LBL-19664
Preprint c.2

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Accelerator & Fusion Research Division

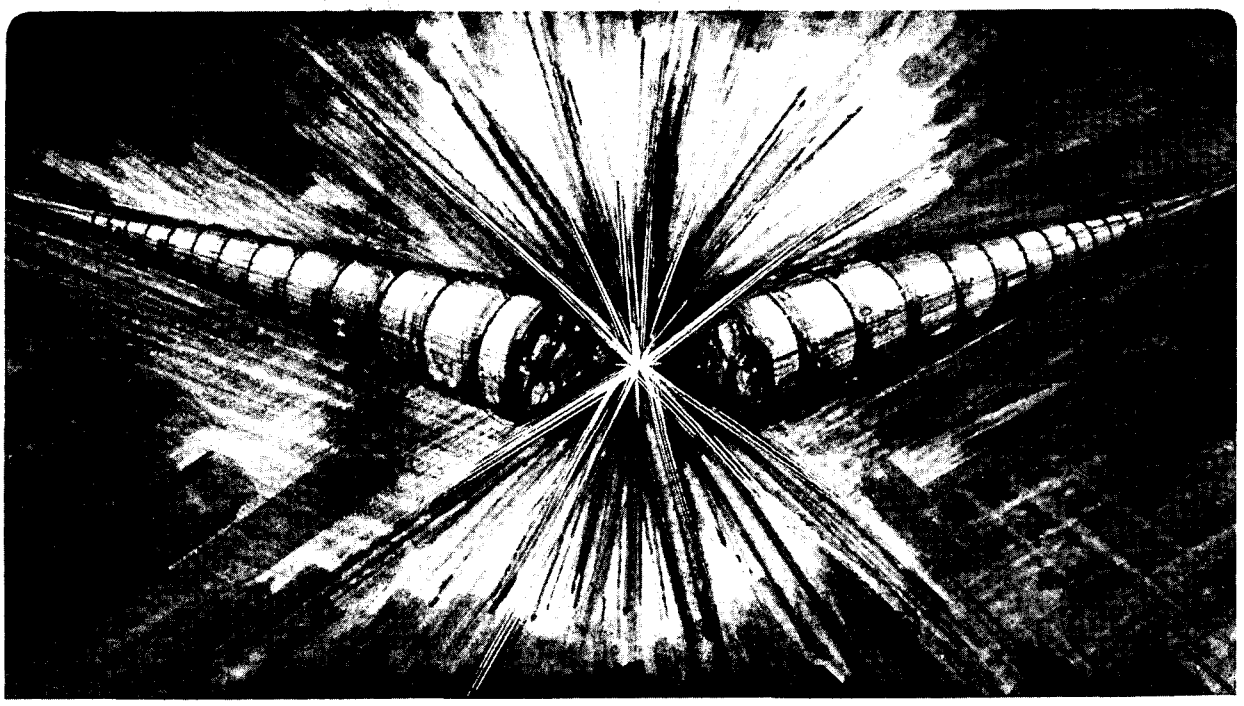
Submitted to Physics of Fluids

RECEIVED
LAWRENCE
BERKELEY LABORATORY
MAR 4 1986
LIBRARY AND
DOCUMENTS SECTION

WAVE ENTROPY:
A DERIVATION BY JAYNES' PRINCIPLE

A.N. Kaufman

December 1985



LBL-19664 c.2

Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

LBL-19664

WAVE ENTROPY: A DERIVATION BY JAYNES' PRINCIPLE*

Allan N. Kaufman
Lawrence Berkeley Laboratory and Physics Department
University of California
Berkeley, CA 94720

Abstract

The Jaynes maximum-entropy principle is used to derive the standard expression for the entropy of a set of weakly interacting waves.

* This work was supported by the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

In discussions of classical wave kinetic equations, use is made of an expression for the wave entropy, as a functional of the mean density of wave action in ray phase space, $\bar{J}(\underline{k},\underline{x})$, and having the property of increasing monotonically in time as $\bar{J}$ evolves. The standard choice[1] is

$$S(\bar{J}) = \int \ln \bar{J}(\underline{k},\underline{x}), \tag{1}$$

with the integration element $d^3x\, d^3k\, /(2\pi)^3$.

It would clearly be desirable to have a classical derivation of this expression. Up to now, one has appealed to the quantum expression,[2] and taken the classical limit of large occupation number. A classical derivation can be based on Jaynes' maximum entropy principle,[3] and provides a striking illustration of its utility. In this note, we review the Jaynes algorithm in performing this derivation.

We begin by recognizing that $(\underline{k},\underline{x})$ space is not a continuum, but rather a set of cells, whose size is determined by Fourier's uncertainty principle. Hence we rewrite Eq. (1) as

$$S(\bar{J}) = \sum \ln \bar{J}_i, \tag{2}$$

where $\bar{J}_i$ is the mean action in the i th cell.

Following Planck, we characterize each cell as an oscillator, possibly nonlinear. With (J_i, Θ_i) as the action-angle variables for the i th oscillator, we introduce the corresponding probability density $\rho_i\ (J_i, \Theta_i)$,

and have

$$\bar{J}_i = \int dJ_i \int d\Theta_i \; \rho_i(J_i,\Theta_i) \; J_i$$
$$= \int dJ_i \; \rho_i(J_i) \; J_i . \tag{3}$$

The Jaynes prescription[3] is to introduce the information-theoretic (Gibbs-Shannon) entropy $S(\rho)$, as a functional of the system distribution function ρ:

$$S(\rho) = - \int \rho \ln \rho , \tag{4}$$

and maximize it with respect to ρ, subject to the constraint (3). This determines the "best" ρ, as a parametric function of the given data $\{\bar{J}_i\}$. When we carry out this procedure, we obtain, not surprisingly, the Gibbs distribution:

$$\rho(J;\bar{J}) = \Pi_i (\bar{J}_i)^{-1} \exp\left(- J_i/\bar{J}_i\right). \tag{5}$$

Since Eq. (4) can be considered as $S = - \langle \ln\rho \rangle$, we form $\langle \ln\rho \rangle$ from (5):

$$\langle \ln\rho \rangle \; (\bar{J}) = \sum_i \left(- \ln \bar{J}_i - 1\right), \tag{6}$$

and finally obtain the desired expression (2), after discarding the constant term in (6). Thus the entropy associated with the data $\{\bar{J}_i\}$ is the information-theoretic entropy of the best distribution consistent with those data.

I am indebted to Steven M. Omohundro, not only for stressing the importance of Jaynes' principle, but also for providing, in his Ph.D. thesis,[4] the basic mathematical underpinning for that principle.

References

1. R.C. Davidson, "Methods in Nonlinear Plasma Theory" (Academic, 1972), Sec. 13.4.4.
2. Landau and Lifshitz, Course of Theoretical Physics, Vol. 5 (Statistical Physics, 3rd Edition, Part 1; Pergamon, 1980), Sec. 55, Eq. (55.8).
3. E.T. Jaynes, "Papers on probability, statistics, and statistical physics" (Reidel, 1983).
4. S.M. Omohundro, "Geometric Perturbation Theory and Plasma Physics," Ph.D. thesis, University of California, Berkeley, 1985 (LBL-20147), Chapter 16.

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

Reference to a company or product name does not imply approval or recommendation of the product by the University of California or the U.S. Department of Energy to the exclusion of others that may be suitable.

LAWRENCE BERKELEY LABORATORY
TECHNICAL INFORMATION DEPARTMENT
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720