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MICROCANONICAL MASTER FIELDS

M.B. Halpern

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MICROCANONICAL MASTER FIELDS

Martin B. Halpern

Lawrence Berkeley Laboratory
and
Department of Physics
University of California
Berkeley, California 94720

Abstract

This paper introduces the concept of microcanonical master fields, a new variety of master field which solves the classical microcanonical equations of motion of very large systems. Examples of the new master fields are obtained explicitly in a number of simple large N systems. A general development of microcanonical equations on manifolds is also given, including a simple set of equations for lattice QCD.

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1. INTRODUCTION

There has been interest recently in the application of microcanonical methods[1,2] to various theories, including lattice gauge theories. In this paper, I discuss, in the microcanonical framework of Callaway et al., [1] the existence of new master fields which I call microcanonical. In this framework, fifth-time classical equations of motion (the microcanonical equations of motion) are associated to a given Euclidean action theory, and time-averages are measured. The new microcanonical master fields are solutions to these equations for very large systems; the solutions, although time-dependent, do not need to be time-averaged when computing global quantities (for example, the traces in large N SU(N)).

I will discuss the microcanonical method and the idea of the new master fields in Section 2, as well as their relation to the "configuration-average" master fields of Ref. (3). In Section 3 I discuss "master momenta". These are in fact a simple example of the configuration-average master fields and play an important role in constructing the new microcanonical master fields. In Sections 4 and 5, explicit microcanonical master fields are obtained in the large N limit of two simple classes of models, quadratic building-block models[4] and one hermitean matrix models. Section 6 is a general development of microcanonical equations of motion on manifolds, including a simple set of equations for SU(N) lattice QCD. Section 7 is an application of the development of Section 6, to a third class of soluble models on a large N SU(N) manifold. Conclusions and directions are discussed in the final Section 8.

2. THE MICROCANONICAL METHOD AND MASTER FIELDS

In the microcanonical method,[1,5] one begins with a D-dimensional Euclidean quantum field theory

$$Z = \int \left(\prod_{a=1}^{N} dx_a\right) e^{-\beta S(x)} \tag{2.1}$$

where S is the action and β is the inverse temperature. The theory is then augmented by the addition of new momenta p_a such that the partition function

$$\begin{aligned} Z &= \int \left(\prod_a dx_a dp_a\right) e^{-\beta H(x,p)} \\ H(x,p) &= \tfrac{1}{2} \sum_a p_a^2 + S(x) \end{aligned} \tag{2.2}$$

is interpretable as the D + 1 dimensional classical statistical mechanics of the classical Hamiltonian H(x, p). Note that the configuration-averages[F1] $< f(x) >_c$ are unaffected by the addition of the momenta, while the momenta themselves are Gaussian

$$< p_a p_b >_c = \beta^{-1} \delta_{ab} \tag{2.3}$$

and decorrelated from the x's.

As an alternative to configuration-averaging, one is interested here in solving the classical (microcanonical) Hamiltonian equations of motion

$$\dot{x}_a = p_a, \quad \dot{p}_a = -\frac{\partial S}{\partial x_a} \tag{2.4}$$

starting from some initial conditions $\{x_a(0), p_a(0)\}$, waiting until equlibrium, and then computing the time-averages $< f(x(t)) >_t$. Here time is the fictitious D + 1 st dimension, often called fifth-time. Sufficient conditions (e.g. ergodicity) are known under which the time-averages will equal the configuration-averages, and the two approaches will be equivalent. The idea is intuitive since configuration-averages are averages over snapshots during the time-development of a system that covers phase space. Remarkably, agreement between time and configuration-averaging has been seen in a number of physically interesting systems, including lattice gauge theories.[1] As an example, I mention only that the behavior

$$\frac{1}{N} < \sum_a p_a p_a >_t = \frac{1}{N} < \sum_a p_a p_a >_c = \beta^{-1} \tag{2.5}$$

is observed in systems of physical interest, and is used to measure the temperature. The precise implications of this phenomenon on notions such as ergodicity deserve further study.

I turn now to a discussion of master fields. In the case of configuration-averaging over very large systems (for example, the large N limit of large groups), it was realized recently[3] that averaging is not necessary for the computation of global[6] quantities (for example, the traces in large N SU(N)). Fluctuations are suppressed by the law of large numbers (large N factorization), so that any one of a very large class of equilibrium configurations, the master fields, will give the correct result. A simple example of these (configuration-average) master fields will be discussed in the following section. In Ref. [3], an explicit construction, via quenched Langevin equations, of large N

(configuration-average) master fields is given for general SU(N) symmetric theories, and this type of master field has been seen numerically in QCD.[6] Further developments in this field are given in Ref. [7].

On similar grounds, microcanonical master fields are expected to exist in very large systems. These master fields are time-dependent solutions to the classical microcanonical equations of motion (2.4) which, however, do not need to be time-averaged for the computation of global quantities. Microcanonical master fields are easy to visualize in a simple example. In computing, for example, the pressure on the wall of your office, a certain amount of time-averaging over molecular collisions with the wall is necessary. However, as N, the number of molecules in the room, increases, the time between collisions decreases, fluctuations in the pressure decrease, and the required amount of time-averaging goes down. In the large N limit, the pressure is independent of time (zero fluctuations is large N factorization), and can be computed at any instant, without time-averaging. To a high degree of accuracy then, the motion of all the molecules in your office forms a microcanonical master field, while a snapshot of a microcanonical master field, at some instant, is a (configuration-average) master field. On the basis of this picture, it is also expected, as in the case of (configuration-average) master fields, that microcanonical master fields will be far from unique, each being an appropriately thermalized (time-dependent) equilibrium configuration. In what follows, I will supplement the intuitive picture above with explicit examples of microcanonical master fields obtained as solutions of simple large N systems.

One of the basic ingredients in my construction is that a snapshot of a microcanonical master field should be a (configuration-average) master field, so that I can require, in the large N limit,

$$p_a(0) \underset{N}{=} p_a^* \tag{2.6}$$

where $\{p_a^*\}$ is a (configuration-average) master field for the momenta, which I call "master momenta". In the next section, I will discuss the construction of the master momenta.

3. MASTER MOMENTA

Consider N free particles in three dimensions, for which the partition function is

$$Z = \int(dp)\exp\left(-\beta \sum_{i=1}^{3N} \frac{p_i^2}{2m}\right) \tag{3.1}$$

where $\beta^{-1} = kT$, and T is the temperature. The average energy is a global quantity which may be computed as

$$< E > = < \sum_{i=1}^{3N} \frac{p_i^2}{2m} > \tag{3.2}$$

by configuration-averaging over all configurations of the momenta. Alternately, I may compute, in the large N limit, with a master field, the master momenta, p_i^*

$$< E > \underset{N}{=} \sum_{i=1}^{3N} \frac{(p_i^*)^2}{2m} \tag{3.3}$$

without averaging. Because

$$< p_i p_j > = m\beta^{-1}\delta_{ij} \tag{3.4}$$

then a correct set of master momenta are obtained as follows. From a table of zero-mean Gaussian random numbers with mean-square deviation $m\beta^{-1}$, pick 3N numbers $\{p_i^*\}$, $i = 1 \ldots 3N$. Since each element p_i^* is an independent Gaussian random variable with mean-square deviation $m\beta^{-1}$, then, in the large N limit, I may use the central limit

evaluate global sums of products of the p_i^*. For example, using the central limit theorem and the law of large numbers, I have

$$\sum_{i=1}^{3N} \frac{(p_i^*)^2}{2m} \underset{N}{=} 3N\left[\frac{<(p^*)^2>}{2m} + 0\left(\frac{1}{\sqrt{3N}}\right)\right] \tag{3.5}$$

$$= \frac{3}{2}NkT + 0(\sqrt{N})$$

which is the correct large N result for $< E >$. In general, as discussed in Ref. 3, the products of p_i^* inside large N global sums may simply be replaced by their large N average, e.g.

$$p_i^* p_j^* \underset{N}{\rightarrow} < p_i^* p_j^* > \underset{N}{=} m\beta^{-1}\delta_{ij} \tag{3.6}$$

so that the master momenta reproduce the large N configuration averages of all global sums.

Of course, there are many such sets of master momenta, each being an appropriately thermalized equilibrium configuration, but each set will give the same result for any global quantity at large N. Entirely analogous properties are given in Ref. [3] for the behavior of Gaussian hermitean master momenta p_{ab}^* (called there master noise) inside large N SU(N) traces.

In what follows, I will always pick the microcanonical momenta $p_a(t)$ to be equal at some reference time, say $t = 0$, to similarly chosen sets of master momenta p_a^*, whose mean-square deviation is always chosen equal to the mean-square deviation of the configuration-average of the momenta. In this way, I start the microcanonical equations in a sufficiently thermalized manner, and will be able to obtain

microcanonical master fields. I should emphasize that the construction yields the new master fields as functions of the p_a^*. Since the latter are Gaussian random variables, I am in fact constructing, as discussed above, a very large class of microcanonical master fields. I defer to Section 8 the important physical question whether, starting from some non-thermalized momenta, the microcanonical equations would thermalize to p^* at a later time.

4. QUADRATIC BUILDING-BLOCK MODELS

As a first soluble model, I consider the O(N) invariant integral

$$Z = \int (d\phi) e^{-\beta N V(z)} \tag{4.1}$$

where $Nz = \phi_a \phi_a$, a is summed from 1 to N, and V(z) is an arbitrary action. The model is easily augmented to a classical partition function by adding the momenta p_a,

$$Z = \int (d\phi\, dp) e^{-\beta H(\phi, p)} \tag{4.2}$$

where

$$H(\phi,p) = \tfrac{1}{2} p_a p_a + NV(z). \tag{4.3}$$

It follows that

$$< p_a p_b > = \beta^{-1} \delta_{ab} \tag{4.4}$$

while the classical equations of motion are

$$\dot{\phi}_a = p_a, \quad \dot{p}_a = -2\phi_a V'(z) \tag{4.5}$$

where prime means differentiation with respect to z.

I wish to find large N solutions to the classical equations of motion such that the (equal time) invariants $\phi_a \phi_a = \phi^2$, $p_a p_a = p^2$, $\phi_a p_a = \phi \cdot p$

are time-independent, which guarantees that no time-averaging will be be necessary. An adequate ansatz for the microcanonical master field is then, in column vector notation,

$$\phi(t) = e^{Kt}\lambda, \quad p(t) = e^{Kt}p^* \tag{4.6}$$

where K is a time-independent antisymmetric matrix which may be thought of as the generator of time translations (or auxiliary Hamiltonian), while

$$\lambda = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_N \end{bmatrix}, \qquad p^* = \begin{bmatrix} p_1^* \\ \cdot \\ \cdot \\ \cdot \\ p_N^* \end{bmatrix} \tag{4.7}$$

are the values of ϕ_a and p_a at $t = 0$. I will specify the master momenta p^* more carefully below. The task is to solve for the unknown quantities K and λ_N.

Substituting the ansatz (4.6) into the classical equations of motion (4.5), I have

$$K\lambda = p^* \tag{4.8a}$$

$$Kp^* = -2\lambda V'(\lambda_N^2/N) \tag{4.8b}$$

From (4.8a) it follows that

$$K_{aN} = \frac{p_a^*}{\lambda_N} \quad (a \neq N) \tag{4.9a}$$

$$p_N^* = 0 \tag{4.9b}$$

so that $\phi \cdot \pi = 0$ already. Equation (4.8b) can be separated into two statements. First $(i, j = 1 \ldots N\text{-}1)$

$$K_{ij}p_j^* = 0 \tag{4.10}$$

to which I shall return shortly, and second

$$K_{Ni}p_i^* = -2\lambda_N V'(\lambda_N^2/N). \tag{4.11}$$

Using (4.9a) and the antisymmetry of K, (4.11) becomes

$$\frac{1}{N}\sum_{i=1}^{N-1}(p_i^*)^2 = \frac{2\lambda_N^2}{N}V'(\lambda_N^2/N). \tag{4.12}$$

As discussed in Section 3, I now choose the master momenta p_i^* to be Gaussian random variables satisfying

$$< p_i^* p_i^* > \underset{N}{=} \beta^{-1}\delta_{ij} \tag{4.13}$$

in the large N limit, as specified by (4.4). Then, at large N,

$$p^2 = \sum_{a=1}^{N} p_a^* p_a^* \underset{N}{=} N\beta^{-1} \tag{4.14}$$

which agrees with the large N configuration-average, and Eq. (4.12) becomes

$$1 \underset{N}{=} 2\beta z_0 V'(z_0) \tag{4.15}$$

where z_0 denotes the value of $z = \lambda_N^2/N$ which solves (4.15). This is the correct large N saddle point (gap) equation of the model (4.2), as is easily seen with other methods.[4] Finally, I must also solve (4.10), for example by choosing $K_{ij} = 0$, so that

$$K_{ab} = \frac{p_a^* \delta_{b,N} - p_b^* \delta_{a,N}}{\sqrt{N z_0}} \tag{4.16}$$

This completes the determination of the time-dependent microcanonical master field (4.6), which gives the correct large N results for the invariants of (4.2) without time-averaging.

Solutions which are more covariant in appearance may be obtained without going to the special frame (4.7). A more general ansatz is

$$\phi(t) = e^{Kt}\phi(0), \quad p(t) = e^{Kt}p^* \tag{4.17}$$

so that

$$K\phi(0) = p^* \tag{4.18a}$$

$$Kp^* = -2\phi(0)V'(z). \tag{4.18b}$$

First choose p^* to be master momenta

$$< p_a^* p_b^* > \underset{N}{=} \beta^{-1}\delta_{ab}. \tag{4.19}$$

The auxiliary Hamiltonian may be taken proportional to the O(N) group generators

$$K_{ab} = \frac{p_a^* \phi_b(0) - p_b^* \phi_a(0)}{\phi^2(0)} \tag{4.20}$$

which solves (4.18a) if I place the single restriction on $\phi(0)$ that $\phi(0)\cdot p^* = 0$. Then (4.18b) becomes a second restriction on $\phi(0)$,

$$2z_0 V'(z_0) = \frac{2\phi^2(0)}{N} V'(\phi^2(0)|N) = \frac{1}{N}\sum_a (p_a^*)^2 \underset{N}{\simeq} \beta^{-1}. \tag{4.21}$$

Any solution (4.17) with these two restrictions on $\phi(0)$ is a microcanonical master field for the large N system. Note finally that all the solutions of this section may be put in the form[F2]

$$\phi_a(t) = \phi_a(0)\cos\omega t + p_a^* \frac{\sin\omega t}{\omega}$$

$$p_a(t) = p_a^* \cos t - \phi_a(0)\omega \sin\omega t \tag{4.22}$$

in terms of the single angular frequency

$$\omega = (\beta z_0)^{-\frac{1}{2}} \tag{4.23}$$

which increases with the square root of the temperature.

5. ONE HERMITEAN MATRIX MODELS

I consider next the SU(N) symmetric models of one hermitean N x N matrix M_{ab}

$$Z = \int (dM)\, e^{-\beta S(M)}$$
$$S(UMU^{\dagger}) = S(M), \quad U \in SU(N). \tag{5.1}$$

Although the method generalizes easily to arbitrary SU(N) symmetric actions S, I will work out the details only for the representative example

$$S(M) = \tfrac{1}{2}\,\mathrm{Tr}\, M^2 + \frac{g}{4N}\,\mathrm{Tr}\, M^4 \tag{5.2}$$

The augmentation to D = 1 classical statistical mechanics employs hermitean momenta π_{ab},

$$Z = \int (dM\, d\pi) e^{-\beta H(\pi, M)}$$
$$H(\pi, M) = \tfrac{1}{2}\,\mathrm{Tr}\, \pi^2 + \tfrac{1}{2}\,\mathrm{Tr}\, M^2 + \frac{g}{4N}\,\mathrm{Tr}\, M^4 \tag{5.3}$$

so that

$$< \pi_{ab}\, \pi_{cd} > = \beta^{-1}\, \delta_{bc}\, \delta_{ad} \tag{5.4}$$

while the microcanonical equations of motion are

$$\dot{M} = \pi$$
$$\dot{\pi} = -M - \frac{g}{N} M^3. \tag{5.5}$$

I wish to obtain the microcanonical master field at large N for the SU(N) invariants of the model (5.3), which are (equal time) traces of functions of M and π. So that these will be time-independent, I ansatz, following the intuition of the previous section,

$$M(t) = e^{iKt} \lambda e^{-iKt}$$
$$\pi(t) = e^{iKt} \pi^* e^{-iKt} \tag{5.6}$$

where K is the time-independent hermitean generator of time-evolution, while the diagonal matrix λ and the hermitean master momentum matrix π^* are also time-independent. With this ansatz, the equations of motion reduce to

$$i[K, \lambda] = \pi^* \tag{5.7a}$$

$$i[K, \pi^*] = -(\lambda + \frac{g}{N}\lambda^3) \tag{5.7b}$$

Equation (5.7a) is easily solved for the off-diagonal elements of the auxiliary Hamiltonian

$$K_{ab} = \frac{i\, \pi_{ab}^*}{\lambda_a - \lambda_b} \quad (a \neq b) \tag{5.8}$$

and diag $\pi^* = 0$. Substitution of (5.8) into (5.7b) results in

$$\sum_c{}' \pi_{ac}{}^* \pi_{cb}{}^* \left(\frac{1}{(\lambda_a - \lambda_c)} + \frac{1}{(\lambda_b - \lambda_c)} \right) - i\,(K_{aa} - K_{bb})\, \pi_{ab}{}^*$$

$$= (\lambda_a + \frac{g}{N} \lambda_a{}^3)\, \delta_{ab}, \tag{5.9}$$

where primed summations do not include singular terms. In particular, the diagonal elements of (5.9) give

$$\sum_b{}' |\pi_{ab}{}^*|^2 \frac{2}{\lambda_a - \lambda_b} = \lambda_a + \frac{g}{N} \lambda_a{}^3. \tag{5.10}$$

As specified by (5.4), I now choose the off-diagonal entries of the master momenta $\pi_{ab}{}^*$ to satisfy

$$< \pi_{ab}{}^* \pi_{cd}{}^* > \underset{N}{=} \beta^{-1} \delta_{ad} \delta_{bc} \tag{5.11}$$

in the large N limit. Then (5.10) becomes

$$2 \sum_b{}' \frac{1}{\lambda_a - \lambda_b} \underset{N}{=} \beta[\lambda_a + \frac{g}{N} \lambda_a{}^3] \tag{5.12}$$

which is the correct large N saddle point equation for the model.[8] Finally, the off-diagonal elements of (5.9), together with (5.11), require only that diag K = 0.

This completes the determination of the microcanonical master field (5.6) for the one-matrix model. As discussed in Ref. (3), the matrix master momenta (5.11) will correctly reproduce the configuration averages of large N traces of functions of π, and, because π^* is decorrelated from M, the large N mixed traces will be obtained correctly as well.

6. MICROCANONICAL EQUATIONS ON MANIFOLDS

A next level of difficulty is the class of models of one unitary matrix, which I will solve in Section 7. In order to obtain the microcanonical equations of motion for this case, I found it easiest to consider the problem on an arbitrary manifold, and then specialize to group manifolds. The approach I develop here is a generalization of the approach of Ref. (5) for the SU(2) x SU(2) non-linear sigma model.

To keep the notation simple, I will work on a single connected manifold without boundary M, with metric tensor $g_{\mu\nu}$ in a coordinate system x^μ, but the method is easily generalized to the case of many disconnected manifolds, such as lattice gauge theory, to which I return below. The coordinate-invariant model is

$$Z = \int (\sqrt{g} \prod_\mu dx^\mu) e^{-\beta S(x)} \tag{6.1}$$

where S(x) is a Euclidean coordinate-invariant action and $g = \det g_{\mu\nu}$. Following Ref. (5), I augment Z by the addition of momenta p_μ

$$Z = \int (\frac{1}{\sqrt{g}} \prod_\nu dp_\nu)(\sqrt{g} \prod_\mu dx^\mu) e^{-\beta H(x,p)}$$

$$= \int (dxdp) e^{-\beta H(x,p)} \tag{6.2}$$

where

$$H(x, p) = \tfrac{1}{2} p_\mu g^{\mu\nu} p_\nu + S(x). \tag{6.3}$$

The augmentation has not changed Z, or averages of functions of x^μ, as is easily seen on integrating out the momenta. The measure (dxdp) is invariant under arbitrary symplectic transformations, which

include coordinate transformations, under which p_μ is covariant. The measure therefore allows interpretation of Z as the classical partition function of the Hamiltonian H(x, p).

The microcanonical equations of motion on M may be obtained directly from H(x, p) via the canonical brackets

$$\{x^\mu, p_\nu\} = \delta_\nu{}^\mu \tag{6.4}$$

or from the associated fifth-time Lagrangian

$$L = \tfrac{1}{2}\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu - S$$
$$\frac{\partial L}{\partial \dot{x}^\mu} = g_{\mu\nu}\dot{x}^\nu = p_\mu. \tag{6.5}$$

The result is

$$\ddot{x}^\mu + \Gamma_{\rho\nu}{}^\mu \dot{x}^\rho \dot{x}^\nu + \nabla^\mu S = 0 \tag{6.6}$$

where $\Gamma_{\rho\nu}{}^\mu$ is the Christoffel connection on M and ∇_μ is covariant derivative. At S = 0, these are the geodesic equations on M. Another interesting form of the equations of motion is obtained in terms of the coordinate-invariant tangent-space momenta

$$p_a \equiv e_a{}^\mu p_\mu = e_\nu{}^a \dot{x}^\nu \tag{6.7}$$

where $e_a{}^\mu$ is the inverse of the vielbein $e_\mu{}^a$ such that $e_\mu{}^a e_\nu{}^a = g_{\mu\nu}$. In terms of these variables, the equations of motion (6.6) become

$$\dot{p}_a - e_c{}^\mu \omega_{\mu ab} p_b p_c + e_a{}^\mu \nabla_\mu S = 0 \tag{6.8}$$

where

$$\omega_{\mu ab} = -e_{a\nu}(\partial_\mu e_b{}^\nu + \Gamma_{\mu\lambda}{}^\nu e_b{}^\lambda) \tag{6.9}$$

is the spin connection on M. The Hamiltonian may be rewritten in terms of p_a

$$H = \tfrac{1}{2} p_a p_a + S \tag{6.10}$$

but it must be kept in mind that the p_a are not canonical variables, since, for example,

$$\{p_a, p_b\} = (e_b{}^\mu \partial_\mu e_a{}^\nu - e_a{}^\mu \partial_\mu e_b{}^\nu)\, e_\nu{}^c p_c. \tag{6.11}$$

The tangent-space momenta are important because their configuration averages are Gaussian

$$< p_a > = 0, \quad < p_a p_b > = \beta^{-1}\delta_{ab} \tag{6.12}$$

and decorrelated from x-averages. To see this, introduce a source $\{J_a\}$ for $\{p_a\}$ and integrate over p_μ,

$$< f(x) e^{J_a p_a} > = \exp\left(\frac{1}{2\beta} J_a J_a\right) < f(x) >. \tag{6.13}$$

Alternately, one may change variables from p_μ to p_a, obtaining

$$Z = \int(\prod_a dp_a)(\sqrt{g}\prod_\mu dx^\mu)e^{-\beta[\frac{1}{2}p_a p_a + S(x)]}. \tag{6.14}$$

The tangent-space momenta are therefore the analogues of the ordinary flat-space momenta, and can be used to measure the temperature of the system. By the same token, Gaussian tangent space master momenta $p_a{}^*$ are natural in seeking microcanonical master fields, as in the following section.

It is now straightforward to specialize the development to group manifolds. With f_{abc} the structure constants of the group, I may choose a basis which obeys the Cartan-Mauer equation

$$\begin{aligned} \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu} &= -f_{abc}e_{b\mu}e_{c\nu} \\ \omega_{\mu ab} &= \tfrac{1}{2}e_{\mu c}f_{cab} \end{aligned} \tag{6.15}$$

which amounts to choosing the basis vectors $e_a{}^\mu\partial_\mu$ to be the killing vectors

$$e_a{}^\mu\partial_\mu e_b{}^\nu - e_b{}^\mu\partial_\mu e_a{}^\nu = f_{abc}e_c{}^\nu. \tag{6.16}$$

It follows from (6.8), (6.11), (6.15) and (6.16) that

$$\begin{aligned} \dot{p}_a &= -e_a{}^\mu\nabla_\mu S = \{p_a, S\} \\ \{p_a, p_b\} &= -f_{abc}p_c \end{aligned} \tag{6.17}$$

so that the tangent space momenta are the classical analogues of the Kogut-Susskind electric fields.

Indeed, in the case of SU(N) or U(N), I may introduce unitary matrices U(x) such that

$$Z = \int(dU)\, e^{-\beta S(U)} \tag{6.18}$$

where dU is Haar measure, and [F3]

$$\begin{aligned} g_{\mu\nu} &= 2\,\mathrm{Tr}\,\partial_\mu U^\dagger\partial_\nu U \\ e_{\mu a} &= -2i\,\mathrm{Tr}(U^\dagger T_a\partial_\mu U). \end{aligned} \tag{6.19}$$

Here $\{T_a\}$ is the (hermitean) adjoint representation of the group, normalized according to

$$\begin{aligned} \mathrm{Tr}T_aT_b &= \tfrac{1}{2}\delta_{ab} \\ (T_a)_{rs}(T_b)_{pq} &= \tfrac{1}{2}(\delta_{rq}\delta_{sp} - N^{-1}\delta_{rs}\delta_{pq}) \end{aligned} \tag{6.20}$$

for SU(N). For U(N), drop the second term of (6.20). As a result, the Lagrangian (6.5) takes the familiar form

$$L = \mathrm{Tr}\,[\dot{U}\dot{U}^\dagger] - S \tag{6.21}$$

while the action of the Killing vectors on U(x) is

$$e_a{}^\mu \partial_\mu U = iT_a U$$

$$e_a{}^\mu \partial_\mu U^\dagger = -iU^\dagger T_a. \tag{6.22}$$

It follows that

$$\{p_a, U\} = -iT_a U$$

$$\{p_a, U^\dagger\} = iU^\dagger T_a \tag{6.23}$$

and so the entire system is seen as a fifth-time classical Kogut-Susskind-like system, with microcanonical equations of motion

$$-i\dot{U}U^\dagger = T_a p_a$$

$$\dot{p}_a = \{p_a, S\}. \tag{6.24}$$

As mentioned earlier, the development is easy to generalize to more complicated situations. In the case, for example, of Wilson lattice QCD,

$$Z = \int(\prod_\ell dU(\ell))\exp(\beta S_W) \tag{6.25}$$

treat each link ℓ as a disconnected unitary manifold with its own momenta etc. The results are

$$L = \sum_\ell \mathrm{Tr}[\dot{U}(\ell)\dot{U}^\dagger(\ell)] + S_W$$

$$H = \tfrac{1}{2}\sum_{a,\ell} p_a(\ell)\, p_a(\ell) - S_W \tag{6.26a}$$

and

$$\{p_a(\ell), p_b(\ell')\} = -f_{abc}p_c(\ell)\,\delta_{\ell\ell'}$$

$$\{p_a(\ell), U(\ell')\} = -i\,T_a U(\ell)\delta_{\ell\ell'}$$

$$\{p_a(\ell), U^\dagger(\ell')\} = i\,U^\dagger(\ell)T_a\delta_{\ell\ell'} \tag{6.26b}$$

which give the microcanonical equations of motion

$$-i\dot{U}(\ell)U^\dagger(\ell) = T_a p_a(\ell)$$

$$\dot{p}_a(\ell) = -\{p_a(\ell), S_W\}. \tag{6.27}$$

The tangent space momenta, or fifth-time electric fields $p_a(\ell)$ also satisfy

$$< p_a(\ell)\, p_b(\ell') > = \beta^{-1}\delta_{ab}\delta_{\ell\ell'} \tag{6.28}$$

as above, and can be used to measure the temperature.

7. ONE UNITARY MATRIX MODELS

As a simple example of the formalism above, consider the one unitary matrix model

$$Z = \int (dU) e^{-\beta \mathrm{Tr}(U + U^\dagger)} = \int (\sqrt{g}\, dx) e^{-\beta \mathrm{Tr}[U(x) + U^\dagger(x)]} \tag{7.1}$$

where (dU) is Haar measure on U(N). In fact there will be no difficulty in generalizing the method to an arbitrary U(N) invariant action which is a function of U. In terms of the hermitean matrix electric field

$$E \equiv 2\, T_a p_a, \tag{7.2}$$

I obtain the classical Hamiltonian system (6.10),

$$H = \tfrac{1}{4} \mathrm{Tr} E^2 + \mathrm{Tr}(U + U^\dagger)$$

$$\{E_{ab}, U_{a'b'}\} = -i\, \delta_{ba'} U_{ab'}$$

$$\{E_{ab}, (U^\dagger)_{a'b'}\} = i (U^\dagger)_{a'b} \delta_{ab'}$$

$$\{E_{ab}, E_{a'b'}\} = i(\delta_{ab'} E_{a'b} - \delta_{ba'} E_{ab'}), \tag{7.3}$$

and therefore the microcanonical equations of motion (6.24),

$$-i\, \dot{U} U^\dagger = \tfrac{1}{2} E$$

$$\dot{E} = \{E, S\} = i\,(U^\dagger - U). \tag{7.4}$$

Also needed is the Gaussian distribution of the electric field

$$< E_{ab} E_{cd} > = 2\beta^{-1} \delta_{ad} \delta_{bc} \tag{7.5}$$

which follows from (6.12) and (7.2).

Following the previous strategy, I ansatz

$$U(t) = e^{iKt} e^{i\theta} e^{-iKt}$$
$$E(t) = e^{iKt} E^* e^{-iKt} \tag{7.6}$$

where the hermitean time-translation generator K, the diagonal matrix $e^{i\theta}$, and E^*, the master electric field, are all time-independent, and so therefore are all the traces of the theory. The equations of motion (7.4) then reduce to

$$(K, e^{i\theta}) e^{-i\theta} + \tfrac{1}{2} E^* = 0 \tag{7.7a}$$

$$(K, E^*) + 2i \sin\theta = 0 \tag{7.7b}$$

From (7.7a) I obtain, as usual, the off-diagonal elements of the generator of time translations,

$$K_{ab} = \frac{e^{-i\theta_{ab}/2} E^*_{ab}}{4i \sin\frac{\theta_{ab}}{2}} \quad (a \neq b) \tag{7.8}$$

where $\theta_{ab} = \theta_a - \theta_b$, $\{\theta_a\} = \text{diag}\,\theta$, and the restriction diag $E^* = 0$. Using (7.8) in (7.7b) results in

$$\tfrac{1}{2} \sum_c{}' E^*_{ac} E^*_{cb} \left[\frac{e^{-i\theta_{ac}/2}}{\sin\frac{\theta_{ac}}{2}} - \frac{e^{-i\theta_{cb}/2}}{\sin\frac{\theta_{cb}}{2}} \right] \tag{7.9}$$

$$+ 2i(K_{aa} - K_{bb}) E^*_{ab} = 4 \sin\theta_a \delta_{ab}$$

where the prime indicates that singular terms are excluded. The relation (7.5) specifies the choice of the (off-diagonal) Gaussian master electric fields,

$$< E^*_{ab} E^*_{cd} > \underset{N}{=} 2\beta^{-1} \delta_{ad} \delta_{bc} \tag{7.10}$$

so that the diagonal elements of (7.9) read

$$\sum_b{}' \cot\frac{\theta_{ab}}{2} \underset{N}{=} 2\beta \sin\theta_a, \tag{7.11}$$

while the off-diagonal elements of (7.9) give $K_{aa} = K_{bb}$. The result (7.11) is the correct saddle point equation of the original model, as is easily checked using the invariant factor of the Haar measure.[9] This completes the determination of the microcanonical master field (7.6). As discussed in Ref. (3), the master matrix electric field E^* will correctly reproduce all the large N configuration averages of traces of functions of E, and the mixed traces as well.

8. DIRECTIONS

In this paper, I have argued intuitively for the existence of microcanonical master fields, and given several explicit examples as solutions of simple large N systems. Some deficiencies, and therefore directions, are clear. First, it would be satisfying to have a formalism to compute the master fields, thus avoiding much of my guess work and ansätze. A possible approach along these lines might employ more or less standard large N Hamiltonian methods[10] on the Liouville operators of the classical systems. This idea has been entertained independently by H. Neuberger. Second, such a formalism might be of utility in extending the idea to the large N limit of realistic quantum field theories.

Other interesting questions could be studied even within these simple models. Are the master fields "limit cycles" of the equations of motion, with some basin of attraction? This question is closely related to the question of thermalization (to p*) over a period of time, and to questions of ergodicity. The answer to these questions is not always yes. In the quadratic building-block example $NV = \frac{1}{2}\phi_a\phi_a$, the system is a collection of simple harmonic oscillators, which is known not to be ergodic, and not to thermalize. I also note that the master fields themselves are not ergodic in the usual sense. In the quadratic building-block models, for example, the large N solution spends its time at the large N radius selected by the gap equation, the rest of phase space being irrelevant in the large N limit.

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FOOTNOTES

F1. By configuration averages, I mean canonical ensemble averages, using (2.2). Although the assumption is not used in this paper, these are presumably equal to microcanonical ensemble averages for large systems.

F2. In fact the forms (4.22) solve the microcanonical equations (4.5) and produce the correct large N results for any initial conditions such that $\phi_a(0)\, p_a(0) = 0$, $p_a(0)p_a(0) = N\beta^{-1}$, while $\phi_a(0)$ solves the gap equation (4.15). Thus, although master momenta $p_a{}^*$ may appear in a measurement of the system at large N, they are not strictly necessary, in this case, for the solution to give the correct large N results. This is an artifact of these simple O(N) models, in which p_a is involved in only two invariants.

F3. These correspondences are discussed for an arbitrary group manifold in Ref. (11).

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