac{3}{2}$ field strength W^A of ten-dimensional Yang-Mills (see the discussion of the four-dimensional case in Section 2 for comparison). In eq. (13) "higher order" stands for

$$(\gamma_a)^{AB} (\gamma^a)^{CD} \left(-\frac{2}{3} A_B [D_C\, A_D , A_A] + \frac{2}{3} A_B [D_C\, A_A , A_D] - \{ A_B , A_C \} \cdot \{ A_A , A_D \} \right), \tag{3.15}$$

derived by requiring invariance under the full nonlinear transformation

$$\delta A_A = [\nabla_A , \Lambda] \quad . \tag{3.16}$$

This is possible in view of the on-shell Bianchi identity satisfied by W,

$$\{ \nabla_A , W^A \} = 0 \quad . \tag{3.17}$$

The use of the ten-dimensional notation is no real restriction, because the smallest seven-dimensional superspace has the same dimensionality as the smallest ten-dimensional one. The result in eq. (9) admits a ten dimensional notation, in the sense that all restrictions to seven dimensions on the indices come from derivatives, which are necessarily seven dimensional objects. The same, of course, can be said for all the other divergences found. Thus, the contradiction with our result comes up because the term in eqs. (13) and (15) has a group theory structure completely fixed by the gauge invariance to be a "tree", in the graphical notation mentioned above. On the other hand, the two-loop divergence in seven dimensions contains a far more complicated group theory factor (see eq. (9)), and such a factor can be shown to be independent of the "tree". The conclusion is that the very essence of the power counting of Grisaru and Siegel [20], the form of the on-shell effective action, is *explicitly violated* beyond N=2 superfields. Whereas this is certainly not a proof that supergravity theories diverge at three loops, it is clearly a strong indication in that sense, because the only available reason for some optimism is removed. What is conclusive in this analysis is the statement that the naive extension of the manipulations made with N=1 superfields to N>2 fails. Whether this is due to the nonexistence of the corresponding off-shell superfield formalisms or to their having a different structure from the N=1 case is, of course, not possible to decide upon at this stage. At any rate, the majority of physicists are not so interested in the details of superspace as in its implications for finiteness, and the results I have discussed do tell us that the power counting cannot be trusted beyond N=2. As to the result in eq. (9), it appears very plausible for two reasons. First of all, it is of the form suggested by the superstring analysis of Ref. [42]. Moreover, one can show that, once it is dimensionally reduced, it can be written in terms of N=1 six-dimensional superfields. This is suggestive of an extension of the available six-dimensional superspace formalism above $d=6$, along similar lines to what is known to be possible for $N=1$ four-dimensional superfields [72].

The work of Refs. [31,32] that I have summarized here involved calculations far more difficult than those appeared earlier in the literature. Success in this enterprise rested heavily on the development of a technique for computing divergent parts of Feynman integrals [33] that I would now like to mention, though in a rather sketchy fashion. I actually believe that this technique is possibly more important than the results themselves, as it may open the way to far

more complicated direct tests of gravity and supergravity. In the following I will illustrate it, together with some minor, but still useful, points, in a series of remarks. This is meant to convey the general ideas behind the methods. For a more detailed discussion the reader is referred to the original literature [32,33].

I. Rather than calculating divergences of S matrix elements, one can calculate the on-shell divergences of the effective action. This implies calculating the parts of the effective action relevant to the given process and enforcing in them the *classical* field equation. Working with the external fields in the coordinate representation, no symmetrization is required. Each individual graph has then to be assigned a combinatoric factor equal to the reciprocal of the dimension of the corresponding discrete symmetry group, obtained allowing interchanges of both internal and external legs.

II. Rather than using counterterms determined by lower order calculations directly, one can more conveniently account for their effect by subtracting subdivergences from Feynman integrals. The minimal subtraction prescription results in a rule for dealing with a pair of mutually contracted Lorentz indices coming from a subtraction. Minkowski metrics generated by any two such indices must be "barred", in the sense that their trace must be taken to be d, rather than $(d-\epsilon)$. The contraction with other indices proceeds as though ϵ were a positive number, *i.e.* with metrics over the "lower" dimensional space dominant, just as projection operators would be. In this approach finiteness is recognized, at a given loop order, by the vanishing of the corresponding divergent contribution of order $\frac{1}{\epsilon}$ to the effective action, once the classical field equations are enforced. Terms determined by lower-order subtractions, however, can provide useful checks of the calculations.

III. Once one has settled to work with subtracted Feynman integrals, one can notice that, on general grounds [73], their divergences are *local* in coordinate space. There follows a very simple algorithm for extracting pole parts from general loop integrals [33], which can be efficiently implemented on computers, thus allowing for large scale divergence calculations at higher loops. Basically, there are two steps in this procedure. First of all, since the divergences are guaranteed to be polynomials in momentum space, say of (integer!) degree a, one can use Euler's theorem to lower the degree of divergence of subtracted Feynman integrals by differentiating with respect

to the external momenta p_i, *i.e.*

$$f^{(a)} = \frac{1}{a} p_i \frac{\partial}{\partial p_i} f^{(a)} \quad . \tag{3.18}$$

Then, once the integrals are reduced to logarithmically divergent ones, the divergent parts of these are guaranteed to be *constant*. They are thus independent of the external momenta, which can be arranged at will, with the only care of not running into fake infrared divergences in the evaluation of the resulting integrals. The conclusion is that computing the pole part of a general n-loop integral is tantamount to computing that of a propagator integral at $(n-1)$ loops. For example, consider the (rather simple) integral

$$\int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{1}{k^2 (k+p)^2 (k-l)^2 l^2 (l+p)^2} \tag{3.19}$$

for the case of six-dimensional space time, where it is quadratically divergent. Upon differentiation, this can be reduced to

$$\int' \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \left(\frac{4 k\cdot p \, l\cdot p}{k^6 l^6 (k-l)^2} + \frac{8 (k\cdot p)^2}{k^6 l^4 (k-l)^2} - \frac{2p^2}{k^6 l^4 (k-l)^2} \right) , \tag{3.20}$$

where the prime is meant to emphasize that each of the terms above has to be calculated together with the corresponding subtraction. These terms are all logarithmically divergent, and the momentum dependence has effectively "factored out". I would like to stress that this is possible only because one is looking at subtracted integrals, which are guaranteed to be local. The remaining part of the calculation is very simple, and yields

$$\frac{p^2}{3\epsilon^2} - \frac{p^2}{9\epsilon} \tag{3.21}$$

Clearly, a far greater simplification is obtained in more complicated cases, where the integrals depend on several external momenta. More details can be found in Ref. [33].

4. Discussion

I have attempted to outline the present understanding of the ultraviolet behavior of supersymmetric theories. Two things emerge clearly from this discussion. We have at our disposal a class of completely finite renormalizable models with extended global supersymmetry, and we have a number of formal

ways of proving their finiteness. However, at present the motivation for looking at such theories is not clear. More precisely, it is not clear why a finite model should be preferred to other infinite, but still renormalizable and predictive, ones. On the other hand, supergravity theories are *a priori* far more interesting, as they offer a perspective for unifying all interactions including gravity. However, their couplings are parametrized by Newton's constant, which is of negative mass dimension. Thus, these theories are all potentially nonrenormalizable. At present it does not seem possible to prove that they are finite along the lines of what has been achieved for supersymmetric Yang-Mills theories. All the available arguments fail, in one way or another, due to the presence of a dimensional coupling. Moreover, the indications of the indirect analysis of Ref. [32] are rather discouraging, and suggest that divergences should really set in at the "obvious" number of loops, three. Of course, explicit calculations in (super)gravity theories would be most illuminating. Hopefully, the integration technique mentioned in Sect. III, together with the development of a suitable computer software and, at least, the completion of the work of Ref. [74], should make this nontrivial task accessible in the near future.

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I am very grateful to Neil Marcus, with whom all the work of Refs. [31-33] was done during a long and enjoyable collaboration. We both received very valuable help from Marc Goroff with the most difficult calculations. I learned a large fraction of what I know about superspace formulations from Warren Siegel. I also long benefited from discussions on these matters with Jeff Koller and Barton Zwiebach. The Appendix is based on the presentation of two-component formalism in Barton Zwiebach's thesis [58]. Finally, I would like to thank Prof. Nicola Cabibbo for his kind invitation to lecture at this School.

Appendix

The two-component notation for four-dimensional space time can be simply arrived at from the more familiar four-component notation. The basic idea behind the two-component formalism is to build up all representations of the Lorentz group starting from its irreducible Weyl spinors. Thus, in some sense it is the most natural formulation. This reflects in the fact that it bypasses the need for explicit γ-matrices altogether. Moreover, it deals with indices running over two values, which can thus be easily symmetrized or antisymmetrized.

Consider an off-diagonal (Weyl) representation of the Dirac algebra (with space time signature (+---)), say

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} . \tag{A.1}$$

These define the matrices σ^μ and $\overline{\sigma}^\mu$ as the internal blocks of the four-dimensional γ matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)^{\alpha\dot\alpha} \\ (\overline{\sigma}^\mu)_{\dot\alpha\alpha} & 0 \end{pmatrix} , \tag{A.2}$$

and the helicity matrix is simply

$$\gamma_5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.3}$$

One writes a four-component spinor as

$$\psi = \begin{pmatrix} \psi^\alpha \\ \chi_{\dot\alpha} \end{pmatrix} . \tag{A.4}$$

The Majorana condition, in the representation of eq. (1), then implies

$$\begin{aligned} \chi^{\dot\alpha} &= \overline{\psi}^{\dot\alpha} = (\psi^\alpha)^\dagger \\ \chi_{\dot\alpha} &= \overline{\psi}_{\dot\alpha} = -(\psi_\alpha)^\dagger \; , \end{aligned} \tag{A.5}$$

where the indices are raised and lowered with the metric tensors $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot\alpha\dot\beta}$ and their inverses, all proportional to the σ^2 Pauli matrix:

$$\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = -\epsilon^{\alpha\beta} = -\epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} . \tag{A.6}$$

The conversion between four-component notation and two-component notation can then be simply achieved using the definitions in eqs. (1), (2) and (3). Thus,

for example, for two anticommuting Majorana spinors χ and ψ,

$$\overline{\chi}\psi = \left((\chi^\alpha)^* \; (\overline{\chi}_{\dot\alpha})^*\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi^\alpha \\ \overline{\psi}_{\dot\alpha} \end{pmatrix} = \chi^\alpha \psi_\alpha + \overline{\chi}^{\dot\alpha} \overline{\psi}_{\dot\alpha} \; . \tag{A.7}$$

Conventionally, one contracts indices from upper left to lower right. Changing convention for a pair of indices introduces a minus sign. The basic identity of the two-component formalism is that any quantity antisymmetric in two (un)dotted indices is proportional to the appropriate ϵ tensor. Thus, for example,

$$A_{\alpha\beta} - A_{\beta\alpha} = -\epsilon_{\alpha\beta} A^\gamma{}_\gamma \; , \tag{A.8}$$

where the coefficient has been fixed using

$$\epsilon^{\alpha\beta}\epsilon_{\alpha\beta} = 2 \; . \tag{A.9}$$

All other identities can be derived using the representation in eqs. (1) and (2). For example, from the γ matrices, one can construct the Lorentz matrices

$$\sigma^{\mu\nu} = \frac{1}{2}\gamma^{[\mu}\gamma^{\nu]} \; . \tag{A.10}$$

Their irreducible blocks are

$$\begin{aligned} (\sigma^{\mu\nu})^\alpha{}_\beta &= \frac{1}{2}(\sigma^{[\mu})^{\alpha\dot\alpha}(\overline{\sigma}^{\nu]})_{\dot\alpha\beta} \\ (\sigma^{\mu\nu})_{\dot\alpha}{}^{\dot\beta} &= \frac{1}{2}(\overline{\sigma}^{[\mu})_{\dot\alpha\beta}(\sigma^{\nu]})^{\beta\dot\beta} \; . \end{aligned} \tag{A.11}$$

These matrices have definite duality properties. Thus

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\sigma_{\rho\sigma})^\alpha{}_\beta = i\,(\sigma^{\mu\nu})^\alpha{}_\beta \; . \tag{A.12}$$

Then, writing an antisymmetric tensor $F_{\mu\nu}$ in terms of these projections as

$$F_{\mu\nu} = (\sigma_{\mu\nu})^\alpha{}_\beta F_\alpha{}^\beta + (\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta} F^{\dot\alpha}{}_{\dot\beta} \; , \tag{A.13}$$

one recognizes the term with undotted indices as corresponding to the self-dual part of $F_{\mu\nu}$, and the term with dotted indices as corresponding to the antiself-dual part. Finally, hermitian conjugation of superfields is obtained most simply by referring to the same operation on strings of θ's and $\overline{\theta}$'s. Thus, for example

$$(A_{\alpha\dot\beta}{}^\gamma)^* \to (\theta_\alpha \overline{\theta}_\beta \overline{\theta}^\gamma)^* = \theta^\gamma \theta_\beta \overline{\theta}_\alpha = \theta_\beta \overline{\theta}_\alpha \theta^\gamma \to A_{\beta\dot\alpha}{}^\gamma \; . \tag{A.14}$$

More details and an extensive list of formulas can be found in Ref. [58].

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