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MICROCANONICAL FORMULATION OF
LATTICE GAUGE THEORIES WITH FERMIONS

A. Iwazaki

July 1985

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Microcanonical Formulation of Lattice Gauge Theories with Fermions

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Abstract

We present a microcanonical formulation of $SU(N_c)$ lattice gauge theories with fermions. In this formulation correlation functions are given by a microcanonical ensemble average of bosonic fields. By use of the weak coupling expansion, we prove the equivalence between this formulation and the standard functional formulation. The standard Schwinger Dyson equations can be found to hold in this formulation.

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In a previous paper,[1] we have shown that a microcanonical formulation of scalar field theories gives the same perturbation series as the standard functional formulation does and that the microcanonical formulation of quantum mechanics can reproduce not only the same perturbation series but also correct results.

This formulation had already been used[2,3] as a practical calculational method in lattice gauge theories and had yielded results which agree well with Monte-Carlo results. However, the validity of the formulation had been superficial and obscure until we proved, in the case of scalar field theories, the perturbative equivalence between the microcanonical and standard functional formulations.

In this paper, we shall discuss a microcanonical formulation of lattice gauge theories with fermions. In pure lattice gauge theories without fermions, we adopt $SU(N_c)$ as a gauge group and prove the equivalence between these two formulations by using the weak coupling expansion. In the case of theories[4] with fermions, we introduce complex boson fields to describe the fermion's determinant. Although the formulation of theories with fermions involves a nonlocal operator, Hamilton's equations derived from the formulation do not include the nonlocal operator. Therefore, assuming ergodicity in the dynamical system described by the formulation, we can solve Hamilton's equations without worrying about the nonlocal character in the microcanonical ensemble.

We shall first briefly review our microcanonical formulation[1] of scalar field theories to clarify the essentials. Then we shall proceed to discussing pure lattice gauge theories and theories with fermions.

In a scalar field theory with Euclidean action $S(\phi)$, we first construct a Hamiltonian of which the microcanonical ensemble density is composed as follows: $H = \sum_{i=1}^{N} \frac{p_i^2}{2} + S(\phi)$. Notice that this Hamiltonian is a function of N field variables (ϕ_i) and their canonical conjugate momentum variables (P_i) put in by hand. As is often the case in statistical mechanics, the field average is performed over an energy surface $E = H$ in the phase space of $\{p_i, \phi_i\}$. On the average, the energy E must be chosen to be equal to N, which consists of N/2 coming from contributions of field variables ϕ_i and of the other N/2 coming from those of momentum variables P_i, namely, energy per degree of freedom must be chosen to be equal to 1/2. This choice of the energy ($E = N$) guarantees the perturbative equivalence between the microcanonical formulation and the standard functional formulation, but does not, in general, guarantee the rigorous equivalence beyond the perturbation theory. Indeed, E should be taken as $E = N/2 + \langle S \rangle$ and $E(g = 0) = N$ where $\langle \ldots \rangle$ implies the microcanonical ensemble average and g is a coupling constant (see ref. 7 and appendix in this paper). Since our perturbation theory is defined only in the small coupling constant, we should take $E = N$ in our argument. Finally, after expanding results into series with regard to g, we take a limit of $N \to \infty$, order by order, keeping an ultraviolet cut off finite. As has been shown in ref. 1, these procedures lead us to a standard perturbation series with an ultraviolet cut off.

Let us now describe our microcanonical formulation of $SU(N_c)$ lattice gauge theories in 4 dimensional Euclidean space (a generalization of the formulation into arbitrary groups and dimensions is straightforward). We consider a finite lattice of N_0^4 lattice sites with periodic boundary conditions. The lattice action is given by

$$S = -\frac{1}{4g^2}\sum_{x,\mu\neq\nu}(P_{x,\mu\cdot\nu} - N_c) \tag{1}$$

with $P_{x,\mu\cdot\nu} = \mathrm{Tr}(U_{x,\mu}U_{x+\mu,\nu}U^{\dagger}_{x+\nu,\mu}U^{\dagger}_{x,\nu})$ and $U_{x,\mu} \in SU(N_c)$

where $U_{x,\mu}$ is a link variable on a link characterized by a lattice site x and a direction μ. In (1), g is a gauge coupling constant. Then the partition function in the microcanonical formulation of lattice gauge theories is defined by

$$Z = \int \prod_{x,\mu} d\vec{P}_{x,\mu} dU_{x,\mu}\delta(E - H) \tag{2}$$

with $H = \frac{1}{2}\sum_{x,\mu}\vec{P}_{x,\mu} + S(u)$, $\vec{P}_{x,\mu} = (P^1_{x,\mu}P^2_{x,\mu}\ldots P^{N_c^2-1}_{x,\mu})$ and $E = N$

where $dU_{x,\mu}$ is the invariant measure and $\vec{P}_{x,\mu}$ plays the role of canonical conjugate momentum corresponding to a link variable $U_{x,\mu}$; $\vec{P}_{x,\mu}$ takes a real value over an infinite range. In the formulation (2), we take E = N as a total energy where N is given by

$$N = 3(N_c^2 - 1)N_0^4 + \frac{1}{2}(N_c^2 - 1)N_0^4 = \frac{7}{2}(N_c^2 - 1)N_0^4 \tag{3}$$

Here, we note that, as previously stated, each of the coordinate variables and of the momentum variables contributes an energy of 1/2 to the total energy E. In lattice gauge theories, we have local gauge invariance so that the number of independent degrees of freedom is much smaller than the number estimated from the number of link variables. Therefore, the energy of $3(N_c^2 - 1)N_0^4$ in

(3) comes from contributions of independent link variables $U_{x,\mu}$ and of their conjugate momenta $\vec{P}_{x,\mu}$. On the other hand, the energy of $\frac{1}{2}(N_c^2 - 1)N_0^4$ comes from contributions of momentum variables only, whose conjugate link variables are redundant. The proof of the perturbative equivalence between the microcanonical formulation and the standard one is accomplished by taking a limit of $N_0 \to \infty$ with relation (3) order by order in a perturbation series.

As we carry out the proof by using weak coupling expansion, we must choose[5] a gauge. We adopt the covariant gauge in which the partition function is given by

$$Z = \int \prod_{x,\mu} d\vec{P}_{x,\mu} dU_{x,\mu} \delta(E - H) e^{-S_{gf} - S_{gh}} \tag{4}$$

where S_{gf} is the gauge fixing term and $e^{-S_{gh}}$ is a corresponding Faddeev-Popov determinant. When the link variables $U_{x,\mu}$ are parametrized as

$$U_{x,\mu} = e^{igA_{x,\mu}} \quad \text{with} \quad A_{x,\mu} = \vec{A}_{x,\mu} \cdot \vec{\lambda} \quad \text{and} \quad \mathrm{Tr}(\lambda^a \lambda^b) = 2\delta_{ab} \quad , \tag{5}$$

the gauge fixing term S_{gf} becomes

$$S_{gf} = \frac{1}{2\alpha} \sum_x \left(\sum_\mu \Delta_\mu \vec{A}_{x-\mu,\mu} \right)^2 \quad \text{with} \quad \Delta_\mu f_x \equiv f_{x+\mu} - f_x \tag{6}$$

where α is a gauge parameter. The Faddeev-Popov term S_{gh} is higher order in powers of g than g^0. Therefore, in the lowest order of g^2, Z can be written as

$$Z = \int \prod_{x,\mu} d\vec{P}_{x,\mu} d\vec{A}_{x,\mu} \delta(E - H_0) \exp\left\{ -\frac{1}{2\alpha} \sum_x \left(\sum_\mu \Delta_\mu \vec{A}_{x-\mu,\mu} \right)^2 \right\} \tag{7}$$

with $H_0 = \frac{1}{2} \sum_{x,\mu} \vec{P}^2_{x,\mu} + \frac{1}{4} \sum_{x,\mu\neq\nu} (\Delta_\mu \vec{A}_{x,\nu} - \Delta_\nu \vec{A}_{x,\mu})^2$.

Now, we wish to prove that the generating functional,

$$Z(J) = \int \prod_{x,\mu} d\vec{P}_{x,\mu} dU_{x,\mu} \delta(E - H) \exp\left\{ -(S_{gf} + S_{gh}) + i \sum_{x,\mu} \vec{J}_{x,\mu} \cdot \vec{A}_{x,\mu} \right\} \tag{8}$$

gives rise to, order by order in weak coupling expansion, a generating functional in the standard functional formulation,

$$Z_s(J) = \int \prod_{x,\mu} dU_{x,\mu} \exp\left\{ -(S + S_{gf} + S_{gh}) + i \sum_{x,\mu} \vec{J}_{x,\mu} \cdot \vec{A}_{x,\mu} \right\} . \tag{9}$$

First, we show that in the lowest order of g^2, the generating functional (8) coincides with the usual one in (9). By imposing periodic boundary conditions, we expand the fields $\vec{A}_{x,\mu}$, $\vec{P}_{x,\mu}$ and $\vec{J}_{x,\mu}$ in Fourier series:

$$\vec{A}_{x,\mu} = \sum_m \frac{e^{iq_m x}}{\sqrt{2N_0^4}} \vec{A}_{m,\mu} \quad , \quad \vec{P}_{x,\mu} = \sum_m \frac{e^{iq_m x}}{\sqrt{2N_0^4}} \vec{P}_{m,\mu} \quad \text{and} \tag{10}$$

$$\vec{J}_{x,\mu} = \sum_m \frac{e^{iq_m x}}{\sqrt{2N_0^4}} \vec{J}_{m,\mu} \quad \text{with } q_m = \frac{2\pi}{N_0 a} (m_1, \ldots, m_4) \quad \text{and} \quad x_\mu = \ell_\mu a$$

where a is a lattice spacing, and ℓ_μ and m_μ are integer $\left(-\frac{N_0}{2} \leq m \leq \frac{N_0}{2}\right)$. Then the generating functional becomes

$$Z_0(J) = \int \prod_{x,\mu} d\vec{P}_{x,\mu} d\vec{A}_{x,\mu} \, (E - H_0) \exp\left\{ -\frac{1}{2\alpha} \sum_m |\vec{A}^L_m|^2 \left(\sum_\mu |k_{m,\mu}|^2 \right) + \right.$$

$$\left. + \frac{i}{2} \sum_m \left(\vec{J}^L_m \cdot \vec{A}^L_m + \sum_\mu \vec{J}^T_{m,\mu} \cdot \vec{A}^T_{m,\mu} \right) \right\} \tag{11}$$

with $$H_0 = \frac{1}{4} \sum_m |\vec{P}^L_m|^2 + \frac{1}{4} \sum_{m,\mu} |\vec{P}^T_{m,\mu}|^2 + \frac{1}{4} \sum_{m,\mu} |\vec{A}^T_{m,\mu}|^2 \left(\sum_\lambda |k_{m,\lambda}|^2 \right) ,$$

$$k_{m,\mu} \equiv 1 - e^{iq_{m,\mu}} \quad \text{and} \quad q_{m,\mu} \equiv \frac{2\pi}{N_0} m_\mu$$

where we have decomposed the fields $\vec{A}_{m,\mu}$, $\vec{P}_{m,\mu}$ and $\vec{J}_{m,\mu}$ into a longitudinal component and transverse components, for example,

$$\vec{A}_{m,\mu} = \frac{k_{m,\mu}}{\sqrt{\sum_\lambda |k_{m,\lambda}|^2}} \vec{A}^L_m + \vec{A}^T_{m,\mu} \quad \text{with} \quad \sum_\mu k^*_{m,\mu} \vec{A}^T_{m,\mu} = 0 \quad . \tag{12}$$

We note that in H_0 there are no potential terms for the longitudinal mode. This fact results from the local gauge invariance of the system and leads us to E = N as the total energy. In other words, if we had potential terms for the longitudinal modes like $\sum_\mu |k_{m,\mu}|^2 |\vec{A}^L_m|^2$, we would need to take $E = 4(N_c^2 - 1)N_0^4$ in order to prove the equivalence.

Changing the integration measure of $\prod_{x,\mu} d\vec{P}_{x,\mu} d\vec{A}_{x,\mu}$ into $\prod_{m,\mu} d\vec{P}_{m,\mu} D\vec{A}_{m,\mu}$, we perform the integrating by using a technique in the previous paper.[1] Then, taking the limit of $N_0 \to \infty$ with the relation (3), we obtain

$$Z_0(J) = C \exp\left\{-\frac{1}{4}\sum_{m,\mu,\nu} \vec{J}^*_{m,\mu}\Delta_{\mu,\nu}(k_m)\vec{J}_{m,\nu}\right\} \tag{13}$$

with $\Delta_{\mu,\nu}(k_m) = \left[\delta_{\mu,\nu}\left(\sum_\lambda |k_{m,\lambda}|^2\right) - (1-\alpha)k_{m,\mu}k^*_{m,\nu}\right] \Big/ \left(\sum_\lambda |k_{m,\lambda}|^2\right)^2$

where C is a constant independent of $\vec{J}_{m,\mu}$. Therefore, in the lowest order of g^2, we obtain the same generating functional as the one in the standard formulation.

Next, we proceed to proving the equivalence in higher order terms of g^2. For this purpose, we must expand the function of $\delta(E - H)$ in (8) with respect to g^2 and carry out the above calculation by using a weight function of $\frac{d^n}{dE^n}\delta(E - H\ )$ instead of $\delta(E - H_0)$. However, it is easy to see[1] that the results obtained by taking the limit of $N_0 \to \infty$ for fixed n are the same as the one in (13) (see (14)) and are independent of n. Hence, we have reached a conclusion that the generating functional (8) in the microcanonical formation coincides precisely, in all orders of weak coupling expansion, with the one in the standard formulation.

A comment is in order. As can be shown easily in perturbation theory, the following formula holds[1]:

$$\lim_{N_0\to\infty} \frac{\int d\mu\ \delta(E - H)W(c)}{\int d\mu\ \delta(E - H)} = \lim_{N_0\to\infty} \frac{\int d\mu\ \frac{d^n}{dE^n}\delta(E - H)W(c)}{\int d\mu\ \delta(E - H)} \tag{14}$$

for arbitrary integer $n > 0$

where $d\mu$ is the integration measure in (2) and W(c) is an arbitrary Wilson loop (the proof is given in the appendix). Using the formula (14), we can

derive the Schwinger-Dyson equations[6] in $U(N_c)$ lattice gauge theories. Indeed, ingredients in the derivation of the equations are 1) use of the invariant measure and 2) use of an ensemble density f(S) such that $(-\frac{d}{dS})f(S) = f(S)$. In the standard formulation, f(S) is provided as $f = e^{-S}$ and the condition 2) is satisfied trivially. On the other hand, in the microcanonical formulation the condition 2) is not satisfied, but the formula (14), instead of 2), ensures the Schwinger-Dyson equations.

We now proceed to discussing the microcanonical formulation of lattice gauge theories with fermions. The formulation has been discussed[4] previously and some computer calculations have been performed in the formulation. However, the equivalence of it to the standard formulation has not yet been proved. Therefore, we wish to show briefly the equivalence in a similar way to that used previously for bosonic systems.

Suppose that a lattice theory with fermions of two flavours is defined by the Euclidean action

$$S = \sum_{i=1}^{2}\sum_{m,n} \bar{\psi}_{i,m}K_{m,n}\psi_{i,n} + S'(\eta) \tag{15}$$

where $S'(\eta)$ is an action of the other bosonic fields η (scalar fields, gauge fields, etc.). $K_{m,n}$ represents the kinetic term which may include the other fields η (m and n indicate both a lattice site and a spinor index). We assume that the determinant det K is real. Then, it follows that

$$(\det K)^2 = \det K^+ \det K = \det(K^+K) = \prod_{s=1}^{N_f} |k_s|^2 \tag{16}$$

where k_s is an eigenvalue of the operator K and N_f is the number of degrees of

freedom of a fermion field. The microcanonical formulation of the theory is defined as follows:

$$Z = \int \prod_{n=1}^{N_f} dP_n^* dP_n d\phi_n^* d\phi_n d\mu' \delta(E - H) \tag{17}$$

with $H = \frac{1}{2}\sum_n |P_n|^2 + \frac{1}{2}\sum_{n,m} \phi_n^*(K^+K)^{-1}_{n,m}\phi_n + H'$

where H' and μ' are the Hamiltonian and the integration measure of fields η, respectively. The total energy E is taken as $E = E' + 2N_f$ with $E' = N'$. Here, we have assumed that the number of degrees of freedom of fields η is N'. The complex fields P_n and ϕ_n in (17) have been introduced to describe the degrees of freedom of fermions. When we calculate green functions of fermions, we make use of the generating functional,

$$Z(J) = \int d\mu\delta(E - H)\exp\left\{\sum_{i=1}^{2}\sum_{n,m} \bar{J}_{i,n}(K^{-1})_{n,m}J_{i,m}\right\} \tag{18}$$

with $d\mu \equiv \prod_{n=1}^{N_f} dP_n^* dP_n d\phi_n^* d\phi_n d\mu'$

where $\bar{J}$ and J are sources of the fermions. In order to show perturbatively that the generating functional is the same as the one in the usual formulation, we expand the fields P_n and ϕ_n with the orthonormal eigenfunctions (q_s) of K as

$$P = \sum_{s=1}^{N_f} a_s q_s \quad , \quad \phi = \sum_{s=1}^{N_f} b_s q_s \quad \text{with} \quad K^+Kq_s = |k_s|^2 q_s \tag{19}$$

and integrate over the complex coefficients a_s and b_s. Writing the Hamiltonian in (17) as

$$H = \frac{1}{2}\sum_{s=1}^{N_f}\left(|a_s|^2 + \frac{|b_s|^2}{|k_s|^2}\right) + H'$$

and inserting the identity

$$\delta(E - H) = \int_0^\infty d\varepsilon\ \delta(E - \varepsilon - H') \int_0^\infty \prod_{s,\alpha} d\varepsilon_{s,\alpha}\delta(\varepsilon_{s,\alpha} - E_{s,\alpha})\ \times$$

$$\times\ \delta(\varepsilon - \sum_{s,\alpha}\varepsilon_{s,\alpha})$$

with $E_{s,\alpha} = \frac{1}{2}\left(a_s^\alpha\right)^2 + \frac{1}{2|k_s|^2}\left(b_s^\alpha\right)^2$, and $\alpha = 1,2$

where $a_s \equiv a_s^1 + ia_s^2$ and $b_s \equiv b_s^1 + ib_s^2$ (both of a_s^α and b_s^α are real), we can perform the integration (see ref. 1). As a result, Z(J) becomes, up to an irrelevant constant, as follows:

$$Z(J) = \int_0^\infty d\varepsilon\ \varepsilon^{2N_f-1}\int d\mu'\delta(E - \varepsilon - H')\det(K^+K)\exp\left\{\sum_{i=1}^{2}\sum_{n,m}\bar{J}_{i,n}(K^{-1})_{n,m}J_{i,m}\right\} . \tag{20}$$

It is easy to demonstrate from (20) in a similar way as in the previous paper[1] that Z(J) gives a perturbation series, which is the same as the one obtained in the standard formulation and which is constructed by expanding H' and K^{-1} with respect to a coupling constant. In such a demonstration, we take a limit of $E \to \infty$, order by order in the series expansion. It is worthwhile to remark

that the limit of $E \to \infty$ corresponds not to a limit of lattice spacing $\to 0$, but to a limit of infinite volume.

In this paper, we have shown that the microcanonical formulation of $SU(N_c)$ lattice gauge theories and theories with fermions is perturbatively equivalent to the standard functional formulation. Furthermore, we have shown that Schwinger Dyson equations of $U(N_c)$ lattice gauge theories can also be derived in the microcanonical formulation. Our results validate the use, in microcanonical simulations, of the standard scaling law between the small bare coupling constant and the lattice spacing.

It should be stressed that the relation $E = N$ should be taken only when we perform the perturbative calculations. In these calculations, we exchange the limit $N \to \infty$ for the summation of infinite series.[7] Beyond the perturbation theory, we must use the more general relation (see appendix) as has been adopted in numerical calculations.[2]

Finally, as an application of the microcanonical formulation, we consider the following possibility: if an energy surface E = constant is ergodic, we can evaluate the correlation functions by the "time" average of fields. When we parametrize the invariant measure as $dU = A(\xi)d\xi^1 \ldots d\xi^{N_c^2-1}$ in $SU(N_c)$ lattice gauge theories and change momentum variables $\vec{P}$ into $\vec{P}' = [A(\xi)]^{1/N_c^2-1} \times \vec{P}$, the Hamiltonian and the integration measure in (2) becomes as follows:

$$H = \sum_{x,\mu} \frac{(\vec{P}'_{x,\mu})^2}{2[A]^{2/N_c^2} - 1} + S(U(\xi)) \quad ,$$

and (21)

$$\prod_{x,\mu} d\vec{P}'_{x,\mu} \; \vec{\xi}_{x,\mu} \quad .$$

Therefore, assuming the ergodicity of the system, we can evaluate correlation functions as

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T Q(\vec{\xi}(\tau))d\tau = \langle Q \rangle \quad . \tag{22}$$

Here, the "time τ" development of the fields $\vec{\xi}_{x,\mu}(\tau)$ is determined by solving Hamilton's equations derived from (21) where $\vec{P}'_{x,\mu}$ and $\vec{\xi}_{x,\mu}$ are regarded as canonical conjugate variables. We remark that this procedure to obtain expectation values depends on a specific parametrization of the group. However, this circumstance can be attributed to an ambiguity in choosing canonical conjugate variables in a given microcanonical formulation: if we change variables $\vec{P}'$ and $\vec{\xi}$ into $\vec{P}''$ and $\vec{\xi}'$ keeping the Jacobian unity, we may regard new variables $\vec{P}''$ and $\vec{\xi}'$ as canonical conjugate and have a new Hamiltonian.

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APPENDIX

We shall show under some reasonable assumptions that the microcanonical formulation coincides precisely with the standard functional formulation and that the relation in eq.(14) holds for an arbitrary n.

Let us consider the microcanonical ensemble average of an arbitrary Wilson loop W(c),

$$\frac{1}{Z}\int d\mu\ \delta(E-H)W(c) \quad , \quad Z \equiv \int d\mu\ \delta(E-H) \quad , \tag{A.1}$$

where $d\mu$ and H are the measure and the Hamiltonian used in eq.(2), respectively. Using the formula $\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\ e^{i\lambda(E-H)} = \delta(E-H)$ and performing the integrations of the momentum variables, we can rewrite eq.(A.1)as

$$\frac{1}{Z'}\int_{-\infty}^{\infty} d\lambda \prod_{x,\mu} dU_{x,\mu} e^{i\lambda(E-S)} W(c)\lambda^{-N/2}$$

$$= \frac{1}{Z'}\int_{-\infty}^{\infty} d\lambda\ \overline{W_\lambda(c)}\ e^{-\frac{N}{2}\log\lambda + i\lambda E + F(\lambda)} \tag{A.2}$$

with

$$F(\lambda) \equiv \log \int \prod_{x,\mu} dU_{x,\mu} e^{-i\lambda S} \quad \text{and} \quad \overline{W_\lambda(c)} \equiv e^{-F(\lambda)} \int \prod_{x,\mu} dU_{x,\mu} W(c) e^{-i\lambda S}$$

Here, we have not specified explicitly an irrelevant normalization factor. Our assumptions are that both of the following limits exist,

$$\lim_{N\to\infty} \overline{W_\lambda(c)} \quad \text{and} \quad \lim_{N\to\infty} \frac{\partial}{\partial\lambda} F/N \quad \text{for} \quad \mathrm{Im}\,\lambda \leq 0 \tag{A.3}$$

Then, by using the steepest descent method as $N \to \infty$ and by choosing the energy such as $E = N/2 + \langle S\rangle_f$ ($\langle\ldots\rangle_f$ denotes the standard functional average), we can easily derive the formula

$$\lim_{N\to\infty} \frac{1}{Z} \int d\mu\, \delta(E - H)W(c) = \lim_{N\to\infty} \int \prod_{x,\mu} dU\, e^{-S}W(c) \Big/ \int \prod_{x,\mu} dU_{x,\mu} e^{-S} \tag{A.4}$$

where the limit $N \to \infty$ is taken, keeping the lattice spacing finite. The stationary point ($\lambda = -i$) in the steepest descent method has been obtained by solving the equations

$$-1/2\,\lambda + iE/N + i\,\frac{\partial F}{\partial\lambda}/N = 0 \quad , \quad \text{and} \quad E = N/2 + \langle S\rangle_f \quad . \tag{A.5}$$

Therefore, we have found that the microcanonical ensemble average is identical with the standard canonical ensemble average.

Next, let us prove the relation in eq.(14). In the above demonstration if we adopt an ensemble density like $\frac{d^n}{dE^n}\delta(E - H)$ in eq.(14), the formula (A.2) is replaced with

$$\frac{1}{Z'} \int_{-\infty}^{\infty} d\lambda (i\lambda)^n\, \overline{W_\lambda(c)}\, e^{-\frac{N}{2}\log\lambda + i\lambda E + F(\lambda)} \quad . \tag{A.6}$$

Since the stationary point is such that $\lambda = -i$, it turns out that the final result does not depend on n. This leads to the relation in eq.(14).

In the above discussion, we have found that the energy E should be taken as $E = N/2 + \langle S\rangle_f$. Here, $\langle S\rangle_f$ may be obtained as

$$\lim_{N\to\infty} \langle S\rangle_f/N = \lim_{N\to\infty} \frac{1}{Z} \int d\mu\ \delta(E - H)S/N \tag{A.7}$$

This is a consistency condition, which has been used in practical calculations.

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