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MAGNETIC CHARGE AND NON-ASSOCIATIVE ALGEBRAS

M. Günaydin and B. Zumino

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Magnetic Charge and non-associative Algebras*

Murat Günaydin
Lawrence Berkeley Laboratory**
Berkeley, CA 94720
and
Lawrence Livermore National Laboratory
Livermore, CA 94550

Bruno Zumino
Physics Department and Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720

Abstract

We consider the possibility that the quantum mechanics of a non-relativistic electron in the magnetic field of a magnetic charge distribution can be described in terms of a non-associative algebra of observables. It appears that the case of a point monopole is excluded, while that of a constant charge distribution is acceptable.

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**Participating Guest at Lawrence Berkeley Laboratory.

1. Introduction

It was pointed out some time ago by Lipkin, Weisberger and Peshkin [1] that the commutators of the velocities of an electron in the field of a point magnetic monopole appear to violate the Jacobi identity at the position where the monopole is located. More recently, a number of authors [2-5] have observed that such a violation of the Jacobi identity could be related to the existence of a 3-cocycle[1] on the three dimensional translation group for the electron and to a violation of associativity for finite translations. The main observation of Refs. 2 - 5 is that, even if the Jacobi identity is violated (at the "Lie-algebra" level), the Dirac quantization condition ensures that finite translations (obtained by exponentiation) are associative: in virtue of this quantization condition the phase which measures the degree of non-associativity of finite translations is always of the form $\exp[in2\pi] = 1$ where n is an integer.

The question remains if it is actually possible that the Jacobi identity is violated. It has been pointed out in Ref. 8 that, in any formulation of the quantum-mechanical monopole problem in which the coordinates and the velocities of the electron are described by operators in a Hilbert space, the Jacobi identity cannot be violated, since it follows from associativity of the operators acting in a Hilbert space (see also Ref. 1). This is described in some detail in Ref. 8 for the Dirac string formulation of the monopole problem as well as for the Wu-Yang formulation in terms of local sections [9]. The Jacobi identity can only be violated if the coordinates and velocities of the electron are not operators but belong instead to a non-associative algebra.[2]

One is then led to formulate the following problem. Given the basic commutation relations

$$[x_a,x_b] = 0,\ [x_a,v_b] = i\,\delta_{ab}$$
$$[v_a,v_b] = i\,B_{ab}(x) = i\,\epsilon_{abc}B_c(x) \tag{1.1}$$

(the indices a, b, c, take the values 1, 2 and 3) consider the algebra consisting of all functions of x_a and v_a, say with complex coefficients. Does the consistency of this algebra imply some restrictions on the magnetic field $B_a(x)$? A simple formal manipulation gives the "Jacobian"

$$[[v_1,v_2],v_3] + [[v_3,v_1],v_2] + [[v_2,v_3],v_1] = -\vec{\nabla}\cdot\vec{B} \tag{1.2}$$

If the divergence of the magnetic field vanishes, the Jacobi identity is satisfied (all other Jacobians vanish in any case) and the algebra can be represented in terms of Hilbert space operators by taking[3]

$$v_a = p_a - A_a, \qquad B_{ab} = \partial_a A_b - \partial_b A_a\ . \tag{1.3}$$

This is the usual case of vanishing magnetic charge. In the case of a monopole located at the origin one has

$$\vec{\nabla}\cdot\vec{B} = -4\pi g\delta^3(\vec{x})\ . \tag{1.4}$$

Is the algebra consistent in this case, or in any other case where the magnetic charge does not vanish identically? With certain reasonable assumptions we shall find that consistency requires

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{B}) = 0 \tag{1.5}$$

i.e., the magnetic charge must be independent of position. This excludes (1.4) but leaves the interesting case in which the magnetic field is linear in x. Indeed the general solution of (1.5) is of this type plus a magnetic field with zero divergence. In Section 3, we show that by redefining the coordinates and the velocities one can always reduce an algebra satisfying (1.5) to the case

$$B_a(x) = x_a\ . \tag{1.6}$$

We shall call the algebra given by (1.1) and (1.6) the "magnetic" algebra. It is an infinite non-associative algebra which seems to deserve some further study. In Section 3, we show that the condition (1.5) follows from an identity, the Malcev identity, which is weaker than the Jacobi identity, but can be expected to be valid in a very large class of algebras, the structurable algebras. The elements of a structurable algebra that close under commutation form a Malcev algebra. Before we must, however, introduce some basic notions in the theory of non-associative algebras.

2. Non-associative Algebras

In the Dirac formulation of quantum mechanics the operators M that act in a complex Hilbert space satisfy the associative law of composition, i.e.

$$(M_1M_2)M_3 = M_1(M_2M_3). \tag{2.1}$$

One may define non-associative algebras among these operators by taking an anti-symmetric or a symmetric product of them. For example, the operators

that close under commutation satisfy the Jacobi identity trivially and form a Lie algebra which is not associative since $[[M_a,M_b],M_c]$ is in general not equal to $[M_a,[M_b,M_c]]$. However, the non-associativity of Lie algebras is not "intrinsic" in the sense that they can always be realized in terms of associative matrices with the product being the commutator.

The hermitian operators corresponding to observables in quantum mechanics close under anti-commutation and form a Jordan algebra [10]. Defining the Jordan product o as 1/2 the anti-commutator it is easy to verify that the hermitian operators in a complex Hilbert space do not satisfy the associative law of composition under the Jordan product, i.e.

$$(H_a o H_b) o H_c \neq H_a o (H_b o H_c) \tag{2.2}$$

where

$$H_a o H_b = 1/2(H_a H_b + H_b H_a) = H_b o H_a. \tag{2.3}$$

Even though one does not have associativity among three arbitrary elements one still has associativity for H_a, H_b and H_a^2

$$(H_a o H_b) o H_a^2 = H_a o (H_b o H_a^2). \tag{2.4}$$

This identity is referred to as the Jordan identity and, together with the symmetry condition on the product, defines a Jordan algebra [10]. In their classic work Jordan, von Neumann and Wigner [11] have shown that with only one possible exception all finite dimensional Jordan algebras are special, i.e., they can always be realized in terms of associative matrices with the Jordan

product being one-half the anti-commutator. The only exception is the exceptional Jordan algebra J_3^O of 3 X 3 hermitian octonionic matrices, the Jordan product being one-half the anti-commutator of the matrices. The exceptionality was proven by A. Albert [12]. Recently, Zelmanov has proved that there are no infinite dimensional Jordan algebras that are exceptional [13]. Thus the algebra J_3^O is the unique exceptional Jordan algebra that has no realization in terms of associative matrices. In spite of its "intrinsic" non-associativity it was later shown [14] how one can formulate quantum mechanics over J_3^O satisfying all the axioms of von Neumann. Furthermore, the Jordan formulation of quantum mechanics has been extended to the quadratic Jordan formulation [15] which extends to the exceptional octonionic quantum mechanics as well. The exceptionality of J_3^O however implies the non-existence of a Hilbert space formulation of the corresponding quantum mechanics a la Dirac.

In studying algebraic structures that do not satisfy the associative law of composition one important concept that is often introduced is that of the associator. Given three elements, a,b,c of an algebra A their associator [a,b,c] is defined as

$$[a,b,c] = (ab)c - a(bc). \tag{2.5}$$

It vanishes identically for associative algebras. An alternative algebra is defined as an algebra in which the following identities hold [16,17]

$$[a,a,b] = 0 \tag{2.6a}$$

$$[b,a,a] = 0. \tag{2.6b}$$

Replacing a by (a + c) in (2.6a) one finds

$$[a + c, a + c, b] = [a,a,b] + [a,c,b] + [c,a,b] + [c,c,b] = 0$$
$$= [a,c,b] + [c,a,b] = 0. \qquad (2.7)$$

Similarly (2.6b) implies

$$[b,a,c] + [b,c,a] = 0. \qquad (2.8)$$

Thus in an alternative algebra the associator [a,b,c] is an alternating function of its arguments. Using this property of the associator, one may derive the following Moufang identities for alternative algebras:

$$(aba)c = a(b(ac)) \qquad (2.9)$$

$$c(aba) = ((ca)b)a \qquad (2.10)$$

$$(ab)(ca) = a(bc)a \qquad (2.11)$$

where (aba) = (ab)a = a(ba) by alternativity. To prove (2.9) one needs only to write the quantity (aba)c - a(b(ac)) in terms of the associators and show that it vanishes as a result of alternating property:

$$(aba)c - a(b(ac)) = [ab,a,c] + [a,b,ac]$$
$$= -[a,ab,c] - [a,ac,b]$$
$$= -(a^2b)c + a((ab)c) - (a^2c)b + a((ac)b)$$

$$= -[a^2,b,c] - a^2(bc) - [a^2,c,b] - a^2(cb) + a((ab)c) + a((ac)b)$$
$$= a(-a(bc) - a(cb) + (ab)c + (ac)b)$$
$$= a([a,b,c] + [a,c,b]) = 0.$$

The identity (2.10) can be similarly proven. To prove (2.11) consider

$$(ab)(ca) - a(bc)a = [a,b,ca] + a(b(ca)) - a(bc)a$$
$$= [a,b,ca] - a[b,c,a]$$
$$= -[a,ca,b] - a[b,c,a]$$
$$= -(aca)b + a((ca)b) - a[b,c,a]$$
$$= -(aca)b + a((ca)b) - a[c,a,b]$$
$$= -(aca)b + a(c(ab)) = 0$$

where the vanishing of the expression in the last line follows from (2.9).

Given an alternative algebra A one can define an algebra A^+ with a symmetric product o defined simply as 1/2 the anti-commutator

$$a \circ b = 1/2(ab + ba).$$

The resulting algebra A^+ is a <u>special</u> Jordan algebra [17]. On the other hand starting from an alternative algebra A if we define a new algebra A^- with the anti-symmetric product (commutator) then A^- is in general <u>not</u> a Lie algebra. This is because the Jacobi identity is not satisfied in A^-. The Jacobian J(a,b,c) defined by

$$J(a,b,c) = [[a,b],c] + [[c,a],b] + [[b,c],a] \qquad (2.12)$$

is simply proportional to the associator

$$J(a,b,c) = 6[a,b,c] \tag{2.13}$$

which does not vanish in general. The algebra A^- thus obtained is a Malcev algebra [18,19]. A Malcev algebra is defined by an anti-symmetric product $*$

$$a * b = -b * a \tag{2.14}$$

and a fourth order identity (Malcev identity)

$$(a * b) * (a * c) = ((a * b) * c) * a + ((b * c) * a) * a + ((c * a) * a) * b. \tag{2.15}$$

To prove that the algebra A^- with the anti-symmetric product $*$ taken as the commutator satisfies the Malcev identity one uses the Moufang identity (2.11). This is most easily done by first showing that the (2.15) is equivalent to the following identity

$$J(a,b,a * c) = J(a,b,c) * a. \tag{2.16}$$

We have

$$J(a,b,a * c) = (a * b) * (a * c) + ((a * c) * a) * b + (b * (a * c)) * a.$$

Replacing for $(a * b) * (a * c)$ the right hand side of (2.15) we have

$$\begin{aligned} J(a,b,a * c) &= ((a * b) * c) * a + ((b * c) * a) * a \\ &\quad + ((c * a) * a) * b + ((a * c) * a) * b \\ &\quad + (b * (a * c)) * a \\ &= J(a,b,c) * a \end{aligned}$$

proving (2.16). Since in A^- the product $*$ is simply the commutator, we have

$$\begin{aligned} &J(a,b,a * c) - J(a,b,c) * a \\ &= J(a,b,[a,c]) - [J(a,b,c),a] \\ &= 6[a,b,[a,c]] - 6[[a,b,c],a] \\ &= 6[a,b,ac] - 6[a,b,ca] - 6[a,b,c]a + 6a[a,b,c] \\ &= -6[ac,b,a] + 6a[b,c,a] - 6[a,b,ca] + 6[a,c,b]a \\ &= -6\{((ac)b)a - (ac)(ba) - a(bc)a + a(b(ca)) + (ab)(ca) \\ &\qquad - a(b(ca)) - ((ac)b)a + a(cb)a\} \\ &= -6\{(ab)(ca) - a(bc)a - (ac)(ba) + a(cb)a\} \\ &= 0 \end{aligned}$$

thus proving the Malcev identity for A^-.

For associative algebras A the associator vanishes identically and hence the corresponding algebra A^- is a Lie algebra. The Malcev identity is trivially satisfied in this case. Therefore Malcev algebras correspond to a generalization of Lie algebras. They arise naturally from alternative algebras under the commutator product. The best known example of an alternative algebra which is not associative is the octonion algebra O of Cayley, Graves and Dickson. The seven imaginary units of the octonions close under commutation and form a Malcev algebra which we denote as O^-. The

algebra O^- is the unique (up to isomorphisms) finite dimensional simple Malcev algebra which is not a Lie algebra [19,20].

As for non-associative algebras that go beyond alternative algebras we should mention skew-alternative algebras [21]. A skew-alternative algebra is an algebra with an involution $x \to \bar{x}$ such that the associator satisfies

$$[s,x,y] = -[x,s,y] = [x,y,s]$$

where s is an odd element (i.e., changes sign under the involution $\bar{s} = -s$) and x,y are arbitrary elements of the algebra. Structurable algebras that have been studied in detail by Allison [21] satisfy the skew-alternativity property. A structurable algebra A is defined as an algebra with an involution that satisfies the following operator identity

$$[T_z, V_{x,y}] = V_{T_z x, y} - V_{x, T_{\bar{z}} y}.$$

The linear operator T_z and the bilinear operator $V_{x,y}$ are defined in terms of multiplication in A:

$$V_{x,y} z = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y$$

$$T_x(z) = xz + zx - z\bar{x} .$$

Structurable algebras are a very general class of non-associative algebras that include associative algebras, Jordan algebras and alternative algebras. Given a structurable algebra A it can be decomposed as a vector space direct sum of its even and odd subspaces

$$A = H + S$$

where $H = \{x \in A : \bar{x} = x\}$ and $S = \{s \in A : \bar{s} = -s\}$. The elements of the odd subspace S close under commutation, i.e. $[s,t] = st - ts \in S$ if $s,t \in S$, and form a Malcev algebra. Thus we see that the concept of a Malcev algebra is very general as long as one is dealing with the commutator product.

3. Consistency Condition

We shall now assume that the commutators given by (1.1) satisfy the Malcev identity which we take in the form (2.16). The only nontrivial consequence of the Malcev identity is obtained if one considers the case of four velocities, for other choices of the basic variables the identity is satisfied. Therefore we require, for instance,

$$- J(v_1, v_2, [v_1, v_3]) + [J(v_1, v_2, v_3), v_1] = 0. \qquad (3.1)$$

Now, the first term on the left hand side is easily seen to vanish. Indeed, it equals

$$[v_1, [v_2, [v_1, v_3]]] + [v_2, [[v_1, v_3], v_1]] + [[v_1, v_3], [v_1, v_2]]]. \qquad (3.2)$$

Using (1.1), we see that the first two terms cancel, because they are respectively equal to $i\partial_1\partial_2 B_2$ and to $-i\partial_2\partial_1 B_2$. The last term vanishes by (1.1) also. Therefore the second term on the left hand side of (3.1) must be zero. Using (1.2) we obtain

$$\partial_1 (\vec{V} \cdot \vec{B}) = 0. \qquad (3.3)$$

The other components of (1.5) can be obtained in a similar way.

We have found that the divergence of the magnetic field must be independent of x

$$\vec{\nabla} \cdot \vec{B} = \text{constant}. \tag{3.4}$$

The most general solution of this equation is given by

$$\vec{B} = \vec{B}' + \vec{\nabla} \times \vec{A} \tag{3.5}$$

where B' is linear in x. If we introduce the new variables

$$v_a' = v_a + A_a(x) \tag{3.6}$$

we see that they satisfy an algebra similar to (1.1) but with B replaced by B'. Finally, it is not hard to see that, combining transformations of the type (3.6) with linear transformations on the coordinates and velocities, one can reduce the algebra to the form (1.1) with (1.6) or, equivalently, to the form (1.1) with

$$B_1 = x_1,\ B_2 = B_3 = 0. \tag{3.7}$$

These transformations, which are invertible, can be interpreted as a change of basis in the infinite algebra consisting of the functions of the basis elements x_a and v_a.

4. Conclusion

We have investigated the possibility that the dynamical variables describing an electron moving in the magnetic field generated by a distribution of magnetic charge form a non-associative algebra which does not satisfy the Jacobi identity. Having dropped the Jacobi identity we have assumed the validity of the weaker Malcev identity, which is known to be valid for a very large class of algebras with an anti-symmetric product. This restricts the magnetic charge distribution to being constant and therefore excludes the case of a point monopole. We have not actually proven that the case of the point monopole is inconsistent, but this seems to be likely, in view of the wide range of validity of the Malcev identity.

On the other hand we expect the magnetic algebra for constant magnetic charge to be consistent. Our attempts to find an explicit representation of the magnetic algebra have been unsuccessful. This infinite algebra would seem to be an interesting subject for further mathematical study.

We have not considered here the question of the quantum mechanical description of the system with constant magnetic charge distribution. However, there seems to be no serious problem in developing such a description in terms of density matrices and projection operators, along lines similar to those followed in the Jordan approach to quantum mechanics [10,15].

Coming back to the question of the 3-cocycle mentioned in the introduction, the outcome of our investigation is that the violation of the Jacobi identity corresponding to a point magnetic monopole is not acceptable, since it leads to an inconsistent algebra, in the sense explained above. It seems that the point monopole must be treated a la Dirac (with a string) or with local sections. On the other hand, the algebra corresponding to constant magnetic charge seems to be consistent. Here we have a violation of the

Jacobi identity, which by exponentiation leads to a true non-associativity of the finite translations as well. Since the amount of magnetic charge contained in a finite volume varies continuously there is no quantization condition which could restore the associativity in exponential form.

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Footnotes

1. Cocycles and cohomology are discussed in Refs. 6 and 7.
2. Here we use the term non-associative to mean not associative. In the mathematics literature the term non-associative algebra is used in a general sense to denote associative as well as not associative algebras.
3. We take the mass and the charge of the electron equal to unity.

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