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INTRODUCTION TO QCD THERMODYNAMICS AND THE QUARK-GLUON PLASMA

M. Gyulassy

July 1985

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Introduction to QCD Thermodynamics and the Quark-Gluon Plasma

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INTRODUCTION TO QCD THERMODYNAMICS AND THE QUARK-GLUON PLASMA

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Abstract:

These lectures review recent theoretical work suggesting that hadronic matter may dissolve into a weakly interacting quark-gluon plasma phase at energy densities only one order of magnitude above the ground state energy density of nuclei. Basic techniques of field theory used for calculating thermodynamic properties of Quantum Chromodynamics (QCD) are introduced. Functional methods are applied to develop QCD perturbation theory at finite temperatures and chemical potentials. The relevance of asymptotic freedom at high T, μ is motivated. We then confront the main skeleton in the QCD closet, namely, the nonperturbative color magnetic sector. Techniques of lattice gauge theories to get beyond the limitations of perturbation theory are then discussed. Recent numerical results are critically assessed.

[Keywords: QCD, Thermodynamics, Perturbation Theory, Lattice Theory, Quark-Gluon Plasma]

1. Introduction

The discovery of the neutron fifty years ago immediately led to speculations that a new form of matter, neutron stars, could exist in nature. Subsequent observations of pulsars confirmed their existence although our understanding of their structure is still far from complete. More recently the discovery of the quark structure of hadrons has fueled speculations that an even more novel form of matter may exist called the quark gluon plasma (QGP). Experiments have already been approved at CERN and BNL to look for this new form of matter in collisions of very high energy nuclei. Since that experimental program is primarily a nuclear science community initiative, it has become important for nuclear physicists to acquire expertise in areas that used to belong exclusively to particle physics. In particular, we need to understand better the standard model of strong interactions, Quantum Chromodynamics QCD. In two major areas of nuclear experimentation, with electron and nuclear beams, the focus of modern nuclear physics has turned to the study of the subhadronic world in a many body context. Since QCD may be the fundamental theory underlying nuclear and hadronic phenomena, it is obviously worthwhile for nuclear physicists to develop a deeper appreciation of its many subtleties. In these lectures we concentrate on some of the recent developments in the area of QCD thermodynamics.

There exists many excellent text books and review articles on this subject, and the serious student should proceed directly to the original sources listed in the reference list. A good place to start is with the reviews in recent conference proceedings[1]-[7]. Refs.[8]-[11] provide more detailed introductions. The goal of these lectures is to introduce nonexperts to the theoretical techniques used in this area and to point out some of the key unresolved problems. There are many subtle aspects of QCD thermodynamics to watch out for, and it is easy to be overly naive. After all we are dealing here with a theory that is supposed to imprison permanently its constituents (quarks and gluons). Therefore, analogies with more familiar many body systems may be misleading.

At first sight QCD thermodynamics looks rather simple because of a property

called asymptotic freedom. This means that the effective coupling, $\alpha_{eff}(q^2)$ between quarks and gluons vanishes at very large momentum tranfers or, equivalently, at short distances. This property suggests that at extreme temperatures and/or densities, where the average distance between quarks and gluons is small, the system should behave as a simple Stefan-Boltzmann gas. However, QCD is devious because even though the coupling vanishes, the singular infrared properties of the theory prevents a rigorous application of perturbation theory. The problem is that large distance or small momentum transfer phenomena must necessarily involve nonperturbative effects. However, as we shall see, nonperturbative calculations suggest that such effects may not disturb too much the simple picture suggested by perturbation theory. Therefore, there is strong theoretical prejudice in favor of the existence of the quark-gluon plasma phase of matter at high energy densities.

These lectures are organized as follows: Section 2 provides an introduction to the physics of dense nuclear matter and the motivation for studying QCD. The cut and paste method is used to guess what the equation of state of high energy density matter might look like. Section 3 introduces basic field theoretic techniques to beginners. Functional methods to compute the partition function are "derived". We show how Feynman rules naturally emerge from such methods. A brief introduction to Grassmann techniques leads us to appreciate fermion determinants. The special problem of quantizing gauge theories such as QCD is treated by introducing the Fadeev-Popov trick. Finally the Feynman rules for computing thermodynamic quantities in QCD are summarized and the lowest order results presented. Section 4 deals with the topic of asymptotic freedom and its relevance at high temperatures and/or densities. Debye screening of color electric fields and the lack of screening of color magnetic fields is discussed. In particular, we show that the absense of perturbative color magnetic mass implies that QCD perturbation theory suffers from a terminal disease associated with uncontrolled infrared singularities. This leads us in section 5 to discuss nonperturbative methods of lattice gauge theory. To preserve gauge invariance on finite lattices, we change of variables from gluon fields to link matricies. We introduce the weird and unfamiliar lattice world of Wilson

actions, plaquettes, and Polyakov loops. The Metropolis Monte Carlo algorithm is described which is used for numerical computation of QCD thermodynamics. The relevance of lattice theory for continuum physics is then addressed, and the elusive asymptotic scaling window is introduced. Finally recent attempts to include quark degrees of freedom are noted. We conclude these lectures on QCD thermodynamics with a better understanding of some of the outstanding theoretical issues remaining to be solved and with a reinforced belief that the transition from the hadronic to quark worlds probably will end up pretty much as we expect on phenomenological grounds.

2. Phenomenology of Hadronic and Quark Matter

2.1 The Hadronic World

A current view[8] of the vacuum is that it is cluttered with condensates and field fluctuations that are responsible for the confinement of quarks and gluons to color neutral bags of radius $R_H \sim 0.8$ Fm. We live in a nonperturbative world where the effective interactions of those bags or hadrons are strong and short range. All we know about the properties of bulk matter formed out of hadronic constituents comes from nuclear physics. However, nuclear matter saturates at a unique baryon density, $\rho_0 \approx 0.145$ Fm^{-3}. We know only that the energy per nucleon, $W(\rho, T)$, has a minimum at $\rho = \rho_0$ at zero temperature ($W(\rho_0, 0) = -16$ MeV), and that the energy density of the ground state is $\epsilon_0 = \epsilon(\rho_0, 0) = \rho_0(m_N + W(\rho_0, 0)) \approx 0.134$ GeV/Fm^3. The curvature or incompressibility constant as determined from giant momonpole resonances is estimated to be[12] $K_\infty = 9\rho_0^2 \partial^2 W(\rho, 0)/\partial\rho^2 \approx 210 \pm 30$ MeV at ρ_0.

Recent heavy ion experiments[13][15] at the BEVALAC in LBL are beginning to extend our knowledge of $W(\rho, T)$ to higher densities and temperatures using nuclear collisions in the energy range $E_{lab} \sim 1$ AGeV. Fig.2.1 shows the zero temperature equation of state, $W(\rho, 0)$, deduced from studying the energy dependence of the pion multiplicity produced in nuclear collisions[13][14][16].

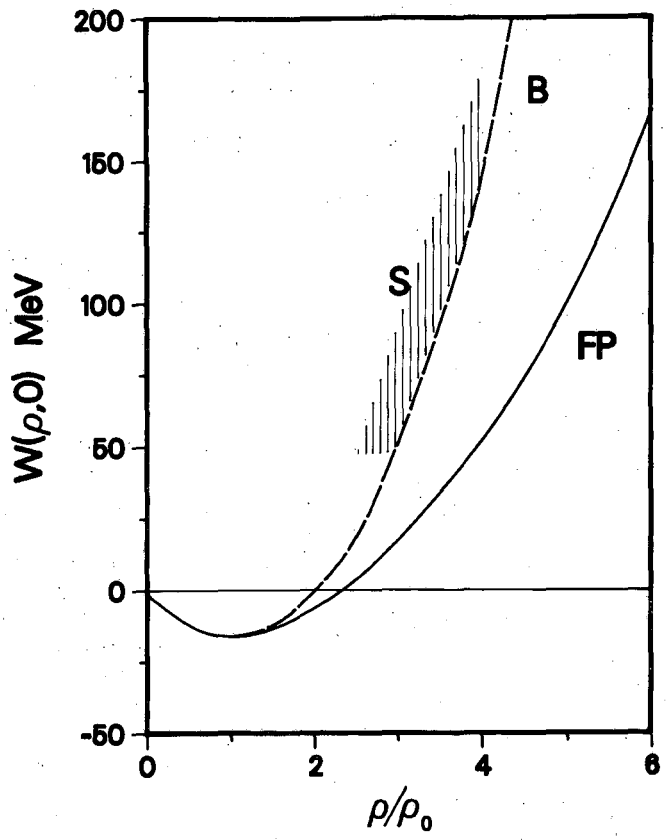

Figure 2.1: Energy per nucleon at T=0 at high densities. Empirical[13][16] (shaded region) from nuclear collision data is compared to theoretical calculations[17][18]

A modern nuclear matter variational calculation[17] is indicated by the curve labeled FP. A non-linear mean field theory calculation[18] is indicated by curve B. The shaded region is the empirical equation of state deduced by Stock *et al*[13] including several corrections[16]. It appears that the nuclear matter equation of state could be much stiffer at densities $\sim 2-4\rho_0$ than expected from conventional nuclear theory[17]. Of course many experimental and theoretical questions about the precise connection between the pion yields and the equation of state remain to be resolved[16]. A very promising new development[19] has been the application of the Vlasov-Uehling-Uhlenbeck equation to the analysis of pion and collective flow data. As new high precision data on heavy nuclear reactions become available,

such analyses will make it possible to extract much more reliable constraints on the nuclear equation of state up to moderate densities and temperatures.

On the theoretical side, a variety of exotic phases at higher densities and/or temperatures have been proposed. These include density isomers[20], pion condensation[21], strange quark drops[22], and skyrmion lattices[23]. Thusfar there is no experimental evidence for or against any such novel states of nuclear matter. The possibilities remain so numerous because the few known properties of nuclear matter near saturation density are not sufficient to constrain the effective models used in hadronic matter studies at high densities[18]. An artistic summary by Siemens[24] of the temperature and density domains where different phases of hadronic matter could exist is shown in Fig.2.2.

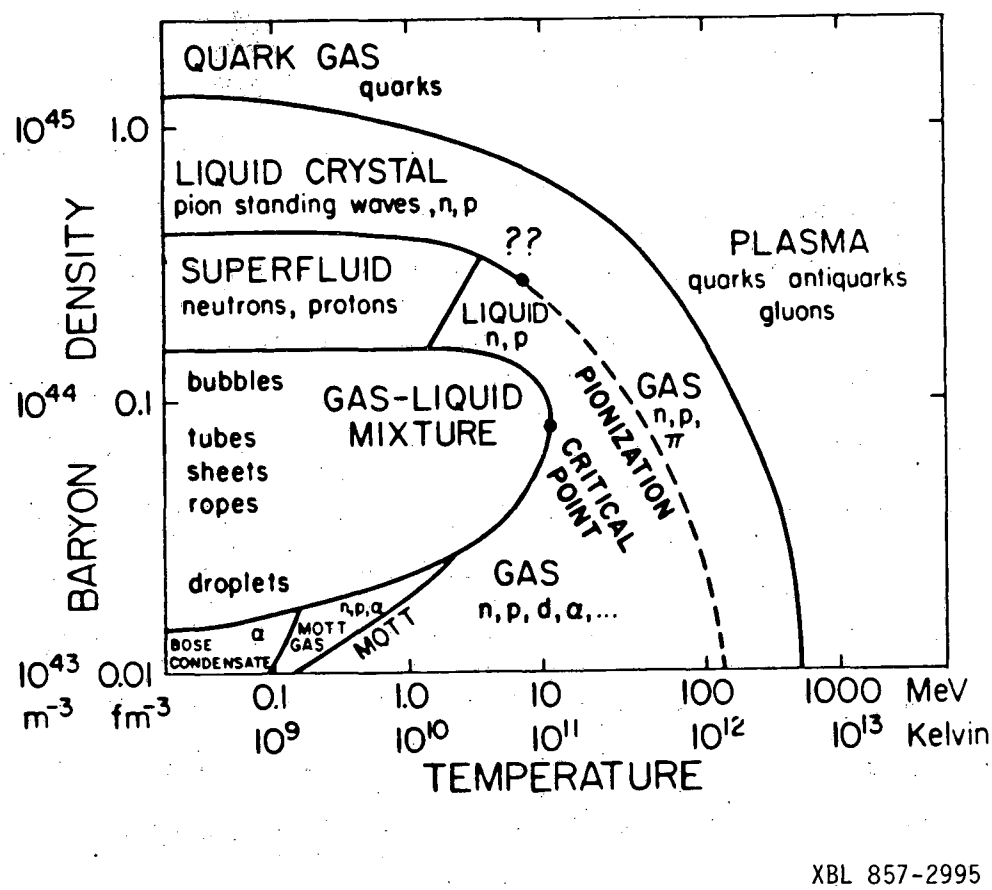

Figure 2.2: Theoretical phase diagram of nuclear matter (artist P. Siemens[24])

The continuing search for signatures of new phases of nuclear matter will remain a central part of the experimental program with nuclear collisions.

Whether exotic phases of *hadronic* matter exist or not, hadron phenomenology already points to something peculiar on the horizon at sufficiently high densities and/or temperatures. The density of hadronic states in free space is consistent with an exponential growth according to the Hagedorn spectrum[25]

$$\rho(m) \sim m^{-a} e^{m/T_0} \quad , \tag{2.1}$$

where $a \approx 5/2$ and $T_0 \approx m_\pi \approx 140$ MeV. If the finite sizes of hadrons could be neglected, then this spectrum would lead to the following energy density of hadronic matter at temperature T:

$$\epsilon(T) = \int dm \rho(m) e^{-m/T} \left(\frac{mT}{2\pi}\right)^{3/2} (m + \frac{3}{2}T + \cdots) \ . \tag{2.2}$$

Obviously, ϵ and also the partition function $Z = \mathrm{tr}\ e^{-H/T}$ have in this case an essential singularity at a finite critical temperature $T = T_0$. However as long as $a < 7/2$ that singularity would only occur at infinite energy density and so would be of no practical concern. The surprise comes when the finite sizes of hadrons are taken into account.

Requiring a covariant thermodynamic formalism, Hagedorn and Rafelski[25] postulated that the volume of a hadron should increase proportional to its mass. As that result also follows from the MIT Bag Model[26], they chose the proportionality constant such that

$$V(m) = \frac{m}{4B} \ , \tag{2.3}$$

where $B \sim 200$ GeV/Fm^3 is the vacuum energy density. As emphasized by Shuryak[8], this B should be considerably larger than the phenomenological Bag constant used to fit hadronic masses. With Eq.(2.3) the energy per unit volume in the excluded volume approximation becomes[25]

$$\epsilon_B(T) = \frac{\epsilon(T)}{1 + \epsilon(T)/4B} \ , \tag{2.4}$$

where $\epsilon_H(T)$ is given by the point particle expression (2.2). The essential singularity occurs now at finite energy density

$$\epsilon_B(T_0) = 4B \sim 1 \text{ GeV/Fm}^3 \ . \tag{2.5}$$

This suggests the end of the hadronic world and the breakdown of hadronic matter theory is in fact just around the corner at an energy density only one order of magnitude above the energy density, ϵ_0, in ground state nuclei.

The above limitation on the hadronic world is a simple consequence of geometry. At low densities and/or temperatures hadrons have plenty of elbow room. But when the hadronic density approches the density of matter within a typical hadron,

$$\rho_H = (4\pi R_H^3/3)^{-1} \approx 3 - 7\rho_0 \ , \tag{2.6}$$

then the hadrons must overlap and it is no longer sensible to continue to describe the properties of matter in terms of degrees of freedom appropriate for isolated hadrons. Close packing of hadrons can be accomplished either by increasing the baryon density or creating new ones to fill the space between old ones by raising the temperature. The critical energy density corresponding to close packing is $\sim m_N\rho_H \sim 3 - 7\epsilon_0$, similar to to Hagedorn's estimate in (2.5). Only with the advent of QCD can we begin to guess what are the properties of matter at higher densities.

Before proceeding to QCD though, we note that for phenomenological considerations a convenient parameterization of the hadronic matter energy density, $\epsilon_h(T)$, and pressure, $p_h(T)$, at zero baryon density and finite temperatures is given by[27] [28]

$$\epsilon_h(T) = \epsilon_H(T/T_0)^{1+c_H^{-2}} \ , \tag{2.7}$$

$$p_h(T) = c_H^2\epsilon_h(T) \ , \tag{2.8}$$

where c_H is the speed of sound in hadronic matter.

2.2 The Quark World

In order to guess what lies beyond the hadronic world we turn to a model of the subhadronic world. The most promising candidate for the theory of subhadronic phenomena is QCD. In this section we review the motivation and "derivation" of QCD.

Part of the motivation for QCD arose out of the need to understand the existence of spin 3/2 baryons and the existence of weakly interacting point like partons inside hadrons. The great success of the SU($N_{flavor} = 3$) classification of mesons and baryons in terms of $q\bar{q}$ and qqq bound states presented a fundamental dilema with regard to the Pauli principle. The three quarks in the $\Omega(\uparrow s \uparrow s \uparrow s), \Delta^{+2}(\uparrow u \uparrow u \uparrow u)$, and$\Delta^{-}(\uparrow d \uparrow d \uparrow d)$ spin 3/2 baryons have to be in a completely symmetric state under interchange of spin or spatial coordinates. The simplest way to reconcile with the Pauli principle was to postulate the existence of a new quantum number, color, and to assume that the color wavefunction of three quarks must be completely antisymmetric. This required the existence of three colors (say blue, green, and red). Thus the quark wavefunctions must carry color(c) in addition to spinor(s) and flavor(f) indices, $\psi_{s,f,c}(x)$. The spinor indices tranform according to the spin 1/2 repesentation of the Lorentz group. The flavor indices transform according to the fundamental representation of the SU(N_{flavor}) group. The guess was that the color indices transform according to the fundamental representation of a new SU(N_{color}) group. That solved the Pauli problem, but the problem of dynamics remained unanswered.

It is at this point that the ideas of Yang and Mills on local gauge invariance came in. Suppose that the quark world is invariant to a symmetry group G such as SU(N_c). This means that the Lagrangian $\mathcal{L}(\psi, \bar{\psi})$ of the quark world is invariant under an infinitesmal transformation

$$\psi_a \rightarrow (1 + \frac{i}{2}\epsilon_\alpha \lambda_\alpha)_{ab}\psi_b \ , \tag{2.9}$$

where $\lambda_\alpha/(2i)$ are the generators of the Lie group G in the fundamental representation, and ϵ_α are small rotation angles. For SU(2) the λ_α, $\alpha = 1, 2, 3$, are the 2×2

Pauli maticies, while for SU(3) λ_α, $\alpha = 1, \cdots, 8$, are the 3×3 Gell-Mann matricies. The group is completely specified by structure constants, $f_{\alpha\beta\gamma}$, that define how the generators commute, $[\lambda_\alpha, \lambda_\beta] = 2if_{\alpha\beta\gamma}\lambda_\gamma$.

The idea of Yang and Mills was that physics should not depend on possibly different conventions used for classifying particles according to the group at *different* space-time points. They argued that only the *local* distinction between quantum numbers should be important. If scientists on Mars see our blue quarks as red, surely that cannot be relevant to the motion of that quark. This principle implies that the theory of quarks should be invariant under a much more general (gauge) transformation , where ϵ_α in (2.9) are replaced an arbitrary functions of space-time, $\epsilon_\alpha(x)$. Under such general gauge transformations the color of quarks can be redefined in an arbitrary manner at every space-time point. However, such gauge invariance is not automatic. In particular a non interacting quark gas is not invariant under such a transformation. To see this note that if $\epsilon_\alpha(x)$ depends on space-time, then that under the transformation (2.9) the free Dirac Lagrangian transforms as

$$i\bar{\psi}_a\gamma_\mu\partial^\mu\psi_a \rightarrow i\bar{\psi}_a(\gamma_\mu\partial^\mu + \frac{i}{2}\lambda_\alpha\gamma_\mu(\partial^\mu\epsilon_\alpha))_{ab}\psi_b \ . \tag{2.10}$$

The solution proposed by Yang and Mills (1954) and adapted to QCD in early 1970's, was to introduce a set of compensating fields $A^\mu_\alpha(x)$ which couple to the quarks according to the famous minimal coupling scheme

$$\partial^\mu \rightarrow \mathcal{D}^\mu = (\partial^\mu + \frac{i}{2}g\lambda_\alpha A^\mu_\alpha(x)) \ . \tag{2.11}$$

In order to compensate for the unwanted extra term in (2.10), these new fields must transform under local gauge transformations as

$$A^\mu_\alpha \rightarrow A^\mu_\alpha - f_{\alpha\beta\gamma}A^\mu_\beta\epsilon_\gamma - \frac{1}{g}\partial^\mu\epsilon_\alpha \ . \tag{2.12}$$

With the above choice for the transformation properties, the minimal quark Lagrangian $\bar{\psi}i\gamma_\mu\mathcal{D}^\mu\psi$ is invariant under local rotations in the symmetry group space. Note that minimal coupling and gauge invariance forces the compensating fields to

be Lorentz vectors (spin 1) and to transform as the generators λ_α of the group (the so called adjoint representation).

To put life into the compensating fields a "kinetic" energy density must also be specified. Guided by QED, a natural guess for the Lagrangian would be $-\frac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha$, in terms of the field tensor $F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu$. However, this does not work because this form would not be invariant under (2.12). The generalization of the field tensor that is invariant can be seen to be

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + g f_{\alpha\beta\gamma} A^\beta_\mu A^\gamma_\nu \ . \tag{2.13}$$

Note that the same coupling g appears above as in (2.11).

We have thus "derived the QCD Lagrangian

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \mathcal{D}_\mu \psi - \frac{1}{4} F^\alpha_{\mu\nu} F^{\mu\nu}_\alpha \ . \tag{2.14}$$

Because of (2.13), F^2 contains cubic and quartic interactions of the fields A^μ_α, which henceforth will be called gluon fields. The requirement of local gauge invariance therefore leads not only to the existence of gluons that can change the color of quarks but also to a definite interaction Lagrangian between quarks and gluons and between gluons and gluons characterized by one coupling g. Note the absence of any scale parameter in $\mathcal{L}$. (It is also worth noting that (2.14) is not the most general renormalizable gauge theory if we allow higher derivatives and couplings with dimensions to enter. See Ref.[29] for an interesting unorthodox possibility.)

The next step of quantizing (2.14) is much harder. The main problem is that for any field configuration $A^\mu_\alpha(x)$ there are infinitely many configurations ${}^G A^\mu_\alpha$ which are physically equivalent to it because they differ only by a gauge transformation G. In the section 3, we will introduce functional integral techniques to handle this problem. For now just consider the naive $g \to 0$ limit of (2.14). Obviously that limit must in reality be very subtle since we do not see free quarks and gluons floating around us. However, as we shall motivate in section 4 that naive limit may hold at very high energy densities because of asymptotic freedom.

Proceeding blindly, if we can set $g = 0$ at high enough temperatures or densities, then the equation of state of the quark world would be quite trivial. It would correspond to a noninteracting gas of N_f flavor quarks that come in N_c colors and $N_c^2 - 1$ spin 1 gluons. In that case the energy density, pressure, and baryon density of the gas are given by the Stefan-Boltzmann expressions

$$\epsilon_{SB}(T,\mu) = \frac{\pi^2}{15}(N_c^2 - 1 + 7N_cN_f/4)T^4 + \frac{N_cN_f}{2}(T^2\mu^2 + \frac{1}{2\pi^2}\mu^4) \ , \tag{2.15}$$

$$p_{SB}(T,\mu) = \frac{1}{3}\epsilon_{SB}(T,\mu) \ , \tag{2.16}$$

$$\rho_{SB}(T,\mu) = \frac{N_cN_f}{9\pi^2}(\mu^3 + \pi^2T^2\mu) \ , \tag{2.17}$$

with T, μ being the quark temperature and chemical potential.

Since we obviously do not live in that world, it is plausible that the vacuum in which the ideal gas of quarks and gluons live differs from ours. Since we are here and 'they' are not, our nonperturbative physical vacuum[8] must have an energy lower then their QCD perturbative vacuum. Phenomenologically, we can try to take this effect into account by adding a constant $Bg_{\mu\nu}$ to the energy momentum tensor of the quark world. This leads to a phenomenological bag model equation of state for the quark world:

$$\epsilon(T,\mu) = \epsilon_{SB}(T,\mu) + B \ , \tag{2.18}$$

$$p(T,\mu) = \frac{1}{3}\epsilon_{SB}(T,\mu) - B \ , \tag{2.19}$$

2.3 Cut and Paste Model for the Equation of State

Now let us try to staple the phenomenological hadronic and quark worlds together at zero baryon density. Fig.2.3 illustrates how this could be done.

As the temperature increases the energy density and pressure increase along the hadronic branch labeled H as parameterized by Eqs.(2.7,2.8). Below a critical

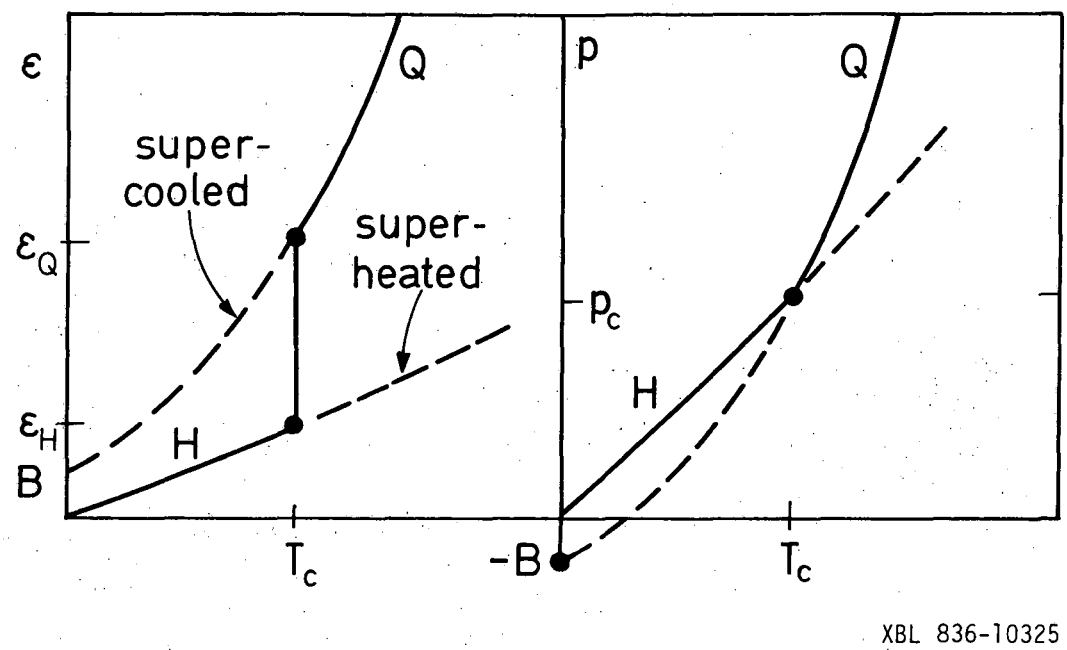

Figure 2.3: Phenomenological equation of state of high temperature hadronic matter with a first order transition to a quark-gluon plasma phase.

temperature T_c the pressure in the quark phase exceeds that in the hadronic phase. At T_c the pressures in the two phases coincide. However, $\epsilon(T_c, 0) \equiv \epsilon_Q$ is greater than $\epsilon_h(T_c, 0) \equiv \epsilon_H$. If we *assume* that the transition between the hadronic and quark worlds is a first order one, then the system would be in a mixed phase for energy densities between ϵ_H and ϵ_Q. In the mixed phase the temperature and the pressure would remain the same. The latent heat per unit volume is $\Delta\epsilon = \epsilon_Q - \epsilon_H \sim 4B$ is what we must supply to melt the nonperturbative vacuum and liberate the quarks. Indicated in Fig.2.3 are also metastable superheated hadronic and supercooled quark phases that may exist. Numerical estimates[49][50][51] with a variety of plausible parameters give

$$\epsilon_Q \sim 1 - 2\mathrm{GeV/Fm}^3 \tag{2.20}$$

From these phenomenological considerations the following picture emerges. Below some energy density $\epsilon_H \sim 3\epsilon_0$ hadronic degrees of freedom are relevant. As the energy density increases the hadrons begin to overlap and the nonperturbative

vacuum confining the quarks into little bags begins to melt. By the time an energy density in excess of one order of magnitude above the ground state energy density of nuclei is reached, the nonperturbative vacuum has evaporated and quarks and gluons freely propagate in the system. This at least is the working hypothesis which we hope to verify by more rigorous methods in the following sections.

3. Field Theory Primer

3.1 Path Integrals and Perturbation Theory

Since the primary focus here is on thermodynamics, we jump directly to the problem of computing the partition function $Z = \mathrm{tr} e^{-\beta H}$ with $\beta^{-1} = T$ being the temperature. To warm up to the problem consider first a simple one dimensional quantum mechanical system described by the Hamiltonian

$$H = \frac{1}{2}p^2 + V(x) \ . \tag{3.1}$$

The partition function for this system is

$$Z = \sum_x \langle x|e^{-\beta H}|x\rangle \ , \tag{3.2}$$

where $|x\rangle$ is a complete basis set of states. The first trick is to note that

$$e^{-\beta H} = \lim_{N\to\infty}(1-\epsilon H)^N \ , \tag{3.3}$$

where $\epsilon = \beta/N$. Following Feynman[34] we insert a complete set of states between each operator $(1-\epsilon H)$ to get

$$Z = \lim_{N\to\infty}\sum_{x_1}\cdots\sum_{x_N}\langle x_1|1-\epsilon H|x_2\rangle\cdots\langle x_N|1-\epsilon H|x_1\rangle \ . \tag{3.4}$$

Next we note that to order ϵ accuracy

$$\begin{aligned}\langle x_i|1-\epsilon H|x_{i+1}\rangle &\approx \langle x_i|e^{-\epsilon p^2/2}|x_{i+1}\rangle\, e^{-\epsilon V(x_i)}(1+O(\epsilon)) \\ &\propto \exp\left[-\frac{(x_i-x_{i+1})^2}{2\epsilon} - \epsilon V(x_i)\right] \ .\end{aligned} \tag{3.5}$$

Defining a *closed* path $x(\tau)$ such that

$$x(\tau) = x_i \text{ at } \tau = i\beta/N \ \textit{and} \ x(\beta) = x(0) \ , \tag{3.6}$$

and defining the path derivative via $\dot{x} = dx/d\tau \approx (x_{i+1} - x_i)/\epsilon$, we can express the partition function as

$$Z \propto \lim_{N\to\infty} \sum_{x_1} \cdots \sum_{x_N} \exp\left(-\epsilon \sum_{i=1}^{N} (\dot{x}^2/2 + V(x))_{x=x_i}\right)$$

$$\equiv \int_{x(0)=x(\beta)} \mathcal{D}x(\tau) \exp(-\int_0^\beta d\tau [\dot{x}^2/2 + V(x)]) \ . \tag{3.7}$$

The partition function is therefore given in terms a sum over all possible paths that close after a "time" β. Each path is weighed by the exponential of minus the "Euclidian action", $S_E(x) = \int_0^\beta d\tau \mathcal{L}_E(x(\tau))$. Note that instead of the true Lagrangian, $\mathcal{L} = \frac{1}{2}\dot{x}^2 - V(x)$, appearing in the exponent, a modified Lagrangian, $\mathcal{L}_E = \frac{1}{2}\dot{x}^2 + V(x)$ appears. To help understand the origin of this modification note the similarity between the statistical operator, $e^{-\beta H}$, and the quantum mechanical time evolution operator, e^{-itH}. Formally the statistical operator just propagates the system into the *imaginary* time direction by an amount $\Delta t = -i\beta$. If we substitute $\beta = it$ in (3.7), then -$\int_0^\beta d\tau \mathcal{L}_E$ changes into $i\int_0^t dt \mathcal{L}$ and we recover Feynman's original path integral expression for the propagator[34] in terms of the true Lagrangian. For statistical mechanics always the "Euclidean rotated" Lagrangian appears in the exponent.

The generalization of the above method to systems with many degrees of freedom is straightforward as long as the Hamiltonian remains quadratic in the p_i. With $x(\tau) \to (q_1(\tau), \cdots, q_n(\tau))$ defining a closed path in an n dimensional space, the partition function is given by (3.7) with the Euclidean action given by $S_E = \int_0^\beta d\tau (\sum_1^n \dot{q}_i^2/2 + V(q_1, \cdots, q_n))$. The final transition to quantum field theory is made by assigning a generalized coordinate to each point in space. This is accomplished by first erecting an artificial lattice scaffolding in space such that the lattice sites can be specified by integers as $\mathbf{x}_i = (i_1, i_2, i_3)a$ in terms of a lattice spacing scale, a. For each lattice site we assign a generalized coordinate , $\phi(\mathbf{x}_i, \tau)$. We further assume that $V(\phi(\mathbf{x}_1, \tau), \phi(\mathbf{x}_2, \tau), \cdots)$ couples only nearest neighbors such that as

the lattice spacing goes to zero

$$V \xrightarrow{a\to 0} \frac{1}{2}|\nabla\phi(x)|^2 + U(\phi(x)) \ . \tag{3.8}$$

This form of V is chosen so that the the field $\phi(\mathbf{x},t)$ obeys the simplest type of wave equation in Minkowski space ($\partial_\mu\partial^\mu\phi = \delta U/\delta\phi$). The partition function for this scalar field theory is then given by

$$Z \propto \int_{\phi(x,0)=\phi(x,\beta)} \mathcal{D}\phi(x,\tau)\exp(-\int_0^\beta d\tau \int d^3x[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}|\nabla\phi|^2 + U(\phi)]) \ . \tag{3.9}$$

Of course, (3.9) is only a formal symbol. It is defined as a limit of a large but finite number of ordinary integrals

$$\int \mathcal{D}\phi = \lim_{\substack{a\to 0 \\ N_x, N_\tau \to \infty}} \prod_{i=1}^{N_x}\prod_{j=1}^{N_x}\prod_{k=1}^{N_x}\prod_{l=1}^{N_\tau}\left\{\int_{-\infty}^{\infty} d\phi(ia,ja,ka,la)\right\} \ , \tag{3.10}$$

with ϕ satisfying periodic boundary conditions. The periodic boundary condition in the temperature direction, $\phi(x,0) = \phi(x,\beta)$, followed because Z involves a trace of the statistical operator. Periodic boundary conditions in the spatial directions are only a matter of convenience since the period goes to infinity at the end. For a fixed temperature, $T = \beta^{-1}$, and lattice spacing, a, the number of steps in the τ direction is constrained via

$$\beta = N_\tau a \ . \tag{3.11}$$

The formal functional integral (3.9) is a very useful starting point to develop perturbation theory. This is done by adding a periodic source current[30][31], $J(x)$, for the ϕ field. The partition function is thus modified to

$$Z(J) \propto \int \mathcal{D}\phi \exp\{-\langle\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + U(\phi) - J\phi\rangle\} \ , \tag{3.12}$$

where $\langle\cdots\rangle = \int_0^\beta d\tau \int d^3x \cdots$, and the Euclidean $\partial_\mu = \partial^\mu = (d/d\tau, \nabla)$. The main trick used to develop perturbation theory is the identity

$$\phi(x)e^{\langle J\phi\rangle} = \frac{\delta}{\delta J(x)}e^{\langle J\phi\rangle} \ . \tag{3.13}$$

Therefore, any function of $\phi(x)$ in the integrand of (3.12) can be expressed in terms of functional derivatives with respect to J. In particular, we can write

$$e^{-\langle U(\phi)-J\phi\rangle} = \exp\{-\langle U(\frac{\delta}{\delta J})\rangle\}e^{\langle J\phi\rangle} \ . \tag{3.14}$$

With this formal trick, we can pull the potential out of the functional integral to obtain

$$Z(J) \propto \exp\{-\langle U(\frac{\delta}{\delta J})\rangle\}\int \mathcal{D}\phi e^{-\langle \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - J\phi\rangle} \ . \tag{3.15}$$

Now we are in business because the remaining functional integral is of gaussian form, i.e., the only type of functional integral that we can readily perform. It is instructive to go over in detail how this particular one can be evaluated. Because $\phi(\mathbf{x},\tau)$ is only defined on a finite $0 \leq \tau \leq \beta$ interval with boundary conditions $\phi(x,\beta) = \phi(x,0)$, we can Fourier decompose ϕ as

$$\phi(\mathbf{x},\tau) = \beta^{-1}\sum_n e^{-i\omega_n\tau}\int \frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{k},\omega_n) \ , \tag{3.16}$$

where the discrete frequencies must be given by

$$\omega_n = \frac{2\pi n}{\beta} \ , \tag{3.17}$$

in terms of integers n. Similarly, we can Fourier decompose $J(\mathbf{x},\tau)$. The exponent in (3.15) can thus be calculated as

$$\begin{aligned}\langle\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - J\phi\rangle &= \beta^{-1}\sum_n\int\frac{d^3k}{(2\pi)^3}[\frac{1}{2}\phi(\mathbf{k},w_n)(\omega_n^2+k^2)\phi(-\mathbf{k},-\omega_n) - J(\mathbf{k},w_n)\phi(-\mathbf{k},-w_n)] \\ &= \beta^{-1}\sum_n\int\frac{d^3k}{(2\pi)^3}[|\chi(\mathbf{k},\omega_n)|^2 - \frac{1}{2}|J(\mathbf{k},w_n)|^2\Delta(\mathbf{k},\omega_n)] \ ,\end{aligned} \tag{3.18}$$

where the last line follows from changing variables to

$$\chi(\mathbf{k},w_n) = \frac{1}{\sqrt{2}}(\Delta(\mathbf{k},w_n)^{-1/2}\phi(\mathbf{k},w_n) - \Delta(\mathbf{k},w_n)^{+1/2}J(\mathbf{k},w_n)) \ , \tag{3.19}$$

with

$$\Delta(\mathbf{k},w_n) = (\omega_n^2+k^2)^{-1} \ . \tag{3.20}$$

The function Δ is the momentum space inverse of the Euclidean operator $\partial_\mu\partial^\mu$ on the space of functions satisfying periodic boundary conditions in the "time" direction. As such it can be called a free field propagator because given an external disturbance $J(\mathbf{k},\omega_n)$ the response is $\phi(\mathbf{k},w_n) = \Delta(\mathbf{k},w_n))J(\mathbf{k},w_n))$. Changing variables from $\phi(\mathbf{x},\tau_j)$ to $\chi(\mathbf{k},\omega_n)$, gives rise to a Jacobian

$$Z_0 = \left|\frac{\partial\phi}{\partial\chi}\right| \propto \prod_n\prod_{\mathbf{k}}\{(\beta^{-2}\Delta(\mathbf{k},w_n)\}^{1/2} = \exp\left\{-\frac{1}{2}\sum_n\int\frac{Vd^3k}{(2\pi)^3}\ln[(2\pi n)^2+(\beta k)^2]\right\} \quad , \tag{3.21}$$

where V is the volume. Unfortunately, the exponent diverges. We can try to isolate that temperature independent divergence by regrouping terms as

$$\frac{1}{2}\sum_{n=-\infty}^{\infty}\ln[(2\pi n)^2+(\beta k)^2] = \aleph+\ln(\beta k)+\sum_{n=1}^{\infty}\ln\left[1+\left(\frac{\beta k}{2\pi n}\right)^2\right] = \aleph+\ln(1-e^{-\beta k})+\frac{1}{2}\beta k \quad . \tag{3.22}$$

(Altenatively, we could go back to the lattice formulation, where $\aleph \sim \ln N_\tau$, and rescale the integration measure to get rid of that constant before taking the continuum limit.) Therefore,

$$Z_0 \propto \exp\left\{-\int\frac{Vd^3k}{(2\pi)^3}[\ln(1-e^{-\beta k})+\frac{1}{2}\beta k]\right\} \quad , \tag{3.23}$$

which is recognized to be the partition function of a non interacting massless boson gas including the zero point energy. That latter term however only shifts the scale of the free energy, $F_0 = -\beta^{-1}\ln Z_0$, by a temperature independent amount.

The Gaussian integral in (2.2.14) can thus be performed by changing variables to $\chi(\mathbf{k},w_n)$. The integrals over the $\chi(\mathbf{k},w_n)$ give for each mode a factor $(\pi\beta)^{1/2}$ to the overall normalization constant. In this way the partition function for anharmonic fields in the presence of an external source can be related to the ideal partition function for harmonic fields via

$$Z(J) = Z_0\exp\{-\langle U(\frac{\delta}{\delta J})\rangle\}\exp\{\frac{1}{2}\beta^{-1}\sum_n\int\frac{d^3k}{(2\pi)^3}|J(\mathbf{k},\omega_n)|^2\Delta(\mathbf{k},\omega_n)\} \quad . \tag{3.24}$$

We have thus "derived" the the generating function for the perturbative expansion of the partition function. The Feynman rules follow from expanding the first exponential in a power series in U, carrying out the functional derivatives, and setting $J = 0$ at the end.

As an example consider $U(\phi(x)) = g\phi(x)^4/4!$. If we chose to work in momentum space, then

$$\langle U(\frac{\delta}{\delta J})\rangle = \frac{g}{4!}\prod_{m=1}^{4}\left\{\beta^{-1}\sum_{n_m}\int\frac{d^3\mathbf{k}_m}{(2\pi)^3}\frac{\delta}{\delta J(\mathbf{k}_m,\omega_{n_m})}\right\}$$

$$\times\left[\beta\delta_{n_1+n_2+n_3+n_4}(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\right] \quad . (3.25)$$

If we chose to work in coordinate space, then the second exponent is expressed as

$$\frac{\beta^{-1}}{2}\sum_n\int\frac{d^3k}{(2\pi)^3}|J(k)|^2\Delta(k) = \frac{1}{2}\int_0^\beta dx_0\int_0^\beta dy_0\int d^3x\int d^3y J(x)\Delta(x-y)J(y) \quad . \tag{3.26}$$

The first order correction to the free energy, $F = -\beta^{-1}\ln Z$, is for example

$$F^{(1)} = \lim_{J\to 0}\frac{g}{4!\beta}\int d^4x'\left(\frac{\delta}{\delta J(x')}\right)^4\exp\{\frac{1}{2}\int d^4x\int d^4y J(x)\Delta(x-y)J(y)\}$$

$$= \frac{3g}{4!}V\Delta(x=0)^2 \quad , \tag{3.27}$$

where $\int d^4x \equiv \int_0^\beta d\tau\int d^3x = \beta V$ is the Euclidean four volume. Higher order terms can be evaluated similarly with exponentially increasing labor. The important point however is that a simple pattern emerges. The higher order terms can always be expressed in as a sum of integrals over products of free propagators, Δ. By associating with each propagator a line, each term in the sum becomes associated with a Feynman graph. The structure of those graphs is completely determined by the form of $U(\phi)$ and are summarized by a small set of Feynman rules that follow from (3.24).

3.2 Fermion Determinants

The main trick in defining the path integral representation of the partition function was to split up $e^{-\beta H}$ into many pieces via (3.3) and inserting a complete set of states between each piece. The main difficulty with treating fermi systems is obviously the necessity of antisymmetrizing the wavefunctions. Let $|\alpha\rangle$ denote a complete set of one particle basis states. Defining anticommuting creation operators b_α^+, such that $b_\alpha b_\alpha^+ + b_\alpha^+ b_\alpha = \delta_{\alpha\beta}$, an antisymmetric n fermion state is written as

$$|\alpha_1 \cdots \alpha_n\rangle_A = b_{\alpha_1}^+ \cdots b_{\alpha_n}^+ |0\rangle \ .$$

The resolution of the identity operator on the space of all antisymmetrized states is thus

$$\mathbf{1}_A = \sum_n \frac{1}{n!} \sum_{\alpha_1 \cdots \alpha_n} |\alpha_1 \cdots \alpha_n\rangle_A \langle \alpha_1 \cdots \alpha_n| \ .$$

However, this is not the most convenient representation of $\mathbf{1}_A$. A more convenient basis would be one specified a set of labels, $\eta = \{\eta_\alpha\}$, with the property

$$b_\alpha |\eta\rangle = \eta_\alpha |\eta\rangle \ . \tag{3.28}$$

If we could construct such states, then they would be analogous to coherent states for bosons and would allow us to compute the matrix elements of any normal ordered operator, $f(b^+, b)$, as $\langle \eta_2 | f | \eta_1 \rangle = f(\eta_2^*, \eta_1)$. Unfortunately, the η_α cannot be ordinary complex numbers because basic relations such as $b_{\alpha_1} b_{\alpha_2} |\eta\rangle = -b_{\alpha_2} b_{\alpha_1} |\eta\rangle$ must be satisfied by the requirement of antisymmetrization. In order retain (3.28) as well as antisymmetrization, the η_α must satisfy a peculiar (Grassmann) algebra

$$\eta_\alpha^2 = 0 \ , \ \eta_{\alpha_1} \eta_{\alpha_2} = -\eta_{\alpha_2} \eta_{\alpha_1} \ . \tag{3.29}$$

By convention the η_α can also be assumed to anticommute with the b_α. With the above rules, the state $|\eta\rangle$ can be constructed as[35]

$$|\eta\rangle = \exp(-\sum_\alpha \eta_\alpha b_\alpha^+)|0\rangle \ . \tag{3.30}$$

A little Grassmann algebra then shows that the overlap of two such coherent fermion states is given by

$$\langle\eta|\eta'\rangle = \exp\left\{\sum_\alpha \eta_\alpha^+ \eta'_\alpha\right\} \ , \tag{3.31}$$

where the adjoint of a Grassmann number is defined by $\langle\eta|b_\alpha^+ = \langle\eta|\eta_\alpha^+$.

To construct a resolution of $\mathbf{1}_A$ in terms of a sum over $|\eta\rangle\langle\eta|$, we need to define an operation, $\int d\eta$, that plays the role of integration. We *define* linear operators, denoted by $\int d\eta_i$, on a set of Grassmann numbers, $\{\eta_i\}$, such that

$$\int d\eta_1 \, z = 0 \ , \ \int d\eta_i \, z\eta_j = z\delta_{i,j} \ , \tag{3.32}$$

where z is a complex number. With this definition and the properties (3.28-3.31), the identity on the space of all antisymmetric states can be written as

$$\mathbf{1}_A = \prod_\alpha \left\{\int d\eta_\alpha^+ d\eta_\alpha e^{-\eta_\alpha^+\eta_\alpha}\right\} |\eta\rangle\langle\eta| \ , \tag{3.33}$$

as can be verified by applying $\mathbf{1}_A$ to an arbitrary antisymmetric state $b_{\alpha_1}^+ \cdots b_{\alpha_n}^+|0\rangle$.

All the machinery is now in place to carry out the program of expressing the partition function in terms of path integrals. For that purpose we need to be able to compute a trace of an operator on the space of coherent fermion states. However, because the η anticommute

$$_A\langle\alpha_1\cdots\alpha_n|\eta\rangle\langle\eta|\beta_1\cdots\beta_n\rangle_A = \langle\eta|\beta_1\cdots\beta_n\rangle_A \ _A\langle\alpha_1\cdots\alpha_n| - \eta\rangle \ .$$

This implies for example that

$$Tr|\eta\rangle\langle\eta|O = \langle\eta|O|-\eta\rangle \ . \tag{3.34}$$

This sign flip is an essential difference between boson and fermion coherent states. As we will see, this leads to antiperiodic boundary conditions rather than periodic ones in the τ direction.

We can now construct a Grassmann path integral representation of the partition function along the same lines that we followed in the last section. For a Fermi system

at temperature, β^{-1}, and chemical potential, μ, specified by a Hamiltonian, H.

$$\begin{aligned} Z_f &= Tr\, e^{-\beta(H-\mu N)} = \lim_{n\to\infty} Tr\, \mathbf{1}_A(1-\epsilon K)\mathbf{1}_A \cdots \mathbf{1}_A(1-\epsilon K) \\ &= \lim_{n\to\infty} \int d\mu(\eta_0)\cdots d\mu(\eta_n)\langle\eta_0|1-\epsilon K|\eta_1\rangle\langle\eta_1|1-\epsilon K|\eta_2\rangle\cdots\langle\eta_n|1-\epsilon K|-\eta_0\rangle \end{aligned} \tag{3.35}$$

where $K \equiv H - \mu N$, $\epsilon = \beta/n$, and the integration measure is

$$d\mu(\eta) = \prod_\alpha d\eta_\alpha^+ d\eta_\alpha e^{-\eta_\alpha^+\eta_\alpha} \ . \tag{3.36}$$

As an interesting example consider the partition function for a relativistic Fermi system in an external $A_\nu(x)$ field. In that case

$$K \equiv \psi^+(\gamma_0 D - \mu)\psi = \int d^3x \psi^+(x)\gamma_0(i\gamma_i\nabla_i + m + g\gamma_\nu A^\nu - \mu\gamma_0)\psi(x) \ , \tag{3.37}$$

where γ_μ are the conventional Dirac matricies[30], and $\psi(x) = \sum_\alpha \langle x|\alpha\rangle b_\alpha$ is the field operator in coordinate space.

Because K is a quadratic form in ψ^+, ψ, the matrix elements of K are quadratic forms in the η_α. Therefore, to order ϵ accuracy

$$\begin{aligned} \langle\eta_i|1-\epsilon K|\eta_{i+1}\rangle &\approx \exp(\eta_i^+\eta_{i+1} - \epsilon\eta_i^+\gamma_0(D-\mu)\eta_i) \\ &= \exp\left(\sum_{\alpha,\beta}\eta_{i,\alpha}^+\left[\delta_{\alpha\beta} - \epsilon\int d^3x\langle\alpha|x\rangle\gamma_0(i\gamma_j\nabla_j + m + g\gamma_\nu A^\nu - \mu\gamma_0)\langle x|\beta\rangle\right]\eta_{i+1,\beta}\right) \end{aligned} \tag{3.38}$$

With this relation (3.35) can be rewritten as

$$Z_f \approx \int d\eta_0^+ d\eta_0 \cdots d\eta_n^+ d\eta_n \exp\left(\sum_{i=0}^n \eta_i^+(\eta_{i+1} - \eta_i) - \epsilon\eta_i^+(\gamma_0 D - \mu)\eta_i\right) \ , \tag{3.39}$$

where $\eta_{n+1} \equiv -\eta_0$ because of (3.34). This form motivates us to define an antiperiodic "path" by

$$\eta(\tau) = \eta_i \text{ at } \tau = i\beta/n \quad \text{such that} \quad \eta(\beta) = -\eta(0) \ , \tag{3.40}$$

where $\eta(\tau)$ is regarded as a column vector with components, $\eta_\alpha(\tau)$. This allows us to recast (3.39) into the form

$$Z_f \propto \int_{\eta(\beta)=-\eta(0)} \mathcal{D}(\eta^+(\tau),\eta(\tau)) \exp\left\{\int_0^\beta d\tau \eta^+(\tau)(\frac{d}{d\tau}+\mu-\gamma_0 D)\eta(\tau)\right\} \quad , \tag{3.41}$$

where

$$\mathcal{D}(\eta^+(\tau),\eta(\tau)) = \lim_{n\to\infty}\prod_{i=1}^{n}\prod_{\alpha} d\eta^+_{i\alpha} d\eta_{i\alpha} \ . \tag{3.42}$$

By a change of variables the exponent can be rewritten as the negative of the Euclidean action as obtained by continuing the Minkowski space action ($iS_M(t = -i\beta) = -S_E(\beta)$). This completes the "derivation" of the path integral repesentation for the fermion partition function.

However, the Grassmann machinery allows us to go one important step further. The integral can be performed because the exponent is just a quadratic form in the variables. Following the Grassmann rules, the master formula we need is

$$\int \prod_{i=1}^{N} d\eta_i^+ d\eta_i \exp\left\{-\sum_{j,k}\eta_j^+ M_{j,k}\eta_k\right\} = \det M \quad , \tag{3.43}$$

which follows because only those terms in the expansion of the exponential contribute which are proportional to $\eta^+_{\sigma_1}\cdots\eta^+_{\sigma_N}\eta_{\sigma'_1}\cdots\eta_{\sigma'_N}$, where σ_i, σ'_i are permutations of N distinct indices. Therefore, the integral is $\frac{1}{N!}\sum_{\sigma,\sigma'} sgn(\sigma)sgn(\sigma')M_{\sigma_1,\sigma'_1}\cdots M_{\sigma_N,\sigma'_N} = \det M$. That $\det M$ rather than $1/\det M$ appears is a characteristic signature of Grassmann gymnastics.

Therefore, the final expression for the partition function becomes

$$Z_f \propto \det\left\{\gamma_0\left(-\frac{d}{d\tau}+\mu\right) - i\gamma_i\nabla_i - g\gamma_\nu A^\nu - m\right\}_A \propto \det\left\{\frac{\delta^2 S_E(\psi^\dagger,\psi)}{\delta\psi^\dagger\delta\psi}\right\} \quad , \tag{3.44}$$

where the subscript A instructs us to evaluate the functional determinant over the space of antiperiodic functions on the interval $0 \le \tau \le \beta$, and S_E is the fermion action in Euclidean space. (We used above the invariance of the the determinant to a change of the sign of in front of $d/d\tau$.) The calculation of the proportionality

constant requires considerable care in the continuum limit. However, we note that the Grassmann numbers are gone! They were only useful to keep track of antisymmetrization in the intermediate steps. The main result is that the partition function can always be expressed as a determinant of an operator that is simply related to the operator that is sandwiched between $\psi^\dagger$ and ψ in the Lagrangian.

The determinant of an operator M, is just the product of its eigenvalues. If $|a\rangle$ are the orthonormal eigenfunctions of M with eigenvalues λ_a, then

$$\det M = \exp\{Tr\ \ln M\} = \exp\{\sum_\alpha \langle\alpha|\ln M|\alpha\rangle\} = \prod_\alpha \lambda_\alpha \ .$$

To get some idea how the determinant in (3.44) can be calculated, consider a fermi gas in an external potential $A_\nu(\mathbf{x})$ that is not so strong as to be able to produce pairs. Let ϵ_α correspond to the single particle energies in that potential. The single particle wavefunctions, $\langle\mathbf{x}|\alpha\rangle$, then satisfy the static Dirac equation

$$\gamma_0 D_x \langle\mathbf{x}|\alpha\rangle \equiv \gamma_0 \left\{i\gamma_i\nabla_i + m + g\gamma_\nu A^\nu(\mathbf{x})\right\} \langle\mathbf{x}|\alpha\rangle = \epsilon_\alpha \langle\mathbf{x}|\alpha\rangle \ . \tag{3.45}$$

In order to calculate the functional determinant in (3.44), we must solve for the eigenvalues of the operator $-d/d\tau + \mu - \gamma_0 D_x$. The eigenfunctions of this operator are just $e^{-i\omega_n\tau}\langle\mathbf{x}|\alpha\rangle$. The antisymmetric boundary condition in the temperature direction however require that the frequencies ω_n must be 'odd'

$$\omega_n = (2n+1)\pi\beta^{-1} \tag{3.46}$$

rather than 'even' as for the Bose case (3.17). The eigenvalues are thus

$$\lambda_{n,\alpha} = i\omega_n + \mu - \epsilon_\alpha \ . \tag{3.47}$$

The fermion determinant is in this case

$$\begin{aligned} Z_f \ &\propto \ \prod_{n=-\infty}^{\infty} \prod_\alpha \lambda_{n,\alpha} = \exp\left\{\sum_\alpha \sum_{n=0}^{\infty} \ln(\omega_n^2 + (\epsilon_\alpha - \mu)^2)\right\} \\ &\propto \ \exp\left\{\sum_\alpha (1 + e^{-\beta(\epsilon_\alpha - \mu)})\right\} \ , \end{aligned} \tag{3.48}$$

which is recognized to be the familiar answer. In arriving at the last line we had to absorb into the proportionality factor an infinite constant, $\prod_{n=0}^{\infty}\omega_n^2$, as well as an infinity associated with the zero point energy. For a more rigorous justification for such steps see Refs.[30][31][33][34].

Dividing Z_f by the partition function, Z_f^0, for a noninteracting gas leads to an expression from which perturbation theory can be defined:

$$Z_f = Z_f^0 \det\{1 + M(A)\} = Z_f^0 \exp\{Tr\ \ln(1 + M(A))\} \equiv Z_f^0 e^{-S_{eff}(A)} \ , \tag{3.49}$$

where $S_{eff}(A)$ is an effective action of the A field due to coupling to the fermion fields. The operator $M(A)$ in coordinate representation is seen from (3.44) to be

$$\langle \mathbf{x}, \tau | M(A) | \mathbf{y}, \tau' \rangle = g\Delta_f(\mathbf{x} - \mathbf{y}, \tau - \tau')\gamma_\nu A^\nu(\mathbf{y}, \tau') \ , \tag{3.50}$$

in terms of the free fermion propagator at temperature, β^{-1}, and chemical potential, μ,

$$\begin{aligned}\Delta_f(\mathbf{x} - \mathbf{y}, \tau - \tau') &= (\gamma_0(-d/d\tau + \mu) - i\gamma_i\nabla_i - m)^{-1} \\ &= \beta^{-1}\sum_n \int \frac{d^3k}{(2\pi)^3} e^{-i\omega_n(\tau-\tau')} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}(\gamma_0(i\omega_n - \mu) - \gamma\cdot\mathbf{k} - m)^{-1}\end{aligned} \tag{3.51}$$

Expanding the exponent in (3.49) in powers of g gives the perturbation series for the thermodynamic potential, $\Omega(\beta, \mu) = -\beta^{-1}\ln Z_f$,

$$\Omega(\beta, \mu) = \Omega_0 - \beta^{-1}\sum_{i=1}^{\infty}\frac{g^i}{i} Tr\ (\Delta_f\gamma_\nu A^\nu)^i \ . \tag{3.52}$$

3.3 Quantizing QCD

What we saw in the previous sections was that the partition function for a field theory can be expressed as a functional integral over the exponential of minus the Euclidean action. The path in "imaginary" time runs over a finite interval, $0 \le \tau \le \beta$, and boson fields must be periodic and fermion fields must be antiperiodic on that interval. For quarks and gluons, the principle of local gauge invariance led

us to the QCD Lagrangian given by Eq.(2.14). Following the proceedure in the previous sections, the QCD partition function would be written as

$$Z = \int \mathcal{D}[A^a_\mu(x,\tau)]\mathcal{D}[\bar{\psi}_{s,a,f}(x,\tau)]\mathcal{D}[\psi_{s,a,f}(x,\tau)]\exp(-S_E(\bar{\psi},\psi,A)) \ , \tag{3.53}$$

in terms of the Euclidean QCD action. Since quark coupling to gluons involves a quadratic form, $\bar{\psi}i\gamma_\nu\mathcal{D}^\nu\psi$, in terms of the covariant derivative (2.11), the integration over the quark fields can be performed giving

$$Z \propto \int \mathcal{D}[A^a_\mu(x,\tau)]\left\{\det(-i\gamma_\nu\mathcal{D}^\nu(A))\right\}^{N_f}\exp(-S_E^{YM}(A)) \ . \tag{3.54}$$

Note that each massless quark flavor gives one power of the determinant. Also $S_E^{YM}(A)$ is just the Yang-Mills action without quarks.

Unfortunately this blind generalization of path integral to gauge theories breaks down. This is because for any configuration, $A^a_\mu(x,\tau)$, there are *infinitely* many other configurations, ${}^GA$, which differ from A by only a guage transformation. A general gauge transformation can be expressed in matrix notation via

$$\frac{\lambda_c}{2i}A^c_\mu \equiv G\frac{\lambda_c}{2i}A^c_\mu G^{-1} + \frac{1}{g}G\partial_\mu G^{-1} \ , \tag{3.55}$$

in terms of an arbitrary matrix of the form $G(x) = \exp(i\lambda_c\epsilon_c(x)/2)$. For infinitesmal gauge transformations ${}^GA$ is given by Eq.(2.12). Every configuration which can be obtained by such a transformation gives the same contribution to the integral because $S({}^GA) = S(A)$. Obviously the problem with the integral in (3.54) is that we are integrating over infinitely too many redundant dergrees of freedom. To remedy this situation we must arrange that only gauge inequivalent configurations are integrated over. This is accomplished by the famous Fadeev-Popov trick. The idea is to insert into the integral a functional delta function that fixes the gauge so that we integrate over only the distinct configurations in a particular gauge.

To see how this works consider the following two dimensional example:

$$Z = \int d^2x e^{-S(\mathbf{x})} \ , \tag{3.56}$$

where the "action", S, is is assumed to be invariant under rotations. In other words, if $R(\theta)$ is the rotation matrix by angle θ, then we assume that $S(\mathbf{x}) = S(R(\theta)\mathbf{x})$.

Obviously we only need to change to polar coordinates to isolate the redundant degrees of freedom, $\int d\theta = 2\pi$. However, we will proceed in a way that is easy to generalize to gauge theories.

Let us define a "gauge" fixing condition, $\mathcal{F}(\mathbf{x}) = 0$, to eliminate the redundant degrees of freedom. For fun, we will call the "$\hat{n}$ gauge" the one specified by

$$\mathcal{F}(\mathbf{x}) = \hat{n} \cdot \mathbf{x} = 0 \ .$$

Now comes the Fadeev-Popov trick. Let us try to find a rotation invariant function, $\Delta_{\mathcal{F}}(\mathbf{x})$, such that

$$1 = \Delta_{\mathcal{F}}(\mathbf{x}) \int_0^{2\pi} d\theta \delta(\mathcal{F}(R(\theta)\mathbf{x})) \ . \tag{3.57}$$

Clearly $\Delta_{\mathcal{F}}$ is just the Jacobian for the complicated change of variables from $\theta \to \mathcal{F}(R(\theta)\mathbf{x})$. That $\Delta_{\mathcal{F}}(\mathbf{x}) = \Delta_{\mathcal{F}}(R(\theta')\mathbf{x})$ for any θ' follows from the group property $R(\theta)R(\theta') = R(\theta + \theta')$, and from the invariance of the integral to arbitratry shifts of the variable θ. The Jacobian is given by

$$\Delta_{\mathcal{F}}(\mathbf{x}) = \left(\sum_i \left| \frac{\partial \mathcal{F}(R(\theta)\mathbf{x})}{\partial \theta} \right|^{-1}_{\theta = \bar{\theta}_i(\mathbf{x})} \right)^{-1} \ . \tag{3.58}$$

where $\bar{\theta}_i$ are the solutions of $\mathcal{F}(R(\bar{\theta}_i)\mathbf{x}) = 0$.

To evaluate (3.58) just chose $\mathbf{x}$ to satisfy the gauge condition. For the example of our "$\hat{n}$ gauge", take $\mathbf{x} = (r\hat{n}_2, -r\hat{n}_1)$. In that case $\bar{\theta}_i = 0$ and π solve the gauge condition, and we need consider only infinitesmal rotations in (3.58):

$$(R(\theta)\mathbf{x})_i = (\delta_{ij} - \epsilon_{ij}\theta)x_j \ ,$$

where $\epsilon_{12} = -\epsilon_{21} = 1$. Thus, to order θ, $\mathcal{F}(R(\theta)x) = \theta \hat{n}_i \epsilon_{ij} x_j = \theta \ r$. Because there are two terms in the sum of (3.58), the Jacobian is given by

$$\Delta_{\mathcal{F}}(\mathbf{x}) = r/2 \ . \tag{3.59}$$

Having found a convenient representation of unity, we insert it into the integral (3.56) to obtain

$$
\begin{aligned}
Z &= \int d^2x \Delta_{\mathcal{F}}(\mathbf{x}) \int_0^{2\pi} d\theta \; \delta(\mathcal{F}(R(\theta)\mathbf{x}))\; e^{-S(\mathbf{x})} \\
&= \left[\int_0^{2\pi} d\theta\right] \int d^2x \Delta_{\mathcal{F}}(\mathbf{x}) \delta(\mathcal{F}(\mathbf{x}))\; e^{-S(\mathbf{x})} \;, \qquad (3.60)
\end{aligned}
$$

where in the second line we used the invariance of d^2x, $\Delta_{\mathcal{F}}(\mathbf{x})$, and $S(\mathbf{x})$. The final answer in the "$\hat{n}$ gauge" is thus

$$
Z = [2\pi] \int d^2x \frac{r}{2} \delta(\hat{n} \cdot \mathbf{x}) e^{-S(\mathbf{r})} \;. \qquad (3.61)
$$

The factor 2π in front is the "gauge volume". What has been accomplished in (3.61) was the elimination of redundant degrees of freedom by inserting a delta function. The price paid for that was the necessity of computing a Jacobian, $r/2$.

The generalization of this trivial example to gauge theories is now easy. To eliminate the gauge degrees of freedom we need a gauge fixing condition, $\mathcal{F}(A) = 0$, such as the one for the Lorentz gauge, $\mathcal{F}(A) = \partial_\nu A^\nu = 0$. In analogy to the above example we construct an invariant integration , $\int \mathcal{D}G$, over the group of all gauge transformation. For SU(3), that integral involves the integration over a different set of eight Eulerian angles at each space-time point. Next the gauge Jacobian $\Delta_{\mathcal{F}}(A)$ is calculated such that

$$
1 = \Delta_{\mathcal{F}}(A) \int \mathcal{D}G \; \delta(\mathcal{F}({}^G A)) \;. \qquad (3.62)
$$

Because the group measure is assumed to by invariant, $\Delta_{\mathcal{F}}(A) = \Delta_{\mathcal{F}}({}^G A)$ is gauge invariant. To compute $\Delta_{\mathcal{F}}(A)$ it is then sufficient to consider configurations satisfying the gauge condition. For such A, only infinitesmal gauge transformations need be considered in (3.62), and thus ${}^G A$ is given by (2.12) in terms of infinitesmal Euler angles $\epsilon_\alpha(x)$. Therefore, the Jacobian is

$$
\Delta_{\mathcal{F}}(A) = \left| \det \left(\frac{\partial \mathcal{F}({}^\epsilon A(x))}{\partial \epsilon_\alpha(x')} \right) \right|_{\epsilon=0} = |\det((f_{\alpha\beta\gamma} A^\mu_\gamma + g^{-1} \delta_{\alpha\beta} \partial^\mu) \partial_\mu)|_{\partial_\nu A^\nu = 0} \;. \qquad (3.63)
$$

where the final expression holds only in the Lorentz gauge.

Inserting (3.62) into (3.54), changing the order of the $\mathcal{D}G$ and $\mathcal{D}A$ intergrations, shifting variables $A \to {}^{G}A$, and using the gauge invariance properties of the action etc., we obtain finally

$$Z \propto \left[\int \mathcal{D}G\right] \int \mathcal{D}A\, \delta(\mathcal{F}(A))\Delta_{\mathcal{F}}(A)\, \{\det(-i\gamma_\nu \mathcal{D}^\nu(A))\}^{N_f} \exp(-S_E^{YM}(A)) \ . \quad (3.64)$$

With Eq.(3.64) we have succeeded in fixing the gauge and removing the infinite gauge volume $[\int \mathcal{D}G]$. The price paid was a new functional determinant (3.63). In some gauges such as the axial gauge, $\hat{n}_\mu A^\mu = 0$, the resulting determinant is independent of A, and so it can be absorbed into the normalization constant. However, in other gauges such as the Lorentz gauge, the determinant does depend on A. In those cases, it is usesful to express the determinant as a Grassmann functional integral. For example,

$$|\det((f_{\alpha\beta\gamma}A^\mu_\beta + g^{-1}\delta_{\alpha\beta}\partial^\mu)\partial_\mu)| \propto \int \mathcal{D}[\bar{\omega},\omega] \exp\left(-\int_0^\beta d\tau \int d^3x \{\bar{\omega}\partial_\mu\partial^\mu\omega + g f_{\alpha\beta\gamma}\bar{\omega}_\alpha\partial_\mu\omega_\gamma A^\mu_\beta\}\right) \ . \quad (3.65)$$

The anticommuting scalar fields $\omega_\alpha(x)$ and $\bar{\omega}_\alpha(x)$ are called Fadeev-Popov ghosts. The Feynmann rules for including ghosts into diagrams follow immediately from (3.65). For example, the ghost propagator in momentum space is $-\delta_{\alpha\beta}/(\omega_n^2 + p^2)$, with ω_n given by the *odd* frequencies (3.46). The ghost-glue interaction vertex has a value $-ig f_{\alpha\beta\gamma} q_\nu$ for a incoming ghost carrying momentum q.

3.4 Perturbative QCD

With the above technique, the Feynmann rules for perturbative QCD thermodynamics in any gauge can be determined[9][36]. In the "α" gauge these rules are summarized in Fig.3.1, from Ref.[36].

To calculate quantities of interest

1. Draw connected diagrams consisting of solid lines (quarks), wavy lines (gluons), and dashed lines (ghosts) connected by one of the four vertices. Each vertex corresponds to a value indicated on the right. Note that lines carry

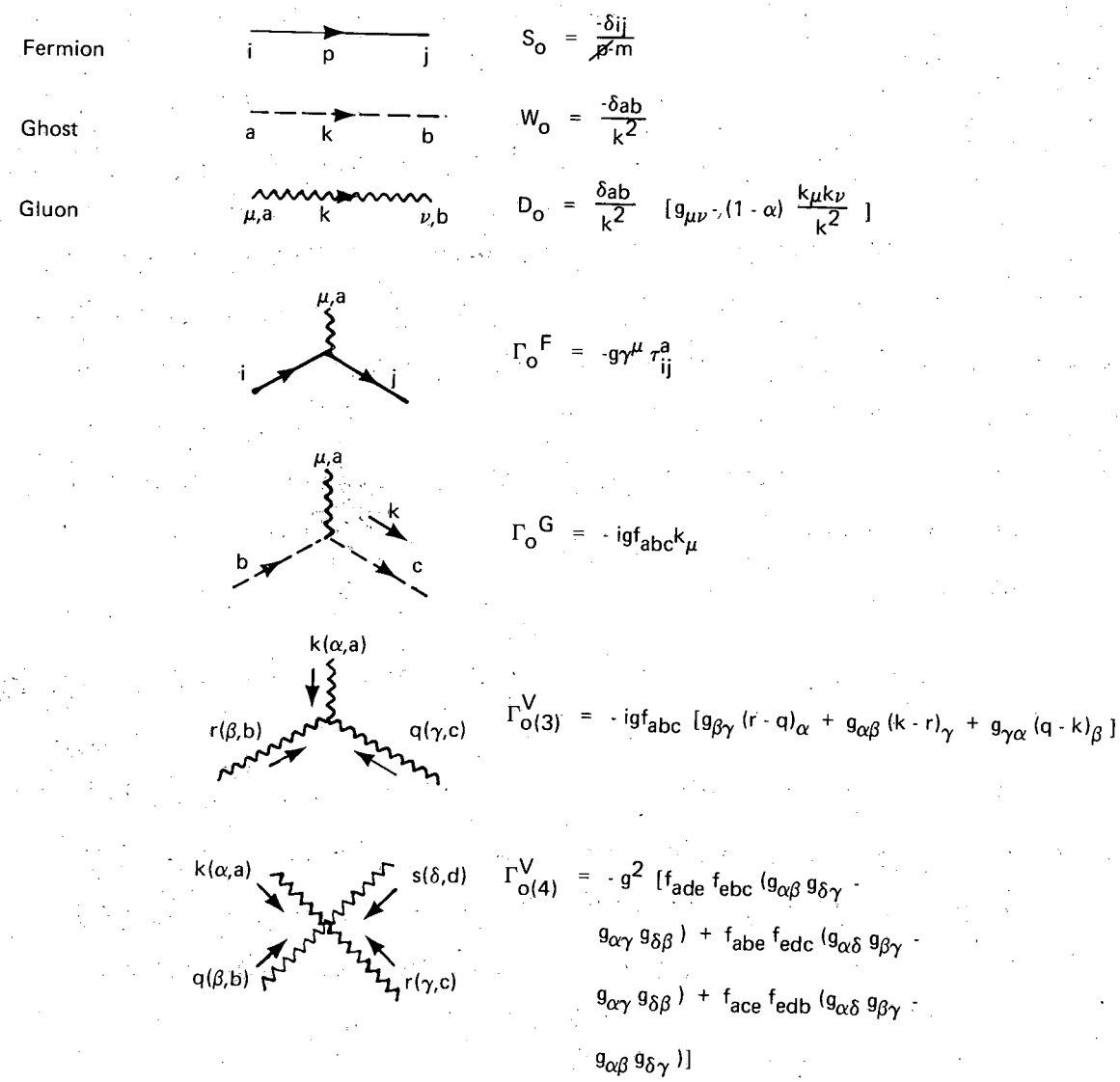

Figure 3.1: Feynman diagrams associated with propagators and verticies in QCD[36].

momentum, color, and Lorentz indices. The diagrams for the thermodynamic potential are those without external lines. Diagrams with external lines correspond to Green's functions.

2. Gluon lines carry even frequencies, $k_0 = i\omega_n = i2\pi n\beta^{-1}$ and repesent gluon propagators. The gauge dependence of the gluon propagator is seen explicitely by its dependence on α.

3. Quark and ghost lines carry odd frequencies, $k_0 = i\omega_n = i(2n+1)\pi\beta^{-1}$.

4. For each closed loop associate a circulating four momentum $(i\omega_n, \mathbf{k})$ and a loop integration
$$\beta^{-1}\sum_n \int \frac{d^3k}{(2\pi)^3}$$
At each vertex energy momentum conservation introduces a delta fuction of the sum of frequency indices n_i and a delta function for the sum of momenta $\mathbf{k}_i$ as in Eq.(3.25). A factor of -1 appears for each closed quark or ghost loop.

5. A combinatorial factor must be calculated for diagrams without external lines.

With these rules an integral over products of vertex functions and propagators is associated with any diagram. For example, the second order contributions to the thermodynamic potential are then given by the sum of the four diagrams in Fig.3.2

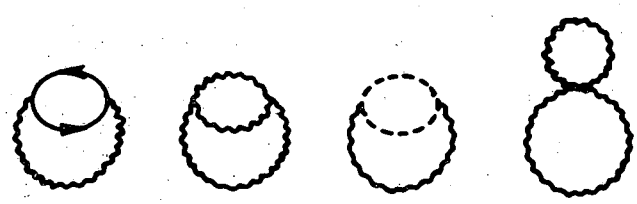

Figure 3.2: Lowest order diagrams for the thermodynamic potential

These diagrams were evaluated by Kapusta[36] to give the following contribution to the free energy density at zero chemical potential:

$$F_2 = \frac{g^2T^4}{16}\frac{1}{9}(N_c^2-1)(N_c+\frac{5}{4}N_f) \ , \tag{3.66}$$

An infinite class of higher order diagrams in Fig.3.3 could also be summed up giving a contribution of order g^3 giving

$$F_3 = -\frac{g^3 T^4}{12\pi}(N_c^2 - 1)(\frac{1}{3}(N_c + N_f/2))^{3/2} \ . \tag{3.67}$$

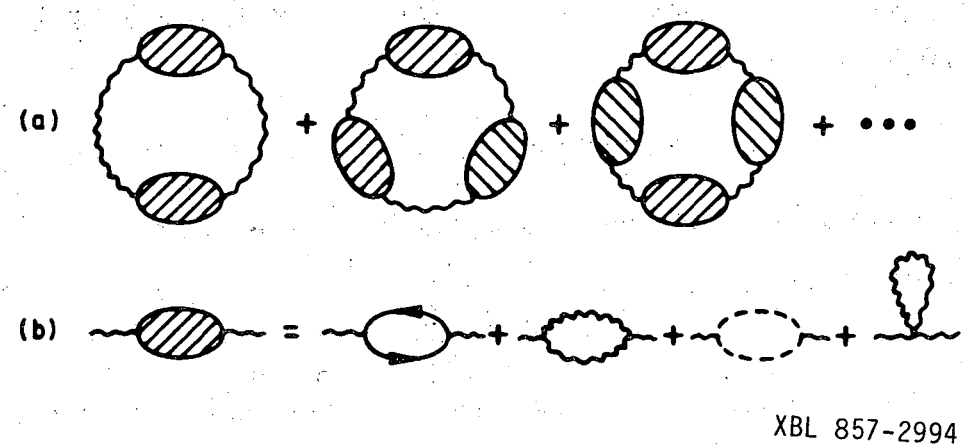

Figure 3.3: (a) Infinite class of ring diagrams contributing to the thermodynamic potential to order g^3. (b) Lowest order gluon self energy diagrams contributing to (a).

Eqs.(3.66,3.67) are the lowest order corrections to the negative of the ideal Stefan-Boltzmann expression for the pressure of a quark-gluon plasma at zero chemical potential. The above rules actually lead to integrals that diverge in these orders. The finite corrections above can be extracted from the infinities only after a "renormalization" program, as illustrated in the next section, is carried out.

4. The Running Coupling

4.1 At zero temperature

Our next problem is to determine what value of g should be used in such perturbative calculations. A complete answer is only provided by renormalization group theory[30]-[33]. However, we can get some insight into the problem by considering the specific example of quark-quark elastic scattering.

Let us define an effective coupling, $\alpha_{eff}(q)$, at momentum transfer, q, by requiring that the $qq \to qq$ scattering amplitude be given by $-4\pi i M(q)\Gamma_{1\mu\alpha}\Gamma_2^{\mu\alpha}$, where

$$M(q) = \frac{\alpha_{eff}(q)}{q^2} \quad , \tag{4.1}$$

and $\Gamma_i^{\mu\alpha} = \bar{u}_i\gamma^\mu\lambda^\alpha u_i'$ are vertex factors. To lowest order $\alpha_{eff}(q) = g^2/4\pi + O(g^4)$. The next lowest order contributions to that amplitude are given by the sum of the following diagrams:

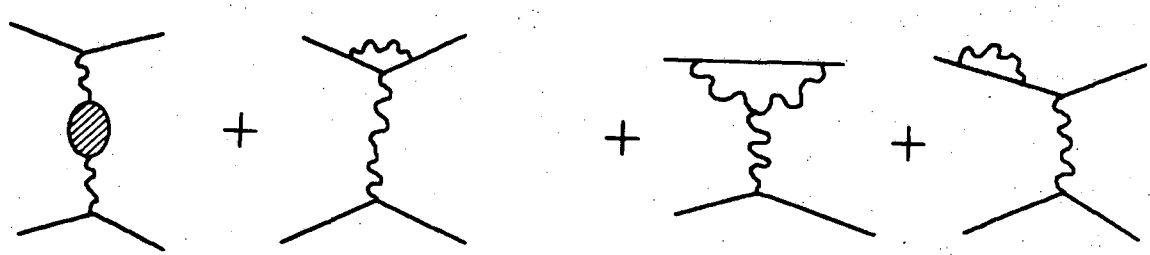

Figure 4.1: Lowest order corrections to $qq \to qq$ amplitude where the gluon self energy is given by the diagrams in Fig. 3.3b

Unfortunately, every diagram containing a closed loop diverges, and we must embark on the program of renormalization. The basic assumption behind renormalization is that physics at a given distance scale q^{-1} should not be sensitive to the physics at some arbitrarily small distance scale, $M_\infty^{-1} \ll q^{-1}$. Thus for example low energy atomic physics does not depend on the microscopic physics that governs whether the electron is a composite particle or not on some scale $\ll m_e^{-1}$. Similarly, scattering of quarks and gluons at momentum transfer q should not depend on the substructure of quarks as long as that substructure is only resolvable on a scale $\ll q^{-1}$. Of course there is no way to know ahead of time whether there is such a clear separation between the scales relevant for quark and subquark physics. Only experiment can tell. Theoretically, we can however propose a renormalizable theory of quark interactions such as QCD and explore its consequences. The essential point is the assumption that there exists some range of momentum transfers or distances where the more microscopic physics is not relevant.

On the other hand, we see that actual calculations with this theory involve integrating over arbitrarily large momentum scales, even beyond the mass of the universe! The assumption that the physics at such scales is irrelevant for the momentum transfers of interest means that we should be able to cut off such momentum integrals at any scale $M_\infty \gg q$. For example we could insert form factor at each vertex which suppress momentum transfers exceding some large scale M_∞. Alternatively we could discretize space-time to cutoff momenta above some inverse lattice scale. This second approach is what underlies the lattice gauge theory formulation of QCD. For continuum perturbation theory some version of the first approach is usually adopted. The actual way that infinite integrals are rendered finite is called the renormalization scheme. One popular scheme is due to Pauli and Villars[32] which involves adding a set of ficticious particles to the theory with masses that are sent to infinity at the end of the calculation and which couple so as to cancel infinities arising in loop integrals.

While any renormalization scheme insures that all Feynman diagrams of the theory are finite, terms such as $\ln(M_\infty^2/q^2)$ appear in the final answer which apparently depend on an arbitrary scale, M_∞. The miracle of renormalizable theories is that it is possible to get rid of those terms by *renormalizing* the finite number of bare coupling constants and masses which appear in the Lagrangian.

To see qualitatively how such a renormalization program works in the Pauli-Villars scheme, consider the higher order contributions to M at an arbitrary space-like momentum scale, $q^2 = -m^2$. That scale will be called the renormalization point. Adding a ficticious heavy quark of mass M_∞, the lowest order correction is found to be[32]

$$\alpha_{eff}(m) = \alpha_0\{1 + \frac{\alpha_0}{4\pi} C \ln(-M_\infty^2/m^2) + O(\alpha_0^2))\} \ , \tag{4.2}$$

where C is

$$C = \frac{11}{3} N_c - \frac{2}{3} N_f \ . \tag{4.3}$$

Note the that the correction would diverge as $\ln M_\infty$ if we let $M_\infty \to 0$ at this point. The trick is to note that we do not really know the value of the bare coupling, α_0,

either. A measurement of $qq \to qq$ could only determine the combination of α_0 and $\ln M_\infty$ in (4.2). Therefore, we must adjust the value of α_0 so that together with the $\ln M_\infty$ correction, a fixed finite value of α_{eff} remains. This is called renormalization of the bare coupling.

The renormalization program thus starts by expessing amplitudes in terms of a renormalized rather than the bare coupling. To see how this works, consider the coupling at a different scale q:

$$\begin{aligned}
\alpha_{eff}(q) &\approx \alpha_0\{1 + \frac{\alpha_0}{4\pi}C(\ln(-M_\infty^2/m^2) - \ln(-q^2/m^2))\} \\
&\approx \alpha_{eff}(m)(1 - \frac{\alpha_{eff}(m)}{4\pi}C\ln(-q^2/m^2) + O(\alpha_{eff}(m)) \\
&\approx \frac{4\pi}{4\pi/\alpha_{eff}(m) + C\ln(-q^2/m^2)} \ , \qquad (4.4)
\end{aligned}$$

The logarithmic infinity has thus disappeared into the the definition of the renormalized coupling, $\alpha_{eff}(m)$. In the last line, we have also "improved" perturbation theory by assuming that the second order correction just represents the first term in a geometric series in the expansion of the denominator in (4.4). Such steps require the full machinery of the renormalization group equations for justification. Unlike in QED where the infinite geometric series follows simply from summing all repeated "bubble" diagrams for the photon propagator, the summation in QCD involves summing *parts* of higher order vertex diagrams in addition to vacuum polarization bubbles.

Eq.(4.4) is still not satisfactory though because it looks like that the effective coupling depends now on the arbitrary renormalization point, m. Since the physical $qq \to qq$ amplitude cannot depend on arbitrary scales, we must impose an additional condition, $d\alpha_{eff}(q)/dm = 0$. This condition implies that we can write

$$4\pi/\alpha_{eff}(m) - C\ln m^2 = -C\ln\Lambda^2 \ ,$$

where Λ is an unknown constant independent of m. With this definition

$$\alpha_{eff}(q) = \frac{4\pi}{C\ln(-q^2/\Lambda^2)} \ . \qquad (4.5)$$

As long as the number of quark flavors is not too large, $N_f < 11N_c/2$, $C > 0$ and $\alpha_{eff} \to 0$ for $q \to 0$. This is the famous asymptotic freedom property of QCD. The remarkable effect of renormalization process was to introduce a dimensional scale Λ into a theory that was initially scale invariant! That scale cannot be calculated but must be determined from experiments. Current data only determine Λ very poorly to be in the range $\Lambda \sim 100 - 400$ MeV.

There are many ways to try to understand the sign of C. None of them are completely satisfying. For example, in a particular gauge we can analyze the sign of each diagram that contributes. Unfortunately, in different gauges the sign and magnitude of diagrams can vary. The theory only guarantees that the sum of *all* diagrams to a given order has a unique value. The following physical picture can nevertheless give a rough idea of the origin of that sign. The sign arises from the competition between ordinary vacuum polarization effects that tend to enhance the coupling at short distances as in QED and antiscreening effects that tend to disperse the color charge. We can view the antiscreening phenomenon as a kind of finite form factor effect. If we tried to concentrate a blue charge onto a heavy quark at the origin, then because the quark can emit a $B\bar{R}$ gluon by becoming red, the blue charge can be distributed over a finite range, $\sim \Lambda^{-1}$. Therefore, if we look for the blue charge in some small volume, $\ll \Lambda^{-3}$, then only a small fraction of the net blue charge will be found there. Qualitatively, it is reasonable to expect then that as the number of colors, N_c, increases, such antiscreening effects should become stronger since there are more ways that a quark can disperse its color by emiting gluons.

In contrast to the antiscreening phenomenon that is unique to non-Abelian theories ($N_c \geq 2$), vacuum polarization always tends to concentrate the charge at the origin. One way to try to understand this effect is to recall that negative energy solutions to both the Dirac and Klein Gordon equations behave opposite to the positive energy solutions. Thus, for example a negative energy electron is *repelled* by a positive charged nucleus. If $\psi_E^Z(x)$ is the electron wavefunction at energy E around a nucleus of charge Z, then for $E < 0$, $|\psi_E^Z(x)|^2 < |\psi_E^0(x)|^2$ in the neighborhood, $\sim m_e^{-1}$, of the origin. The vacuum polarization charge density, which measures the

change in distribution of the negative energy sea,

$$\rho_{VP} = -|e| \sum_{E<0} \left\{ |\psi_E^Z|^2 - |\psi_E^0|^2 \right\}$$

is thus positive near the nucleus[37]. Exactly at the origin the bare charge is reduced by an infinite amount, $\propto \ln M_\infty/m_e$, but after renormalization the residual VP density remains positive. Clearly this effect must increase as the number of fermion flavors, N_f, increases.

In summary, we have the following qualitative picture for the competition between screening and antiscreening: The effective QCD charge is enhanced $\propto N_f$ due to conventional vacuum polarization phenomenon, and it is reduced $\propto N_c$ due to the ability of quarks and gluons to disperse their charge. The proportionality constants for QCD are given in Eq.(4.3). Asymptotic freedom hold only when the antiscreening effects dominates.

While (4.5) exhibits the asymptotic property of QCD, it also shows that the effective coupling could grow to be arbitrarily large at large distances. In fact, (4.5) has a singularity at finite momentum transfers $t = -\Lambda^2$. This is an artifact of perturbation theory. Nonperturbative analysis[38] suggests that α_{eff} should have a simple pole at $t = 0$ instead. Such a singularity would be consistent with the hoped for confinement property of QCD since in coordinate space it would imply a linearly rising potential at large distances[29]. An approximate phenomenological formula for $t = q^2 < 0$ incorporating both asymptotic freedom and "infrared slavery " is thus

$$\alpha_{eff}(t) = \frac{4\pi}{C \log(1 - t/\Lambda^2)} \quad . \tag{4.6}$$

4.4.2 At high temperatures

Why is asymptotic freedom relevant at high temperatures or densities? The answer has to do with color electric and magnetic shielding in the QGP. To see this, we consider again the scattering of quarks or gluons in the Born approximation[39].

The amplitude (4.1) together with the effective coupling (4.6) obviously lead to a divergent total cross section because of the singular small momentum transfers behavior. What saves the day is the modification of the gluon propagator due to the polarizability of the many body medium. In analogy to ordinary electromagnetic plasmas, the exchanged gluon can interact with quarks and gluons already in the plasma in addition to the quantum fluctuations discussed above. Figure[?] illustrate such many body modifications of the effective qq scattering amplitude:

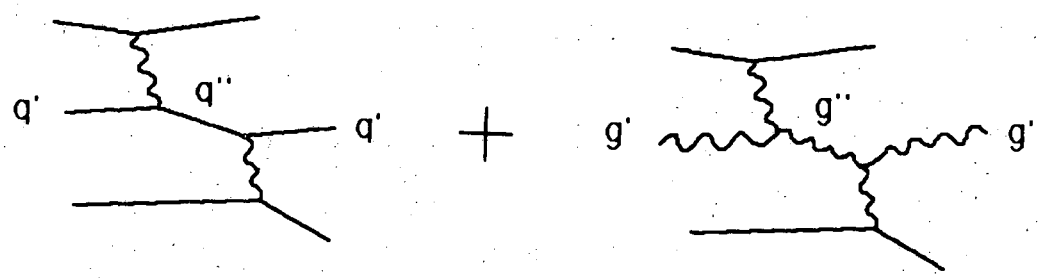

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Figure 4.2: Examples of many body modifications of the effective scattering amplitude

In the random phase approximation[41]-[44] a special subclass of diagrams of this sort can be summed to infinite order (see next section). It is then found that the $\mu = \nu = 0$ part of the static ($\omega_0 = 0$) propagator is modified such that in the small t limit as

$$D^{00}(t) \approx \frac{1}{t - m_E^2} \ , \tag{4.7}$$

where the color electric mass is given by

$$m_E^2 \approx g^2 \left[(N_c + N_f/2)T^2/3 + (1/2\pi^2) \sum_{f=1}^{N_f} \mu_f^2 \right] \ , \tag{4.8}$$

in terms of the termperature T and flavor chemical potentials μ_f. This modification of the gluon propagator implies that static color electric fields are screened on a length scale m_E^{-1} as is evident from linear response theory[48] ($A_0^a(x) = \int d^3y D_{00}^{ab}(x-$

$y)\rho^b_{ext}(y))$. The scale m_E^{-1} is also called the Debye length. The physical origin of such screening is clear. Since the particles in the plasma carry color electric charge, a cloud of particles of opposite charge tend to accumulate around any external charge put into the plasma.

With Debye screening, large distance or low momentum transfer scatterings between quarks is thus suppressed. The dominant momentum transfers, $t \sim -m_E^2$, associated with color electric scattering in fact *increases* then with either temperature or baryon density. Therefore, only the effective coupling at short distances can matter. Obviously at distances much smaller than the average distance $\sim 1/T, 1/\mu$ between quarks in the plasma many body effects cannot be relevant and the effective coupling reduces to its free space form. This is why asymptotic freedom is relevant at high energy densities.

There is however a major catch. That concerns the yet unsolved magnetic shielding problem. There are no elementary quanta in QCD that carry color magnetic charge. Therefore, there cannot be any perturbative mechanism that screens static color magnetic fields. This poses a problem because quarks and gluons interact not only through color electric interactions $\propto D_{00}(q)$, but also also though color magnetic or current-current interactions $\propto D_{ij}(q)$. In ordinary QED plasmas there is no magnetic shielding either, but the sign of the photon self energy, $\Pi_{ij}(0,\mathrm{q}) = O(\alpha q T)$, as $q \to 0$ is such that the magnetic part of the propagator, $D_{ij} = (D^{(0)} - \Pi)^{-1}_{ij}$, has no (tachyon) pole at $q^2 < 0$. Therefore in QED magnetic scattering leads to finite cross section even without magnetic screening. In QCD though[45][46], the sign of $\Pi_{ij}(0,q)$ is *opposite* and D_{ij} acquires a tachyon pole at $q \sim g^2 T$. This means that perturbation theory must break down at low momentum transfers $q < g^2 T$. In QCD the finite electric mass is not sufficient to solve the small momentum transfer, i.e., infrared singularity problem.

The current hope is that a finite magnetic mass is generated in QCD in some nonperturbative fashion[45][46]. That could arise for example if a magnetic glueball condensate were generated somehow. Very preliminary Monte Carlo studies[47] for SU(2) Yang-Mills theory on small lattices seem to indicate that static color magnetic

fields may indeed be screened at high temperatures. It is therefore possible that there does exist a color magnetic mass scale m_M^{-1}, where

$$m_M \approx C_M g^2 T \tag{4.9}$$

with $C_M \sim 1$ necesarily being a nonperturbative constant[46]. Note that unlike m_E which has a nonvanishing contribution to order gT, m_E must be at least $O(g^2T)$, (see further discussion in next sebsection).

With color electric and magnetic screening, the scattering cross section in the plasma could be roughly expessed as[39]

$$\frac{d\sigma^{ab}}{dt} \approx c_{ab}\frac{4\pi\alpha_{\rm eff}^2(t)}{t^2} \ , \tag{4.10}$$

where the effective screened coupling at finite temperatures and chemical potentials is

$$\alpha_{\rm eff}(t) \approx \frac{\alpha_0(t)}{2}\left\{\frac{t}{t-m_E^2}+\frac{t}{t-m_M^2}\right\} \ , \tag{4.11}$$

and $\alpha_0(t)$ is the effective coupling at zero temperature (4.6). In Eq.(4.10) $c_{ab} = 9/4, 1, 4/9$ are color factors relevant to $ab = gg, qg$, and qq scatterings respectively. Screening causes the effective coupling to vanish at low momentum transfers. Asymptotic freedom causes it to vanish at large momentum transfers. The maximum occurs for $\mid t \mid \sim \min(m_E^2, m_M^2)$, and the therefore the maximum value α_{eff} decreases logarithmically with increasing temperature. The color electric screening length also decreases at high baryon densities, but it is not known how magnetic screening behaves at finite chemical potentials. QCD may hold a surprise in this connection.

As the temperature decreases, higher order corrections to the screening lengths must also be considered. In Ref.[48] it was shown that a self consistent treatment of electric screening led to a nonperturbative reduction of m_E to order $\alpha^{3/2}$ as

$$m_E^2 \to m_E^2(1-(\gamma g^2/4\pi)^{1/2}) \ , \tag{4.12}$$

where $\gamma \sim 1$. Therefore, higher order corrections work against screening.

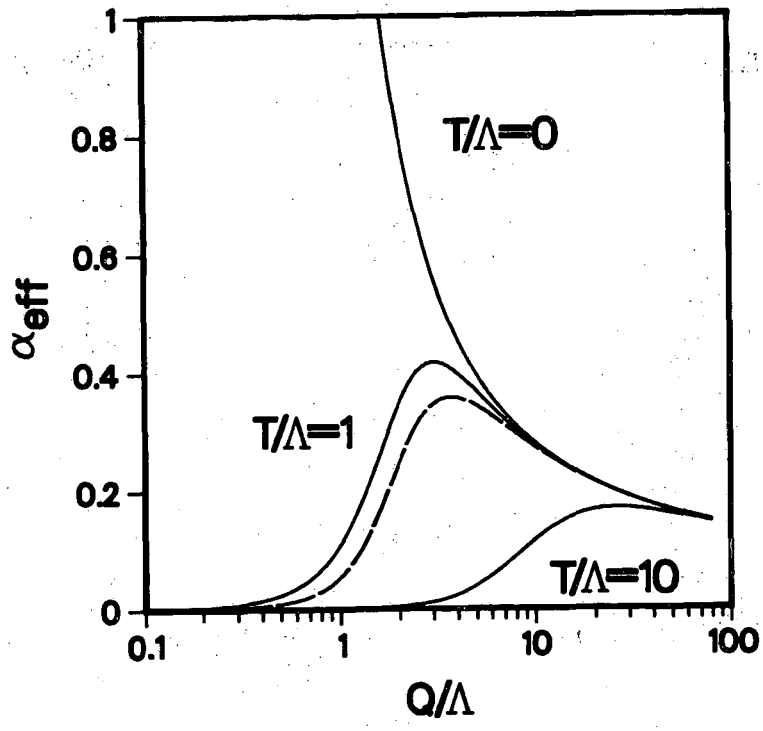

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Figure 4.3: Dependence of effective coupling on momentum transfer for various temperatures.

In Fig.4.3 the effective screened coupling is shown including a rough estimate of possible nonlinear effects from Eq.(4.12).

In this numerical example[39], we have approximated $g^2/4\pi$ in (4.12) by $\alpha_0(t \sim -s)$, where $s = \langle (p_1 + p_2)^2 \rangle \approx 17T^2$ is the average center of mass energy squared for binary collisions in a relativistic quark-gluon gas. Furthermore, we used $\alpha_0(t)$ in place of $g^2/4\pi$ in (4.8,4.9). The dashed curve corresponds to neglecting the nonlinear correction above and the difference between the solid and dashed curves gives some indication for the order of magnitude uncertainty in the effective coupling in the interesting temperature range $T \sim \Lambda$. The value $C = 1/4$ was used[39] for the nonperturbative constant in Eq.(4.9). The qualitative behavior of the screened coupling as a function of transverse momentum are well illustrated in this figure. Note in particular that the screened coupling maximizes near the average momentum, $q \approx 3T$, of quarks and gluons in the QGP.

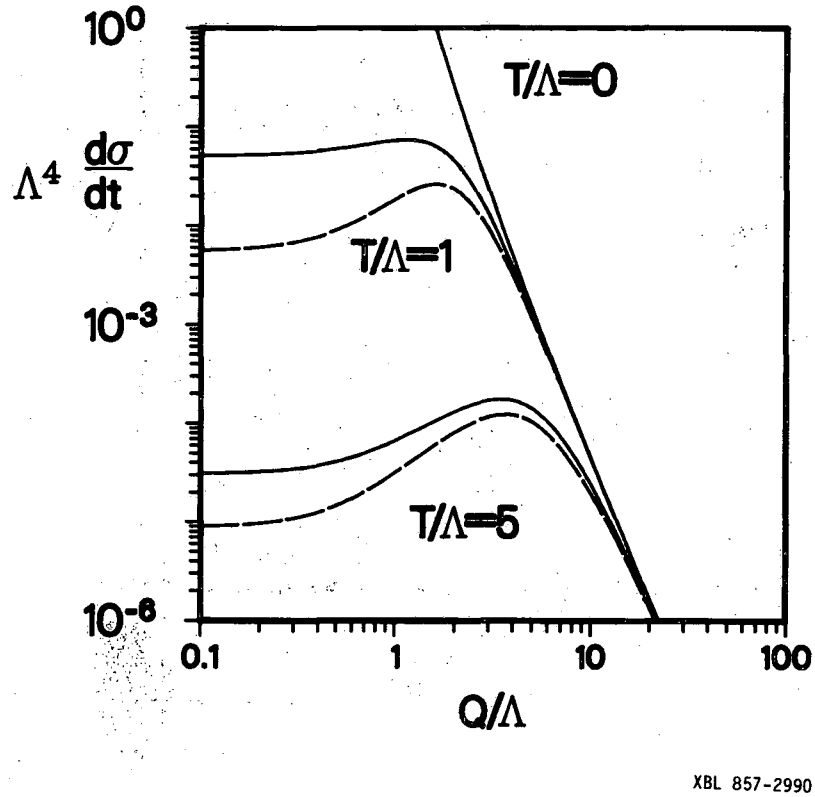


Figure 4.4: Effective quark-quark differential cross section.

In Fig.4.4 the corresponding screened differential cross sections are shown. Because the amplitude is heavily weighed toward low t, the uncertainties associated with the nonlinear modifications of the screening lengths are considerably amplified for $T \sim \Lambda$. At high temperatures that uncertainty decreases, but unfortunately in the temperature range relevant to nuclear collisions the effective quark gluon cross sections are rather uncertain. Note that we have neglected corrections[40] of order t/s to the differential cross sections due to spin effects. It is clear from Fig.4.4 that the total cross section decreases as T^{-2}. However, since the density of quarks and gluons increases as T^3, the mean free paths decrease $\propto T^{-1}$. Therefore, asymptotic freedom is consistent with thermal equilibration on an ever decreasing spatial scale. Unfortunately, for $T \sim \Lambda$, which may be most easily accessible experimentally, there is great uncertainty as to value of the mean free paths. This translates into considerable uncertainty as to whether local equilibration can be attained in nuclear collisions via conventional kinetic effects.

How does all this relate to the perturbative calculations of the thermodynamic

potential? We were trying to figure out what value to use for $g^2/4\pi$. The above arguments hopefully make it plausible that by suitable rearrangement and partial summation of perturbation theory, we should be able to use an effective value of α that depends on temperature and density. Since $\alpha_{eff}(t)$ maximizes at $t \sim -T^2$, at high temperatures we should be able to use an effective coupling

$$\alpha_T \approx \alpha_{eff}(q^2 = -bT^2) \approx \frac{4\pi}{C\ln(bT^2/\Lambda^2)} \quad . \tag{4.13}$$

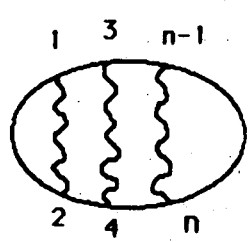
Exactly what constant b should be used above is not clear. At sufficiently large T/Λ it does not matter. Where the value of b does matter, α_T is too large to believe in perturbative results anyway. For phenomenological applications a value $b \sim 10$ could be used since α_{eff} maximizes near the average thermal momentum $\sim 3T$.

Since $\alpha_T \to 0$ for $T/\Lambda \gg 1$, it is then plausible that the perturbative corrections (3.66,3.67) to the free energy density (negative of the pressure at $\mu = 0$) may become arbitrarily small. QCD matter at high energy densities would then correspond to an ideal quark-gluon plasma, as characterized by Eqs.(2.17).

4.3 Breakdown of Perturbation Theory

Unfortunately, QCD is not that simple. In the previous section we had to appeal to some nonperturbative mechanism to generate a magnetic mass, m_M. In this section we look more carefully at this problem. This will allow us to fully appreciate the limitations of perturbative analysis of QCD thermodynamics.

Consider the higher order diagram for the thermodynamic potential in Fig.4.5.



Shattered Egg

Figure 4.5: n^{th} order graph, Ω'_n, contributing to the thermodynamical potential. This one involves one fermion loop with many glue exchanges.

We will study the infrared behavior of such diagrams[45]. According to the Feynman rules discussed in section 3, each wavy line corresponds to a gluon propagator, $D_{\mu\nu}$, and solid lines correspond to quark propagators, S_q. If the number of vertices in the above diagram is n, then there are n quark propagators and $n/2$ gluon propagators. There are then also $1+n/2$ loop integrals, each giving rise to a sum of the form $T\sum_m \int d^3p_i/(2\pi)^3$. The infrared behavior of that diagram can be studied by considering that region of the $3(1+n/2)$ dimensional momentum loop integral where all p_i are approximately equal to p and $p \to 0$. Since both D and S_q decrease with increasing frequency ω_n it is furthermore sufficient to study only the lowest frequency contribution, $n = 0$.

Because gluon fields are periodic, the lowest discrete frequency allowed for gluons is $\omega_0 = 0$ as evident from Eq.(3.17). However, recall that fermion fields must be antiperiodic in the temperature direction. This means that the lowest frequency, (3.46), which contributes to fermion propagators is finite, $\omega_0 = \pi T$. Therefore, for $p \to 0$

$$D(\omega_0, p) \to 1/p^2 \ , \tag{4.14}$$

while

$$S_q(\omega_0, p) \to 1/(\pi T) \ , \tag{4.15}$$

where we have suppressed color and Lorentz factors. In other words, the gluon propagators diverge at small momenta, whereas quark propagators remain finite[45].

We can now check the infrared behavior of our higher order diagram for the thermodynamic potential as

$$\begin{aligned}
\Omega'_n &\sim g^n \underbrace{\left(T\int d^3p\right)^{1+\frac{n}{2}}}_{1+\frac{n}{2}\ loops} \underbrace{\left(\frac{1}{p^2}\right)^{\frac{n}{2}}}_{\frac{n}{2}\ gluons} \underbrace{\left(\frac{1}{T}\right)^{n}}_{n\ quarks} \\
&\sim g^n T^{1-\frac{n}{2}} \int_\lambda^T dp p^{2+\frac{n}{2}} \\
&\sim g^n T^4 \ .
\end{aligned} \tag{4.16}$$

Note that the answer does not depend on the infrared cutoff scale λ, and therefore this type of diagram is infrared finite and stable.

In QCD there is however another class of diagrams shown in Fig.4.6 that look like Fig.4.5 except that all quark lines are replaced by gluon lines.

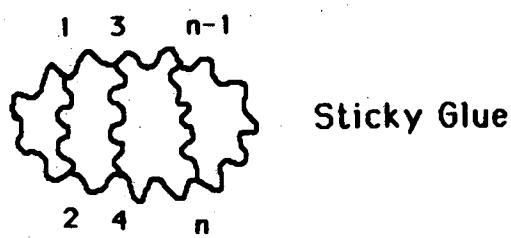


Figure 4.6: n^{th} order graph, Ω_n^{glue}, causing the death of perturbation theory

This type of contribution to the thermodynamic potential is unique to non-Abelian theories because gluons are required to interact with each other. Since the three gluon interaction involves a derivative coupling, each vertex brings in a factor of the loop momentum. Proceeding as above, the infrared behavior of such diagrams is then given by

$$\begin{aligned}
\Omega_n^{glue} &\sim g^n \left(T \int d^3p\right)^{1+\frac{n}{2}} \underbrace{\left(\frac{1}{p^2}\right)^{\frac{3n}{2}}}_{\frac{3n}{2}\ gluons} \underbrace{p^n}_{n\ vertices} \\
&\sim g^n T^{1+\frac{n}{2}} \int_\lambda^T dp p^{2-n/2} \\
&\sim \begin{cases} g^n T^4 & for\ \ n \leq 5 \\ g^n T^4 \ln(T/\lambda) & for\ \ n = 6 \\ g^n T^4 (T/\lambda)^{n/2-3} & for\ \ n > 6 \end{cases} .
\end{aligned} \tag{4.17}$$

We see explicitly that for orders $n \geq 6$, the thermodynamic potential diverges as the infrared cutoff $\lambda \to 0$. It is also true that these diagrams diverge in the high frequency limit, but there exists a renormalization theory cure for those divergences. The divergences above are due to the singular long wavelength properties of QCD.

In the previous subsection we noted to a possible solution to such infrared divergence via partial summation of higher order diagrams. The problem above was due to divergence of the static gluon propagator at low momenta as seen in Eq.(4.14). Let us try to sum a class of diagrams that modify the gluon propagator via the Dyson equation

$$\bar{D}(q) = D(q) + D(q)\Pi(q)\bar{D}(q) = (D^{-1}(q) - \Pi(q))^{-1} \ , \tag{4.18}$$

which corresponds to summing all "bubble" diagrams of the form

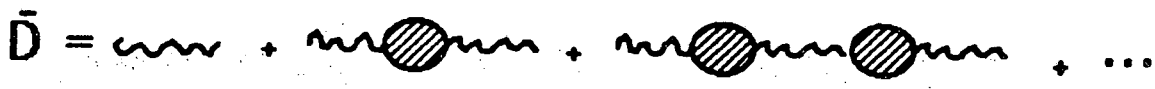


Figure 4.7: Dressed gluon propagator

where $= \Pi(q)$ denotes the gluon self energy as shown in Fig.3.3b.

As shown in Refs.[36]-[44], the lowest order self energy has the property that $\Pi_{00}(0, k \to 0) = m_E^2 \sim g^2T^2$ as given by (4.8). Unfortunately, to lowest order the spatial part of the self energy, Π_{ij}, vanishes in the static infrared limit. This is why we had to assume that m_M^2 must be at least $\sim O(g^4)$.

Lets try to calculate m_M^2 perturbatively by considering the fourth order self energy diagram in Fig.4.8.

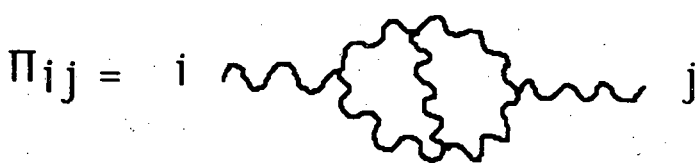


Figure 4.8: Divergent fourth order gluon self energy.

The infrared analysis of that contribution gives

$$m_M^2 \approx \Pi_{ii}^{(4)}(0, k \to 0) \sim g^4 \left(T \int d^3p\right) \left(\frac{1}{p^2}\right)^5 (p)^4 \sim g^4T^2 \ln(T/\lambda) \to \ln\infty \ . \tag{4.19}$$

We see therefore that perturbation theory for the self energy breaks down at order g^4 already. Similar considerations show that higher order self energy diagrams diverge even faster $(T/\lambda)^{n/2-2}$ as a function of the infrared cutoff λ.

Is it possible that by first summing over all orders and then taking the $\lambda \to 0$ limit, that a finite answer for m_M could result? Suppose that $m_M = C_M g^2 T$. Then $1/p^2$ is replaced by $1/(p^2 + m_M^2)$ in (4.19). In that case we can send λ to zero and the nth order self energy becomes for $n > 4$

$$\Pi_{ii}^{(n)} = a_n g^n T^2 (T/m_M)^{n/2-2} = a_n g^4 T^2 (1/C_M)^{n/2-2} \ , \tag{4.20}$$

where a_n is a computable constant. This is indeed a peculiar result. Because we tried to cutoff the low wavelengths at order g^2T, all higher order contributions to the self energy reduced to the *same* order (g^4). Summing all orders we get a self consistency equation

$$m_M^2 \equiv C_M^2 g^4 T^2 = \sum_{n=4}^{\infty} \Pi_{ii}(0, 0^+) = g^4 T^2 f(C_M) \ , \tag{4.21}$$

where $f(x) = a_4 \ln(1/C_M g^2) + \sum_{n=6}^{\infty} a_n / C_M^{n/2-2}$ is a function that we can only determine after calculating the coefficients a_n to all orders in perturbation theory. Herein lies the rub! Every order of perturbation theory for this problem contributes equally. Had nature been kind, and $m_M \propto gT$ as for the electric mass, then the nth order self energy would only be reduced to order $g^{n/2+2}$, and thus at sufficiently high temperatures only the first few terms in the perturbation expansion would have been sufficient. Because in fact $m_M \propto g^2T$, *all* orders in perturbation theory are needed to determine the proportionality constant.

Not only does $m_M \propto g^2T$ ruin any attempt to calculate the proportionality constant perturbatively, but it also ruins any attempt to calculate the thermodynamic quantities perturbatively. If we replace λ by m_M in Eq.(4.17), then we see that for orders $n > 6$ the infrared sensitivity of the thermodynamic potential reduces all higher orders to the same order, g^6. Thus a priori it is not possible to calculate within perturbation theory the corrections to the thermodynamic potential beyond

sixth order. On the positive side, this breakdown of perturbation theory occurs at perhaps high enough order that the nonperturbative corrections never amount to much. If this turns out to be the case, then our picture of an ideal quark-gluon plasma would still hold at high energy densities. Obviously, we must turn to nonperturbative techniques to find out. The rapid development of lattice gauge theory holds out the promise that this question may be answered in the near future.

5. The Lattice World

In section 3, we started the program of quantizing QCD by first considering a finite set of coupled oscillators defined on a space-time lattice and then taking the continuum limit (3.10). However, we got into some technical difficulties in section 3.3, associated with the infinite gauge volume in the continuum limit. With the Faddeev-Popov trick, that difficulty was surpassed (3.54) and we embarked on perturbation theory. Unfortunately, we found that perturbation theory for QCD is terminally ill beyond some finite order of g due to infrared singularities. To make further progress, we must therefore consider nonperturbative techniques. Currently, the most promising handle on nonperturbative problems is based on the "*brutus forcus*" techniques of lattice gauge theory.

The cult of the lattice is, however, enshrouded by special incantations such as "plaquettes", "Polyakov loops", "Euclidean Wilson Action","scaling window", etc., which often create fear in the hearts of nonpractitioners of the faith. This section offers a layman's tour of this occult.

5.1 Link Variables and the Wilson Action

The first step is to go back to the the original definition of path integrals on a finite space-time lattice. Let a denote the lattice spacing between the lattice sites. The sites are labelled by four integers (n_x, n_y, n_z, n_τ). Consider the finite lattice with $1 \leq n_i \leq N_x$ and $1 \leq n_\tau \leq N_\tau$. Recall that the temperature is related to a and N_τ by eq.(3.11). At each site of the lattice we assume that there are

$4 \times (N_c^2 - 1)$ oscillators denoted by the gluon field, $A_\mu^c(x_i, \tau_j)$. The requirement of periodic boundary conditions imply that $A_\mu^c(x_i, \tau_1) = A_\mu^c(x_i, N_\tau a)$.

The path integral for the partition function involves then $4 \times (N_c^2 - 1) \times (N_x - 1)^3 \times (N_\tau - 1)$ ordinary infinite integrals over the values of $A_\mu^a(x_i, \tau_j)$ at the lattice points. Unfortunately though, this definition of the path integral for QCD fails to preserve its the most sacred property, gauge invariance. The discretized version of $F_{\mu\nu}^2$ is simply not invariant to gauge transformation to all orders in the lattice spacing a.

To preserve gauge invariance to all orders in a, Wilson and Polyakov proposed a more sophisticated version of the lattice theory, where instead of integrating over the gluon fields at each lattice site, they chose to integrate over SU(N_c) rotation matricies associated with the $4 \times (N_x - 1)^3 \times (N_\tau - 1)$ links connecting the lattice sites. These new "link" variables are related to the gluon fields by a path ordered integral

$$\begin{aligned} U_{i,\mu} &= \lim_{n\to\infty} \left(1 - i\frac{a}{n}\frac{\lambda_c}{2}A_\mu^c(x_i + \hat{\mu}\frac{a}{n})\right) \cdots \left(1 - i\frac{a}{n}\frac{\lambda_c}{2}A_\mu^c(x_i + n\hat{\mu}\frac{a}{n})\right) \\ &= P\exp\left(-i\int_{x_i}^{x_i+a\hat{\mu}} dx \frac{\lambda_c}{2}A_\mu^c(x)\right) \end{aligned} \tag{5.1}$$

where $\lambda_c/2i$ are the generators of SU(N_c). Thus, $U_{i,\mu}$ is an SU(N_c) matrix associated with the link on the lattice between x_i and $x_i + \hat{\mu}a$ in the direction $\hat{\mu}$. The inverse of $U_{i,\mu}$ is then given by $U_{i+a\hat{\mu},-\mu}$.

These objects are contructed so that under a gauge transformation, $G(x) = \exp(i\lambda_c\epsilon_c(x)/2)$, by which A_μ^c transforms via (3.55) the link matricies transform as

$$U_{i,\mu} \to G^{-1}(x_i)U_{i,\mu}G(x_i + a\hat{\mu}) \ . \tag{5.2}$$

The important point is that the transformation property (5.2) of $U_{i,\mu}$ is exact to all orders in a, whereas we can define a discretized gauge transformation of A_μ from (3.55) that is only accurate to a finite order in a. With these new variables it is therefore possible to construct a lattice theory that has exact gauge invariance.

The effective action, $S(U)$, in terms of the link variables must be chosen so that it reduces to the QCD action in the continuum limit, $a \to 0$. Since $U_{i,\mu}$ are matricies while $S(U)$ is a number, S must involve traces of products of U's. The simplest action that does the job looks rather wierd and unfamiliar:

$$S_W(U) = \frac{2N_c}{g^2} \sum_{i,\mu,\nu} [1 - \frac{1}{N_c} \mathrm{tr}(U_{i,\mu} U_{i+\mu,\nu} U^\dagger_{i+\nu,\mu} U^\dagger_{i,\nu})] \ . \tag{5.3}$$

Eq.(5.3) instructs us to sum over the trace of a product of four rotation matricies associated with the four sides of elementary squares (plaquettes) of dimensions $a \times a$ that can be formed anywhere on the lattice. That $S_W(U)$ is gauge invariant to all orders in a follows from (5.2) and the cyclic property of traces. By expanding the $U_{i+a\hat{\nu},\mu} \approx 1 + a(\lambda_c/2i)A_\mu(x_i + a\hat{\nu}) + O(a^2)$, and approximating $\partial_\nu A_\nu(x_i) \approx (A_\mu(x_i + a\hat{\nu}) - A_\mu(x_i))/a$, etc., $S_W(U)$ can also be shown to reduce to the continuum QCD action in the $a \to 0$ limit. Obviously, the choice of $S(U)$ is not unique since we could add terms involving a trace of products of more than four U's that mantains local gauge invariance and approximates the continuum action even better for finite a. What Wilson found was that Eq.(5.3) happens to be the simplest action that does the job.

5.2 Monte Carlo Method

With $S_W(U)$ so defined, the partition function for pure Yang-Mill theory can be calculated on a computer by integrating $\exp(-S_W(U))$ over all possible SU(N_c) rotation matricies associated with the links of the lattice. In practice, we want to evaluate expectation values of operators, $A(U)$, that correspond to interesting quantities in the continuum limit. Such expectation values are given in lattice theory by

$$\langle A \rangle = \frac{1}{Z} \int \prod_{\text{links}} [dU_{i,\mu}] A(U) e^{-S_W(U)} \ , \tag{5.4}$$

with $[dU]$ being the SU(N_c) group measure (for SU(N_c) that measure involves an integration over $N_c^2 - 1$ Eulerian angles associated with every link). The stategy in

lattice theory is to brute force such integrals on a computer using the Monte-Carlo Metropolis technique.

In that method a random set of SU(N_c) rotation matricies are assigned to each link on the lattice. Each matrix can be defined in terms of $N_c^2 - 1$ angles, $\theta_{i,\mu}^j$, $j = 1, N_c^2 - 1$. An entire lattice configuration is then specified in terms of

$$\mathcal{N} = 4 \times (N_c^2 - 1) \times (N_x - 1)^3 \times (N_\tau - 1) \tag{5.5}$$

angles, $\Theta = \{\theta_{i,\mu}^j\}$, that specify the $\mathcal{N}/(N_c^2 - 1)$ rotation matricies. For a $10^3 \times 6$ lattice for example, the integral (5.4) involves $\sim 10^5$ ordinary finite range $(0, 2\pi)$ intergals!

Monte Carlo methods are particularly powerful to handle very high dimensional integrals. The idea is to take a random walk in the $\mathcal{N}$ dimensional space of angles. Suppose $\Delta\Theta$ is a small step taken in a random direction in the compact $\mathcal{N}$ dimensional angle space spanned by the θ_i. If $\Delta S = S(U(\Theta + \Delta\Theta)) - S(U(\Theta)) < 0$, then the step should be accepted since the random walk is toward regions of higher probability (less action). If the $\Delta S > 0$, then another random number $0 < \varsigma < 1$ is thrown and the step is accepted or rejected according to whether $e^{-\Delta S} > \varsigma$ or $< \varsigma$. Continuing this random walk $N_\$$ steps until your computer budget is spent, a sequence of points Θ_α is generated such that in the $N_\$ \to \infty$ limit the $U(\Theta_\alpha)$ become distributed according to the probability distribution $e^{-S(U)}/Z$. With this ensemble of configurations, averages of interesting quantities can thus be approximated by

$$\langle A \rangle \approx \frac{1}{N_\$} \sum_\alpha A(U(\Theta_\alpha)) \ . \tag{5.6}$$

The accuracy of this approximation increases $\propto 1/\sqrt{N_\$}$ independent of the dimension of the integral. Herein lies the power of this '*brutus forcus*' method. The disadvantage is obviosly the need for large computers and large computer budgets. From the physics point of view a month of CRAY running time furthermore provides no insight into *why* the answer came out the way it did. Lattice workers must proceed as experimentalists measuring a variety of variables on the lattice to try to formulate an overall picture of what the important dergrees of freedom and physics

may be. At this early stage, however, the main goal is to determine a few crucial answers to such questions as whether there is a deconfinement temperature and what is the nature of the effective quark-atiquark potential in QCD. The why's can come later.

5.3 Observables

To determine whether there is a deconfinement transition in QCD, the quantity that is most fequently measured is the expectation value of the thermal Wilson line or Polyakov loop, $\langle L \rangle$. This quantity measures the change in the free energy of the system if a static (infinitely heavy) quark is put into the system at some point x_0. An external quark can be put into the system by applying[52] the quark creation operator, $\psi_a^\dagger(x_0)$ to an arbitrary gluon configuration, $|\alpha\rangle$. The free energy of that isolated quark averaged over its N_c possible colors is

$$e^{-\beta F_q} = \frac{1}{N_c}\sum_{a=1}^{N_c}\sum_{\alpha}\langle\alpha \mid \psi_a(x_0)e^{-\beta H}\psi_a^\dagger(x_0) \mid \alpha\rangle \ . \tag{5.7}$$

The Hamiltonian is modified by the presence of a static quark by the addition of

$$H_{ex} = g\int d^3x\psi^\dagger(x)\frac{\lambda_c}{2}\psi(x)A_0^c(x) \ . \tag{5.8}$$

In order to express $e^{-\beta F_q}$ in terms of a path integral we proceed as before using (3.3) and inserting complete sets of states. We encounter matrix elements as

$$\langle\alpha_i|\psi(x(i))(1-\frac{\beta}{N_\tau}(H_0+H_{ext}))\psi^\dagger(x(i+1))|\alpha_{i+1}\rangle \approx$$

$$\approx e^{-a\langle S_{YM}(A(x_i))\rangle}\left[e^{-ag\frac{\lambda_c}{2}\langle A_0^c(x_i)\rangle_i}\right](1+O(a^2)) \tag{5.9}$$

where $\beta/N_\tau = a$ is the lattice spacing. The expression in the square brackets almost looks like the link matrix $U_{x_0,\hat{\tau}}$. By replacing the Minkowski four vector field by its Euclidean conterpart $A_0 \longrightarrow iA_0$ and rescaling the fields by g, the expression in the brackets does become the link matrix. We end up then with the following formula

for the free energy of a static quark[52]:

$$
\begin{aligned}
e^{-\beta F_Q} &= \langle L(x_0) \rangle \\
&= Z^{-1} \int \prod dA e^{-S_{YM}(A)} \mathrm{tr} \left\{ P \ \exp \left[- \int_0^\beta d\tau g \lambda^\alpha A_0^\alpha(x_0, \tau) \right] \right\} \\
&= Z_W^{-1} \int \prod dU e^{-S_W(U)} \mathrm{tr} \left\{ P \prod_{n=1}^{N_\tau - 1} U_{x_0,\hat{\tau}}(\tau = na) \right\} \ ,
\end{aligned}
\tag{5.10}
$$

where $L(x_0)$, as given by the trace over color indices of the product of U's in the timelike direction, is called the thermal Wilson line or Polyakov loop. In terms of $L(x_i)$, the free energy of a collection of static quarks and antiquarks can similarly be shown to be given by

$$
e^{-\beta F_{m\bar{q},n\bar{q}}} = \langle L(x_1) \cdots L(x_m) L^\dagger(x_1') \cdots L^\dagger(x_n') \rangle \ . \tag{5.11}
$$

To test for confinement, we need only check that the free energy of an isolated quark diverges, i.e., $\langle L \rangle \to 0$. The deconfinement transition in QCD at a critical temperature, T_c, would show up as a sudden jump of $\langle L \rangle$ from 0 to some finite value. Another way to test for confinement would be to look for a linearly increasing quark-antiquark potential through $-T \ln \langle L(0) L^+(x) \rangle \approx \sigma x$.

A quantity of great interest is the energy density,

$$
\epsilon = \frac{-1}{V} \frac{\partial \ln Z}{\partial \beta} = \frac{1}{N_x^3 N_\tau a^3} \langle \frac{\partial S}{\partial a_\tau} \rangle \ , \tag{5.12}
$$

which involves the expectation value of the variation of the action with respect to variation in the lattice scale in the temperature direction. In practice this quantity involves the difference between the expectation value of plaquettes oriented in the space dimensions and those with two of the sides oriented in the temperature direction[53]. Because this quantity involves the difference of two large numbers, reliable numbers require especially large computer runs. In addition, the divergent vacuum energy density has to subtracted. Finally, a correction on the order of a factor of two has to be estimated to compensate for finite size effects on currently accessible lattices. Current numerical results therefore entail large systematic errors. Nevertheless, those results look rather encouraging.

5.4 QCD without Quarks

For QCD without dynamical quarks, a general consenses is slowly beginning to emerge. In order to better appreciate the significance of those results, we must first discuss how the temperature is fixed. Suppose that we want to calculate $\langle L(0)\rangle$ on a $N_x^3 \times N_\tau$ lattice at a temperature $T = 100$ MeV. The temperature is related to the lattice spacing by (3.11). But the computer only knows about N_τ and the value of the bare coupling g in the action. QCD has no intrinsic scale.

Recall from section 4 that a scale only arises because of the necessity of renormalization. The renormalized coupling $g(a)$ must then depend on the scale at which the coupling is to be evaluated. That dependence is given by the renormalization group equation

$$dg/d\ln a = \bar{\beta}(g) \approx \beta_1 g^3 + \beta_2 g^5 + \cdots \quad , \tag{5.13}$$

where the coefficients β_i are computed from the i-loop contributions to the renormalized effective coupling. In section 4, we indicated how the one loop corrections to the gluon propagator and vertex functions modify the effective coupling. Renormalization group theory shows how to include systematically higher order quantum fluctuations. For SU(3), the coefficients were found to be $\beta_1 = (33 - 2N_f)/(24\pi^2)$ and $\beta_2 = (102 - 38N_f)/3)/(256\pi)^4$. The solution of (5.13) in the weak coupling limit $g(a) \to 0$ is clearly

$$g^2(a) = F(a\Lambda_L) \approx \left(\beta_1 \ln(1/a\Lambda_L)^2 + \beta_2/\beta_1 \ln\ln(1/a\Lambda_L)^2 + \cdots\right)^{-1} \quad , \tag{5.14}$$

where Λ_L is an integration constant. Because g is dimensionless, a momentum scale Λ_L had to enter so that g becomes a function of the dimensionless quantity $a\Lambda_L$. The value of Λ_L is not the same as the value of Λ that results from renormalizing the theory via the Pauli-Villars scheme in section 4. Estimates for $\Lambda_L \approx \Lambda/83.5$ typically give a value ~few MeV. Eq.(5.14) exhibits the characteristic logarithmic decrease of g as the distance scale is reduced.

We can use (5.14) to estimate the ratio of T to Λ_L as a function of g. Since

$Ta = 1/N_\tau$, and

$$a\Lambda_L = F^{-1}(g) \approx (\beta_1 g^2)^{-\beta_2/2\beta_1^2} \exp(-1/(2\beta_1 g^2)) \ . \tag{5.15}$$

Therefore, T/Λ_L is approximately given by

$$T/\Lambda_L = (N_\tau F^{-1}(g))^{-1} \ . \tag{5.16}$$

Eqs.(5.15,5.16) show how the coupling constant changes the lattice spacing, a, and the temperature, T, in units of a scale, Λ_L.

Instead of expressing physical quantities in terms of the unkown scale Λ_L, we could compute ratios of physical quantities. This can be done because all physical quantities, $M(a,g)$, must approach a value independent of the artificial lattice spacing and g as $a \to 0$. In the continuum limit we must have then[33]

$$\frac{dM}{da} = (\frac{\partial}{\partial a} + \frac{\partial g}{\partial a}\frac{\partial}{\partial g})M(a,g) \to 0 \ . \tag{5.17}$$

If M has dimensions of mass, then we are able to write $M(a,g) = f(g)/a$. The solution of (5.17) is then

$$f(g) = f(g_0)\exp(\int_{g_0}^{g} dg\bar{\beta}^{-1}(g)) = \kappa_M \left[F^{-1}(g)\right]^{-1} \ , \tag{5.18}$$

where $F^{-1}(g)$ is the *same* function as in (5.15,5.16), and κ_M is a nonperturbative constant. Therefore, M scales with g exactly as a^{-1} does if we are close to the continuum limit. The range of coupling constants, g, for which such scaling holds is called the "scaling window". Calculations with g in that range gaurantee that ratios of masses $M/M' = \kappa_M/\kappa_{M'}$ do not depend on a, Λ_L, *or* g.

In practice, we only know the approximate form of $F^{-1}(g)$ in the asymptotic limit $g \to 0$. That is sufficient though since (5.17) only holds in that limit. In that asymptotic limit, the two loop approximation leading to (5.15) should be accurate enough. We therefore see that ratios of physical quantities reflect true continuum physics only if we calculate those quantities with a g in the asymptotic scaling window. For a finite fixed lattice, $N^3 \times N_\tau$, we can only hope that asymptotic

scaling occurs for a range of g such that $N_x a$ is not too small and T/Λ is in an interesting range of temperatures

For illustration, consider a lattice calculation for pure SU(3) Yang-Mills ($N_f = 0$) on a $8^3 \times 3$ lattice[53]. In this case $N_\tau = 3$, $\beta_1 \approx 0.07$, and $\beta_2 \approx 0.004$. A calculation with $6/g^2 = 5.5531$ corresponds to a lattice spacing, $a \approx 0.004\Lambda_L^{-1}$, and a temperature, $T \approx 86\Lambda_L$. Assuming that $\Lambda_L \sim 2$ MeV, we would be working at temperature $T \sim 172$ MeV and on a lattice of spatial size $N_x a \sim 3.2$ fm. If we were lucky enough to find that this g happened to fall into the asymptotic scaling window, then indeed we would be in an interesting region from the point of view of the expected deconfinement transition.

The numerical results of the Bielefeld group[53] for this $8^3 \times 3$ lattice are shown in Fig.5.1.

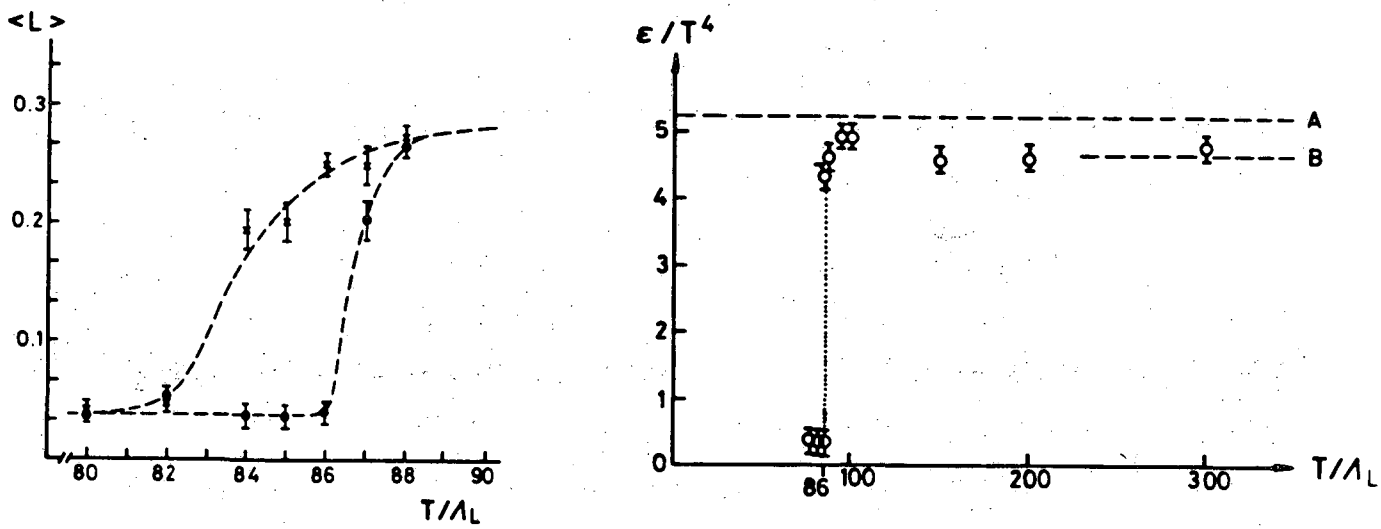


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Figure 5.1: Lattice gauge calculations[53] of the Wilson line and energy density on an $8^3 \times 3$ lattice for SU(3) without quarks as a function of temperature assuming asymptotic scaling.

In plotting the quantities as a function of T/Λ_L instead of g, it has been as-

sumed that asymptotic scaling holds for the corresponding values of g. While new results[55] show that asymptotic scaling does not hold for such small lattices with $g \gtrsim 1$, the above results are nevertheless qualitatively interesting.

The numerical results for the expectation value of the Wilson line show a stricking hysteresis curve. As the temperature is increased the free energy of an isolated static quark remains infinite, $\langle L \rangle \approx 0$, until a critical temperature $T_c \approx 86\Lambda_L$ is reached. Then very suddenly the free energy becomes finite indicating a transition into a phase where color is no longer confined. However, if we reverse the proceedure and cool the system, then instead of following the same path, the quark remains unconfined until a lower temperature, $T_c' \approx 82\Lambda_L$, is reached. Between $82 \lesssim T/\Lambda_L \lesssim 87$, there are two metastable states of the system: one that confines and the other that does not. This indicates that the deconfinement transition at least without dynamical quarks is a strong first order transition. Further evidence for this is seen in the energy density plot. At the critical temperature, there is a sharp discontinuity of $\epsilon(T)$. That discontinuity is presumably the latent heat per unit volume required to melt the nonperturbative vacuum that confines color. Above the critical temperature, ϵ/T^4, becomes constant with a value close to the Stefan-Boltzmann value A (2.17) appropriate for $N_c = 3$, $N_f = 0$. The value B includes an estimate for the finite size color neutrality correction. Calculations by other groups[54] on similar size lattices give similar results.

5.5 The Continuum Limit

Now nonsider in more detail how $T_c(a,g)/\Lambda_L$, scales with g. Fig.5.2 shows the scaling property of the critical temprature as reported in Ref.[55].

The solid curve shows how $aT_c(N_\tau, g) = 1/N_\tau$ must depend on g if the two loop asymptotic scaling holds. Note that while the results for small lattices, $2 \leq N_\tau \leq 4$ seem to obey scaling, larger lattice results, with $6 \leq N_\tau \leq 10$ do not. Therefore the miracle of precocious scaling at rather large couplings, $g > 1$, appears to be a fluke. Remember that in the continuum limit, we expect $g \to 0$. This calculation

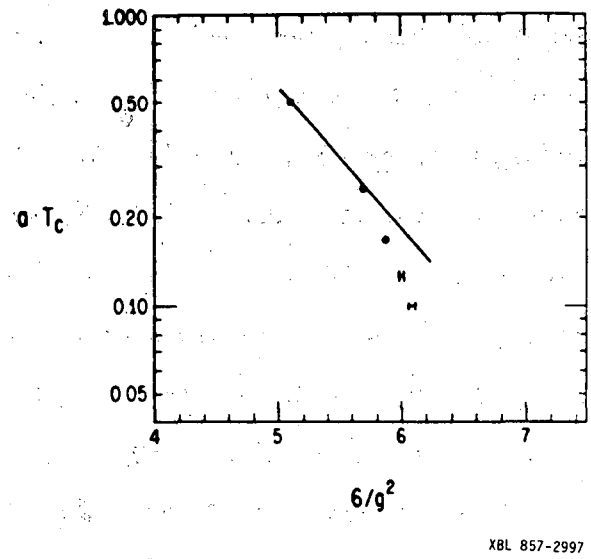

Figure 5.2: Scaling of the critical temperature[55].

therefore calls into question the relevance of numerical studies on small lattices to continuum physics. There seems to be no doubt about the existence of a first order transition in lattice QCD without dynamical quarks, but without scaling we have no way of translating the lattice spacing or T_c into physical units. The latest results[55] suggest that asymptotic two loop scaling may hold for $6/g^2 \gtrsim 7$. In that case, $a \lesssim 0.0008\Lambda_L^{-1}$ and $T \gtrsim 1300\Lambda_L/N_\tau$. This means that the same temperature and spatial volume as was assumed to be studied on the smaller $8^3 \times 3$ lattice now requires an enourmous $32^3 \times 15$ lattice!. Clearly even a small delay of the onset of asymptotic scaling hurts very much.

Until very large lattice become more feasible, there is a less rigorous way to proceed. Even though there is no guarantee that ratios of physical quantities should be independent of g outside the asymptotic scaling window, we may luck out anyway. Consider the ratio of the critical temperature to the square root of the string tension. Fig.5.3 summarizes the available 'data' on such a comparison[6].

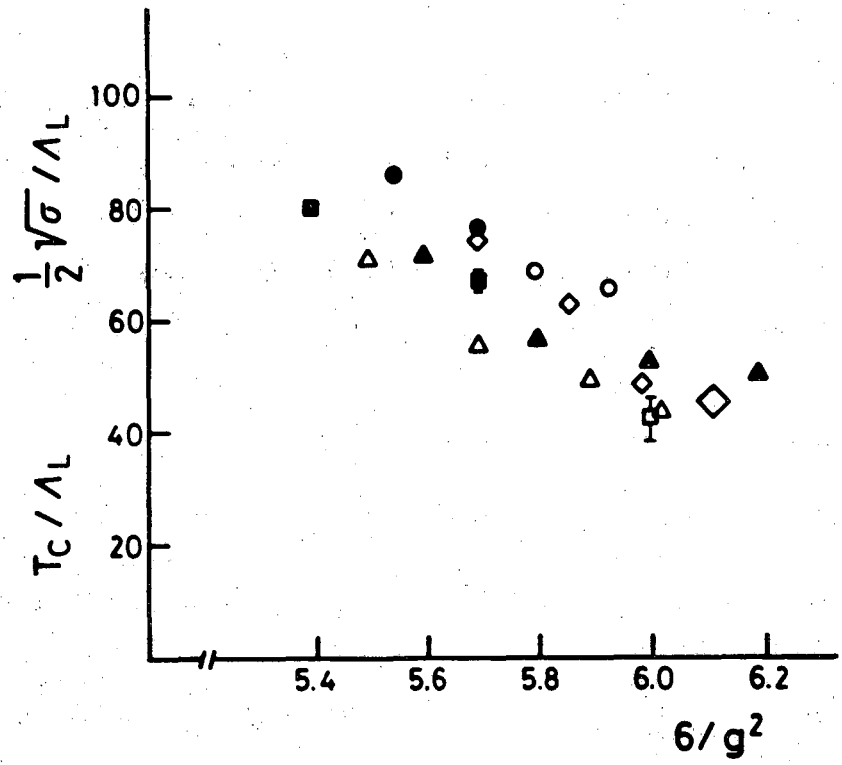

Figure 5.3: Comparison of $\sqrt{\sigma}$ and T_c in the nonscaling region.

The string tension is obtained by studying the correlation function $\langle L(0)L^\dagger(r)\rangle \propto e^{-\beta\sigma r}$. The square root of the zero temperature string tension, which empirically is $\sqrt{\sigma} \approx 400$ Mev, is compared to the critical temperature at which $\epsilon(T)$ makes a jump, for different values of the coupling. The large open diamond at $6/g^2 \approx 6.1$ corresponds to the recently published[55] value of T_c on a $11^3 \times 10$ lattice. These results suggest that the ratio $T_c/\sqrt{\sigma}$ may infact be roughly independent of g even in this nonscaling region with

$$T_c \sim \frac{1}{2}\sigma \ . \tag{5.19}$$

It is interesting to note that numerical studies[56] of SU($N_c \gg 3$) also give a similar estimate. It should be kept in mind is that none of these calculations include effects of dynamical quarks.

5.6 QCD with quarks

We come finally to the most challenging frontier of the lattice world: the inclusion of dynamical quarks. In the past few years, there has been very rapid progress in this area. However, the dust has not even begun to settle. The inclusion of dynamical quarks requires us to calculate the treacherous fermion determinant (3.64). The problem is that determinants of enormous dimensioned matricies are very difficult to brute force on a computer.

The Euclidean continuum quark action (see (3.41)) for finite T and μ is,

$$S_f(\bar{\psi},\psi) = \int_0^\beta d\tau \int d^3x \bar{\psi}(i\gamma_\nu^E \mathcal{D}^\nu - i\mu\gamma_0^E - m)\psi \ , \tag{5.20}$$

where these Euclidean $\gamma_0^E = i\gamma_0$, $\gamma_j^E = \gamma_j$. To dicretize this action while preseving gauge invariance to all orders in the lattice spacing requires the use of link variables again. An effective action that reduces to (3.28) and is gauge invariant is given by[57]

$$S_f = \sum_{sites\ n,m} \bar{\psi}_n (1-\kappa M)_{nm} \psi_m \ , \tag{5.21}$$

where M is the matrix connecting adjacent sites

$$\begin{aligned} M_{nm} &= \sum_{\nu=1}^{3} [(1-i\gamma_\nu^E)U_{nm}\delta_{n,m-\hat{\nu}} + (1+i\gamma_\nu^E)U_{nm}^\dagger \delta_{n,m+\hat{\nu}}] \\ &\quad + e^{\mu a}(1-i\gamma_0^E)U_{nm}\delta_{n,m-\hat{\tau}} + e^{-\mu a}(1+i\gamma_0^E)U_{nm}^\dagger \delta_{n,m+\hat{\tau}} \ , \end{aligned} \tag{5.22}$$

where ψ_n is the spinor Grassmann field at site n, $U_{nm} = U_{n,\hat{m}-\hat{n}}$ is the link matrix between sites n, m, and the parameter κ is related to the quark mass approximately as $\kappa \approx 1/8(1-ma/4)$. The way in which the chemical potential μ enters is not unique, and is chosen so as to cancel a quadratic divergence in the energy density at $T=0$ for finite μ. See additional discussion in Ref.[11].

Because the lattice action is still quadratic in the quark fields, the integral over them changes (5.4) into

$$\langle A \rangle = \frac{1}{Z} \int \prod_{\text{links}} [dU_{i,\mu}] A(U) \{det(1-\kappa M(U))\}^{N_f} e^{-S_W(U)} \ , \tag{5.23}$$

The effect of dynamical quarks is thus the introduction of a determinant of a matrix with dimensions $4 \times (N_x - 1)^3 \times (N_\tau - 1)$, which even for very small lattices is too big to handle numerically.

The first attempts to include some effects from dynamical quarks involved the so called hopping parameter expansion[53]:

$$det(1 - \kappa M) \approx \exp\left(-\sum_{l=1}^{N} Tr(\kappa M)^l / l\right) \quad . \tag{5.24}$$

In Ref.[57] the approximation of keeping only the lowest contributing order ($N = 3$) in the expansion and setting the phase of the complex determinant equal to zero led to the numerical results shown in Fig.5.4 for the critical temperature as a function of chemical potential. This is the first attempt to estimate the phase diagram of quark matter at finite baryon densities nonperturbatively.

The scale $\Lambda_L^{(2)}$ is the lattice parameter relevant for two quark flavors. Asymptotic scaling has been assumed on this $8^3 \times 3$ lattice. Of course, it is at present impossible to estimate the uncertainty in these numbers due to the many uncontrolled approximations. Nevertheless, from the phenomenological point of view these results reinforce our prejudice that there should be a deconfinement transition at high baryon densities in addition to high temperatures. Obviously, these calculations will have to be greatly improved before we can be sure though.

Another technique that has been applied to calculate the fermion determinant is called the pseudofermion method[58]. That is a Monte-Carlo method based on

$$(det(1 - \kappa M))^{-1} \propto \int \mathcal{D}(\phi, \phi^*) \exp(-\phi^*(1 - \kappa M)\phi) \quad . \tag{5.25}$$

Thus one has to perform a Monte-Carlo calculation within a Monte-Carlo calculation in terms of an artificial complex scalar field, ϕ. The primary disadvantage of this method is the slow convergence of the Monte-Carlo method requires very long runs in order that accurate results are obtained. Furthermore, this technique is limited to zero chemical potentials so that the fermion determinant is real.

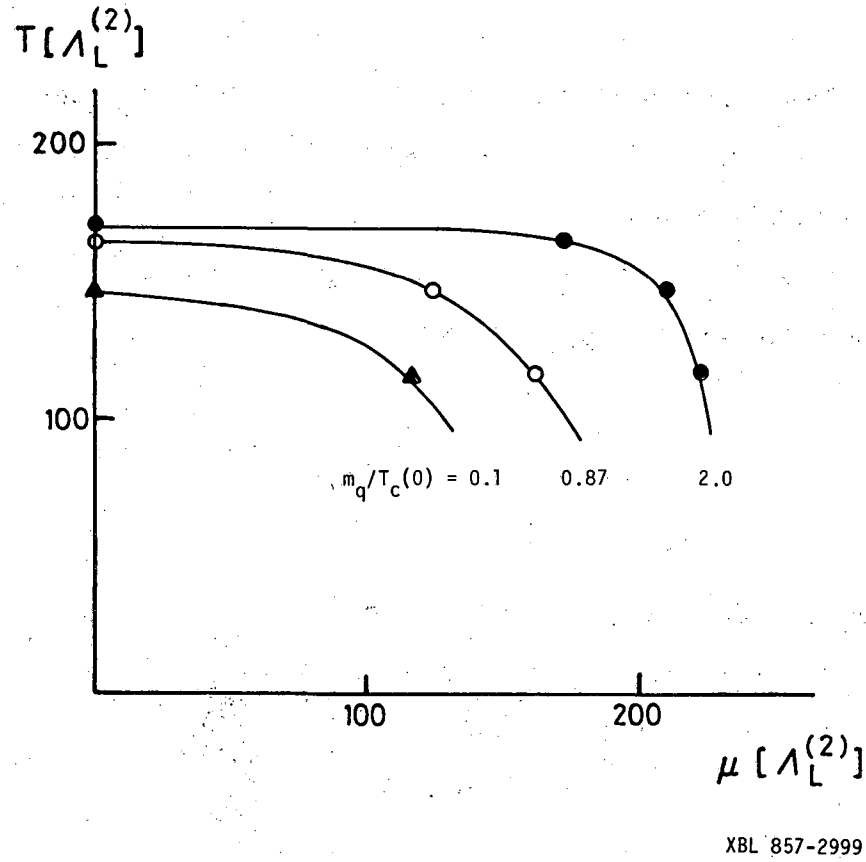

Figure 5.4: Very preliminary phase diagram from QCD lattice calculations[57] marking the transition form hadronic matter to a quark-gluon plasma.

Finally, we mention a novel methods that looks promising at the present. That is the so called microcanonical method[59]. The basic idea is to calculate expectation values of quantities using a microcanonical rather than canonical ensemble. For illustration consider the thermodynamics of a system with an action $S(Q,\dot{Q})$. The canonical expectation value of an operator A is of the form

$$\langle A\rangle_C = \frac{1}{Z_C}\int \mathcal{D}Q\; A(Q)e^{-S(Q,\dot{Q})} \quad . \tag{5.26}$$

The factor e^{-S} plays the role of the Boltzmann factor. We could also consider a microcanonical ensemble average via

$$\langle A\rangle_M = \frac{1}{Z_M}\int \mathcal{D}Q\mathcal{D}P\; A(Q)\delta(E - H(Q,P)) \quad , \tag{5.27}$$

where H is the Hamiltonian and E is some fixed energy. Statistical mechanics assures us that in the infinite volume limit, these two averages should be the same if $E = \mathcal{N}_f T/2 + \langle S \rangle$, with $\mathcal{N}_f$ corresponding to the total number of degrees of freedom.

In (5.27) every configuration (Q, P) with energy E is given equal *a priori* probability in the ensemble. The final step is to invoke the ergodic hypothesis, namely that the microcanonical ensemble is equivalent to a long time average

$$\lim_{V\to\infty} \langle A \rangle_C = \langle A \rangle_M = \lim_{t\to\infty} \frac{1}{t} \int_0^t dt' A(Q(t')) \ , \tag{5.28}$$

where the 'time' evolution of $Q(t), P(t)$ obeys the Hamilton equations $dQ/dt = \partial H/\partial P$ and $dP/dt = -\partial H/\partial Q$. The time here is only some ficticious parameter that allows us to sample all accessible configurations by following these classical equations of motion. We have to assume that the dynamics of our system is chaotic so that we are not locked up in some periodic type orbit.

Armed with these heavy duty theorems in statistical mechanics, the strategy in applications to QCD is to invent an artificial classical system that evolves in a fifth 'time' dimension in a way that will generate expectation values identical to the canonical method. Obviously, there is room for a great deal of technical gymnastics here in the choice of the Hamiltonian for that artificial system. In Ref.[59] one choice was adopted, but the technical gymnastics are too strenuous to record here. The classical equations obtained were solved numerically to obtain a path in configuration space that is hoped to samples a large enough area of the available phase space. Time averages (5.28) are then computed. Fig.5.5 summarizes their results.

The calculation was done on a $8^3 \times 4$ lattice for four quark flavors. Asymptotic scaling was assumed. We see that the Wilson line still exhibits a sharp discontinuity at $6/g^2 \approx 5.1$ corresponding to $T/\Lambda_L^{(4)} \approx 280$ that is also reflected in the energy density. Furthermore, the quantity $\langle \bar{\psi}\psi \rangle$, that is the order parameter for chiral symmetry decreases rapidly to zero at the same point. These results suggest

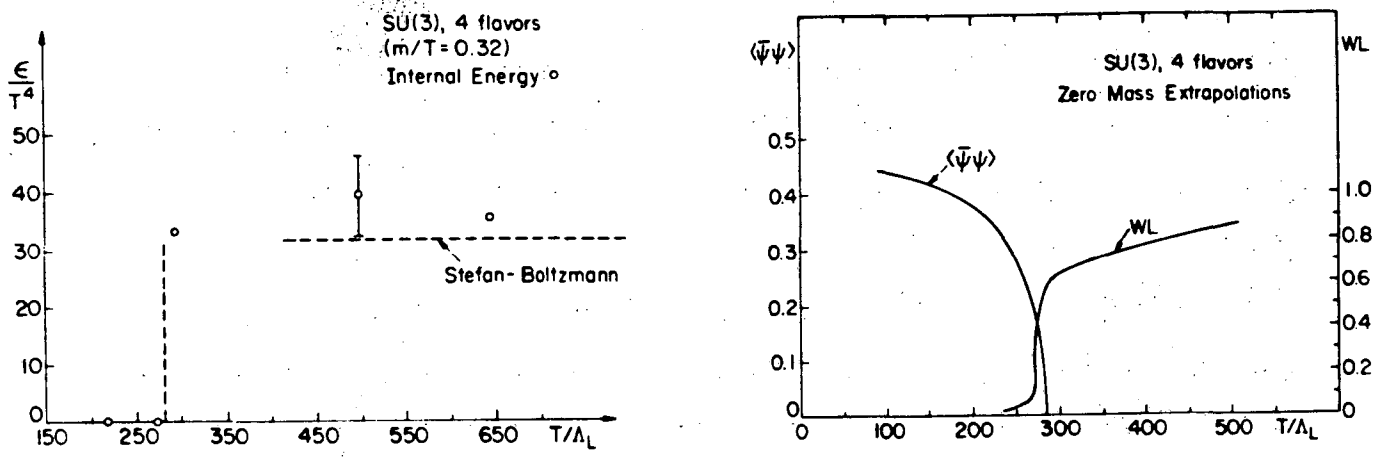

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Figure 5.5: Energy density, Wilson line, and $\langle\bar{\psi}\psi\rangle$ as a function of temperature at zero chemical potential for SU(3) including dynamical quarks via the microcanonical method[59].

then that the first order character of the phase transition may not be drastically altered by the inclusion of dynamical quarks. Also it appears that the deconfinement transition is closely associated with chiral symmetry restoration. We must however await calcaulations on much bigger lattices to confirm these finding.

We conclude that current numerical results are consistent with our prejudices concerning the existence of a qualitative change of the thermodynamic properties of QCD matter at temperatures and chemical potentials $\sim$ few hundered MeV. With the very rapid technical progress today, we can look forward to increasingly reliable 'data' in the near future.

6. Final remarks

These lectures could only provide a brief introduction into the many novel theoretical problems and techniques in QCD thermodynamics. Nevertheless, I hope to have covered enough material to enable nonexperts to follow this rapidly evolving area in the literature. We have gone into some detail in showing the limitations of perturbative methods. We have seen that the color magnetic sector of QCD

could spoil our simple picture of the quark-gluon plasma. We have seen how lattice theory can in principle provide insight into the interesting nonperturbative effects. Nevertheless, we also saw that the connection between the lattice world and the continuum real world is tenuous at present because of limited computing power. While there are as yet no firm final answers, the tentative results thusfar look very promising. Clearly the most exciting quasi "prediction" is the occurrence of a deconfinement transtion at energy densities roughly one order of magnitude above the ground state energy of nuclear matter. The details of that transition are not yet settled, but it appears that thermodynamic quantities may change rather suddenly as the large number of quark-gluon degrees of freedom become liberated.

From the phenomenological point of view, these tentative results are very exciting because current estimates[4][5][6] suggest that such energy densities could be easily achieved with collisions of heavy nuclei at relatively "low" ($\gtrsim$ 10 AGeV) and also at much higher ($\gtrsim$ 1 ATeV) energies. At the lower energies, heavy nuclei are expected to stop each other[60], and therefore high energy densities are expected to be accompanied by high baryon densities. At the high energies, low baryon density but high energy density matter is expected to be produced in the midrapidity regions. Therefore, it may be possible to investigate experimentally the deconfinement transition in Fig.5.4 over a wide temperature and density domain with nuclear collisions. Thus the study of QCD thermodynamics need not be limited to gedanken or digital experimentation.

Of course, there remains the formidable challenge of relating experimental observables such as inclusive cross sections to thermodynamic quantities such as the equation of state. In the BEVALAC area we saw in section 2 that after a decade of experimental and theoretical work on dynamical reaction theories, tentative estimates for the nuclear equation of state are finally emerging. In the search for the quark-gluon plasma much more sophisticated dynamical theories await development. Those theories must be able to describe nonequilibrium effects and the hadronization process in addition to hydrodynamic phenomena that may result if local equilibrium is achieved. The lattice world on the computer is at best rele-

vant to the nuclear collision world in the lab only for that phase of the collision after local equilibrium is achieved and before the system disintegrates. In order to assess whether local equilibrium can be attained at all, transport properties[39] of the QGP have to be known also. These quantities can only be obtained rigorously only by studying real time correlation functions in the QGP- a task out of reach at present. Clearly, there will be no shortage of fascinating problems in this frontier area of physics for quite some time.

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