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THE NUCLEAR DROPLET MODEL FOR ARBITRARY SHAPES

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THE NUCLEAR DROPLET MODEL FOR ARBITRARY SHAPES*

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W. D. Myers and W. J. Swiatecki

Abstract

The Droplet Model of nuclear masses and density distributions, introduced in Ref. [1] for spherical configurations, is generalized to arbitrary shapes. Equations in closed form are given for the neutron and proton density non-uniformities induced by the electric forces, and also for the dependence of the neutron skin thickness on position on the nuclear surface. The formulae for the corrections to the nuclear energy associated with these effects are derived and this leads to a Droplet Model atomic mass formula which is presented with a preliminary set of coefficients adjusted to nuclear ground state masses and fission barriers.

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I. INTRODUCTION

The Droplet Model was introduced in Ref. [1] as a refinement of the Liquid Drop Model of average nuclear properties. It is a model which takes into account effects associated with the deviations of neutron and proton densities from constant bulk values, and the deviations of the effective boundaries of the neutron and proton distributions from a common surface. The theory developed in Ref. [1] was specialized from the outset to spherical shapes. The purpose of the present paper is to derive the equations of the Droplet Model in their full generality, for arbitrary shapes. (An informal account of such a generalization was given in an unpublished report, UCRL-19543, 1970. A number of applications of the Droplet Model approach have been made in Refs. [2-8].)

In Refs. [1,9-12] the background and motivation for the Droplet Model have already been discussed. We will not repeat that material here except to stress once again that the Droplet Model deals only with smooth, average nuclear properties and shell effects are outside the scope of the model.

In Sec. II of this work the degrees of freedom of the model are defined, together with the constraints arising from the conservation of neutrons and protons. In Sec. III the potential energy of the Droplet Model is written down as a function of the degrees of freedom (in terms of Taylor expansions

about the Liquid Drop Model configuration as starting point). In Sec. IV this energy is minimized with respect to all degrees of freedom except the geometrical shape of the system. This yields a Droplet Model mass formula (and expressions for constructing the predicted neutron and proton density distributions) expressed as a function of the neutron and proton numbers and the shape. Section V provides additional information on the shape dependences of certain key functions entering the mass formula.

II. DEGREES OF FREEDOM

In general two functions of position $\rho_N(\underset{\sim}{r})$ and $\rho_Z(\underset{\sim}{r})$, whose spatial integrals are N and Z, can specify the time-averaged distribution of N neutrons and Z protons in a nucleus. In the idealized case of distributions with sharp surfaces, i.e. such that ρ_N and ρ_Z are almost constant within two surfaces Σ_N and Σ_Z and zero outside, the shapes of the surfaces Σ_N and Σ_Z and the functions ρ_N and ρ_Z together constitute the degrees of freedom describing the configuration of the system. Under certain assumptions this remains true for leptodermous systems in which the density distributions are no longer sharp, but the degree of diffuseness is small, so that the densities fall from their bulk values to zero in a thin region around the surfaces Σ_N, Σ_Z (thin compared to the dimensions of the system). Such thin-skinned density distributions may be thought of as derived from the sharp <u>generating</u> density distributions by a normal shift of neutrons (or protons) from layers just inside the surface Σ_N (or Σ_Z) to layers just outside, according to some diffuseness prescription. The diffuseness prescription might be, for example, in the form of a folding or convolution operation in which a short-range function is folded into the sharp generating distribution to produce a diffuse density distribution. (Such an approach has already been employed for the generation of diffuse surface optical model or single-particle (shell model) potential wells, see Refs. [13,14] for example. The influence of the diffuseness of the surface and of the range of nuclear forces on the nuclear surface energy can also be approximated in a similar way (see Ref. [15]).) If now the degrees of freedom associated with the neutron and proton density profiles in the surface are imagined eliminated by the requirement that the energy of the system be stationary with respect to variations of

these profiles, no new degrees of freedom are introduced by the optimized diffuseness and the resulting leptodermous system continues to be described by the effective sharp surfaces Σ_N, Σ_Z and the generating functions ρ_N, ρ_Z.

It is important to bear in mind the distinction between the actual leptodermous density distributions ρ_N^{actual}, ρ_Z^{actual}, which are diffuse functions without unique sharp surfaces, and the generating functions ρ_N, ρ_Z which, together with the effective sharp surfaces Σ_N, Σ_Z, serve as the degrees of freedom of the leptodermous system. In what follows ρ_N, ρ_Z will always stand for the generating functions and not for the actual density distributions ρ_N^{actual}, ρ_Z^{actual}.

With the diffuseness of the actual density distributions taken care of by an optimized diffuseness or convolution operation, it is now possible to insist that only generating functions ρ_N, ρ_Z need be considered that deviate little from standard nuclear matter values <u>all the way up</u> to the surfaces Σ_N, Σ_Z. (This is in contrast to ρ_N^{actual}, ρ_Z^{actual} which exhibit a drastic change of behavior--a decrease towards zero--on approaching the surfaces Σ_N, Σ_Z.) The smallness of the non-uniformities of the generating functions enables one to make use of Taylor expansions in powers of these deviations from constancy.

More specifically the Droplet Model of saturating, leptodermous, two-component systems (such as nuclei) results from making expansions of properties of interest (such as the energy) in powers of three small quantities (functions of position). These are as follows: the small deviations of the two generating densities ρ_N and ρ_Z from the corresponding standard nuclear matter values, and the small (variable) separation between the effective sharp surfaces Σ_N and Σ_Z.

With these expansions in mind it is convenient to introduce a mean effective sharp surface Σ and a (thin) neutron skin thickness t in place of the surfaces Σ_N and Σ_Z. The neutron skin thickness t is simply the outward normal distance from the surface Σ_Z to the (neighboring) surface Σ_N. (It is a function of position on these surfaces.) The mean surface Σ is introduced to represent the mean location of the surfaces Σ_N, Σ_Z, the mean being appropriately weighted with the surface values of the generating densities ρ_N^s and ρ_Z^s. The weighting is such that inside any cylinder erected in the normal direction on a surface element, the excess of neutrons in the generating distribution between Σ and Σ_N (assuming, for definiteness, that Σ_N is outside Σ) is the same as the deficiency of protons between Σ_Z and Σ. (That is, the number of protons that would be needed to extend the proton generating function from Σ_Z to Σ.) This means that if n denotes a normal coordinate across the neighboring surfaces Σ_N and Σ_Z then the position of Σ is given by

$$n = \frac{n_N \rho_N^s + n_Z \rho_Z^s}{\rho_N^s + \rho_Z^s} , \tag{1}$$

where n_N and n_Z are the values of n locating Σ_N and Σ_Z. The neutron skin thickness is

$$t = n_N - n_Z , \tag{2}$$

which may now be regarded as a (small) function of position on the mean surface Σ.

We shall further introduce the following small functions in place of the generating densities $\rho_N(\underset{\sim}{r})$ and $\rho_Z(\underset{\sim}{r})$:

$$\varepsilon(\underset{\sim}{r}) = -\frac{1}{3}\left(\frac{\rho(\underset{\sim}{r}) - \rho_o}{\rho_o}\right) \tag{3}$$

and

$$\delta(\underset{\sim}{r}) = \frac{\rho_N(\underset{\sim}{r}) - \rho_Z(\underset{\sim}{r})}{\rho(\underset{\sim}{r})} \quad , \tag{4}$$

where $\rho(\underset{\sim}{r}) = \rho_N(\underset{\sim}{r}) + \rho_Z(\underset{\sim}{r})$, and ρ_o is the equilibrium density of standard nuclear matter (related to the standard nuclear radius constant r_o by $\rho_o = (\frac{4}{3}\pi r_o^3)^{-1}$. Inverting the above relations we find

$$\rho_N = \frac{1}{2}\rho_o(1 - 3\varepsilon)(1 + \delta) \tag{5}$$

$$\rho_Z = \frac{1}{2}\rho_o(1 - 3\varepsilon)(1 - \delta) \quad . \tag{6}$$

We also find it convenient to introduce the averages $\bar{\varepsilon}$ and $\bar{\delta}$ (numbers) and the functions of position $\tilde{\varepsilon}$ and $\tilde{\delta}$ defined by

$$\tilde{\varepsilon}(\underset{\sim}{r}) \equiv \varepsilon(\underset{\sim}{r}) - \bar{\varepsilon}$$

$$\tilde{\delta}(\underset{\sim}{r}) \equiv \delta(\underset{\sim}{r}) - \bar{\delta} \quad . \tag{7}$$

The averages are taken over the volume enclosed by the mean surface Σ. (This implies that the generating functions ρ_N and ρ_Z, originally defined only within the respective surfaces Σ_N and Σ_Z, are imagined to be extrapolated, when necessary, to the neighboring mean surface Σ.)

The following conservation conditions must be satisfied by the quantities t, $\tilde{\varepsilon}$, $\tilde{\delta}$, $\bar{\varepsilon}$, $\bar{\delta}$. The total number of neutrons is given by

$$N = \iiint \rho_N^{actual} = \iiint_{\Sigma_N} \rho_N$$

$$= \iiint_{\Sigma} \rho_N + \iint_{\Sigma} \frac{t\rho_Z^s}{\rho^s} \rho_N^s + \text{ higher powers of } t \quad . \tag{8}$$

Note that since ρ_N^{actual} is obtained from ρ_N by moving a certain number of neutrons from inside Σ_N to outside Σ_N the volume integral of the generating function ρ_N taken over the inside of Σ_N represents correctly the total number of neutrons. This integral is then written as an integral over the inside of the mean surface Σ plus the excess neutrons between Σ and Σ_N. This excess is an integral over the surface Σ of the distance between Σ_N and Σ (equal to $t\rho_Z^s/\rho^s$) times the relevant surface density ρ_N^s. Thus

$$N = N_r + \int_S \left(\frac{\rho_Z \, \rho_N}{\rho} \right) t \quad , \tag{9}$$

where N_r is the "reference neutron number" (the number of neutrons inside the reference surface Σ). We shall use the symbol $\int_V$ to denote a volume integral over the inside of Σ, and $\int_S$ to denote the surface integral over Σ. In such surface integrals it is unnecessary to keep the superscripts s on ρ_N and ρ_Z. Similarly the total proton number is

$$Z = Z_r - \int_S \left(\frac{\rho_N \, \rho_Z}{\rho} \right) t \quad , \tag{10}$$

where Z_r is the number of protons that the surface Σ would contain if the reference function ρ_Z were extrapolated up to Σ.

Using Eqs. (5) and (6) we find

$$N = N_r + \frac{1}{4}\rho_o \int_S t(1 - 3\varepsilon)(1 - \delta^2)$$

$$Z = Z_r - \frac{1}{4}\rho_o \int_S t(1 - 3\varepsilon)(1 - \delta^2) \quad , \tag{11}$$

where

$$N_r = \frac{1}{2}\rho_o \int_V (1 - 3\varepsilon)(1 + \delta)$$

$$Z_r = \frac{1}{2}\rho_o \int_V (1 - 3\varepsilon)(1 - \delta) \quad . \tag{12}$$

The total number of particles is given by

$$A = N + Z = N_r + Z_r = \rho_o \int_V (1 - 3\varepsilon)$$

$$= (1 - 3\bar{\varepsilon})\rho_o V \quad , \tag{13}$$

in virtue of the definition of the average $\bar{\varepsilon}$. Here V stands for the volume inside the reference surface Σ. If we define a standard volume V_o as the volume that A particles would occupy at standard nuclear matter density, and if we introduce equivalent reference and standard radii R and R_o by

$$\frac{4}{3}\pi R^3 = V \ , \qquad \frac{4}{3}\pi R_o^3 = V_o \quad , \tag{14}$$

we find that

$$\left(\frac{R_o}{R}\right)^3 = 1 - 3\bar{\varepsilon} \tag{15}$$

or

$$R = R_o(1 + \bar{\varepsilon} \ + \ \text{higher powers of} \ \bar{\varepsilon}) \quad . \tag{16}$$

Thus $\bar{\varepsilon}$ is to first order a scale parameter relating the linear scale of the mean reference shape Σ of volume V to a similar shape with standard volume.

From the quantities N_r, Z_r we may form the reference relative neutron excess I_r, defined as $(N_r - Z_r)/(N_r + Z_r)$. Thus

$$\begin{aligned} I_r &= \frac{1}{2A}\rho_o \int_V (1 - 3\varepsilon)(2\delta) \\ &\approx \bar{\delta} \end{aligned} \tag{17}$$

to lowest order in small quantities. The actual relative neutron excess is $I = (N - Z)/(N + Z)$, which may be written as

$$\begin{aligned} I &= \frac{1}{A}\left[\frac{1}{2}\rho_o \int_V (1 - 3\varepsilon)(2\delta) + \frac{1}{2}\rho_o \int_S t(1 - 3\varepsilon)(1 - \delta^2)\right] \\ &= I_r + \frac{\rho_o}{2A}\int_S t(1 - 3\varepsilon)(1 - \delta^2) \quad . \end{aligned} \tag{18}$$

This results in a relation between $\bar{\delta}$ and a surface integral over t which, to lowest order in small quantities, may be written as

$$\bar{\delta} \approx I - \frac{\rho_o r_o}{2A}\int_S \tau \quad , \tag{19}$$

where $\tau = t/r_o$. This equation gives the average of the relative neutron excess function $\bar{\delta}$ (the average bulk neutron excess) in terms of the actual relative neutron excess I and a surface integral over the neutron skin thickness t. Introducing the surface average of τ, denoted by $\bar{\tau}$, and the deviation $\tilde{\tau}$ defined by

$$\tilde{\tau} = \tau - \bar{\tau} \quad ,$$

we may re-write Eq. (19) as

$$\bar{\tau} = \frac{2A}{\rho_o \, r_o \, S} (I - \bar{\delta}) \tag{20}$$

where S is the area of the surface Σ. This is a proportionality relation between the surface average of the neutron skin thickness τ and the difference between the actual relative neutron excess I and the bulk average neutron excess $\bar{\delta}$.

The net result of the transformations discussed in this section is that the original degrees of freedom Σ_N, Σ_Z, ρ_N, ρ_Z have been transformed to the following set

Σ, the mean shape of the system,

$\bar{\varepsilon}$, a scale factor,

$\tilde{\varepsilon}$, the density nonuniformity function,

$\bar{\delta}$, the average relative neutron excess in the bulk,

$\tilde{\delta}$, the neutron excess nonuniformity function,

$\tilde{\tau}$, the neutron skin nonuniformity function.

The average neutron skin thickness $\bar{\tau}$ is not an independent degree of freedom but is given uniquely in terms of $\bar{\delta}$ by Eq. (20). (Alternatively one

could regard $\overline{\tau}$ as an independent degree of freedom and $\overline{\delta}$ as a derived quantity.)

Conservation of neutron and proton numbers is ensured by the conditions

$$\int_V \tilde{\varepsilon} = 0 \quad , \quad \int_V \tilde{\delta} = 0 \quad . \tag{21}$$

III. THE ENERGY

In the Droplet Model, as in the Liquid Drop Model, there are three components in the energy: a volume energy E_V, a surface energy E_S and a Coulomb energy E_C. The volume energy is assumed to be an integral over the volume inside Σ of an energy density which is a function of the generating densities ρ_N and ρ_Z (or equivalently of ε and δ). The surface energy is assumed to be an integral over the surface Σ of a surface energy density γ which is a function of the conditions prevailing in the immediate neighborhood of the relevant point on the surface. The possibility of splitting up the main part of the nuclear energy into a volume and a surface term has its roots in the approximately leptodermous nature of most nuclei, as a result of which the major modifications of the energy caused by the presence of the surface can be localized to a relatively thin surface region. A discussion of the precise conditions under which a given system of interacting particles may be treated as leptodermous, and the division of the energy into a volume term and a surface term can be achieved, is outside the scope of the present work and will not be discussed here. Examples of the limitations of a leptodermous approach are afforded by situations where the range of interactions is not small compared to the size of the system (e.g. atoms), or when quantum conditions involving the whole extension of the system are the focus of attention (e.g. shell effects in nuclei). The degree of validity of the leptodermous approach when the interactions do have a short range (and shell effects are absent) has been illustrated in Ref. [1]. The limitations imposed by shell effects have been illustrated in Ref. [11]. Methods to supplement a leptodermous treatment of nuclei by incorporation of shell effects are reviewed in Refs. [14,24], and are a subject of continued study.

For the present we assume the validity of the leptodermous approach and write the total energy, including Coulomb energy, as

$$E = E_V + E_S + E_C \quad ,$$

where

$$E_V = \int_V \rho\, e \quad , \tag{22}$$

$$E_S = \int_S \gamma \quad , \tag{23}$$

$$E_C = \frac{1}{2} e^2 \iiint_{\Sigma_Z} \iiint_{\Sigma_Z} \frac{P(r_{12})\, \rho_Z(\underset{\sim}{r}_1)\, \rho_Z(\underset{\sim}{r}_2)}{r_{12}} \quad . \tag{24}$$

The three terms will be discussed one by one.

A. Volume Energy

In the volume energy we have written the energy density as a product of the generating density ρ times a local energy per particle e (not to be confused with the unit of charge e in Eq. (24)). This energy per particle, considered as a function of ρ_N and ρ_Z, has a minimum for ρ_N, ρ_Z corresponding to standard nuclear matter, i.e. for $\rho_N = \rho_Z = \frac{1}{2}\rho_o$ (or $\varepsilon = 0$, $\delta = 0$). We write a Taylor expansion to second order in ε and δ^2 as follows

$$e = -a_1 + J\delta^2 + \frac{1}{2} K\varepsilon^2 - L\varepsilon\delta^2 + \frac{1}{2} M\delta^4 + \cdots \quad . \tag{25}$$

(The terms retained are necessary and sufficient for the resulting Droplet Model mass formula to be accurate to terms of relative order I^4, $I^2A^{2/3}$ and $A^{1/3}$--see Ref. [1], p. 416.) The coefficients a_1, J, K, L, M are discussed, and their magnitudes estimated, in Ref. [1].

Inserting the expressions for e, Eq. (25), into Eq. (22) we find

$$E_V = \int_V \rho \left[-a_1 + J\delta^2 + \frac{1}{2} K\varepsilon^2 - L\varepsilon\delta^2 + \frac{1}{2} M\delta^4\right]$$

$$\approx \left[-a_1 + J\overline{\delta}^2 + \frac{1}{2} K\overline{\varepsilon}^2 - L\overline{\varepsilon}\overline{\delta}^2 + \frac{1}{2} M\overline{\delta}^4\right] A + \rho_o \int_V (J\tilde{\delta}^2 + \frac{1}{2} K\tilde{\varepsilon}^2)$$

$$+ \text{ higher order terms} \quad . \tag{26}$$

In arriving at Eq. (26) we replaced ε by $\overline{\varepsilon} + \tilde{\varepsilon}$ and δ by $\overline{\delta} + \tilde{\delta}$ used Eq. (21), neglected terms beyond $\overline{\varepsilon}^2$, $\overline{\varepsilon}(\overline{\delta})^2$ and $(\overline{\delta}^2)^2$, and kept only the leading terms in $\tilde{\varepsilon}$ and $\tilde{\delta}$.

B. Surface Energy

As mentioned above, the surface energy coefficient γ will be assumed to be a function only of local properties of the surface. These include the values of ε and δ on the surface Σ, and the local neutron skin thickness t. In addition, γ may depend on the mean local curvature κ of the surface (the sum of the reciprocals of the principal radii of curvature, i.e., $\kappa = (R_1^{-1} + R_2^{-1})$.

We accordingly write the following Taylor expansion for γ.

$$\gamma = \gamma_o + \gamma_\varepsilon\, \varepsilon + \frac{1}{2}\gamma_{tt}\, t^2 + \gamma_{t\delta}\, t\delta + \frac{1}{2}\gamma_{\delta\delta}\, \delta^2 + \gamma_\kappa\, \kappa + \cdots \quad . \tag{27}$$

Suffixes denote differentiations with respect to the quantities indicated, evaluated at $\varepsilon = t = \delta = \kappa = 0$. In the above ε, δ stand for ε_s, δ_s, the values of ε, δ on the surface Σ. The terms retained in the expansion are again necessary and sufficient for our purposes (Ref. [1]).

It is more convenient in what follows to use the expansion of the quantity $4\pi r_o^2\gamma$ (with dimension of an energy). Thus we write

$$4\pi r_o^2\gamma = a_2 + F\varepsilon_s + H\tau^2 + 2P\tau\delta_s - G\delta_s^2 + \frac{1}{2}\, a_3 r_o \kappa \quad , \tag{28}$$

where we have used r_o for the unit of length ($\tau = t/r_o$) and have introduced the coefficients a_2 and a_3 to establish contact with the standard notation for the nuclear surface energy coefficient and the curvature correction coefficient. (For estimates of all the coefficients see Ref. [1]. It will turn out that F is identically zero.)

Inserting the expression for γ, Eq. (28), into Eq. (23) we find

$$\begin{aligned} E_S &= \frac{1}{4\pi r_o^2} \int_S \left[a_2 + F\varepsilon + H\tau^2 + 2P\tau\delta - G\delta^2 + \frac{1}{2}\, a_3 r_o \kappa \right] \\ &= a_2\, A^{2/3}\, B_s(1 + 2\bar{\varepsilon}) + \frac{1}{3}\, \rho_o r_o \int_S \left[F\varepsilon + H\tau^2 + 2P\tau\delta - G\delta^2 \right] \\ &\quad + a_3\, A^{1/3}\, B_k \;+\; \text{higher order terms} \quad . \end{aligned} \tag{29}$$

In Eq. (29) we made use of the transformations

$$\frac{1}{4\pi r_o^2} \int_S a_2 = \frac{S a_2}{4\pi r_o^2} = a_2 \cdot \frac{4\pi R_o^2}{4\pi r_o^2} \cdot \left(\frac{R}{R_o}\right)^2 \cdot \frac{S}{4\pi R^2}$$

$$= a_2 \cdot A^{2/3} (1 + 2\bar{\varepsilon}) \cdot B_s + \text{higher order terms} \quad , \tag{30}$$

where S is the area of the reference surface Σ and B_s is this area in units of $4\pi R^2$, (the area of the sphere of the same volume) and $(R/R_o)^2 \approx 1 + 2\bar{\varepsilon}$ in virtue of Eq. (16).

Similarly

$$\frac{1}{4\pi r_o^2} \int_S \frac{1}{2} a_3 r_o \kappa = \frac{a_3}{8\pi r_o} \int_S \kappa$$

$$= a_3 A^{1/3} B_k + \text{terms of higher order in } \bar{\varepsilon} \quad , \tag{31}$$

where B_k is the total integrated curvature of the surface Σ in units of $8\pi R$, the total integrated curvature of the sphere of the same volume.

C. Coulomb Energy

The Coulomb energy in Eq. (24) is the usual double integral over the inside of Σ_Z of the proton densities ρ_Z at points 1 and 2 divided by the distance r_{12}. A probability of approach function $P(r_{12})$ is included to allow for the anticorrelation of protons in identical spin states required by the exclusion principle. This leads to the "exchange correction" to the Coulomb energy, which may be written as[1] $-c_4 Z^{4/3}/A^{1/3}$, with $c_4 = \frac{5}{4} (3/2\pi)^{2/3} c_1$, where $c_1 = \frac{3}{5} \frac{e^2}{r_o}$. (See, for example, Ref. [1], p. 407.) The diffuseness correction that should be applied to the Coulomb energy of a sharp, but arbitrarily shaped, distribution of charge can be shown to be a multiplicative factor $(1 - \frac{5}{2} \beta^2 + \cdots)$, where $\beta = b/R_Z$ (see Refs. [17,18]). Here b is the "surface width" defined by Süssmann (Ref. [19]), and R_Z is the radius of the effective

sharp sphere (the sphere with the same volume as the sharp distribution in question). We shall disregard the slight modification of the diffuseness correction caused by the redistribution of charge (See Ref. [1], p. 496) and we shall also disregard the difference between R_Z and $r_o A^{1/3}$. To this order the diffuseness correction may be written as $-c_3 Z^2/A$, where $c_3 = \frac{5}{2} c_1 (b/r_o)^2$. For a Fermi (or Woods-Saxon) charge profile, given by $\rho_Z \propto \{1 + \exp[(r - c)/a]\}^{-1}$, the width b is related to the parameter $\underline{a}$ by $b = (\pi/\sqrt{3})a$. We shall use the value $b = 0.99$ fm which corresponds to $a = 0.55$ fm. Both the diffuseness and exchange corrections are independent of the shape of Σ_Z (to first order in the diffuseness). With these corrections out of the way the formula for the Coulomb energy is derived most simply by first writing down a "reference Coulomb energy", which is what the Coulomb energy would be if the generating density ρ_Z were extended to the reference surface Σ, and then correcting for the excess of charge in the layer between Σ_Z and Σ. Thus we write

$$E_C = \frac{3}{5} \frac{Z_r^2 e^2}{R} B_c + e \int_V v \tilde{\rho}_Z - e \int_S t \frac{\rho_N \rho_Z}{\rho} v_s + \text{corr.} \qquad (32)$$

Here Z_r is the reference proton number given by Eq. (10), and the first term gives the Coulomb energy of a <u>uniform</u> distribution of Z_r protons inside the shape Σ. The quantity B_c stands for the Coulomb energy of this shape in units of $\frac{3}{5} \frac{Z_r^2 e^2}{R}$, the Coulomb energy of a sharp sphere of the same volume. The second term corrects this Coulomb energy for the (small) deviation of the proton distribution ρ_Z from its average $\bar{\rho}_Z$ ($\tilde{\rho}_Z$ stands for $\rho_Z - \bar{\rho}_Z$). It is a volume integral of $e\tilde{\rho}_Z$, times the electric potential of the uniform charge distribution $e\bar{\rho}_Z$ inside Σ, which we denote by v.

The third term corrects the "reference Coulomb energy" for the layer of protons between Σ_Z and Σ, which experience a potential v_s (the value of v at the surface Σ), and which have to be "shaved off" in order to arrive at the sharp charge distribution bounded by Σ_Z. The last term stands for the sum of the exchange and diffuseness corrections.

Using Eq. (10) the first term may be re-written as follows

$$\frac{3}{5}\frac{Z_r^2 e^2}{R} B_c \approx \frac{3}{5}\frac{Z^2 e^2}{R} B_c + \frac{6}{5}\frac{e^2 Z}{R} B_c \int_S t\left(\frac{\rho_N \rho_Z}{\rho}\right)$$

$$\approx \frac{3}{5}\frac{Z^2 e^2}{R_o}(1-\bar{\varepsilon}) B_c + e\int_S t\left(\frac{\rho_N \rho_Z}{\rho}\right)\bar{v} \quad . \tag{33}$$

Here $\bar{v}$ stands for the average of v taken over the volume of Σ which average, to sufficient accuracy, is equal to $\frac{6}{5}\frac{eZ}{R} B_c$ (because half the average potential times the total charge is the Coulomb energy). Combining all terms we find

$$E_C = \frac{3}{5}\frac{Z^2 e^2}{R_o}(1-\bar{\varepsilon}) B_c + e\int_V v\,\tilde{\rho}_Z + e\int_S t\,\frac{\rho_N \rho_Z}{\rho}(\bar{v} - v_s) + \text{corr.}$$

Expressing ρ_N, ρ_Z, t in terms of ε, δ, τ we obtain

$$E_C = c_1 \frac{Z^2}{A^{1/3}}(1-\bar{\varepsilon}) B_c - \frac{1}{2} e\,\rho_o \int_V (3\tilde{\varepsilon} + \tilde{\delta})\, v - \frac{1}{4}\rho_o r_o e \int_S \tau\,\tilde{v}$$

$$- c_3 \frac{Z^2}{A} - \begin{cases} c_4 Z^{4/3}/A^{1/3} \\ \text{or}^1\ (c_4/2^{1/3})Z \end{cases} \quad . \tag{34}$$

Here $\tilde{v}$ stands for $v - \bar{v}$.

D. Total Energy

The total Droplet Model energy of N neutrons and Z protons, as a function of the degrees of freedom Σ, $\bar{\varepsilon}$, $\tilde{\varepsilon}$, $\bar{\delta}$, $\tilde{\delta}$, $\tilde{\tau}$ now reads

$$E(N,Z;\ \Sigma,\ \bar{\varepsilon},\ \tilde{\varepsilon},\ \bar{\delta},\ \tilde{\delta},\ \tilde{\tau}) = \left[-a_1 + J\bar{\delta}^2 + \frac{1}{2}K\bar{\varepsilon}^2 - L\bar{\varepsilon}\bar{\delta}^2 + \frac{1}{2}M\bar{\delta}^4\right] A$$

$$+ \rho_o \int_V (J\tilde{\delta}^2 + \frac{1}{2}K\tilde{\varepsilon}^2) + A^{2/3}\, a_2 B_s (1 + 2\bar{\varepsilon})$$

$$+ \frac{1}{3}\rho_o\, r_o \int_S (F\varepsilon + H\tau^2 + 2P\tau\delta - G\delta^2) + A^{1/3}\, a_3\, B_k$$

$$+ c_1 \frac{Z^2}{A^{1/3}} B_c (1 - \bar{\varepsilon}) - \frac{1}{2} e\, \rho_o \int_V (3\tilde{\varepsilon} + \tilde{\delta})\, v - \frac{1}{4} e\, \rho_o r_o \int_S \tau\tilde{v}$$

$$- c_3 \frac{Z^2}{A} - \begin{cases} c_4\, Z^{4/3}\, A^{-1/3} \\ \text{or } (c_4/2^{1/3})Z \end{cases} . \qquad (35)$$

In the above ε stands for $\bar{\varepsilon} + \tilde{\varepsilon}$, δ stands for $\bar{\delta} + \tilde{\delta}$ and τ stands for $\bar{\tau} + \tilde{\tau}$, with $\bar{\tau}$ given by Eq. (20), which may now be written as

$$\bar{\tau} = (I - \bar{\delta})D \quad , \qquad (36)$$

where

$$D = \frac{2A}{\rho_o\, r_o\, S} = \frac{2\, A^{1/3}}{3\, B_s} \quad ,$$

since

$$\rho_o r_o = \frac{3}{4\pi r_o^2} \qquad \text{and} \qquad S = 4\pi r_o^2 \, A^{2/3} \, B_s \quad .$$

IV. MINIMIZATION OF THE ENERGY

We shall now write down the conditions for the energy of a fixed shape Σ to be stationary with respect to arbitrary small variations $\delta\tilde{\varepsilon}$, $\delta\tilde{\delta}$, $\delta\tilde{\tau}$, $\delta\bar{\delta}$, $\delta\bar{\varepsilon}$ (in that order).

A. Variation with Respect to $\tilde{\varepsilon}$

The variation with respect to $\tilde{\varepsilon}$ gives

$$\delta E = \int_V (\rho_o K\tilde{\varepsilon} - \frac{3}{2} e\rho_o v)\, \delta\tilde{\varepsilon} + \frac{1}{3} \rho_o r_o F \int_S \delta\tilde{\varepsilon} \tag{37}$$

The above expression consists of two distinct terms and, in general, the vanishing of the sum for arbitrary $\delta\tilde{\varepsilon}$ implies the vanishing of the two terms separately. This is because one cannot--except for a singular type of $\delta\tilde{\varepsilon}$ discussed below--balance off a surface integral by a volume integral.

The vanishing of the first term in Eq. (37) for all particle-conserving $\delta\tilde{\varepsilon}$ implies that

$$K\tilde{\varepsilon} - \frac{3}{2} ev = \text{constant} \quad . \tag{38}$$

The vanishing of the second term for all particle conserving $\delta\tilde{\varepsilon}$ can only be ensured if $F = 0$.

The above argument may be made more precise by formally converting the surface integral in Eq. (37) into a volume integral. Thus, introducing the Dirac delta function $\delta(n)$ of the normal across the surface Σ, we may write

$$\int_V \delta(n)\, \delta\tilde{\varepsilon} = \int_S \int_{-\infty}^{0} dn\, \delta(n)\, \delta\tilde{\varepsilon} = \frac{1}{2} \int_S \delta\tilde{\varepsilon} \quad . \tag{39}$$

Hence Eq. (37) becomes

$$\delta E = \rho_o \int_V \left[K\tilde{\varepsilon} - \frac{3}{2} ev + \frac{2}{3} r_o F \, \delta(n) \right] \delta\tilde{\varepsilon} \quad . \tag{40}$$

If this is to vanish for arbitrary particle-conserving $\delta\tilde{\varepsilon}$ the expression in the square brackets must be a constant. Hence

$$K\tilde{\varepsilon} - \frac{3}{2} ev = \text{constant} - \frac{2}{3} r_o F \, \delta(n) \quad . \tag{41}$$

This equation shows that, unless F is zero, the condition for equilibrium against changes in $\tilde{\varepsilon}$ would formally lead to an equilibrium distribution for $\tilde{\varepsilon}$, and thus for the generating density ρ, which has a δ-type singularity at the surface. (It is clear that the presence of the term $\frac{3}{2}$ ev does not change the argument: the electric potential is not singular even if the density is singular at the surface and so it cannot cancel out the δ-function singularity.) Consequently, if the energy expression (35) is to have nonsingular equilibrium solutions for the generating function ρ, it is necessary that in the Taylor expansion of the surface energy there should be no linear term in ε.

Equivalent proofs of the vanishing of F are described in Ref. [1], p. 420. The prediction of the vanishing of F has also been verified numerically in a Thomas-Fermi treatment of the nuclear surface, Ref. [1], p. 467.

With F set equal to zero we may determine the constant in Eq. (38) by taking an average of this equation over the volume inside ε. Subtracting this average from Eq. (38) then leads to the following equation for $\tilde{\varepsilon}$

$$\tilde{\varepsilon} = \frac{3e}{2K} \tilde{v} \quad , \tag{42}$$

where $\tilde{v} = v - \bar{v}$.

Thus the net result of demanding the vanishing of the variation in Eq. (37) has resulted in <u>two</u> equations, Eq. (42) and Eq. (43) below:

$$F = 0 \quad . \tag{43}$$

Equation (42) is an explicit solution for the equilibrium distribution $\tilde{\varepsilon}$ since, for the small nonuniformities considered here, one may evaluate the right-hand side of Eq. (42) by using the unperturbed (uniform) distribution of the protons.

B. Variation with Respect to $\tilde{\delta}$

Next we take a variation of Eq. (35) with respect to $\tilde{\delta}$,

$$\delta E = \rho_o \int_V \left[2J\tilde{\delta} - \frac{1}{2} ev \right] \delta(\tilde{\delta}) + \frac{1}{3} \rho_o r_o \int_S (2P\tau - 2G\delta)\, \delta(\tilde{\delta}) = 0 \quad . \tag{44}$$

As before the separate vanishing of the volume and surface terms (or a more formal argument involving a Dirac delta function) leads to two equations

$$2J\tilde{\delta} - \frac{1}{2} ev = \text{constant} \tag{45}$$

and

$$P\tau - G\delta_s = 0 \quad , \tag{46}$$

where δ_s is the value of δ on the surface Σ.

Subtracting from Eq. (45) its volume average leads to

$$\tilde{\delta} = \frac{e}{4J}\,\tilde{v} \quad . \tag{47}$$

Again $\tilde{v}$ may be evaluated for a uniform proton distribution.

C. Variation with Respect to $\tilde{\tau}$

The variation with respect to $\tilde{\tau}$ gives

$$\delta E = \int_S \left[\frac{1}{3}\,\rho_o r_o\,(2H\tau + 2P\delta) - \frac{1}{4}\,e\rho_o r_o \tilde{v}\right]\delta\tilde{\tau} \quad . \tag{48}$$

If this expression is to vanish for all variations $\delta\tilde{\tau}$ which satisfy $\int_S \delta\tilde{\tau} = 0$, the expression in square brackets must be a constant:

$$\frac{2}{3}\,\rho_o r_o\,(H\tau + P\delta_s) - \frac{1}{4}\,e\rho_o r_o \tilde{v}_s = \text{constant} \quad . \tag{49}$$

Subtracting the surface average of this equation all constant terms drop out and we find

$$\frac{2}{3}\,\rho_o r_o\left[H\tilde{\tau} + P(\tilde{\delta}_s - \bar{\tilde{\delta}})\right] - \frac{1}{4}\,e\rho_o r_o\,(\tilde{v}_s - \bar{\tilde{v}}) = 0 \quad , \tag{50}$$

where $\bar{\tilde{\delta}}$ stands for the surface average of $\tilde{\delta}_s$ (which is the value on the surface of the deviation of δ from its <u>bulk</u> average). Similarly $\bar{\tilde{v}}$ is the surface average of $\tilde{v}_s$. Using Eq. (47) and its surface average we may express $\tilde{\delta}_s - \bar{\tilde{\delta}}$ in terms of $\tilde{v}_s - \bar{\tilde{v}}$

$$\tilde{\delta}_s - \bar{\tilde{\delta}} = \frac{1}{4}\,\frac{e}{J}\,(\tilde{v}_s - \bar{\tilde{v}}) \quad . \tag{51}$$

Inserting this in Eq. (50) leads to the following explicit solution for $\tilde{\tau}$

$$\tilde{\tau} = \frac{3}{8}\frac{e}{Q}(\tilde{v}_s - \bar{\tilde{v}}) \quad , \tag{52}$$

where Q is given by

$$Q = \frac{HJ}{J - \frac{2}{3}P} \quad . \tag{53}$$

Before proceeding with the remaining variations note the following. If from Eq. (46) we subtract its surface average we find

$$\tilde{\tau} = \frac{G}{P}(\tilde{\delta}_s - \bar{\tilde{\delta}})$$

$$= \frac{1}{4}\frac{eG}{JP}(\tilde{v}_s - \bar{\tilde{v}}) \quad . \tag{54}$$

This is a second relation between $\tilde{\tau}$ and $\tilde{v}_s - \bar{\tilde{v}}$. If it is to be consistent with the previous one, Eq. (52), the following relation between the coefficients Q, G, J, P must hold

$$Q = \frac{3}{2}\frac{JP}{G} \quad . \tag{55}$$

Eliminating Q between Eqs. (53) and (55) we find the following relation[2] between H, P, G, and J:

$$\frac{G}{P} = \frac{3}{2}\frac{J}{H}\left(1 - \frac{2}{3}\frac{P}{J}\right) \quad , \tag{56}$$

or

$$\frac{GH + P^2}{P} = \frac{3}{2} J \quad .$$

This condition on the coefficients in the Taylor expansions of the volume and surface energies is analogous to the condition F = 0, discussed previously. It is related to the separate vanishing of the surface term in the variation (44), which ensures the existence of nonsingular solutions for the generating densities. As with the vanishing of F, alternative proofs of Eq. (56) are given in Ref. [1], p. 418, and the relation has been verified numerically in Ref. [1], p. 466.

D. Variation with Respect to $\bar{\delta}$

We next take the variation with respect to $\bar{\delta}$. Since $\frac{d\bar{\tau}}{d\bar{\delta}} = -D$ in virtue of Eq. (36), we find

$$\delta E = \delta(\bar{\delta}) \left\{ (2J - 2L\bar{\varepsilon} + 2M\bar{\delta}^2)\, \bar{\delta}A \right.$$

$$\left. + \int_S \left[-\frac{2}{3} \rho_o r_o\, D(H\tau + P\delta) + \frac{2}{3} \rho_o r_o\, (P\tau - G\delta) + \frac{1}{4} \rho_o r_o\, D\, e\tilde{v} \right] \right\} \quad . \qquad (57)$$

The terms in L and M are of higher order in small quantities than the leading term 2J in the first line, and may be disregarded. In the second line $P\tau - G\delta$ vanishes on account of Eq. (46). The remaining terms in the square bracket add up to a constant in virtue of Eq. (49), and so the square bracket may be replaced by its surface average. The surface integral then reduces to this average times the surface area S, equal to $4\pi r_o^2 A^{2/3} B_s$. The vanishing of the coefficient of $\delta(\bar{\delta})$ then leads to

$$2AJ\bar{\delta} + SD\,\rho_o r_o \left[-\frac{2}{3}(H\bar{\tau} + P\bar{\delta}) - \frac{2}{3}P\bar{\tilde{\delta}}_s + \frac{1}{4}e\bar{\tilde{v}} \right] = 0 \quad .$$

Expressing $\bar{\tilde{\delta}}$ in terms of $\bar{\tilde{v}}$ (Eq. (47)) and $\bar{\tau}$ in terms of $\bar{\delta}$ (Eq. (36)), and using the definitions of S and D (note that $\rho_o r_o SD = 2A$) leads to

$$\bar{\delta}(HD + \frac{3}{2}J - P) - HID + \frac{3}{8}e\bar{\tilde{v}}\,(1 - \frac{2}{3}\frac{P}{J}) = 0 \quad . \tag{58}$$

Using the definition of Q (Eq. (53)) we find the following explicit solution for $\bar{\delta}$:

$$\bar{\delta} = \frac{I - \frac{3}{8}\frac{e\bar{\tilde{v}}}{QD}}{1 + \frac{3}{2}\frac{J}{QD}} = \frac{I + \frac{3}{16}(c_1/Q)ZA^{-2/3}\,B_v}{1 + \frac{9}{4}(J/Q)A^{-1/3}\,B_s} \quad . \tag{59}$$

In the last expression we introduced the quantity B_v, defined as the integral $\int_S \tilde{v}$ in units of its value for a spherical shape of the same volume.

E. Variation with Respect to $\bar{\varepsilon}$

The last variation of the energy in Eq. (35) is with respect to $\bar{\varepsilon}$.

$$\delta E = (KA\bar{\varepsilon} - LA\bar{\delta}^2 + 2A^{2/3}\,a_2 B_s - c_1 \frac{Z^2}{A^{1/3}}\,B_c)\,\delta\bar{\varepsilon} \quad .$$

Its vanishing gives

$$\bar{\varepsilon} = (-2a_2 A^{-1/3}\,B_s + L\bar{\delta}^2 + c_1 Z^2\,A^{-4/3}\,B_c)/K \quad . \tag{60}$$

We now have explicit equations (Eqs. (42), (47), (52), (59), (60)) for the values of $\tilde{\varepsilon}$, $\tilde{\delta}$, $\tilde{\tau}$, $\bar{\delta}$, and $\bar{\varepsilon}$ that make the energy in Eq. (35) stationary. It remains to substitute these equilibrium solutions in the energy equation.

F. The Equilibrium Energy Expression

The result of substituting the equilibrium values for $\bar{\varepsilon}$, $\tilde{\varepsilon}$, $\tilde{\delta}$ may be written down at once by remembering that the value of a function consisting of a linear and a quadratic term is, after minimization, equal to minus the quadratic term evaluated at the equilibrium point. Thus all the terms containing $\bar{\varepsilon}$ may be combined into a contribution $-\frac{1}{2}KA\bar{\varepsilon}^2$, where $\bar{\varepsilon}$ is given by Eq. (60). Note that this energy decrease is the result of an increase of $\frac{1}{2}KA\bar{\varepsilon}^2$ in the bulk nuclear energy and a decrease of magnitude $-KA\bar{\varepsilon}^2$ in the surface, Coulomb, and "density-symmetry" energies (the last one is the term $-LA\bar{\varepsilon}\bar{\delta}^2$).

Similarly substitution of the equilibrium values of $\tilde{\varepsilon}$ and $\tilde{\delta}$ gives an energy decrease

$$-\rho_o \int_V (J\tilde{\delta}^2 + \frac{1}{2}K\tilde{\varepsilon}^2) = -\rho_o e^2 \left(\frac{1}{16J} + \frac{9}{8K}\right) \int_V \tilde{v}^2 \qquad (61)$$

which consists of an increase of this amount in the bulk nuclear energy and a decrease of double this amount in the Coulomb energy. By introducing the shape-dependent quantity B_r, defined as the value of $\int_V \tilde{v}^2$ in units of the corresponding integral for the sphere, we may write Eq. (61) as

$$-c_2 Z^2 A^{1/3} B_r \quad ,$$

where

$$c_2 = \frac{c_1^2}{336}(1/J + 18/K) \quad .$$

To evaluate the result of substituting the equilibrium values of $\overline{\delta}$ and $\widetilde{\tau}$ in Eq. (35) we first simplify the surface integrals in that equation as follows:

$$\int_S (F\varepsilon + H\tau^2 + 2P\tau\delta - G\delta^2) = \int_S (H + \frac{P^2}{G})\,\tau^2 \quad \text{(using Eqs. (43), (46))}$$

$$= \int_S Q\tau^2 \quad \text{(using Eqs. (55), (56)).} \quad (62)$$

The two surface integrals in Eq. (35) can then be written as

$$\frac{1}{3}\rho_o r_o \int_S \tau(Q\tau - \frac{3}{4}e\widetilde{v})$$

$$= \frac{1}{3}\rho_o r_o \int_S \frac{G}{P}\delta(\frac{QG}{P}\delta - 3J\widetilde{\delta}) \quad \text{(using Eqs. (46), (47))}$$

$$= \frac{1}{3}\rho_o r_o \int_S \frac{3}{2}\frac{J}{Q}(\overline{\delta} + \widetilde{\delta})(\frac{3}{2}J(\overline{\delta} + \widetilde{\delta}) - 3J\widetilde{\delta}) \quad \text{(using Eq. (55))}$$

$$= \frac{1}{3}\rho_o r_o (9/4)(J^2/Q) \int_S (\overline{\delta}^2 - \widetilde{\delta}^2)$$

$$= \frac{9}{4}(J^2/Q)\overline{\delta}^2 A^{2/3} B_s - \frac{3}{64}\rho_o r_o \frac{e^2}{Q} \int_S \widetilde{v}^2$$

$$= \frac{9}{4} (J^2/Q)^{-2} A^{2/3} B_s - c_5 Z^2 B_w \quad , \tag{63}$$

where B_w is the value of $\int_S \tilde{v}^2$ in units of the same integral evaluated for a sphere, and $c_5 = \frac{1}{64} (c_1^2/Q)$. The term $c_5 Z^2 B_w$ is a surface redistribution energy associated with the nonuniformity of the neutron skin thickness caused by the electrostatic forces.

Collecting together all contributions the final minimized Droplet Model energy formula may be written in the following form

$$\begin{aligned} E(N,Z;\text{ shape}) = & \left[-a_1 + J\bar{\delta}^2 - \frac{1}{2} K\bar{\varepsilon}^2 + \frac{1}{2} M\bar{\delta}^4 \right] A \\ & + \left[a_2 + \frac{9}{4} (J^2/Q)\bar{\delta}^2 \right] A^{2/3} B_s + a_3 A^{1/3} B_k \\ & + c_1 Z^2 A^{-1/3} B_c - c_2 Z^2 A^{1/3} B_r - c_5 Z^2 B_w - c_3 Z^2 A^{-1} \\ & - \begin{cases} c_4 Z^{4/3} A^{-1/3} \\ \text{or } (c_4/2^{1/3}) Z \end{cases} , \end{aligned} \tag{64}$$

where

$$\bar{\delta} = \left[I + \frac{3}{16} (c_1/Q) Z A^{-2/3} B_v \right] \Big/ \left[1 + \frac{9}{4} (J/Q) A^{-1/3} B_s \right] \tag{65}$$

$$\bar{\varepsilon} = \left[-2a_2 A^{-1/3} B_s + L\bar{\delta}^2 + c_1 Z^2 A^{-4/3} B_c \right] / K \quad . \tag{66}$$

For convenience we collect together the definitions of the constants $c_1 \cdots c_5$

$$c_1 = \frac{3}{5} \frac{e^2}{r_o} \quad , \qquad c_2 = (c_1^2/336)(1/J + 18/K) \quad ,$$

$$c_3 = \frac{5}{2}\, c_1\, (b/r_o)^2 \;, \quad \text{where} \quad b = 0.99 \text{ fm}$$

$$c_4 = \frac{5}{4}\, c_1\, (\frac{3}{2\pi})^{2/3} \;, \quad c_5 = \frac{1}{64}\, (c_1^2/Q) \quad . \tag{67}$$

The shape-dependent functionals B_s, B_c, B_k, B_r, B_v, B_w will be discussed further in the next section.

In using the energy formula (64) in practice one first calculates $\bar{\delta}$ given in terms of known quantities by Eq. (65), then $\bar{\varepsilon}$ given in terms of $\bar{\delta}$ by Eq. (66). One then inserts these values of $\bar{\delta}$ and $\bar{\varepsilon}$ in Eq. (64).

G. The Mass Formula

A preliminary, but illustrative, set of values for the adjustable parameters entering the theory is,

$$\begin{aligned}
a_1 &= 15.986 \text{ MeV}, \\
a_2 &= 20.76 \text{ MeV}, \\
a_3 &= 0 \text{ MeV}, \\
J &= 36.5 \text{ MeV}, \\
Q &= 17 \text{ MeV}, \\
K &= 240 \text{ MeV}, \\
L &= 100 \text{ MeV}, \\
M &= 0 \text{ MeV}, \\
r_o &= 1.175 \text{ fm} \\
b &= 0.99 \text{ fm}
\end{aligned} \tag{68}$$

hence

$$c_1 = 0.73531 \quad ,$$
$$c_2 = 0.00016477 \quad ,$$
$$c_3 = 1.30501 \quad ,$$
$$c_4 = 0.56149^{\,3} \quad ,$$
$$c_5 = 0.00049695 \quad . \tag{69}$$

The coefficients given above result in a mass formula that reproduces fairly well nuclear ground state masses and fission barriers after these have been corrected for three non-smooth terms

1) Shell effects

2) An even-odd term of the form

$$(\text{even-odd}) = \begin{cases} 11\ \text{MeV}/\sqrt{A} & \text{for odd nuclei} \\ 0 & \text{for odd mass nuclei} \\ -11\ \text{MeV}/\sqrt{A} & \text{for even nuclei} \end{cases}$$

3) A "Wigner term" of the form

$$(\text{Wigner}) = 30\ \text{MeV}\left[\,|I| + \begin{cases} A^{-1} & \text{for } N = Z = \text{odd} \\ 0 & \text{otherwise} \end{cases}\right]$$

For a discussion of one form of the Wigner term see Ref. [18]. The form used above is suggested by more recent considerations.

Thus a preliminary Droplet Model atomic mass formula (without shell effects) is

$$M(N,Z;\ \text{shape}) = M_n N + M_H Z + \text{Eq. (64)}^{\,3} + (\text{odd-even}) + (\text{Wigner}) \quad . \tag{70}$$

In the above

$$M_n = 8.07169 \text{ MeV}$$

and

$$M_H = 7.28922 \text{ MeV}$$

are the respective masses of the neutron and of the hydrogen atom on the carbon scale, Ref. [22]. The odd-even and Wigner terms are taken to be independent of shape (except that when--as in fission--a nucleus divides into two fragments, these terms are calculated for the separate fragments).

We hope to give a fuller discussion of the relation of the Droplet Model theory to experimental data on a future occasion, but one important feature should be pointed out. The value $r_o = 1.175$ fm found for the radius constant of standard nuclear matter leads (through Eqs. (65), (66), (71), (73)) to effective sharp radii for the proton distributions which agree closely with the experimental result of approximately $1.128 \cdot A^{1/3}$ fm (Ref. [20]). Thus, it appears that the use of the mass formula Eq. (70) removes the disturbing discrepancy noted in Ref. [21] between experimental charge radii and radii deduced from nuclear masses by means of a Liquid Drop Model mass formula.

H. Density Distributions

If, in addition to the energy, one is interested in displaying the equilibrium density distributions of the N neutrons and Z protons associated with the assumed shape Σ, one proceeds as follows. First one draws the shape Σ, making sure that its volume V is related to the volume V_o of A particles at standard nuclear matter density by $(V_o/V)^3 = 1 - 3\bar{\varepsilon}$ (Eq. (13)), where $\bar{\varepsilon}$ is given by Eq. (66). Next one constructs the generating densities ρ_N, ρ_Z using Eqs. (5,6). In these equations $\varepsilon = \bar{\varepsilon} + \tilde{\varepsilon}$ and $\delta = \bar{\delta} + \tilde{\delta}$, where $\bar{\varepsilon}$, $\tilde{\varepsilon}$, $\bar{\delta}$, $\tilde{\delta}$ are given by Eqs. (66), (42), (65), and (47), respectively. From the value of δ

evaluated at the surface Σ one now calculates the neutron skin thickness τ using Eq. (46), which in virtue of Eq. (55) may also be written as

$$\tau = \frac{3}{2}\frac{J}{Q}\delta_s \quad ,$$

or

$$t = \frac{3}{2}\frac{J}{Q} r_o \delta_s \quad . \tag{71}$$

One is now in a position to construct the effective sharp surfaces Σ_N, Σ_Z by normal displacements n_N, n_Z from the surface Σ, where n_N, n_Z are obtained by inverting Eqs. (1), (2). Note that if we take the origin of the normal coordinate to be on Σ then Eq. (1) becomes

$$n_N \rho_N^s + n_Z \rho_Z^s = 0 \quad . \tag{72}$$

Hence

$$n_N = \frac{1}{2}\frac{\rho_Z^s}{\rho^s} t = \frac{1}{2} t(1 - \delta_s)$$

$$n_Z = -\frac{1}{2}\frac{\rho_N^s}{\rho^s} t = -\frac{1}{2} t(1 + \delta_s) \quad . \tag{73}$$

The generating functions ρ_N, ρ_Z are extrapolated, when necessary, to the surfaces Σ_N, Σ_Z. We now have the complete picture of the generating distributions for the neutrons and protons. In order to construct the actual diffuse neutron or proton density distribution it is necessary to diffuse the generating distribution in a way that leaves the position of the effective sharp surface unchanged (i.e., in such a way that the number of particles removed from inside the surface equals the number of particles placed outside it). This could be achieved by folding in a short-range function into the generating densities, or simply by constructing an appropriate fall-off profile whose effective sharp surface coincides with Σ_N or Σ_Z, as the case may be. The surface width b of this profile should be about 0.99 fm, to agree with experiment.

V. THE SHAPE DEPENDENCES

When Eqs. (65), (66) are substituted into Eq. (64) a Droplet Model energy expression is obtained which is a function of N, Z and the nuclear shape only. Regardless of the complexity of the shape the only relevant characteristics are the six shape dependences B_s, B_c, B_k, B_r, B_v, and B_w whose definitions are summarized in Table I. Some of these dependences (B_s and B_c) are well known, others (B_k and B_r) are less familiar, and some of them (B_v and B_w) are new. (For references to the earlier work see Ref. [5].) In an appendix to a recent paper that investigates the influence of curvature and compressibility effects on fission barriers (Ref. [5]) Hasse gives an extremely useful compilation of formulae for calculating these shape dependences. The relationship between his notation and ours is given in Table II along with the formulae for calculating the B's when symmetric shapes close to a sphere are being considered.

Table III contains a listing of the numerical values of the B's for a one parameter family of shapes, of special relevance to fission. A member of this family is specified by a parameter y, which is equal to one minus the fissility parameter x at which that member is a saddle-point shape in the Liquid Drop Model. (See Ref. [23].) For $y = 0$ the shape is a sphere, for small values of y it is a prolate spheroid, for $y \approx 0.2$ it is an approximate cylinder with rounded ends, for $y \approx 0.4$ it is an hour-glass figure which, with increasing y, tends to two equal tangent spheres at $y = 1$.

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FOOTNOTES

[1]For Z not too different from A/2 one may write the exchange correction simply as $-(c_4/2^{1/3})Z$.

[2]In Ref. [1] in Eq. (3.7) on p. 418, and again on p. 426, this relation was written with the factor $(1 - \frac{2}{3}\frac{P}{J})$ on the wrong side of the equation.

[3]The set of coefficients given above was determined using the second form of the exchange correction (i.e., $-(c_4/2^{1/3})Z$) and must <u>not</u> be used in conjunction with the first form.

REFERENCES

1. W. D. MYERS and W. J. SWIATECKI, Ann. Phys. (N.Y.) 55 (1969) 395.

2. W. D. MYERS, Phys. Letters 30B (1969) 451; Nucl. Phys. A145 (1970) 387; Nucl. Phys. A204 (1973) 465.

3. J. W. TRURAN, A. G. W. CAMERON, and E. HILF, "Proceedings of the International Conference on the Properties of Nuclei Far from the Region of Beta-Stability," Leysin, Switzerland, August 1970.

4. P. A. SEEGER, "Proceedings of the International Conference on the Properties of Nuclei Far from the Region of Beta-Stability," Leysin, Switzerland, August 1970.

5. R. W. HASSE, Ann. Phys. (N.Y.) 68 (1971) 377.

6. P. A. SEEGER, in "Atomic Masses and Fundamental Constants 4," Plenum Press, London-New York, 1972, Proceedings of the Fourth International Conference on Atomic Masses and Fundamental Constants, Teddington, England, 1971.

7. S. LUDWIG, et al., Nucl. Phys. A203 (1973) 627.

8. W. M. HOWARD and J. R. NIX, IAEA-SM-174/60; A. J. SIERK and J. R. NIX, IAEA-SM-174/74; P. MÖLLER and J. R. NIX, IAEA-SM-174/202; all to be published in the "Proceedings of the Third International Symposium on the Physics and Chemistry of Fission," Rochester, New York, August 1973.

9. W. D. MYERS, "Proceedings of the Third International Conference on Atomic Masses," Winnipeg, 1967.

10. W. D. MYERS and W. J. SWIATECKI, "Contributions to the International Conference on Nuclear Structure," Tokyo, September 1967.

11. C. F. TSANG, Ph.D. Thesis, Lawrence Radiation Laboratory Report UCRL-18899, 1969.

12. W. D. MYERS, in "Dynamic Structure of Nuclear States," Proceedings of the 1971 Mont Tremblant International Summer School, University of Toronto Press, 1972.

13. G. W. GREENLESS, G. J. PYLE, and Y. C. TANG, Phys. Rev. 171 (1968) 1115.

14. J. R. NIX, Ann. Rev. Nucl. Sci. 22 (1972) 65.

15. H. J. KRAPPE and J. R. NIX, IAEA-SM-174/12, "Proceedings of the Third International Symposium on the Physics and Chemistry of Fission," Rochester, New York, August 1973.

16. M. BRACK, et al., Rev. Mod. Phys. 44 (1972) 320.

17. B. C. CARLSON, Iowa State J. Sci. 35 (1961) 319.

18. W. J. SWIATECKI, "Nuclidic Masses," Proceedings of the Second International Conference on Nuclidic Masses," Vienna, 1963; W. D. MYERS and W. J. SWIATECKI, Nucl. Phys. 81 (1966) 1.

19. G. SÜSSMANN, Lawrence Berkeley Laboratory Report LBL-1615.

20. W. D. MYERS, Nucl. Phys. A204 (1973) 465.

21. W. D. MYERS and W. J. SWIATECKI, Ark. Phys. 36 (1967) 343.

22. A. H. WAPSTRA and N. B. GOVE, Nucl. Data Tables 9 (1971) 265.

23. M. BOLSTERLI, et al., Phys. Rev. C5 (1972) 1050.

24. S. G. NILSSON, et al., Nucl. Phys. A131 (1969) 1.

25. J. R. NIX, Nucl. Phys. A130 (1969) 241.

TABLE I

Definitions of shape dependences for a uniform sharp surfaced density distribution of arbitrary shape.[a,b]

$$B_s = \int_S d\sigma/(4\pi R^2)$$

$$B_c = \int_V W(\underset{\sim}{r})\, d\tau/(\frac{32}{15}\pi^2 R^5)\ ^{c}$$

$$B_k = \int_S \kappa\, d\sigma/(8\pi R)^{d}$$

$$B_r = \int_V [\tilde{W}(\underset{\sim}{r})]^2\, d\tau/(\frac{64}{1575}\pi^3 R^7)\ ^{e}$$

$$B_v = \int_S \tilde{W}(\underset{\sim}{r})\, d\sigma/(-\frac{16}{15}\pi^2 R^4)$$

$$B_w = \int_S [\tilde{W}(\underset{\sim}{r})]^2\, d\sigma/(\frac{64}{225}\pi^3 R^6)$$

[a]See also Ref. [5].

[b]As was discussed earlier, there are now four new B's in addition to the two familiar ones B_s and B_c. They are defined in the usual way so as to have the value unity for a sphere. In each case the quantity B_i is calculated by performing the indicated integral over the volume V or surface S of the given shape and dividing by the value that the integral would have for a sphere of equal volume. As written here the denominators are in terms of R, the radius of such a sphere.

[c]For notational convenience the quantity $W(\underset{\sim}{r})$, which is proportional to the Coulomb potential, is defined by the expression $W(\underset{\sim}{r}) = \int_V \frac{1}{r_{12}}$. For a sphere $W(r) = 2\pi R^2[1 - \frac{1}{3}(r/R)^2]$ for $r < R$.

[d]The symbol κ represents the local curvature of the surface, which is defined in terms of the principal radii of curvature, R_1 and R_2, by the expression $\kappa = R_1^{-1} + R_2^{-1}$.

(continued)

TABLE I (continued)

[e]The quantity $\widetilde{W}(\underset{\sim}{r})$, which is proportional to the deviation of the Coulomb potential from its average value, is defined by the expression $\widetilde{W}(\underset{\sim}{r}) = W(\underset{\sim}{r}) - \overline{W}$, where $\overline{W} = \int_V W(\underset{\sim}{r})/\frac{4}{3}\pi R^3$. For a sphere $\overline{W} = \frac{8}{5}\pi R^2$ and $\widetilde{W}(r) = \frac{2}{5}\pi R^2[1 - \frac{5}{3}(r/R)^2]$.

TABLE II

Shape dependences for a slightly distorted sphere.[a]

Our notation	Hasse's notation	General expression
		$r = \lambda_0^{-1}[1 + \alpha_2 P_2(\cos\theta) + \alpha_4 P_4(\cos\theta)]$
		$\lambda_0^{-1} = 1 - \frac{1}{5}\alpha_2^2 - \frac{2}{105}\alpha_2^3 + \frac{2}{25}\alpha_2^4 - \frac{2}{35}\alpha_2^2\alpha_4 - \frac{1}{9}\alpha_4^2 \ldots$
B_s	B_{surf}	$1 + \frac{2}{5}\alpha_2^2 - \frac{4}{105}\alpha_2^3 - \frac{66}{175}\alpha_2^4 - \frac{4}{35}\alpha_2^2\alpha_4 + \alpha_4^2 \ldots$
B_c	B_{Coul}	$1 - \frac{1}{5}\alpha_2^2 - \frac{4}{105}\alpha_2^3 + \frac{51}{245}\alpha_2^4 - \frac{6}{35}\alpha_2^2\alpha_4 - \frac{5}{27}\alpha_4^2 \ldots$
B_k	B_{curv}	$1 + \frac{2}{5}\alpha_2^2 + \frac{16}{105}\alpha_2^3 - \frac{82}{175}\alpha_2^4 + \frac{2}{35}\alpha_2^2\alpha_4 + \alpha_4^2 \ldots$
B_r	B_{red}	$1 + \frac{2}{5}\alpha_2^2 + \frac{16}{105}\alpha_2^3 - \frac{34043}{13475}\alpha_2^4 + \frac{668}{385}\alpha_2^2\alpha_4 + \frac{1330}{891}\alpha_4^2 \ldots$
B_v	$\sqrt{B}_{srl}$ [b]	$1 - \frac{1}{5}\alpha_2^2 - \frac{2}{105}\alpha_2^3 - \frac{253}{1225}\alpha_2^4 - \frac{4}{105}\alpha_2^2\alpha_4 + \frac{4}{9}\alpha_4^2 \ldots$
B_w	B_{sr2}	$1 + 0\,\alpha_2^2 + 0\,\alpha_2^3 - \frac{243}{245}\alpha_2^4 + \frac{4}{7}\alpha_2^2\alpha_4 + \frac{91}{81}\alpha_4^2 \ldots$

[a]Based on Ref. [5].

[b]Hasse tabulates formulae for the quantity B_{srl}, where $B_v = \sqrt{B}_{srl}$.

TABLE III

Shape dependences for y-family shapes.[a]

y	B_s	B_c	B_k	B_r	B_v	B_w
0.00	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.02	1.00086	0.99957	1.00087	1.00085	0.99957	1.00000
0.04	1.00338	0.99827	1.00352	1.00341	0.99826	0.99995
0.06	1.00750	0.99609	1.00799	1.00746	0.99605	0.99977
0.08	1.01319	0.99303	1.01433	1.01267	0.99285	0.99927
0.10	1.02044	0.98905	1.02265	1.01857	0.98855	0.99819
0.12	1.02927	0.98409	1.03306	1.02446	0.98298	0.99619
0.14	1.03974	0.97807	1.04576	1.02944	0.97591	0.99278
0.16	1.05195	0.97088	1.06099	1.03232	0.96706	0.98736
0.18	1.06604	0.96239	1.07910	1.03146	0.95604	0.97908
0.20	1.08224	0.95238	1.10056	1.02469	0.94238	0.96685
0.22	1.10085	0.94060	1.12603	1.00906	0.92546	0.94915
0.24	1.12229	0.92667	1.15651	0.98048	0.90450	0.92390
0.26	1.14717	0.91008	1.19348	0.93335	0.87854	0.88812
0.28	1.17623	0.89017	1.23915	0.86045	0.84669	0.83771
0.30	1.20963	0.86664	1.29590	0.75626	0.80959	0.76880
0.32	1.24296	0.84250	1.35951	0.63714	0.77505	0.68918
0.34	1.26532	0.82584	1.41013	0.55370	0.75660	0.62984
0.36	1.27619	0.81749	1.44103	0.51320	0.75132	0.59866
0.38	1.28126	0.81347	1.46026	0.49390	0.75123	0.58281
0.40	1.28362	0.81155	1.47339	0.48414	0.75297	0.57442
0.42	1.28458	0.81073	1.48308	0.47907	0.75532	0.56995
0.44	1.28477	0.81057	1.49068	0.47657	0.75783	0.56774
0.46	1.28450	0.81081	1.49692	0.47559	0.76032	0.56694
0.48	1.28394	0.81134	1.50226	0.47557	0.76272	0.56706
0.50	1.28320	0.81206	1.50694	0.47620	0.76497	0.56783
0.52	1.28235	0.81294	1.51114	0.47730	0.76709	0.56906
0.54	1.28141	0.81394	1.51502	0.47875	0.76906	0.57065
0.56	1.28042	0.81503	1.51864	0.48048	0.77090	0.57252
0.58	1.27941	0.81622	1.52208	0.48245	0.77260	0.57463
0.60	1.27837	0.81748	1.52539	0.48463	0.77419	0.57696
0.62	1.27732	0.81882	1.52861	0.48703	0.77566	0.57948
0.64	1.27627	0.82024	1.53177	0.48964	0.77703	0.58220
0.66	1.27522	0.82174	1.53490	0.49246	0.77830	0.58513
0.68	1.27418	0.82333	1.53803	0.49553	0.77949	0.58827

(continued)

TABLE III (continued)

y	B_s	B_c	B_k	B_r	B_v	B_w
0.70	1.27314	0.82501	1.54117	0.49886	0.78059	0.59166
0.72	1.27210	0.82679	1.54473	0.50248	0.78163	0.59531
0.74	1.27108	0.82869	1.54762	0.50645	0.78259	0.59929
0.76	1.27006	0.83072	1.55096	0.51082	0.78350	0.60364
0.78	1.26906	0.83291	1.55440	0.51568	0.78435	0.60843
0.80	1.26806	0.83528	1.55798	0.52111	0.78516	0.61377
0.82	1.26707	0.83788	1.56173	0.52728	0.78594	0.61979
0.84	1.26610	0.84074	1.56567	0.53436	0.78669	0.62669
0.86	1.26514	0.84396	1.56980	0.54265	0.78742	0.63472
0.88	1.26418	0.84763	1.57413	0.55256	0.78816	0.64430
0.90	1.26325	0.85190	1.57860	0.56475	0.78892	0.65606
0.92	1.26233	0.85699	1.58301	0.58024	0.78973	0.67097
0.94	1.26147	0.86321	1.58688	0.60062	0.79063	0.69056
0.96[b]						
0.98[b]						
1.00[c]	1.25992	0.89244	1.58740	0.72236	0.79370	0.80816

[a]The shape parameterization is that of Nix (Ref. [14]), and the values of B_s and B_c are from Ref. [25]. The quantity B_k was calculated with a computer program provided by Hasse (private communication). We are indebted to Peter Möller for actually carrying out the calculation of B_r, B_v and B_w.

[b]The shape parameterization fails in this region.

[c]Touching spheres.

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