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RECURSION RELATIONS FOR SOLUTIONS TO THE SCHRÖDINGER EQUATION

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RECURSION RELATIONS FOR SOLUTIONS TO THE SCHRÖDINGER EQUATION[1]

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RECURSION RELATIONS FOR SOLUTIONS TO THE SCHRÖDINGER EQUATION

ABSTRACT

We will consider the general eigenfunction for a 2nd-order differential operator in one variable. For many well-known elementary functions, we can also find a three-term recursion relation in the eigenvalue parameter. For practical computation, this is a very desirable property. Examples include Legendre functions, Bessel functions, etc.

In reference [2], Bochner showed that the only polynomial solutions to this problem were the well-known ones. This paper will look for solutions that may not be polynomials.

It will be shown that, unfortunately, for the simple recursion relation considered here, no really new examples exist.

1. Notation

Let L be a 2nd order differential operator in one variable x,

$$L\varphi(x) \equiv g_1(x)\,\frac{d^2\varphi}{dx^2} + g_2(x)\,\frac{d\varphi}{dx} + g_3(x)\varphi\,, \tag{1}$$

and let B be a 2nd order difference operator in k, acting on a sequence $\{l_k\}$ by

$$(Bl)_k \equiv a_k l_{k+1} + b_k l_k + c_k l_{k-1}\,. \tag{2}$$

For later convenience, subscripts like k will range over $\mathbf{Z}+c$ for some fixed $c \in \mathbf{C}$; that is, over all complex numbers differing from a given one by an integer. Constants will normally be complex; functions will be: $\mathbf{C} \to \mathbf{C}$ and as smooth as necessary.

2. Basic Problem

We seek a sequence of functions $\{\varphi_k(x)\}$ s.t.

$$L\varphi_k(x) = \lambda_k \varphi_k(x) \tag{3$_k$}$$

and

$$(B\varphi(x))_k = a_k\varphi_{k+1}(x) + b_k\varphi_k(x) + c_k\varphi_{k-1}(x) = \vartheta(x)\varphi_k(x) \tag{4$_k$}$$

for all $k \in \mathbf{Z}+c$ and some L, $\{a_k, b_k, c_k\}$, $\{\lambda_k\}$, $\vartheta(x)$. Of course, we really seek the whole sextuple $(L, B, \lambda, \vartheta, \varphi, c)$.

We can allow some (not all) of the φ_k's to be zero. This will enable us to include the classical orthogonal polynomials. We will see in Chapter 5, though, that all φ_k's must be nonzero, either for each k big enough, or else for each k small enough. $\vartheta(x)$ should be nonconstant in x, and λ_k nonconstant in k. Note that L and ϑ are independent of k, and that B and λ are independent of x. This will allow us to commute them in several formulas.

3. Symmetries

Given a solution to our Basic Problem, many obvious changes yield new sextuples which will also be solutions. Nine of these changes will be listed, to be used extensively later. Assume, then, that $c' \in \mathbf{C}$ is an arbitrary constant, and that $(L, B, \lambda, \vartheta, \varphi, c)$ solves the Basic Problem. The following is a list of new solution sextuples:

(a)

$$(c'L, B, c'\lambda, \vartheta, \varphi, c) \tag{S1}$$

This scales L by a constant. Since $(3)_k$ implies $(c'L)\varphi_k = (c'\lambda_k)\varphi_k$, (S1) is

also a solution.

(b)

$$(L, B, \lambda, \vartheta, c'\varphi, c) \tag{S2}$$

φ is scaled by a constant here. (S2) clearly follows from multiplying $(3)_k$ and $(4)_k$ by c'.

(c)

$$(L + c', B, \lambda + c', \vartheta, \varphi, c) \tag{S3}$$

(S3) transfers a constant between L and λ_k. We see it by modifying $(3)_k$ to be $(L+c')\varphi_k = (\lambda_k + c')\varphi_k$.

(d)

$$(L, B, \lambda, \vartheta, \varphi, c - c') \tag{S4}$$

This new solution just has a shifted subscript. The $(3)_k$ corresponding to it is now $L\varphi_{k+c'} = \lambda_{k+c'}\varphi_{k+c'}$ for $k \in \mathbf{Z} + c - c'$ and the new $(4)_k$ is similar.

(e)

$$(L, c'B, \lambda, c'\vartheta, \varphi, c) \tag{S5}$$

(S5) scales the recursion relation by a constant. The new $(4)_k$ will be $((c'B)\varphi)_k = (c'\vartheta(x))\varphi_k$.

(f)

$$(L, B + c', \lambda, \vartheta + c', \varphi, c) \tag{S6}$$

In (S6), a constant is transferred between B and ϑ, so that $(4)_k$ is now

$(B\varphi)_k + c'\varphi_k = (\vartheta(x) + c')\varphi_k.$

(g) Symmetry seven is under the change of independent variable x, say by $x = x(x')$. If L' is the new differential operator resulting from the variable change, we derive the new solution (S7):

$$(L', B, \lambda, \vartheta(x(x')), \varphi(x(x')), c) \tag{S7}$$

$(3)_k$ and $(4)_k$ change in the obvious ways. We will use this transformation most commonly for the simple shift $x = x' + c'$.

(h) Let $\varphi_k(x) = f(x)\psi_k(x)$, for some function $f(x)$ and all k. Then

$$(f^{-1}Lf, B, \lambda, \vartheta, \psi(x), c) \tag{S8}$$

is a solution allowing us to multiply φ by an arbitrary function of x. $(4)_k$ will look similar, with ψ in place of φ, and $(3)_k$ will be $(f^{-1}Lf)\psi_k(x) = \lambda_k\psi_k(x)$.

(i) Let $\varphi_k(x) = f_k\psi_k(x)$, for all k and for f_k independent of x. This multiplication of φ_k by an arbitrary constant depending only on k gives the new solution

$$(L, f^{-1}Bf, \lambda, \vartheta, \psi(x), c). \tag{S9}$$

Here $(3)_k$ will be similar, and $(4)_k$ becomes $f_k^{-1}(B(f\psi))_k = \vartheta(x)\psi_k(x)$.

We will be using these to transfer to simpler-looking solutions, and eventually to reduce to a few fundamental ones. Clearly, the space of possible variations of these fundamental solutions will be extremely large.

4. Commutation Relations

L is 2nd order, and we can think of the operator of multiplying by $\vartheta(x)$ as 0th order. Then

$$\begin{aligned} [\vartheta, L] &\equiv \vartheta L - L\vartheta && \text{is 1st order,} \\ [\vartheta,[\vartheta, L]] & && \text{is 0th order, and} \\ [\vartheta,[\vartheta,[\vartheta, L]]] & && = 0. \end{aligned}$$

Thus:

$$[\vartheta,[\vartheta,[\vartheta, L]]]\varphi_k = 0.$$

A typical term of this last expression is

$$\begin{aligned} & -3\vartheta\vartheta L\vartheta\varphi_k, && \\ = \quad & -3\vartheta\vartheta L(B\varphi)_k && \text{by } (4)_k\ , \\ = \quad & -3\vartheta\vartheta(B(L\varphi))_k && \text{as } L \text{ and } B \text{ commute,} \\ = \quad & -3\vartheta\vartheta(B(\lambda\varphi))_k && \text{by } (3)_k\ , \\ = \quad & -3(B\lambda\vartheta\vartheta\varphi)_k && \text{since } B \text{ and } \vartheta \text{ also commute,} \\ = \quad & -3(B\lambda BB\varphi)_k && \text{once again using } (4)_k\ . \end{aligned}$$

Continuing in this way, we soon see that

$$0 = [\vartheta,[\vartheta,[\vartheta, L]]]\varphi_k = ([B,[B,[B,\lambda]]]\varphi)_k\ .$$

This last expression is a linear combination of seven terms, multiplying $\varphi_{k-3}, \varphi_{k-2}, \ldots, \varphi_{k+3}$, respectively. If we assume the φ_k's linearly independent for different k's (as, for example, if $\lambda_k \neq \lambda_{k'}$, for $k \neq k'$), each of the seven terms must vanish. In particular, let us examine the terms containing φ_{k+3} and φ_{k-3}, which each must be zero.

The term with φ_{k+3} is

$$a_{k+2}a_{k+1}a_k(\lambda_{k+3} - 3\lambda_{k+2} + 3\lambda_{k+1} - \lambda_k)\varphi_{k+3}\ .$$

The term with φ_{k-3} is

$$c_k c_{k-1} c_{k-2}(\lambda_k - 3\lambda_{k-1} + 3\lambda_{k-2} - \lambda_{k-3})\varphi_{k-3}\ .$$

These must be zero for all $k \in \mathbf{Z} + c$. Thus, for a given k,

$$\begin{aligned} &\text{Either} \quad (a_k a_{k+1} a_{k+2}\varphi_{k+3} = c_{k+1}c_{k+2}c_{k+3}\varphi_k = 0) \\ &\text{or} \qquad \lambda_k - 3\lambda_{k+1} + 3\lambda_{k+2} - \lambda_{k+3} = 0\ . \end{aligned} \tag{5}$$

5. Pinning down λ

(Theorem)

At least one of the following statements is true:

$$\begin{aligned} &1)\ c_k \neq 0 \text{ and } \varphi_k \neq 0 \text{ for all } k \text{ sufficiently large negative} \\ \text{or}\quad &2)\ a_k \neq 0 \text{ and } \varphi_k \neq 0 \text{ for all } k \text{ sufficiently large positive.} \end{aligned} \tag{6}$$

(Recall that the a's and c's are the coefficients in (4).)

Proof: Pick some $\varphi_l \neq 0$. If, for all $k < l$, c_k and $\varphi_k \neq 0$, the theorem is proven. Otherwise, there is some largest $m \leq l$ such that

$$c_m \cdot \varphi_{m-1} = 0.$$

Since this m is maximal, $\varphi_m \neq 0$. Then $(4)_m$ becomes

$$0 + (b_m - \vartheta(x))\varphi_m + a_m \varphi_{m+1} = 0.$$

Thus $\lambda_m \varphi_{m+1} \neq 0$ and $\varphi_{m+1} = \varphi_m \cdot$ (polynomial in $\vartheta(x)$, degree 1). Now $(4)_{m+1}$ implies

$$\varphi_m \cdot (\text{polynomial in } \vartheta(x),\ \text{degree } 2) + \lambda_{m+1}\varphi_{m+2} = 0.$$

Therefore $\lambda_{m+1}\varphi_{m+2} \neq 0$ and $\varphi_{m+2} = \varphi_m \cdot$ (polynomial in ϑ, degree 2). We continue this way to generate the φ_k's for $k > m$. Since ϑ is not constant, the nth degree polynomial in ϑ at the nth step can't be zero. At the nth step we'll get $\lambda_{m+n}\varphi_{m+n+1} \neq 0$, so that we've shown $\lambda_n \neq 0$ and $\varphi_{n+1} \neq 0$ for all $n \geq k$. ∎

We will write $k \in K$ as a synomym for "k sufficiently large negative" or "k sufficiently large positive", respectively. It will turn out in Chapter 7 that all we'll need is that the theorem be true for many contiguous k's. This explains why we can be so ambiguous about the finite endpoint of K.

From this theorem and (5) we get that

$$\lambda_k - 3\lambda_{k+1} + 3\lambda_{k+2} - \lambda_{k+3} = 0 \quad \text{for } k \in K .$$

Therefore, $\lambda_k = r_1 k^2 + r_2 k + r_3$ for $k \in K$; $\quad r_1, r_2, r_3$ fixed.

$\lambda_k \neq$ constant, so $r_1 \neq 0$ or $r_2 \neq 0$.

If $r_1 \neq 0$, use (S1) to scale λ_k so that the leading coefficient is 1. Then (S4) can shift k and kill the linear term. Finally, (S3) transfers the constant term into L, yielding $\lambda_k = k^2$.

If $r_1 = 0$, $r_2 \neq 0$, (S1) can be used to scale the leading coefficient of λ_k to be 1. (S4) will kill the constant term by shifting k, giving $\lambda_k = k$. We can keep (S3) in reserve this time for possible need later.

We have reduced to either of two cases

$$\lambda_k = k \quad \text{or} \quad \lambda_k = k^2 \quad \text{for} \quad k \in K . \tag{7}$$

6. Some Convenient Formulas

(Lemma)

$$(L-\lambda_k)^n(\vartheta\varphi_k) = \underbrace{[L,[L,[\cdots,[L,\vartheta]\cdots]}_{<----n\ L's---->}\varphi_k \qquad n \geq 0 \tag{8}$$

Proof: For $n = 1$, the proof is

$$(L-\lambda_k)\vartheta\varphi_k = (L\vartheta - \vartheta\lambda_k)\varphi_k = (L\vartheta - \vartheta L)\varphi_k .$$

Larger n's can be shown similarly by induction. ■

(Lemma)

$$(L-\lambda_{k+1})(L-\lambda_k)(L-\lambda_{k-1})(\vartheta\varphi_k) = 0.$$

Proof: $\vartheta\varphi_k = (B\varphi)_k =$ a linear combination of $\varphi_{k-1}, \varphi_k, \varphi_{k+1}$. ■

Call $\Delta_k \equiv \lambda_k - \lambda_{k+1}$, $\nabla_k \equiv \lambda_k - \lambda_{k-1}$. Then from the previous lemma,

$$\begin{aligned} 0 &= (L-\lambda_{k+1})(L-\lambda_k)(L-\lambda_{k-1})(\vartheta\varphi_k) \\ &= (L-\lambda_k+\Delta_k)(L-\lambda_k)(L-\lambda_k+\nabla_k)(\vartheta\varphi_k) \\ &= ((L-\lambda_k)^3 + (\Delta_k+\nabla_k)(L-\lambda_k)^2 + \Delta_k\nabla_k(L-\lambda_k))(\vartheta\varphi_k)\,. \end{aligned}$$

By (8), $\quad = ([L,[L,[L,\vartheta]]] + (\Delta_k+\nabla_k)[L,[L,\vartheta]] + \Delta_k\nabla_k[L,\vartheta])\varphi_k = 0\,.$

Call $A_n \equiv$ the operator $\underbrace{[L,[L,\cdots[L,\vartheta]]\cdots]}_{<----n\ L's---->}$. Then

$$A_3\varphi_k + (\Delta_k+\nabla_k)A_2\varphi_k + (\Delta_k\nabla_k)A_1\varphi_k = 0 \tag{9}$$

which is often a convenient form.

In [3], Grünbaum and Duistermaat attack the problem of finding a differential equation, (instead of a recursion relation), in the spectral parameter, and arrive at a simpler but similar formula.

7. Finding Possible Potentials

We need not work with the general 2nd order operator for L. Using (S7) to set the leading coefficient to 1, and (S8) to kill the $\frac{d}{dx}$ term, we can use

$$L = \frac{d^2}{dx^2} + V(x), \qquad \text{without loss of generality}\,.$$

Hand calculations show: $\left(\text{use } D = \frac{d}{dx}\right)$

$$A_1 = 2\vartheta' D + \vartheta'' \quad , \quad A_2 = 4\vartheta'' D^2 + 4\vartheta''' D + \vartheta'''' - 2\vartheta' V'\,,$$

$$\begin{aligned} A_3 &= 8\vartheta''' D^3 + 12\vartheta'''' D^2 + (6\vartheta^{(5)} - 4\vartheta' V'' - 12\vartheta'' V')D \\ &\quad + (\vartheta^{(6)} - 6\vartheta''' V' - 8\vartheta'' V'' - 2\vartheta' V''') \end{aligned}$$

Case I) $\lambda_k = k$, $k \in K$ (from (7)). Here $\Delta_k = -1$, $\nabla_k = +1$. Then by (9)

$$(A_3 - A_1)\varphi_k = 0 \quad \text{for } k \in K.$$

Since the finite order operator $A_3 - A_1$ has here infinitely many independent solutions (by the Addendum to Thm (6), $\varphi_k \neq 0$ for all $k \in K$), it must be identically zero:

$$A_3 - A_1 \equiv 0. \tag{10}$$

We will use this argument again. Since we only need an infinite number of solutions, the sloppy definition of K in Chapter 5 is sufficient. Take (10), and equate to zero each coefficient of a power of D.

D^3 : $8\vartheta''' = 0 \quad => \quad \vartheta = r_1 x^2 + r_2 x + r_3$ for some fixed r_1, r_2, r_3
D^2 : $12\vartheta'''' = 0 \quad => \quad$ nothing new.
D^1 : $6\vartheta^{(5)} - 4\vartheta' V'' - 12\vartheta'' V' = 2\vartheta'$. Using the D^1 equation $=>$
$-4(2r_1 x + r_2)V'' - 24 r_1 V' = 2(2r_1 x + r_2)$.
D^0 : nothing new.

There are now two subcases

(a) $r_1 = 0$, so $-4r_2 V'' = 2r_2$. $\vartheta \neq$ constant, so r_2 cannot be zero. Then

$$\begin{cases} V = -\dfrac{1}{4}x^2 + q_1 x + q_2 & \text{for some fixed } q_1, q_2 \\ \vartheta = r_2 x + r_3 \end{cases}$$

Use (S7) to shift $x \to x + 2q_1$, and (S3)+(S4) to move a constant into λ_k, giving $V = -\dfrac{1}{4}x^2 - \dfrac{1}{2}$ $\left[-\dfrac{1}{2} \text{ for later convenience}\right]$. Use (S5),(S6) giving $\vartheta = x$.

(b) $r_1 \neq 0$. Using (S7),(S5),(S6) yields $\vartheta = x^2$ and $-8r_1 V'' - 24 r_1 V' = 4r_1 x$ for our two equations. The second one can be easily solved:

$$2xV'' + 6V' = -x\ ,$$

$$(2x^3 V')' = -x^3\ ,$$

$$2x^3 V' = -\frac{1}{4}x^4 - s_1\ ,$$

$$V' = -\frac{1}{8}x - \frac{s_1}{2x^3},$$

$$V = -\frac{1}{16}x^2 + \frac{s_1}{x^2} + s_2 .$$

Here s_1, s_2 are arbitrary constants.

Thus Case I) $\lambda_k = k$ has two possible kinds of solution:

(a) $\vartheta = x$, $V = -\frac{1}{4}x^2 - \frac{1}{2}$, and

(b) $\vartheta = x^2$, $V = -\frac{1}{16}x^2 + \frac{s_1}{x^2} + s_2$.

Solutions to Case I): If $R_n(x)$ solves $y'' - 2xy' + 2ny = 0$ (Hermite eqn), then $R_n\left(\frac{1}{\sqrt{2}}ix\right)e^{\frac{1}{4}x^2}$ is an eigenfunction for (a).

For (b), pick an m s.t. $\frac{1}{4} - m^2 = s_1$. Using (S3) and (S4) to move a constant into λ_k, we can get $V = -\frac{1}{16}x^2 + \frac{1-4m^2}{4x^2} - \frac{m+1}{2}$. Then if $L_n^{(m)}(x)$ solves

$$xy'' + (\lambda + 1 - x)y' + ny = 0 \tag{11}$$

(the Generalized Laguerre eqn), one can check that $L_n^{(m)}\left(-\frac{x^2}{4}\right) \cdot e^{\frac{1}{8}x^2} x^{m+\frac{1}{2}}$ is an eigenfunction for (b), eigenvalue n.

For the sake of completeness, the recursion relations for Hermite and Laguerre solutions are included (see [1], p. 252, 241): If R_n is a solution to the Hermite equation, then

$$R_{n+1} - 2xR_n + 2nR_{n-1} = 0 .$$

If $L_n^{(m)}$ solves the Generalized Laguerre equation, then

$$(n+1)L_{n+1}^{(m)} = (2n+m+1-x)L_n^{(m)} - (n+m)L_{n-1}^{(m)} .$$

These relations are well known. They don't only work for the orthogonal

polynomials. Instead, for any element of the solution space for a given n, one can find elements of the solutions spaces for $n-1$ and $n+1$ such that the recursion relation will hold.

Case II) $\lambda_k = k^2$ for $k \in K$ (from (7)).

In this case $\Delta_k = -2k-1$, $\nabla_k = 2k-1$. Then by (9)

$$(A_3 - 2A_2 + (1-4k^2)A_1)\varphi_k = 0 \quad \text{for } k \in K.$$

Here $k^2\varphi_k = \lambda_k\varphi_k = L\varphi_k$, so $(A_3 - 2A_2 + A_1(1-4L))\varphi_k = 0$ for $k \in K$. Again, since this operator has too many solutions,

$$A_3 - 2A_2 + A_1(1-4L) = 0.$$

We'll use $L = D^2 + V(x)$ again, and set coefficients to zero. Only the D^3 and D^1 coefficients give new data.

D^3: $8\vartheta''' - 8\vartheta' = 0 \implies \vartheta = r_1 e^x + r_2 e^{-x}$, r_1, r_2 constant.

Using (S7) to shift x, and (S5), we get to

$$\vartheta = e^x \quad \text{or} \quad \vartheta = \cosh x \,. \tag{12}$$

D^1: $6\vartheta^{(5)} - 12\vartheta''V' - 4\vartheta'V'' - 8\vartheta''' + 2\vartheta' - 8\vartheta'V = 0$.

From (12), we can replace ϑ'' by ϑ, so

$$-12\vartheta V' - 4\vartheta' V'' - 8\vartheta' V' = 0 \,.$$

Now a little algebra:

$$(\vartheta' V' + 2\vartheta V)' = 0 \,,$$

$$(\vartheta' V' + 2\vartheta V) = s_1 \,,$$

$$((\vartheta')^2 V)' = s_1 \vartheta' \,,$$

$$(\vartheta')^2 V = s_1 \vartheta + s_2 \,,$$

$$V = \frac{s_1\vartheta + s_2}{(\vartheta')^2}. \tag{13}$$

We can list the possible solutions to (12) and (13):

(a) $\vartheta = e^x$, $V = e^{-2x}$ ($s_1 = 0$, use (S7) to shift x)

(b) $\vartheta = e^x$, $V = e^{-x}$ ($s_2 = 0$, use (S7) to shift x)

(c) $\vartheta = e^x$, $V = -\nu\omega e^{-x} - \omega^2 e^{-2x}$ ($s_1 = -\nu\omega$, $s_2 = -\omega^2$)

(d) $\vartheta = \cosh x$, $V = \dfrac{(\beta^2-\alpha^2)\cosh x + \frac{1}{16}(1-4\alpha^2)}{\sinh^2 x}$ $\left[\begin{array}{l} s_1 = \beta^2-\alpha^2 \\ s_2 = \frac{1}{16}(1-4\alpha^2) \end{array}\right]$

where we've set the constants for later convenience.

Solutions to Case II):

(a) For $V = e^{-2x}$: If $D_n(x)$ solves

$$x^2y'' + y' + (x^2-n^2)y = 0 \qquad \text{(Bessel eqn)}$$

then $D_n(e^{-x})$ is an eigenfunction for (a), eigenvalue n^2.

(b) $V = e^{-x}$: $D_{2n}(2e^{-x/2})$ is eigenfunction, eigenvalue n^2.

(c) $V = -\nu\omega e^{-x} - \omega^2 e^{-2x}$. If $L_n^{(m)}(x)$ is a solution of (11), the Generalized Laguerre eqn, for given n and m, then

$$z^n e^{-z/2} L^{(2n)}_{-\frac{\nu+1}{2}-n}(z), \quad [\text{ with } z = 2\omega e^{-x}]$$

is an eigenfunction to (c) with eigenvalue n^2.

(d) $V = \dfrac{(\beta^2-\alpha^2)\cosh x + \frac{1}{16}(1-4\alpha^2)}{\sinh^2 x}$, α,β constants. If $P_n^{\alpha\beta}(x)$ solves

$$(1-x^2)(P_n^{\alpha\beta})'' + [\beta-\alpha-(\alpha+\beta+2)x](P_n^{\alpha\beta})' + n(\alpha+\beta+n+1)P_n^{\alpha\beta} = 0$$
(Jacobi eqn)

then $\left(\sinh\frac{x}{2}\right)^{\alpha+\frac{1}{2}}\left(\cosh\frac{x}{2}\right)^{\beta+\frac{1}{2}} P^{\alpha\beta}_{n-\frac{\alpha+\beta+1}{2}}(\cosh x)$ is an eigenfunction to (d) with eigenvalue n^2. (See [1], p. 214)

For Bessel functions $D_n(x)$, the recursion relation is (see [1], p. 67):

$$D_{n-1} + D_{n+1} = \frac{2n}{x} D_n \ .$$

This relation can be used to derive a recursion relation also for the Bessel functions that solve (b). For the recursion relation for the Jacobi functions P, see [1], p. 213. The relation for the Generalized Laguerre function in (c) is not obvious. However, the following relation

$$\frac{(n+1)(n-m)}{(m+1)(m+2)} L_{n-m}^{(m)}(y) + \left[\frac{2n-m+1}{(m+1)(m+3)} - \frac{1}{y} \right] y L_{n-m-1}^{(m+2)}(y) + \frac{1}{(m+2)(m+3)} y^2 L_{n-m-2}^{(m+4)}(y) = 0$$

is sufficient, and can be derived by repeated use of

$$L_n^{(m)} = L_n^{(m+1)} - L_{n-1}^{(m+1)}$$

and

$$y L_{n-2}^{(m+2)} - (m+1-y) L_{n-1}^{(m+1)} + n L_n^{(m)} = 0$$

found in [1], p. 241-2. Here we use the standard normalizations for the $L_n^{(m)}$'s.

8. Conclusion

All the possible solutions found correspond to shifts, using the symmetries from Section 3, of well-known functions; i.e., solutions to the Bessel, Generalized Laguerre, Hermite, and Jacobi differential equations. Note again that those solutions need not be the standard orthogonal polynomials, but can come from the full solution spaces. Since these functions are already well-known, no attempt was made to catalog them precisely: Some are special cases of others.

9. Final Note

Though certain steps may be harder, many of these techniques should work for higher-order L's or B's.

Some examples of nontrivial recursion relations with more than three terms are given by Grünbaum in [4]. The examples he gives are of soliton-like functions.

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