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CONVERGENCE OF A RANDOM METHOD WITH
CREATION OF VORTICITY

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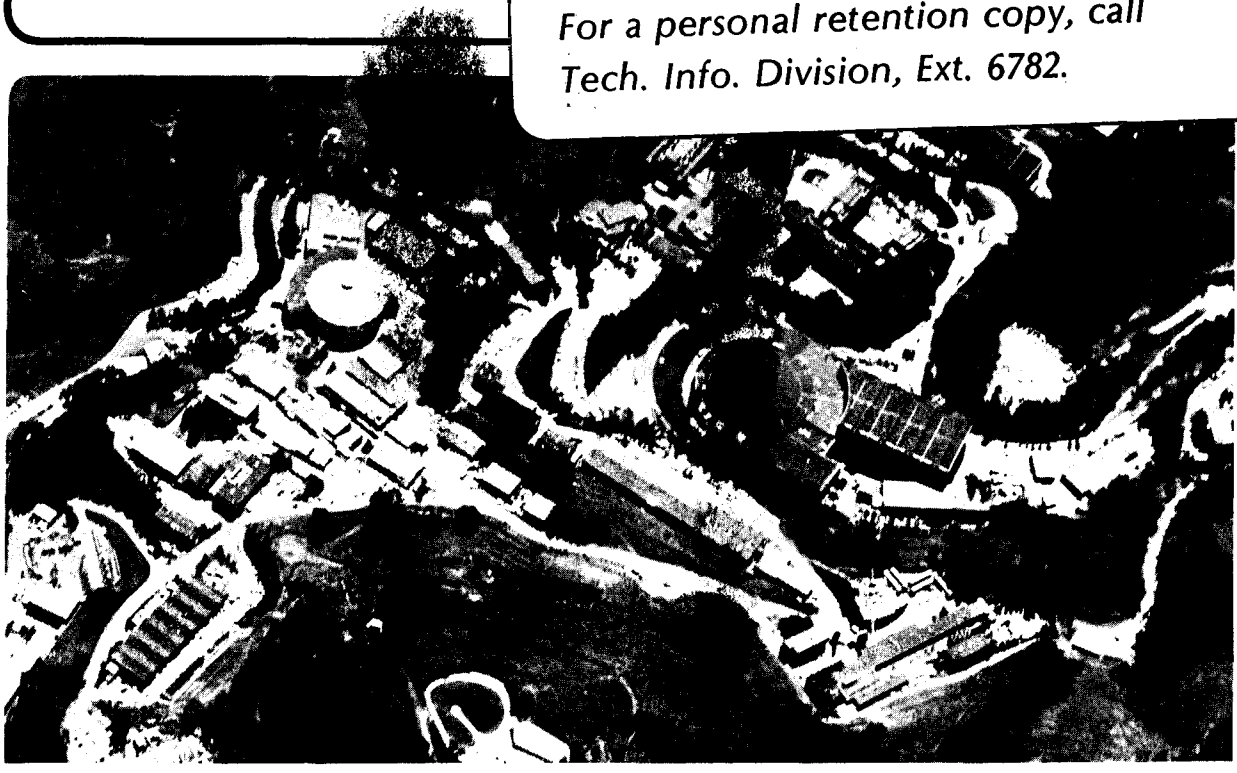

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CONVERGENCE OF A RANDOM METHOD WITH CREATION OF VORTICITY

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Abstract. A system of diffusion equations modeling free convection near a wall is solved by a grid free random walk method that involves creation of the vorticity at the boundary. We prove that the pointwise error and the least squares error of the computed solution tend to zero in probability as the time step tends to zero and the number of particles in the random walk increases.

Key words. convergence, random walk, vorticity.

Introduction. The purpose of this paper is to prove the convergence of a random method for solving a coupled system of diffusion equations. The equations describe free convection of an incompressible fluid. The flow is parallel to a hot vertical wall and is caused by the uneven heating of the fluid. This problem is of interest because it involves creation of vorticity at the boundary. The mathematical solution can be written down explicitly and this fact allows us to estimate the rate of convergence for the random method, both for the expected value and for the variance.

The algorithm was presented by Ghoniem and Sherman in their survey paper on random methods as an illustration of the basic approach, (see [9]). We regard the free convection problem as a model for more complicated problems in fluid dynamics. In increasing order of complexity we have random methods for the reaction-diffusion equation, the Prandtl boundary layer equations and the Navier-Stokes equations, (see Chorin [5,4,3]). The basic idea in all three cases is to use a fractional step method. In the first step we solve the nonlinear part of the equations by a deterministic method and in the second step we simulate the diffusion part of the equations by a random walk technique. If the problem involves boundaries we must also satisfy the boundary conditions. This is done in an intermediate step by creating vorticity at the boundary.

The analysis of the convergence of random methods is less advanced than the understanding of the underlying differential equations. In addition, the methods have been applied to physical problems that are so complicated that they - at present - defy analysis. A good example is turbulent combustion, (see Ghoniem, Chorin and Oppenheim [8]). However, Marchioro and Pulvirenti [14] have proved that in two dimensions the random vortex method converges weakly to a weak solution of the Navier-Stokes equations. A similar statement for the Prandtl boundary layer equations can be found in Benefatto and Pulvirenti [1]. For the reaction-diffusion equation Hald [10] has proved convergence of an

inefficient version of Chorin's method, whereas Brenier [2] has generalized Chorin's method to nonlinear advection-diffusion equations and proved the convergence of a deterministic version of the algorithm. The convergence of the random methods is not really in doubt but because of the statistical error it is difficult to check the dependence on the various parameters carefully, (see Milinazzo and Saffman [15] and Roberts [16]).

This paper is organized as follows. In section one we discuss the physical problem, give the differential equations and write the solution down explicitly. One of the terms in the solution can be interpreted as coming from the creation of vorticity at the boundary. In section two we present the numerical method. Since the ordering of the intermediate steps in a fractional step method is arbitrary we have several choices. Here we follow Ghoniem and Sherman [9] as their choice leads to a simple proof, but the other versions have similar accuracies. To prove that the random method converges we must establish two facts. First that the expected value of the computed solution tends to the exact solution as the time step tends to zero, and secondly that the variance goes to zero as we use more and more particles. By approximating the gradient of the solution by a finite number of particles (or vortex sheets) and following these we in effect reduce the variance of the computed solution, which we obtain via integration. This idea is due to Chorin. It is stated in [5], but used implicitly in the random vortex method [3].

In section three we will prove the convergence of the method, but only pointwise. Our estimates are very crude, but uniform in space and time as long as our final time is bounded. In the proof we use only elementary probability theory. Two interesting questions remains, namely: "Is the estimate of the variance of the correct order of magnitude?" and "What happens as the viscosity and the thermal diffusivity tend to zero?" In section four we will show that the variance of the solution does not go to zero just because the time step goes to zero,

even though the number of computational elements at a fixed time increases. This is not as surprising as it sounds. It corresponds to the fact that a finite difference scheme will not converge unless both the mesh length and the time step tend to zero. Finally in section 5 we use nonuniform bounds for the expected value and for the variance to estimate the global error of the computed solution in a least squares sense. We can then study the effect of the viscosity and diffusivity.

1. The differential equations. In this section we will present the differential equations for a one dimensional model of free convection and give the mathematical solution. We consider an incompressible fluid in the half space $x>0$ and assume that the velocity u and the temperature T depend only on the distance x to the wall (at $x=0$). Our differential equations are (see [9])

$$u_t = \nu u_{xx} + g\beta' T \tag{1.1}$$

$$T_t = \alpha T_{xx} . \tag{1.2}$$

Here ν is the kinematic viscosity, g is the acceleration due to gravity, β' is the coefficient of thermal expansion of the fluid and α is the coefficient of thermal diffusivity. The first equation is derived from the Navier-Stokes equations in a gravitational field while the second is a simplified version of the equation for heat transfer, (see [13, p. 189 and p. 213]). To complete the description of (1.1) and (1.2) we need the initial conditions

$$u(x,0) \equiv 0, \qquad T(x,0) \equiv 0$$

for $0<x<\infty$. For $t>0$ we adopt the boundary conditions

$$u(0,t) = 0 \qquad u(x,t) \to 0 \quad as\ x \to 0$$

$$T(0,t) = 1 \qquad T(x,t) \to 0 \quad as\ x \to 0.$$

Throughout this paper we assume that $g\beta'=1$. This simplifies the writing. The solution of the heat equation (1.2) with the appropriate initial and boundary conditions is

$$T(x,t) = \int_x^\infty \frac{2\, e^{-y^2/4\alpha t}}{\sqrt{4\pi\alpha t}}\, dy \,. \tag{1.3}$$

We can now find the solution of the inhomogeneous equation (1.1). By using Duhamel's principal and the Green's function for the heat equation on a half line and with a Dirichlet boundary condition at the origin (see Hellwig [11,p. 25 and p. 53]), we obtain

$$u(x,t) = \int_0^t \int_0^\infty \left(\frac{e^{-(x-y)^2/4\nu(t-s)}}{\sqrt{4\pi\nu(t-s)}} - \frac{e^{-(x+y)^2/4\nu(t-s)}}{\sqrt{4\pi\nu(t-s)}} \right) T(y,s)\, dy\, ds \,. \tag{1.4}$$

This formula shows that the velocity u is positive throughout the fluid. To analyze the solution further we introduce the vorticity ξ and the heat flux q by

$$u(x,t) = \int_x^\infty \xi(y,t)\, dy \qquad T(x,t) = \int_x^\infty q(y,t)\, dy \,.$$

Thus $-\xi$ and $-q$ are the gradients of u and T. Let $\varphi(x,\sigma^2)$ be the normal density with mean 0 and variance σ^2, i.e. $\varphi=(2\pi\sigma^2)^{-1/2}\exp(-x^2/2\sigma^2)$. Since $\partial_x\varphi(x-y)=-\partial_y\varphi(x-y)$ it follows from (1.4) that

$$\xi(x,t) = \int_0^t \int_0^\infty \partial_y[\varphi(x-y,2\nu(t-s))+\varphi(x+y,2\nu(t-s))]\, T(y,s)\, dy\, ds \,.$$

After integrating by parts we arrive at

$$\xi(x,t) = -\int_0^t 2\varphi(x,2\nu(t-s))\, ds + \int_0^t 2\varphi(x,2\alpha s+2\nu(t-s))\, ds \,. \tag{1.5}$$

Here we have used the identity

$$\varphi(x-z,a+b)+\varphi(x+z,a+b) = \tag{1.6}$$

$$\int_0^\infty [\varphi(x-y,a)+\varphi(x+y,a)]\,[\varphi(y-z,b)+\varphi(y+z,b)]\,dy$$

with $z=0$. This follows from the fact that the convolution of two normal densities with means x,z and variances a,b is the normal density with mean $x+z$ and variance $a+b$, (see [7,p. 45]). Finally, by integrating both sides in equation (1.5) with respect to x we get

$$u(x,t) = -\int_0^t\int_x^\infty \frac{2\,e^{-y^2/4\nu(t-s)}}{\sqrt{4\pi\nu(t-s)}}dy\,ds + \int_0^t\int_x^\infty \frac{2\,e^{-y^2/(4as+4\nu(t-s))}}{\sqrt{4\pi as+4\pi\nu(t-s)}}dy\,ds\,. \quad (1.7)$$

It turns out that the expected value of the computed velocity is a discretized version of the right hand side of (1.7). Moreover, the numerical method reveals that the first term in the solution is due to the creation of vorticity at the boundary. After several changes of variables in (1.7) we find that $u(x,t)=tf(x/\sqrt{2at},\nu/a)$, where f can be given explicitly. This result is close but not identical to Illingworth's form of the solution, (see [12]). It shows that the maximum of u is a linear function of time and depends only on the Prandtl number ν/a.

2. The random method. In this section we present the numerical method following Ghoniem and Sherman [9]. The basic idea is to approximate the gradients of u and T by a finite number of particles and let these particles undergo random walk. Thus we solve the differential equations

$$\xi_t = \nu\xi_{xx} + q \quad (2.1)$$

$$q_t = aq_{xx}\,. \quad (2.2)$$

We begin with the heat equation (2.2). At time $t=0$ we place N particles at the origin. We assume that each particle has mass $1/N$. In each step we let the

particles jump by an amount which is drawn from a Gaussian distribution with mean 0 and variance $2\alpha\Delta t$. If a particle lands at a point $x<0$ we reflect it across the wall to $|x|$. In probability language this is called a reflecting barrier and corresponds to a Neumann boundary condition for the differential equations. To describe the process mathematically we let X_j^n be the position of the j'th particle at time $t=n\Delta t$. Let $\{X\}$ be a collection of independent normal distributed random variables with mean 0 and variance $2\alpha\Delta t$. Then

$$X_j^n = |X_j^{n-1} + X|. \tag{2.3}$$

with $X_j^0=0$ for $j=1,\cdots,N$. We denote the approximations to the heat flux and the temperature by Q and Θ ($\vartheta\acute{\varepsilon}\rho\mu\eta$=heat). To express these approximations we let δ be the Dirac delta function and let H be the Heaviside function. Here $H(x)=1$ if $x\geq 0$ and 0 otherwise. At $t=n\Delta t$ we set

$$\begin{aligned} Q(x,t) &= \frac{1}{N}\sum_{j=1}^{N}\delta(X_j^n - x) \\ \Theta(x,t) &= \frac{1}{N}\sum_{j=1}^{N}H(X_j^n - x). \end{aligned} \tag{2.4}$$

Thus for each (x,t), Q is a random measure and Θ is a random variable. The temperature at a point x is the number of particles to the right of x times the mass of a single particle. Note that our definition of the Heaviside function ensures that the approximate temperature distribution satisfies the boundary condition at $x=0$ exactly.

The complete fractional step method for (2.1) and (2.2) consists of four intermediate steps

$$q_t = \alpha q_{xx}, \quad \xi_t = \nu\xi_{xx}, \quad \xi_t = q, \quad \int_0^\infty \xi = 0.$$

We have already discussed the first step. The last three constitute a fractional step method for solving (2.1). Here $\int_0^\infty \xi=0$ is shorthand for the creation of vorti-

city at the boundary. At time $t=n\Delta t$ we approximate the vorticity ξ by nN vortex sheets of strength $\Delta t/N$ and an equal number of strength $-\Delta t/N$. Here a vortex sheet of strength s is a plane parallel to the wall such that the velocity decreases by the amount s when we cross the sheet in the direction of increasing values of x. The velocity $u(x,t)$ is approximated by the total strength of the vortex sheets in the interval $[x,\infty)$. Thus $u(0,t)=0$. Note the change in language: We use particles to calculate the temperature and sheets to calculate the velocity. We simulate the diffusion $\xi_t=\nu\xi_{xx}$ by a random walk. The displacements are drawn from a Gaussian distribution with mean 0 and variance $2\nu\Delta t$. If a vortex sheet lands at $x<0$ we reflect it back into the fluid. Next we solve $\xi_t=q$ by Euler's method, i.e.

$$\xi(x,t+\Delta t) = \xi(x,t) + \Delta t\, q(x,t) .$$

At the position of each heat particle we introduce a vortex sheet of strength $\Delta t/N$. The velocity u at $x=0$ will therefore be Δt. To satisfy the boundary condition $u=\int_0^\infty \xi=0$ we create N vortex sheets of strength $-\Delta t/N$ at the wall. This completes the description of the algorithm. Note that we have no sheets in the beginning of the first step and in the end of the last step we have N sheets at the origin.

To describe the solution of (2.1) more precisely we let $Y_j^{m,n}$ be the position at time $t=n\Delta t$ of one of the N vortex sheets that were created at the origin at time $m\Delta t$. Similarly we let $Z_j^{m,n}$ be the positions at time $n\Delta t$ of one of the N vortex sheets that were introduced into the flow at time $m\Delta t$. Thus $Y_j^{m,m}=0$ and we set $Z_j^{m,m}=X_j^m$ for $j=1,\cdots,N$ and $m=1,\cdots,n$. Let $\{Y\}$ be a collection of independent normal distributed random variables with mean 0 and variances $2\nu\Delta t$. Then

$$Y_j^{m,n} = |\, Y_j^{m,n-1}+Y\,| \tag{2.5}$$

$$Z_j^{m,n} = |\, Z_j^{m,n-1} + Y \,| \tag{2.6}$$

for $n > m$. Let n be fixed. Since the Y's are independent we conclude that all the $Y_j^{m,n}$ are independent. Moreover, the class $\{Y_j^{m,n}\}$ is independent of the class $\{Z_j^{m,n}\}$. However, two random variables $Z_j^{m,n}$ with the same j but different m's are not independent, because the two vortex sheets were spawned by the same heat particle, albeit at different times. We illustrate the dependence in Figure 1. On the other hand, the classes $\{Z_1^{m,n}\}, \cdots, \{Z_N^{m,n}\}$ are mutually independent. If Ξ and U denote our approximations to the vorticity and the velocity then

$$\Xi(x,t) = \frac{1}{N}\sum_{j=1}^{N}\Big(-\Delta t \sum_{m=1}^{n} \delta(Y_j^{m,n} - x) + \Delta t \sum_{m=1}^{n} \delta(Z_j^{m,n} - x) \Big)$$

$$U(x,t) = \frac{1}{N}\sum_{j=1}^{N}\Big(-\Delta t \sum_{m=1}^{n} H(Y_j^{m,n} - x) + \Delta t \sum_{m=1}^{n} H(Z_j^{m,n} - x) \Big). \tag{2.7}$$

This shows that the computed velocity U at (x,t) is really the number of vortex sheets to the right of x times the strength of a single vortex sheet. Here the vortex sheets are counted as $+1$ if they were introduced into the flow and as -1 if they were created at the origin. Since the random variables with a fixed n but different j's are independent we see that U is an average of N independent, identically distributed random variables. Thus instead of using N heat particles with mass $1/N$ we may as well use one particle with unit mass and average over N trials. We can therefore assume that $N=1$ and this simplifies our analysis considerably.

3. Pointwise convergence. In this section we will prove the convergence of the random method for the temperature and for the velocity. Because of the dependence between some of the particles we shall be satisfied with very crude estimates for the expected value and for the variance (here denoted by E and var).

THEOREM 1. *Let* $u(x,t)$ *and* $T(x,t)$ *be the solution of* (1.1) *and* (1.2) *with* $g\beta=1$. *Let* $U(x,t)$ *and* $\Theta(x,t)$ *be the random solutions given by* (2.7) *and* (2.4). *If* (x,t) *is fixed and* $t=n\Delta t>0$ *then*

$$|E\Theta - T| = 0 \qquad\qquad |EU - u| \le \Delta t$$

$$var\,\Theta \le \frac{1}{4N} \qquad\qquad var\,U \le \frac{t^2}{3N}.$$

Remark. The factor t^2 in the variance corresponds to a factor t in the standard deviation. Since the maximum velocity is a linear function of t we expect that the error bound is at least proportional to t. By using Chebyshev's inequality (see [7,p.149]) we can estimate the error in probability. For each $k>1$ and each (x,t) the inequalities

$$|\Theta(x,t) - T(x,t)| \le \frac{k}{\sqrt{4N}}$$

$$|U(x,t) - u(x,t)| \le \Delta t + \frac{kt}{\sqrt{3N}}$$

are satisfied with a probability greater than $1-k^{-2}$. If $k=3$ the bounds should hold in roughly 8 out of 9 trials. Note that our estimates do not depend upon the viscosity or the diffusivity. A similar phenomenon has been observed by Sethian and Ghoniem [17] in their simulation of two dimensional viscous flow over a half step. The amount of work for our algorithm is roughly $Nt^2/\Delta t^2$. Thus we get the smallest error bound for the computed velocity if N is proportional to $t^2/\Delta t^2$. To obtain high accuracy in a particular run we need that N is large. If we only have a few particles then there is no point in taking small time steps.

Proof. We begin with the convergence for the temperature. This is the easiest part of the proof. We assume that $N=1$ and drop the subscript j. We will show that the probability that a heat particle lies in the interval $[0,x]$ at time $m\Delta t$ is

$$P(X^m \le x) = \int_0^x 2\varphi(y, 2\alpha m \Delta t)\, dy. \tag{3.1}$$

This is proved by induction. For $m=1$ $X^m = |X|$. Thus (3.1) follows from the symmetry of the density function for X. We assume next that (3.1) holds for some $m \ge 1$. It follows from (2.3) that the density for X^{m+1} is $h(x)+h(-x)$, where h is the density for $X^m + X$. Since X^m and X are independent we have

$$h(x) = \int_0^\infty \varphi(x-y, a)\, 2\varphi(y, b)\, dy$$

where $a = 2\alpha m \Delta t$ and $b = 2\alpha \Delta t$, (see [7,p. 7]). By using (1.6) with $z=0$ we see that the density function for X^{m+1} is $2\varphi(x, a+b)$. This completes the induction.

We can now prove that the temperature converges. Let $t = n\Delta t$ be fixed and consider the random variable $H(X^n - x)$. It is one if $X^n \ge x$ and zero otherwise. Thus

$$E\, H(X^n - x) = P(X^n \ge x), \qquad var\, H(X^n - x) = P(X^n \ge x) P(X^n < x). \tag{3.2}$$

Since Θ consists of N independent random variables and the variance of $H(X^n - x)$ is always less than $1/4$ we conclude from (2.4), (3.1) and (1.3) that

$$E\Theta(x,t) = \frac{1}{N} \sum_{j=1}^{N} \int_x^\infty \frac{2\, e^{-y^2/4\alpha t}}{\sqrt{4\pi\alpha t}} dy = T(x,t)$$

$$var\, \Theta(x,t) = \frac{1}{N^2} \sum_{j=1}^{N} \int_x^\infty \frac{2\, e^{-y^2/4\alpha t}}{\sqrt{4\pi\alpha t}} dy \int_0^x \frac{2\, e^{-y^2/4\alpha t}}{\sqrt{4\pi\alpha t}} dy \le \frac{1}{4N}. \tag{3.3}$$

Our estimate for the variance is the best possible uniform bound. In fact we have equality at the point x where $P(X^n \ge x) = P(X^n < x)$. On the other hand it is a severe overestimate if $x/\sqrt{2\alpha t}$ is either very small or very large.

We will now discuss the approximation of the velocity. We observe first that it is not essential in the derivation of (3.1) that all X's have the same density. What matters is that the X's are independent and normal distributed. Under this assumption the variances are additive. By using this observation we

conclude from (2.5) and (2.6) that

$$P(Y^{m,n} \le x) = \int_0^x 2\,\varphi(y, 2\nu(n-m)\Delta t)\,dy \tag{3.4}$$

$$P(Z^{m,n} \le x) = \int_0^x 2\,\varphi(y, 2\alpha m\Delta t + 2\nu(n-m)\Delta t)\,dy\,. \tag{3.5}$$

Here we have suppressed the subscript j as the distributions do not depend on j. The first result holds for $m=1,\cdots,n-1$, the second for $m=1,\cdots,n$. In addition $Y^{n,n}=0$. We can now calculate the expected value of the velocity. Let $x>0$ and $t=n\Delta t$ where $n\ge 2$. By using (2.7) and (3.2) with $Y^{m,n}$ and $Z^{m,n}$ instead of X^n we conclude from (3.4) and (3.5) that

$$E\,U(x,t) = -\Delta t \sum_{m=1}^{n-1} \int_x^\infty \frac{2\,e^{-y^2/(4\nu(n-m)\Delta t)}}{\sqrt{4\pi\nu(n-m)\Delta t}}\,dy \tag{3.6}$$

$$+\Delta t \sum_{m=1}^{n} \int_x^\infty \frac{2\,e^{-y^2/(4\alpha m\Delta t + 4\nu(n-m)\Delta t)}}{\sqrt{4\pi\alpha m\Delta t + 4\pi\nu(n-m)\Delta t}}\,dy.$$

The sums in (3.6) approximate the integrals in the exact solution (1.7). Since the $Y_j^{m,n}$ are created at the origin we can now understand why the first term in (1.7) must be regarded as due to the creation of vorticity at the boundary. To estimate the error in the expected value of U we let $f(s)$ and $g(s)$ be the integrals in (3.6) as functions of $s=m\Delta t$. Note that we suppress the dependence on x. By using a change of variables we see that

$$f(s) = \int_{x/\sqrt{2\nu(t-s)}}^\infty \frac{2\,e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

$$g(s) = \int_{x/\sqrt{2(\alpha-\nu)s+2\nu t}}^\infty \frac{2\,e^{-y^2/2}}{\sqrt{2\pi}}\,dy. \tag{3.7}$$

We consider the two cases $\nu\le\alpha$ and $\nu>\alpha$ separately. In the first case f is a decreasing function of s while g is increasing. This implies that the first sum in (3.6) is a lower Riemann sum for the integral $\int_0^t f(s)ds$ while the second sum is an upper Riemann sum for the integral $\int_0^t g(s)ds$. We remember now that the

error in approximating an integral by a Riemann sum is less than the mesh length times the total variation of the function. But our functions are monotone. Hence

$$0 \le E\,U - u \le \Delta t f(0) + \Delta t(g(t) - g(0)) = \Delta t g(t)$$

where we have used that $f(0)=g(0)$. If $\nu>\alpha$ then g is a decreasing function of s. The second sum in (3.6) is therefore a lower Riemann sum for $\int_0^t g(s)ds$. Consequently

$$-\Delta t g(0) < -\Delta t(g(0) - g(t)) \le E\,U - u \le \Delta t f(0).$$

By combining our estimates and using that $0<g<1$ we conclude that

$$|E\,U(x,t) - u(x,t)| \le \Delta t \max(g(0), g(t)) < \Delta t. \tag{3.8}$$

To complete the proof we must estimate the variance of the computed velocity. For simplicity we set $N=1$ and drop the index j. Since the $Y^{m,n}$ are mutually independent and independent of the $Z^{m,n}$ it follows from (2.7) and Cauchy-Schwarz inequality that if $x>0$ then

$$\begin{aligned} var\,U(x,t) &= var\,-\Delta t \sum_{m=1}^{n-1} H(Y^{m,n} - x) + var\,\Delta t \sum_{m=1}^{n} H(Z^{m,n} - x) \\ &\le \Delta t^2 \sum_{m=1}^{n-1} var\,H(Y^{m,n} - x) + \Delta t^2 n \sum_{m=1}^{n} var\,H(Z^{m,n} - x). \end{aligned} \tag{3.9}$$

By using that the variance of a zero-one random variable is less than 1/4 we conclude that $var\,U \le 5t^2/16$ if $N=1$. If $N>1$ then U consists of N independent, identically distributed random variables and in this case $var\,U(x,t)$ is less than $5t^2/(16N)$. The proof simplifies if $n=1$. Note that our estimates also hold at $x=0$ because the computed temperature and the computed velocity satisfy the boundary conditions exactly. This completes the proof.

4. Improvements and limitations. Our bound for the variance of the random velocity does not tend to zero as the time step tends to zero. This is not due to an imperfect technique. In this section we will show that all the covariances of the $Z^{m,n}$ are positive and this excludes cancellation in the expression for the variance. The problem can be circumvented by modifying the algorithm, but the modification is not in the spirit of random solution of differential equations.

We begin with a reexamination of (3.9). The variance of the sum of the dependent random variables in (3.9) is equal to

$$\Delta t^2 \sum_{m=1}^{n} var\, H(Z^{m,n}-x) + 2\Delta t^2 \sum_{l=1}^{n-1} \sum_{m=l+1}^{n} cov\,(\, H(Z^{l,n}-x)\, H(Z^{m,n}-x)\,), \quad (4.1)$$

(see [6,p. 216]). We denote the covariances by $C^{l,m,n}$. Let $r=l\Delta t$, $s=m\Delta t$ and $t=n\Delta t$. We will show that

$$C^{l,m,n} = C(2\alpha r, 2\nu(t-r), 2\alpha(s-r)+2\nu(t-s))$$

where C is a continuous positive function. This implies that the double sum in (4.1) tends to the double integral

$$2\int_0^t \int_r^t C(2\alpha r, 2\nu(t-r), 2\alpha(s-r)+2\nu(t-s))\, ds\, dr\ . \quad (4.2)$$

This expression is positive. Since the first sum in (3.9) and the first sum in (4.1) are $O(\Delta t)$ we conclude that the variance of the computed velocity tends to the expression in (4.2) times $1/N$ as the mesh length tends to zero. We are therefore lead to

OBSERVATION 1. *The bounds for the variances in Theorem 1 have the correct dependence on the number of particles.*

The calculation of the covariances $C^{l,m,n}$ can best be understood by an example. Let

$$Z^l = \,|\,|X+Y_1|+Y_2|\,, \qquad Z^m = \,|\,|X+V_1|+V_2|\,.$$

where the Y_i and the V_i are independent normal distributed random variables with mean 0 and variances a_i and b_i. We assume that X has density $2\varphi(x,c)$ for $x\geq 0$. If $a_1=a_2=b_2=2\nu\Delta t$ and $2b_1=c=4\alpha\Delta t$ then we have the situation indicated in Figure 1 with $Z^l=Z^{2,4}$ and $Z^m=Z^{3,4}$. Finally we introduce the functions

$$F_l(z) = E\ H(|\,|z+Y_1|+Y_2|-x) \tag{4.3}$$
$$F_m(z) = E\ H(|\,|z+V_1|+V_2|-x)$$

which we shall evaluate explicitly. Note that F_l and F_m are positive. Since X, Y_i and V_i are independent it follows from the definition of the covariance that

$$C^{l,m,n} = \int_0^\infty 2\varphi(z)F_l(z)F_m(z)\,dz - \int_0^\infty 2\varphi(y)F_l(y)\,dy \int_0^\infty 2\varphi(z)F_m(z)\,dz$$
$$= 2\int_0^\infty\int_0^\infty \varphi(y)\varphi(z)\,[F_l(y)-F_l(z)][F_m(y)-F_m(z)]\,dy\,dz. \tag{4.4}$$

Here we have suppressed c in $\varphi(z,c)$. We will show that F_l and F_m are increasing functions of z. The product [..][..] in (4.4) is therefore positive and this implies that $C^{l,m,n}>0$ as claimed. Since the Y's are independent we can compute the expected value in (4.3) as

$$F_l(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(|\,|z+y_1|+y_2|-x)\varphi(y_1,a_1)\varphi(y_2,a_2)\,dy_1\,dy_2.$$

We attack the innermost variable first. Let $z+y_1=\eta_1$. By splitting the integral into $\eta_1>0$ and $\eta_1<0$ we find after a change of variables that

$$F_l(z) = \int_{-\infty}^{\infty}\int_0^\infty H(|\eta_1+y_2|-x)\,[\varphi(\eta_1-z_1,a_1)+\varphi(\eta_1+z,a_1)]\,\varphi(y_2,a_2)\,d\eta_1\,dy_2.$$

Next we interchange the order of integration, set $\eta_1+y_2=\eta_2$, split the integral into $\eta_2>0$ and $\eta_2<0$ and get

$$F_l(z) = \int_0^\infty\int_0^\infty H(\eta_2-x)[\varphi(\eta_2-\eta_1,a_2)+\varphi(\eta_2+\eta_1,a_2)][\varphi(\eta_1-z_1,a_1)+\varphi(\eta_1+z,a_1)]\,d\eta_2 d\eta_1$$

By interchanging the order of integration, using the definition of the Heaviside function and (1.6), we obtain

$$F_l(z) = \int_z^\infty [\varphi(\eta - z, a) + \varphi(\eta + z, a)]\, d\eta \tag{4.5}$$

where $a = a_1 + a_2$. Since x and z are positive (4.5) implies that $F_l(z)$ is an increasing function of z. The same result holds for $F_m(z)$. Equation (4.5) gives the general result. If we return to the original variables then it follows from (2.3) and (2.6) that $Z^{l,n}$ and $Z^{m,n}$ depend on X^l if $l < m$. Consequently $F_l(z)$ is given by (4.5) with $a = 2\nu(n-l)\Delta t$. For $F_m(z)$ the variance a is replaced by $b = 2\alpha(m-l)\Delta t + 2\nu(n-m)\Delta t$. Finally (3.1) shows that $c = 2\alpha l \Delta t$. By combining these observations we can interpret the double integral in (4.4) as the evaluation of a positive function at the grid points of a two dimensional grid. This completes the proof of Observation 1.

Our analysis shows that the variance of the computed velocity would be of the order $\Delta t / N$ if the vortex sheets were independent. This can be arranged. At the mth time step we let the particles start at the origin and jump according to a Gaussian distribution with mean 0 and variance $2am\Delta t$. Those that land at $x < 0$ are reflected across the boundary. The costs for the two algorithms are identical. The modified version of the algorithm uses that we can simulate the solution of the heat equation by taking one large step instead of many small. However, to take one large step is contrary to the spirit of numerical solution of differential equations. We will therefore not discuss this variant any further.

5. Convergence in L^2. Our estimates in Theorem 1 say nothing about what happens at several points at the same time. From a practical point of view we would like to know that the maximum error in the computed solution tends to zero in probability. I have been unable to obtain such a result. In this section we will show that it is unlikely that the computed solution differs from the exact solution by a moderate amount on a large set or by a large amount on a moderate set. The error may still be large on a small set. To establish this result we need

LEMMA 1. *If $a>0$ and $b>0$ then*

$$\int_0^\infty \int_{x/\sqrt{a}}^\infty \frac{2e^{-y^2/2}}{\sqrt{2\pi}}dy \int_0^{x/\sqrt{b}} \frac{2e^{-y^2/2}}{\sqrt{2\pi}}dy\, dx = \sqrt{2/\pi}(\sqrt{a+b}-\sqrt{b})$$

$$\int_0^\infty \int_{x/\sqrt{a}}^\infty \frac{2e^{-y^2/2}}{\sqrt{2\pi}}dy \int_{x/\sqrt{b}}^\infty \frac{2e^{-y^2/2}}{\sqrt{2\pi}}dy\, dx = \sqrt{2/\pi}(\sqrt{a}+\sqrt{b}-\sqrt{a+b}).$$

Proof. Let I_1 denote the first integral. Change the variable x to $x=z\sqrt{a}$ and set $f(z)=\int_0^{\beta z}\exp(-y^2/2)dy$ where $\beta=\sqrt{a/b}$. By interchanging the order of integration in I_1 we get

$$I_1=\frac{4\sqrt{a}}{2\pi}\int_0^\infty e^{-y^2/2}\int_0^y f(z)dz\, dy = \frac{2\sqrt{a}}{\pi}\int_0^\infty e^{-y^2/2}[y\int_0^{\beta y} e^{-\eta^2/2}d\eta+\frac{1}{\beta}e^{-\beta^2y^2/2}-\frac{1}{\beta}]\,dy.$$

To compute the double integral we integrate by parts. This yields

$$I_1 = \frac{2\sqrt{a}}{\pi}\left[\int_0^\infty e^{-y^2/2}\beta e^{-\beta^2y^2/2}dy + \frac{1}{\beta}\int_0^\infty e^{-(1+\beta^2)y^2/2}dy - \frac{1}{\beta}\int_0^\infty e^{-y^2/2}dy\right].$$

The value of I_1 can now be computed. To evaluate the second integral in Lemma 1 we note that $\int_{x/\sqrt{b}}^\infty 2\varphi = 1-\int_0^{x/\sqrt{b}}2\varphi$ and use the value of I_1. This completes the proof.

We will estimate the relative error in the computed solution. We must therefore find lower bounds for the L^2 norm of the temperature and of the velocity. Since T and $t^{-1}u$ are functions of $x/\sqrt{2at}$ it follows that the L^2 norms of

T and u are proportional to $(2\alpha t)^{1/4}$ and $t(2\alpha t)^{1/4}$. This simple argument does not give the value of the constants nor does it indicate the dependence on the Prandtl number ν/α. By using Lemma 1 we can calculate the L^2 norms explicitly.

LEMMA 2. *Let $u(x,t)$ and $T(x,t)$ be the solution of* (1.1) *and* (1.2) *with* $g\beta=1$. *Then*

$$\|T\|^2 = \frac{2(\sqrt{2}-1)}{\sqrt{\pi}}\sqrt{2\alpha t}$$

$$\|u\|^2 = \frac{32t^{\frac{5}{2}}}{15\sqrt{\pi}}\alpha^2\frac{(\sqrt{2}-1)(\sqrt{\alpha}+\sqrt{\nu})^2+\sqrt{2}(\sqrt{\alpha}+\sqrt{\nu})\sqrt{\alpha+\nu}+\sqrt{2}(\alpha+\nu)}{(\sqrt{\alpha}+\sqrt{\nu}+\sqrt{\alpha+\nu})^2(\sqrt{\alpha}+\sqrt{\nu})^2(\sqrt{\alpha}+\sqrt{\nu}+\sqrt{2}\sqrt{\alpha+\nu})}.$$

Proof. We begin with the temperature. By using (1.3) and Lemma 1 with $a=b=2\alpha t$ we find that

$$\|T\|^2 = \int_0^\infty\Big(\int_{x/\sqrt{2\alpha t}}^\infty \frac{2e^{-y^2/2}}{\sqrt{2\pi}}dy\Big)^2dx = \sqrt{2/\pi}\,(2\sqrt{2\alpha t}-\sqrt{2\cdot 2\alpha t})$$

from which the first statement follows. Let next $u=-A+B$ where A and B are the integrals in (1.7). We replace the variable $t-s$ in A by s and shall compute $\|u\|^2$ as $\int A^2+B^2-2AB$. Fubini's theorem and Lemma 1 yield

$$\int_0^\infty A^2dx = \int_0^t ds\int_0^t d\sigma\int_0^\infty dx\int_{x/\sqrt{2\nu s}}^\infty\frac{2e^{-y^2/2}}{\sqrt{2\pi}}dy\int_{x/\sqrt{2\nu\sigma}}^\infty\frac{2e^{-y^2/2}}{\sqrt{2\pi}}dy$$

$$= \frac{2}{\sqrt{\pi}}\int_0^t ds\int_0^t d\sigma[\sqrt{\nu s}+\sqrt{\nu\sigma}-\sqrt{\nu s+\nu\sigma}].$$

A similar calculation can be carried out for $\int B^2$ and $\int 2AB$ and by combining the results we get

$$\|u\|^2=\frac{2}{\sqrt{\pi}}\int_0^t ds\int_0^t d\sigma[2(\nu s+(\alpha-\nu)\sigma+\nu t)^{\frac{1}{2}}-(\nu s+\nu\sigma)^{\frac{1}{2}}-((\alpha-\nu)s+\nu t+(\alpha-\nu)\sigma+\nu t)^{\frac{1}{2}}]$$

$$= \frac{8t^{\frac{5}{2}}}{15\sqrt{\pi}}\left[2\frac{(\alpha+\nu)^{\frac{5}{2}}-\alpha^{\frac{5}{2}}-(2\nu)^{\frac{5}{2}}+\nu^{\frac{5}{2}}}{\nu(\alpha-\nu)}-\frac{(2\nu)^{\frac{5}{2}}-2\nu^{\frac{5}{2}}}{\nu^2}-\frac{(2\alpha)^{\frac{5}{2}}-2(\alpha+\nu)^{\frac{5}{2}}+(2\nu)^{\frac{5}{2}}}{(\alpha-\nu)^2}\right]$$

It is not obvious from this formula what happens as $\alpha-\nu$ tends to zero or even that $\|u\|^2$ is positive. The rest of the proof therefore consists in transforming the expression into a form which we can work with. Since $\nu^{-1}+(\alpha-\nu)^{-1}=\alpha\nu^{-1}(\alpha-\nu)^{-1}$ it follows that

$$\|u\|^2 = \frac{16t^{\frac{5}{2}}}{15\sqrt{\pi}}\frac{\alpha}{\nu}\left[\frac{(\alpha+\nu)^{\frac{5}{2}}-2\sqrt{2}\alpha\nu(\sqrt{\alpha}+\sqrt{\nu})}{(\alpha-\nu)^2}-\frac{\alpha\sqrt{\alpha}-\nu\sqrt{\nu}}{\alpha-\nu}\right].$$

By writing $(\alpha+\nu)^2$ as $(\alpha-\nu)^2+4\alpha\nu$ and using that $\alpha-\nu$ is equal to $(\sqrt{\alpha}-\sqrt{\nu})(\sqrt{\alpha}+\sqrt{\nu})$ we obtain

$$\|u\|^2 = \frac{16t^{\frac{5}{2}}}{15\sqrt{\pi}}\frac{\alpha}{\nu}\left[\sqrt{\alpha+\nu}+\frac{2\sqrt{2}\alpha\nu}{\sqrt{\alpha}+\sqrt{\nu}}\frac{1}{(\sqrt{\alpha+\nu}+\sqrt{2\alpha})(\sqrt{\alpha+\nu}+\sqrt{2\nu})}-\frac{\alpha+\sqrt{\alpha}\sqrt{\nu}+\nu}{\sqrt{\alpha}+\sqrt{\nu}}\right]$$

We can now put all terms in [..] over a common denominator. Since $2\sqrt{\alpha\nu}$ equals $(\sqrt{\alpha}+\sqrt{\nu})^2-(\sqrt{\alpha+\nu})^2$ we can express the numerator as $(\alpha+\nu)^2$ times a fourth order polynomial in $\gamma=(\sqrt{\alpha}+\sqrt{\nu})/\sqrt{\alpha+\nu}$. After factoring this polynomial we have

$$\|u\|^2 = \frac{16t^{\frac{5}{2}}}{15\sqrt{\pi}}\frac{\alpha}{\nu}\sqrt{\alpha+\nu}\,(\gamma-1)^2\,\frac{(\sqrt{2}-1)\gamma^2+\sqrt{2}\gamma+\sqrt{2}}{2\gamma^2(\gamma+\sqrt{2})}. \tag{5.1}$$

The formula in Lemma 2 can now be derived from (5.1) by rationalizing the numerator in $(\gamma-1)^2$. This completes the proof.

The expression for $\|u\|^2$ in Lemma 2 is too complicated to be useful. It is better to have a simple approximation. Since $1<\gamma\leq\sqrt{2}$ it follows that the last quotient in (5.1) is greater than 3/8 and less than $\frac{1}{2}(7-4\sqrt{2})$. We can therefore estimate the L^2 norm of u by

$$\frac{8t^2}{5\sqrt{\pi}}\sqrt{\alpha t+\nu t}\;(1+\beta)^{-2}\left(1+\frac{1+\sqrt{\beta}}{\sqrt{1+\beta}}\right)^{-2}\leq\|u\|^2\leq 2\cdot\textit{the lower bound}. \tag{5.2}$$

Here $\beta=\nu/\alpha$ is the Prandtl number and we have simplified $\beta^{-1}(\gamma-1)^2$ by rationalizing the numerator. The lower bound in (5.2) is best possible because we have equality for $\alpha=\nu$.

THEOREM 2. *Let* $u(x,t)$ *and* $T(x,t)$ *be the solutions of* (1.1) *and* (1.2) *with* $g\beta=1$. *Let* $U(x,t)$ *and* $\Theta(x,t)$ *be the random solutions given by* (2.7) *and* (2.4). *Set* $\beta=\nu/\alpha$. *If* $t=n\Delta t>0$ *and* $k>1$ *then*

$$1-\frac{1}{k^2}\le P\left(\frac{\|\Theta-T\|}{\|T\|}\le\frac{k}{\sqrt{N}}\right)$$

$$1-\frac{1}{k^2}\le P\left(\frac{\|U-u\|}{\|u\|}\le(1+\beta)\left(1+\frac{1+\sqrt{\beta}}{\sqrt{1+\beta}}\right)\left(\frac{\Delta t}{t}+\frac{k}{\sqrt{N}}\right)\right).$$

Remark. The factor that depends an the Prandtl number β is an increasing function. It is always greater than 2 and less than 5 if $\beta\le1$. It is less than 20 if $\beta\le7.75$. If $\Delta t=t/n$, $N=n^4$, $k=n$ and $\beta\le1$ then we have

$$P(\,\|U-u\|>10\|u\|/n\,)\le 1/n^2.$$

It is surprising that our bound improves as t increases. If Δt is small and t is comparable to Δt then the absolute error is small with high probability, but the relative error could be large.

Proof. We begin with the temperature. Since $T=E\Theta$ it follows from Chebyshev's inequality (see [7,p. 149]) that

$$P\left(\|\Theta-T\|>\frac{k\|T\|}{\sqrt{N}}\right)\le\frac{E(\|\Theta-E\Theta\|^2)}{k^2\|T\|^2N^{-1}}.$$

To compute the expected value we use Fubini's Theorem, (3.3) and Lemma 1 with $a=b=2\alpha t$. This yields

$$E(\|\Theta-E\Theta\|^2)=\int_0^\infty var\,\Theta\,dx=\frac{1}{N}\sqrt{2/\pi}\,(\sqrt{2}-1)\sqrt{2\alpha t}$$

$$=\frac{\|T\|^2}{\sqrt{2}N}.$$

Here we have used Lemma 2. Since the probability of an event is one minus the probability of the complementary event we have proved the first claim in the theorem. We will now study the velocity. Let δ be a constant such that $\|EU-u\|\le\delta\|u\|$. Then

$$\{ \| U-u \| > (\varepsilon+\delta) \| u \| \} \subset \{ \| U-E\,U \| > \varepsilon \| u \| \} .$$

Thus it follows from Chebyshev's inequality that

$$1-P\left(\frac{\| U-u \|}{\| u \|} \le \varepsilon+\delta\right) \le P\left(\frac{\| U-E\,U \|}{\| u \|} > \varepsilon\right) \le \frac{E(\| U-E\,U \|^2)}{\varepsilon^2 \| u \|^2} \tag{5.3}$$

We will choose the value of ε such that the right hand side is less than $1/k^2$. To estimate the expected value we use Fubini's theorem and (3.9) with the extra factor $1/N$. By combining (3.2) with the distributions in (3.4) and (3.5) and using Lemma 1 we get

$$\begin{aligned} E(\| \Theta-E\,\Theta \|^2) &\le \frac{1}{N}\Delta t^2 \sum_{m=1}^{n-1} \sqrt{2/\pi}\,(2-\sqrt{2})\,\sqrt{2\nu(n-m)\Delta t} \\ &\quad + \frac{1}{N}\Delta t^2\, n \sum_{m=1}^{n} \sqrt{2/\pi}\,(2-\sqrt{2})\,\sqrt{2\alpha m\Delta t+2\nu(n-m)\Delta t} \\ &\le \frac{5t^2}{4N}\frac{2}{\sqrt{\pi}}(2-\sqrt{2})\,\sqrt{\max(\alpha,\nu)t} . \end{aligned}$$

By inserting this result in (5.3) and using our lower bound (5.2) for $\| u \|^2$ it follows that the last term in (5.3) is less than k^{-2} if

$$\varepsilon = (1+\beta)\left(1+\frac{1+\sqrt{\beta}}{\sqrt{1+\beta}}\right)\frac{k}{\sqrt{N}} .$$

To find a convenient choice of δ we use the inequality (3.9). From definition (3.7) and Lemma 1 with $a=b=2\max(\alpha,\nu)t$ we conclude that

$$\begin{aligned} \| E\,U-u \|^2 &\le \Delta t^2 \int_0^\infty \left(\int_{x/\sqrt{a}}^\infty \frac{2e^{-y^2/2}}{\sqrt{2\pi}}dy\right)^2 dx \\ &\le \Delta t^2 \sqrt{2/\pi}\,(2-\sqrt{2})\,\sqrt{2\max(\alpha,\nu)t} . \end{aligned}$$

Combining this result with the lower bound for $\| u \|^2$ we obtain

$$\frac{\| E\,U-u \|^2}{\| u \|^2} \le (1+\beta)^2\left(1+\frac{1+\sqrt{\beta}}{\sqrt{1+\beta}}\right)^2 \frac{\Delta t^2}{t^2} = \delta^2 .$$

This completes the proof.

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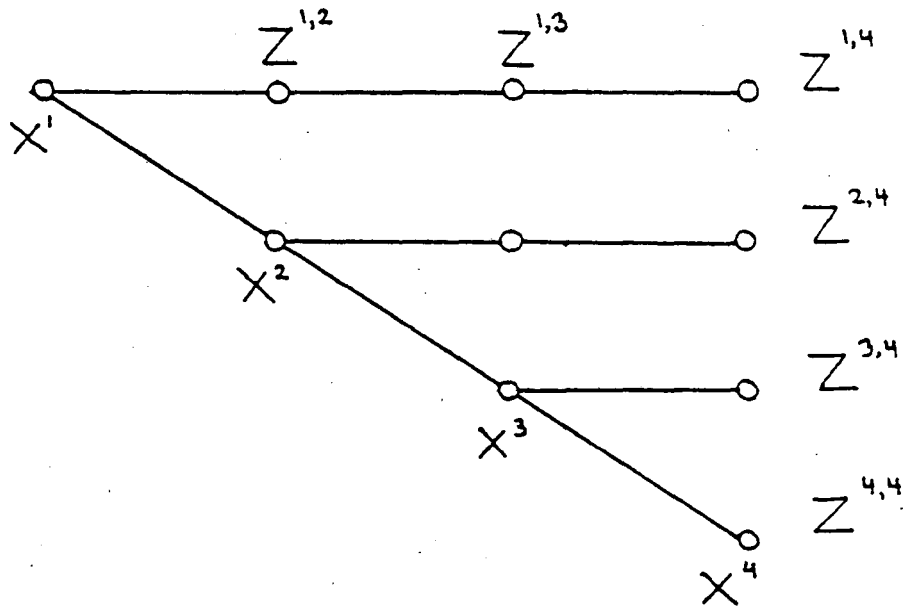

FIG. 1. Dependence between the vortex sheets that are introduced into the flow.

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