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HAMILTONIAN SYSTEMS IN ACCELERATOR PHYSICS

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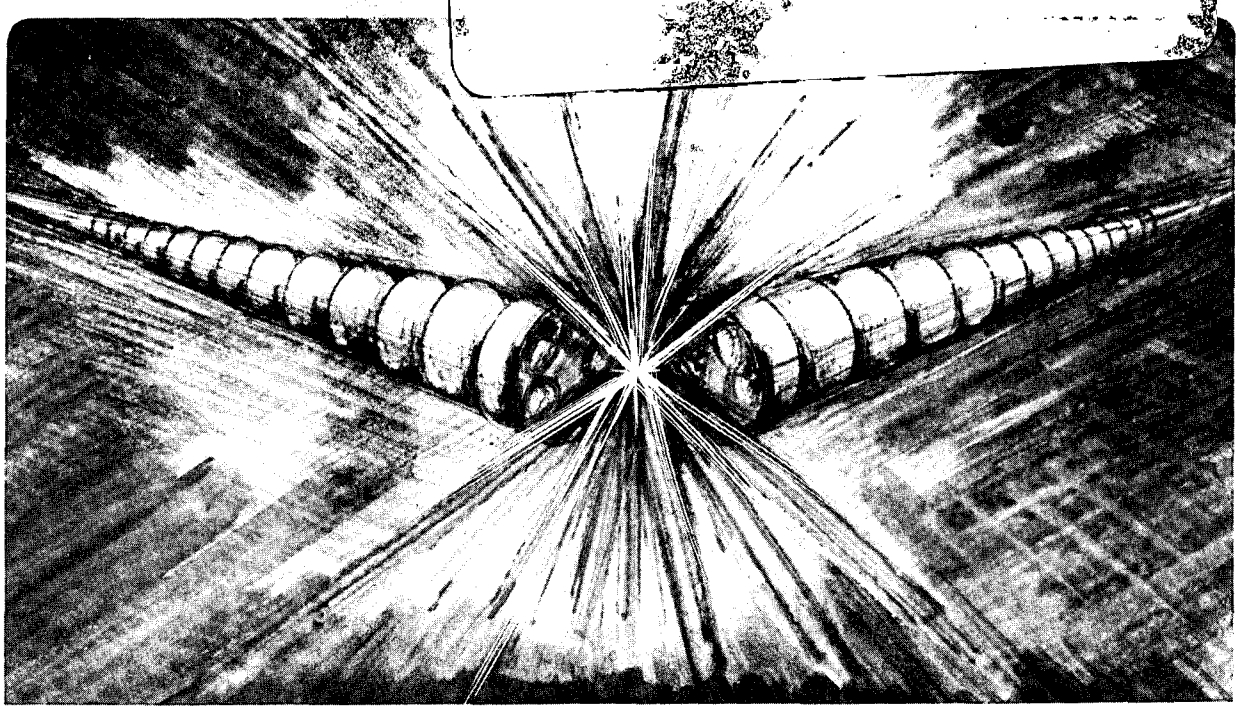

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HAMILTONIAN SYSTEMS IN ACCELERATOR PHYSICS*

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HAMILTONIAN SYSTEMS IN ACCELERATOR PHYSICS*

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Abstract General features of the design of annular particle accelerators or storage rings are outlined and the Hamiltonian character of individual-ion motion is indicated. Examples of phase plots are presented, for the motion in one spatial degree of freedom, of an ion subject to a periodic nonlinear focusing force. A canonical transformation describing coupled nonlinear motion also is given, and alternative types of graphical display are suggested for the investigation of long-term stability in such cases.

In our work directed to the design of new accelerators we are driven more and more strongly into contact with the difficult and challenging problem of nonlinear orbital dynamics. As the machines become more advanced in their required performance and cost, it becomes increasingly important to steer a balanced course between designs that are excessively conservative or insufficiently so. It also is noteworthy that as the overall size becomes larger there is an understandable trend to use magnetic elements of small cross section, with current-carrying conductors close to the region reserved for the beam, so that the dynamical consequences of small constructional errors require particularly careful evaluation.

It is my intent in these remarks to outline in very general and simplified terms the type of device with which we in the accelerator community are concerned and to suggest the type of Hamiltonian

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formulation one may adopt to characterize ion motion in such devices. It is my hope that such an outline will be helpful in assisting participants from other disciplines to make suggestions in the course of the Workshop that will prove helpful to us in our work. It will be noted that, perhaps mistakenly, I will be directing attention almost exclusively to issues of individual-particle motion and neglecting inter-particle interaction effects that also merit careful examination.

One may visualize the device as an annular structure within which charged particles circulate, being confined by focusing forces to remain close to a closed equilibrium orbit. Some nonlinearities will be introduced into the dynamics deliberately, in order better to accommodate particles of somewhat different energies, and other nonlinearities will be present to some extent as a result of inadvertent errors in design or construction. We require well-focussed beam confinement to continue for hours or days in a storage-ring device, while the ions move with virtually the speed of light, so questions of long-term stability present issues of great importance.

An individual ion circulating in the accelerator is acted on by <u>spatially periodic</u> guide and focusing fields. The basic period of such fields is one revolution (save for possible slow adiabatic changes), and ideally (save for construction errors) one revolution may involve passage in a periodic manner through several identical segments of the full ring. A Hamiltonian function to represent the particle motion thus conveniently may employ as an independent variable a distance s (measured around the ring, on an equilibrium or reference orbit) rather than the time. The structure of the accelerator accordingly may be described by terms that are s-dependent in a periodic manner. The action of R.F. electric fields, as would be required to maintain the bunching of a stored beam, then may be included through use of a dependent-variable pair such as particle energy and time.

Use of such a single-particle Hamiltonian function suggests:

> the neglect of radiation (although radiation is beginning to become important for proton

accelerators at the higher energies now being considered),

the omission of beam-gas and of beam-beam collisions, and

deferred treatment of external time-dependent perturbations ("noise").

Adoption of a Hamiltonian then implies that the evolution of the canonical dependent variables should be in accord with the rules of Hamiltonian mechanics -- validity of Poisson-bracket conditions, conservation of phase-space volume, etc. Even if the detailed character of the Hamiltonian does not describe precisely the physical system of interest, we still may expect that the systems do have a Hamiltonian character and that the consequences thereof should follow. It thus would be unfortunate and misleading if, for example, an integration algorithm were adopted (for numerical solution of the differential equations) that generates erroneously a small but consistent damping of phase-space area or volume. The Hamiltonian character of the motion thus may be best described by transformations that inherently are strictly canonical in character (save for register-length round-off errors in numerical computations), and it is gratifying that Dr. Ruth has exhibited some explicit integration algorithms of this nature.

Computed results may be presented at "homologous points" in the structure -- perhaps at intervals of one revolution -- and working variables other than the canonical variables of our Hamiltonian of course may be employed for plotting. The issue remains, however, how best to present output data for problems involving more than a single pair of conjugate variables and it is attractive to consider to what extent we in practice can make use of the "surface-of-section" concept.

Others from the accelerator community will present later in this Workshop some designs and illustrative results for a projected high-energy super-conducting "collider" (or 2-beam storage ring) now under active study, so I shall sketch here only very briefly the general characteristics of such a facility. Dipole magnets provide the required bending fields and

the focusing forces arise from the fields of magnetic quadrupole lenses. Such fields should be incorporated into our Hamiltonian description in such a way that they also are described in a manner that is rather realistic with respect to their Maxwellian character. Thus quadrupole fields that are focusing in one transverse direction are necessarily defocusing in the other, and to obtain a net focusing in the transverse plane a sequence of quadrupole lenses of suitable strength and alternating sign must be employed. Such lenses act to provide focusing (or de-focusing) forces that ideally might be regarded as having a linear dependence upon distance from the equilibrium orbit and, if this were strictly true, the transverse oscillations then would have the simple character described by the Floquet theory for solutions to linear differential equations with periodic coefficients.

To accommodate simultaneously ions of even moderately different energies, however, it usually is desirable to supplement the quadrupole focusing by the inclusion at points around the ring of sextupole lens elements -- and possibly also by the inclusion of elements of still higher multipole order. The transverse motion of the ions thus becomes susceptible to the many nonlinear resonances and potential stochastic behavior that can occur due to the presence of nonlinear elements deliberately introduced or inadvertently present in the focusing lattice.

An additional complication arises from the fact that the kinetic energy of an individual ion may vary with time as a result of applied longitudinal RF fields provided by RF cavities situated at one or more locations around the ring. In the case of an accelerator such cavities of course are necessary to bring the ion energy up to the desired final value, but in a storage ring similar cavities may be desired in order that the beam may be confined longitudinally into individual bunches. Under these conditions individual ions will execute energy oscillations (typically at a rate that is quite slow compared to the frequency of the transverse oscillations). Additional dependent variables to be included in our s-dependent Hamiltonian function thus may become, as noted above, the

particle energy and time. Additional opportunities for coupling effects then arise through the mechanism of synchroton-betatron resonances.

An early schematic configuration for a lattice formed of three super-periods (Figure 1) has been prepared by Dr. Garren to indicate the several types of insertions that must be included in a complete design and that act to break the short wavelength periodicity of a simple quadrupole sequence:

Bend-free crossing insertions.
Bend-free utility insertions.
Regular insertions of sextupole lenses.
Dispersion suppressors.
Lumped compensatory corrections.
R.F. stations.

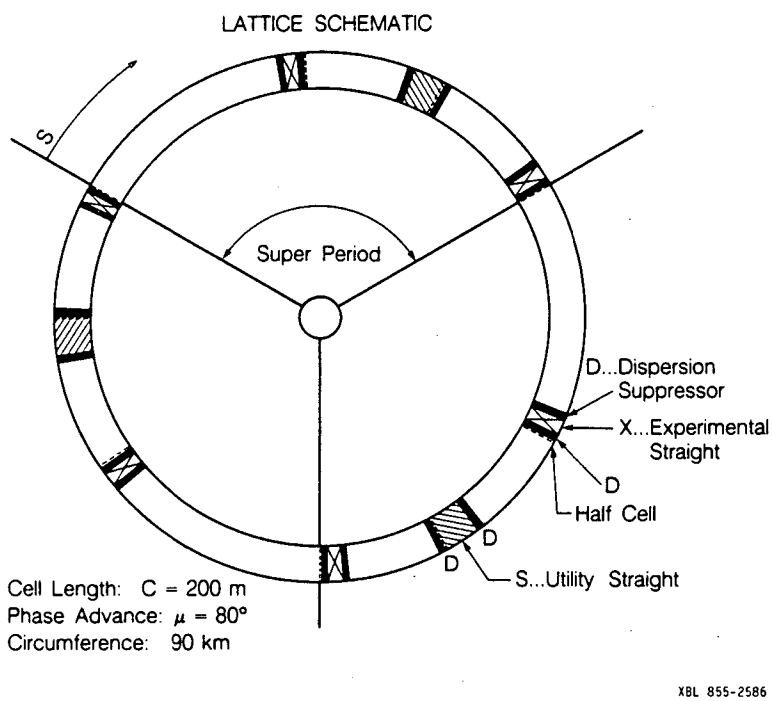

FIGURE 1 Schematic diagram of a lattice with three super-periods.

"Tracking studies" are undertaken to determine orbit characteristics -- such as aperture requirements and stability -- that may be expected for ions with various initial conditions moving in such devices. This

approach will be recognized as essentially deductive, in that the specific design characteristics of a particular device are provisionally assumed and the attempt then made to determine and evaluate the performance that might be expected. Interpretation of the performance so found in such a study of course then can be used to suggest desirable refinements in the design. It is desirable that the computations reflect accurately the Hamiltonian character of the particle motion, in that the evolution of the phase-space variables should be canonical. It also is desirable to include orbit computations for ions in a ring that is subject to some errors in its construction. Such computations should be carried forward for a sufficiently large number of turns in each case that the long-term character of the trajectories may be inferred, if possible, with some confidence.

If the particle motion is confined to a single spatial degree of freedom, signficant orbit characteristics can be displayed appropriately by means of phase plots of the type we illustrate here for trajectories governed by the differential equation

$$\frac{d^2x}{dz^2} = -A\,x + \frac{1}{8}\,x^2 \quad \cos z\ . \tag{1}$$

Plots of $p_x = dx/dz$ vs. x at one-period intervals (say at $z = 0$, mod. 2π in this instance) then assume the form illustrated by Figure 2 for several different initial conditions, when the coefficient A is assigned such a value that oscillations of infinitesimal amplitude advance in phase by 60° per period. The tune for larger amplitude oscillations is reduced from this value and one notes on the diagram the presence of a pronounced order-7 island system ($\sigma = 360/7 \cong 51.43°$). An apparently smooth phase trajectory may be present surrounding this island system, but a grossly stochastic behavior is seen to develop in association with additional island systems at somewhat greater amplitude. Additional structure may be expected to be present, of course, in such a diagram. Thus the presence of an order-13/2 system ($\sigma = 2\text{x}360/13 \cong 55.3846°$) is readily established in the interior (Figure 3) and a portion of the island

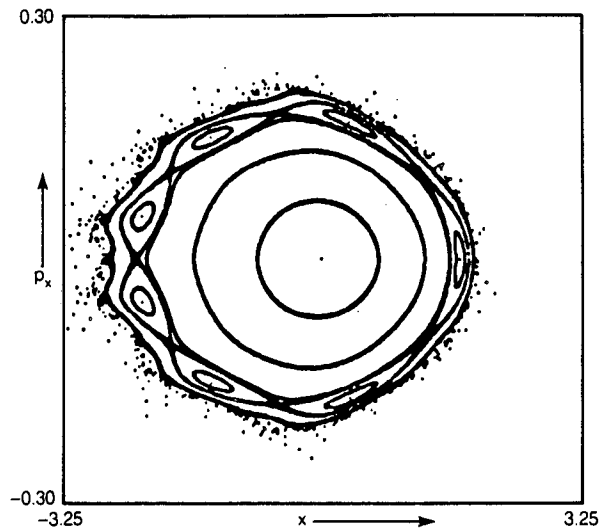

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FIGURE 2 Phase plot for solutions to Eq. (1) at $z = 0$ mod. 2π, with the coefficient A assigned a value such that $\sigma_0 = 60°$. Note the presence of a prominent order-7 system and gross stochastic behavior beginning at somewhat greater amplitude.

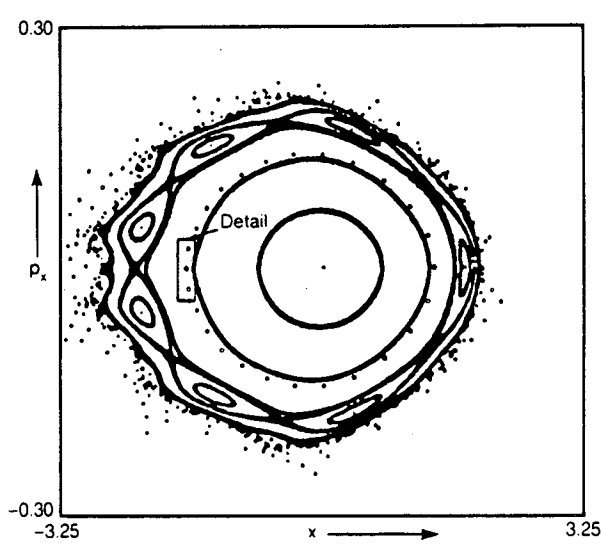

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FIGURE 3 Phase plot similar to Figure 2, with inclusion of stable and unstable fixed points of order 13/2.

structure for this system is illustrated on a detailed plot constructed to a suitable scale (Figure 4).

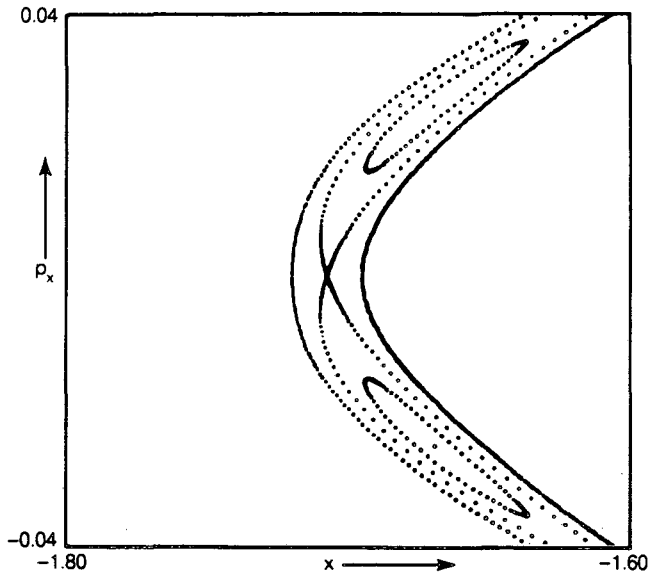

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FIGURE 4 Detail showing phase trajectories in the neighborhood of some of the order 13/2 fixed points plotted in Figure 3.

The presentation of orbit characteristics becomes more difficult when the motion takes place in a phase space with a greater number of dimensions. In such cases one is interested not only in the ranges of excursion of the relevant variables but also will be concerned with the issue of long-term stability for coupled motion. Graphical presentations that provide a clear indication of gradual amplitude growth, or "diffusion," accordingly would be particularly useful. (A method of presentation that frequently has been adopted by other workers investigating mappings in multi-dimensional phase space is that which employs one or more surfaces of section, but this technique may not be directly applicable here.) One must recognize that the process of diffusion in certain problems involving coupled motion unfortunately sometimes can be sufficiently slow as to become evident computationally only in runs of very long duration.

An illustration of coupled motion, for which it may be informative to investigate the merits of possible alternative graphical techniques, is provided by the canonical mapping

$$x_{n+1} = x_n \cos \sigma_x + p_{x_n} \sin \sigma_x + \frac{1}{2} y_n^2 \sin \sigma_x$$

$$p_{x_{n+1}} = -x_n \sin \sigma_x + p_{x_n} \cos \sigma_x + \frac{1}{2} y_n^2 \cos \sigma_x$$

$$y_{n+1} = y_n \cos \sigma_y + p_{y_n} \sin \sigma_y + x_n y_n \sin \sigma_y$$

$$p_{y_{n+1}} = -y_n \sin \sigma_y + p_{y_n} \cos \sigma_y + x_n y_n \cos \sigma_y \ . \qquad (2a\text{-}d)$$

(This mapping may be regarded as representing ion transport through one period of a periodic structure formed by a thin nonlinear lens and a subsequent channel with linear focusing. Behavior similar to that to be reported here also is found if, to represent Maxwellian forces more adequately, the quadratic factors y^2 in the first two equations are each replaced by y^2-x^2.) This mapping is responsive to, amongst others, the sum and difference resonances

$$\sigma_x + 2\sigma_y = 2\pi \quad \text{and} \quad \sigma_x - 2\sigma_y = 0 \ ,$$

and it is of interest here to consider operation at a point $\sigma_x = 0.210(2\pi)$, $\sigma_y = 0.120(2\pi)$ that is close to the line $\sigma_x - 2\sigma_y = 0$ but is rather remote from other resonances. Proximity to the difference resonance results in a behavior such that x and y motions can exchange amplitudes repeatedly. One finds, however (e.g., with double-precision computations on the LBL CDC-6600 computer), that such runs may ultimately become grossly unstable -- although perhaps only after several hundred thousand iterations of the mapping.

A sequence of figures was presented at the Workshop to illustrate various ways in which such behavior (and potential instability) might be depicted. The various runs were initiated with

x_0: various (0.50, 0.55, 0.60, 0.64, and 0.65)

$p_{x_0} = 0$

$$y_0 = 1.0\times10^{-6}$$

$$p_{y_0} = 0 \ .$$

Runs were continued, in each case for which it was possible, through 1.6×10^6 applications of the mapping, but the run launched with $x_0 = 0.65$ was found to become grossly unstable after a smaller number of iterations. One is able to observe, moreover, indications of diffusion in some of the runs initialized with smaller values of x_0 and may experience doubt concerning the ultimate stability of such cases.

Shown here (Figures 5-7) are three types of plot as examples for depicting the motion, initiated with $x_0 = 0.65$, through (a) 450 000 and (b) 900 000 applications of the transformation:

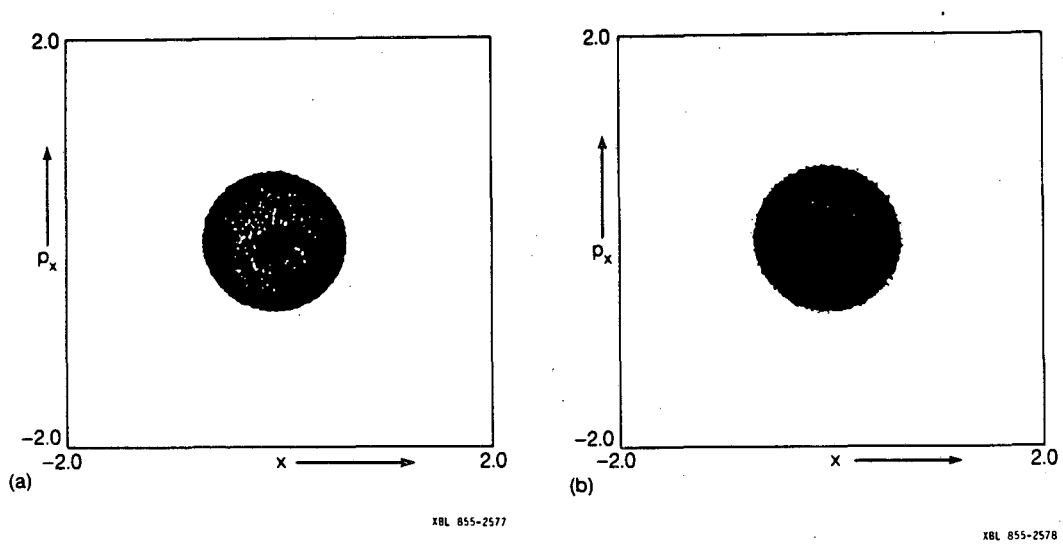

FIGURE 5 p_x vs. x. Such a plot reveals the y motion only through its influence on the x motion. This influence of course can be very substantial, as is seen, and indeed can lead at times to repeated virtual suppression of the x amplitude. For economy in plotting one may elect to plot, as here, only values computed for (say) every 20th iteration.

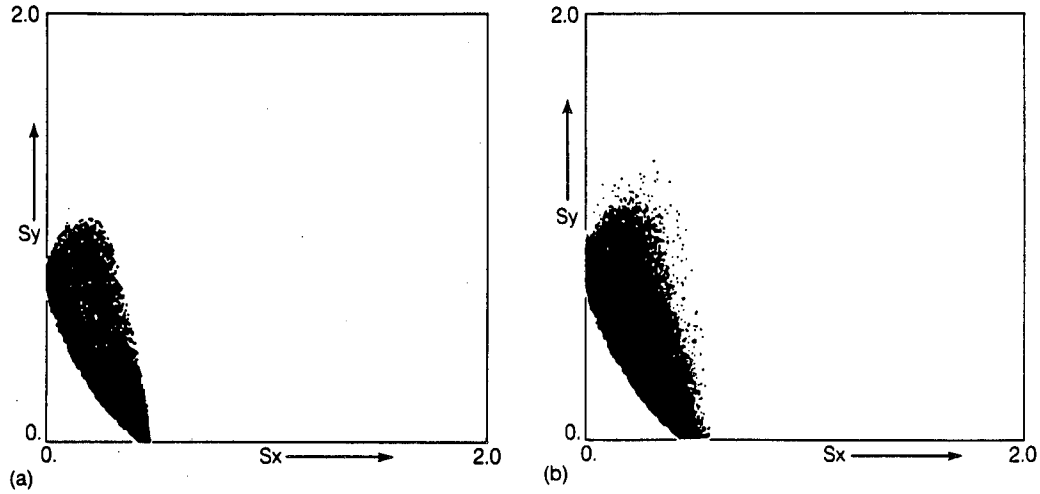

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FIGURE 6 $S_y = p_y^2 + y^2$ vs. $S_x = p_x^2 + x^2$. The quantities S_x and S_y are quadratic forms that for the linearized mapping would individually be constants of the motion. They thus respectively represent in a sense the squared x and y amplitudes and may be regarded as approximately proportional to action variables J_x and J_y. A trend toward progressively greater amplitudes is indicated by these results. For economy in plotting one again may elect to plot only values computed for (say) every 20th iteration.

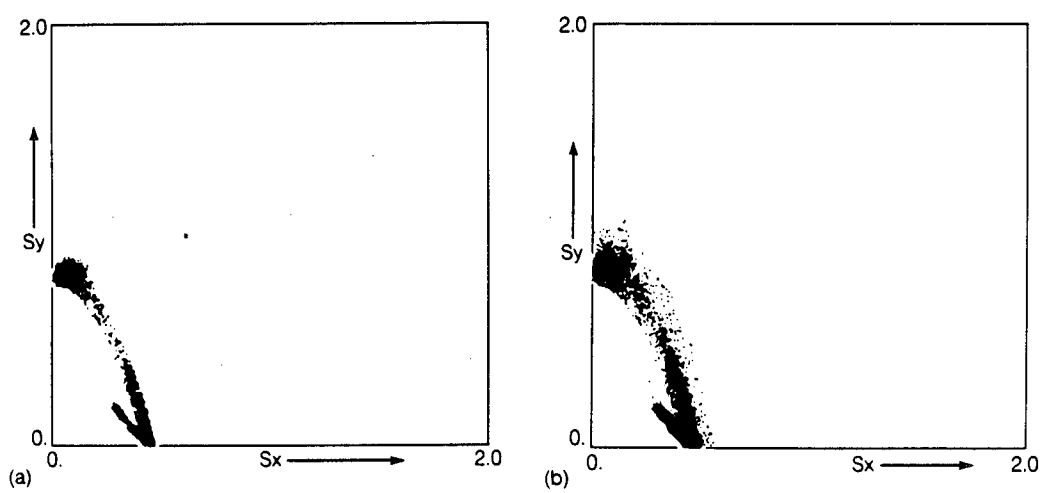

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FIGURE 7 Similar to Figure 6, save that points are plotted only if the contributions of p_x^2 and p_y^2 to the respective totals S_x and S_y are each relatively small. Plotting is omitted if $p_x^2 > S_x/64$ or if $p_y^2 > S_y/64$.

The quantities S_x and S_y may be regarded as having a closer relationship to the squared amplitudes or to action variables when plotted selectively in the manner illustrated by Figure 7, and the resulting displays are seen to have a more localized or slender character. With adoption here of a strong restriction on the magnitudes of p_x and p_y, it is advisable to plot <u>every point</u> for which the imposed selection criteria are fulfilled. It is appropriate to point out that adoption of this method of selection unfortunately may lead to the omission of much relevant amplitude data if the phases of p_x and p_y fail to become small simultaneously -- as indeed has been observed to occur in some coupling runs performed in the neighborhood of the $\sigma_x \cong \sigma_y$ resonance.

The particular mapping adopted here for illustration of course is only one example of a mechanism for obtaining coupling, and the initial conditions selected also are somewhat special. It is hoped, however, that this example and the figures that are presented will stimulate further consideration of issues concerning the long-term stability of coupled motion.

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