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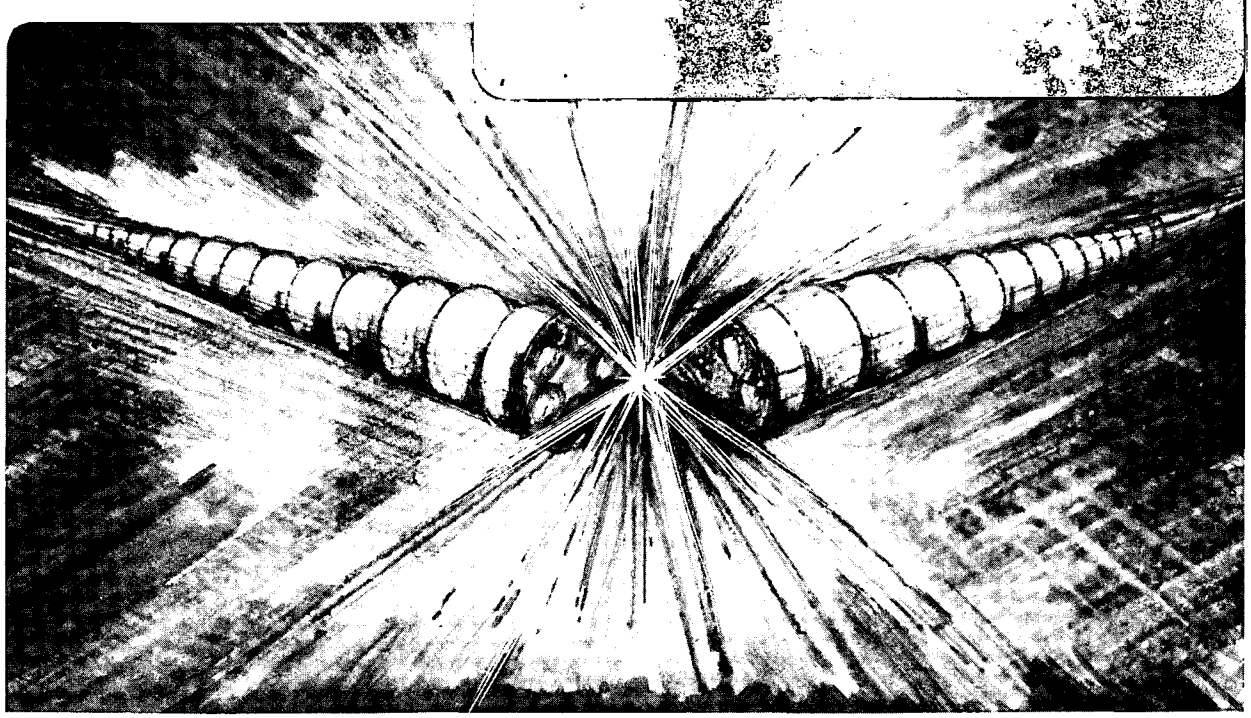

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Theory of Beat-Resonant Coupling of Electrostatic Modes

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Abstract

We derive a general expression for the beat-resonant coupling of electrostatic modes in a Vlasov plasma. The result for the coupling of two modes has a simple structure: the appropriate momentum gradient of the equilibrium particle distribution is weighted by a positive coupling coefficient and averaged over the resonance surface in momentum space. The contributions of all the resonance surfaces are then summed. This basic structure had been previously exhibited only for specific homogeneous plasma models. The present theory, which unifies and greatly simplifies these individual treatments, is based on a variational formulation of the Vlasov-Poisson equations. Using Lie transforms, we re-express the variational principle in oscillation center variables, and then obtain the nonlinear wave dynamics from the independent variations of the wave phase and the wave amplitude. The power of the method is then applied to a strongly magnetized, strongly inhomogeneous, nonneutral plasma model.

I. Introduction

We present a derivation of the beat-resonant coupling between two electrostatic waves in a nonuniform Vlasov plasma. This process, which is also known as nonlinear Landau damping or induced scattering, has been studied in less general settings by various authors.[1-10] Our derivation involves two ingredients. We use a variational formulation of the Vlasov-Poisson model[11-14] which describes the particle dynamics via a phase space Lagrangian.[15-17] The second ingredient is the use of Lie transforms[18-22] to construct a canonical transformation to oscillation center variables. This change of variables is facilitated by the form invariance of the particle Lagrangian under canonical transformations.

Theoretical work on beat-resonant coupling dates to the earliest studies of quasi-linear theory and the development of weak turbulence theory. Most previous studies have treated infinite homogeneous plasma models, and accordingly the modes were described as plane waves.[1-5,7,8,10] Discussions of inhomogeneous equilibria have assumed sufficiently weak spatial dependence to justify an eikonal form for the waves.[6,9] (An exception is the unpublished work[23] of Johnston which discusses nonuniform plasma quite generally.)

Our derivation is considerably more general, allowing us to treat waves in plasmas of arbitrary inhomogeneity, including, in particular, equilibria of bounded extent. Accordingly, we do not assume that the modes involved are of plane-wave form; rather, their spatial structure is left unconstrained. In particular, we are able to treat both the global eigenmodes

of the system (satisfying realistic boundary conditions) and complicated disturbances driven into the plasma by external sources.

We do make several key assumptions which limit the generality of the results. The waves are assumed to have sufficiently small amplitude that nonlinear effects may be treated perturbatively and that the time scale of the linear frequencies is short compared to the nonlinear time scale. We neglect the linear damping of the waves in the derivation of the beat-resonant coupling. These approximations are familiar from previous theories of nonlinear wave interactions.[10] Beyond these assumptions, the particle motion in the unperturbed plasma (no waves) is assumed to be integrable. This requires three invariants; the specification of these invariants will depend on the characteristics of the equilibrium. If the unperturbed particle trajectories are non-integrable, then even the study of the linear wave dynamics presents major difficulties.[31,32]

The oscillation center description has been previously used to study various nonlinear plasma problems.[7,18,24-28] Our work stands in closest relation to Johnston's oscillation center theory of induced scattering[7] in homogeneous plasma and its extension to the inhomogeneous case.[23] The salient distinction between the present derivation and Johnston's work is the use of a variational principle to obtain the dynamical equations for the wave action. Johnston directly calculated only the energy change of the beat-resonant particles. The equations for the wave amplitudes were then inferred by positing appropriate Manley-Rowe relations. We are able to derive these relations as consequences of the wave dynamics.

The variational principle we employ is similar in spirit to Low's variational principle,[11] but differs by utilizing a phase-space Lagrangian to describe the particle motion.[16,17] This phase space formulation provides a Hamiltonian description of the particle dynamics and allows one to exploit Lie transform methods without sacrificing the self-consistency required between the fields and particles. Recent appreciation of this advantage has motivated applications of the variational principle to gyrokinetic formalism,[14] covariant pondermotive theory,[29] and relativistic guiding-center plasma theory.[30]

We now discuss the organization of the paper. The next section formulates the variational principle (1). In section III, we choose a form for the electrostatic potential which tailors the variational principle to the situation of weakly interacting waves and explicitly introduces the wave amplitudes and phases as fields to be varied. In section IV, we define the oscillation center transformation which eliminates the primary wave oscillation in the particle motion. The variational principle is then re-expressed in oscillation center variables (30). In section V, we carry out the variation in the oscillation center representation. This yields the slow-time scale, nonlinear wave dynamics (54), and we specifically focus on the beat-resonant coupling (53). In section VI, the general formula (53) is evaluated for a simple model of a magnetically confined, pure electron plasma.[33] Complications introduced by bounded equilibria are discussed. Section VII summarizes.

II. The Variational Principle

Our theory is based on the variational principle $\delta S = 0$ defined by the action functional:

$$S(\underset{\sim}{I}, \underset{\sim}{\theta}, \Phi) = \int_{t_1}^{t_2} dt \left\{ \int d^6 z_o \, f_o(z_o) \left[\underset{\sim}{I}(z_o, t) \cdot \dot{\underset{\sim}{\theta}}(z_o, t) - H(\underset{\sim}{I}, \underset{\sim}{\theta}, \Phi) \right] + \int d^3 x \, |\nabla \Phi|^2 / 8\pi \right\} . \quad (1)$$

The notation in (1) is as follows: z_o denotes a point in the six dimensional single particle phase space. An initial or reference distribution of particles is described by $f_o(z_o)$. The phase space Lagrangian for the particles, $\underset{\sim}{I} \cdot \dot{\underset{\sim}{\theta}} - H$, is expressed in terms of the action-angle variables $(\underset{\sim}{I}, \underset{\sim}{\theta})$, their derivatives $\dot{\underset{\sim}{\theta}} = d\underset{\sim}{\theta}/dt$, and the single particle Hamiltonian $H(\underset{\sim}{I}, \underset{\sim}{\theta}, \Phi)$. The action-angle variables in (1) are regarded as functions of (z_o, t). The field Φ is a function of $(\underset{\sim}{x}, t)$ where $\underset{\sim}{x}$ denotes a point in 3-dimensional (physical) space and t is the time coordinate; we assume Φ satisfies some prescribed boundary conditions.

The choice of action-angle variables for the single particle Lagrangian is motivated by the foresight that we will be assuming a plasma equilibrium with integrable single particle dynamics. However, for the purposes of formulating the variational principle, action-angle variables are not required. Any other canonical set of variables could be used without changing the form of S, or more generally, the particle Lagrangian could be expressed in terms of noncanonical variables.[34]

The Hamiltonian is the sum of kinetic and potential terms:

$$H(\underset{\sim}{I}, \underset{\sim}{\theta}, \Phi) = H_{kin}(\underset{\sim}{I}, \underset{\sim}{\theta}) + \int d^3x \; \rho(\underset{\sim}{x}; \underset{\sim}{I}, \underset{\sim}{\theta}) \, \Phi(\underset{\sim}{x}, t) \tag{2}$$

Here $\rho(\underset{\sim}{x}; \underset{\sim}{I}, \underset{\sim}{\theta})$ is the single particle charge density, $\rho = q\,\delta^3(\underset{\sim}{x} - \underset{\sim}{R}(\underset{\sim}{I}, \underset{\sim}{\theta}))$, where $R(\underset{\sim}{I}, \underset{\sim}{\theta})$ is the spatial position of a particle with phase space coordinates $(\underset{\sim}{I}, \underset{\sim}{\theta})$. For multiple species, the particle action in (1) must be summed over species.

The independent variation of $\underset{\sim}{I}(z_o, t)$ and $\underset{\sim}{\theta}(z_o, t)$, subject to the constraint $\delta\underset{\sim}{\theta} = 0$ at $t = t_1$ and $t = t_2$, yields Hamilton's equations for a particle,

$$\begin{aligned} \dot{\underset{\sim}{\theta}} &= \partial H/\partial \underset{\sim}{I} \\ \underset{\sim}{I} &= -\partial H/\partial \underset{\sim}{\theta} \, . \end{aligned} \tag{3}$$

In the variation of S with respect to Φ, we assume that $\delta\Phi = 0$ on the boundaries and obtain Poisson's equation

$$\nabla^2 \Phi = -4\pi \int d^3\theta \; d^3I \; \rho(\underset{\sim}{x}; \underset{\sim}{I}, \underset{\sim}{\theta}) \, f(\underset{\sim}{I}, \underset{\sim}{\theta}, t) \; . \tag{4}$$

The Vlasov distribution f in (4) is defined in terms of $f_o(z_o)$ and the solution to (3) by

$$f(\underset{\sim}{I}', \underset{\sim}{\theta}', t) \equiv \int d^6 z_o \; f_o(z_o) \; \delta^3(\underset{\sim}{\theta}' - \underset{\sim}{\theta}(z_o, t)) \; \delta^3(\underset{\sim}{I}' - \underset{\sim}{I}(z_o, t)) \; ; \tag{5}$$

this implies the Vlasov equation

$$\partial f/\partial t + \{f, H\} = 0 \; , \tag{6}$$

in terms of the canonical Poisson bracket $\{f, H\}$.

III. Weakly Nonlinear Waves

We now consider a plasma characterized by a slowly varying equilibrium supporting small amplitude, weakly damped, weakly interacting waves. In applying the variational principle, we assume a form for $\Phi(\underset{\sim}{x}, t)$ appropriate for such a plasma; the potential is the sum of an unperturbed potential $\phi_e(\underset{\sim}{x})$ ("e" = equilibrium), the potential of the primary modes $\phi_m(\underset{\sim}{x}, t)$ ("m" = modes), and the potential of the nonlinear beats $\phi_b(\underset{\sim}{x}, t)$ ("b" = beats):

$$\Phi(\underset{\sim}{x}, t) = \phi_e(\underset{\sim}{x}) + \phi_m(\underset{\sim}{x}, t) + \phi_b(\underset{\sim}{x}, t) \quad . \tag{7}$$

The unperturbed potential $\phi_e(\underset{\sim}{x})$ is independent of the modes (zero order in the mode amplitudes). In reality there are, of course, nonlinear, quasi-static corrections to $\phi_e(\underset{\sim}{x})$, but these are not required for the beat-resonant coupling and we neglect them.

The primary modes are represented by

$$\phi_m(\underset{\sim}{x}, t) = \sum_n A_n(t)\, e^{-i\alpha_n(t)}\, \psi_n(\underset{\sim}{x}) + \text{c.c.} \quad , \tag{8}$$

where $A_n(t)$ is a slowly varying amplitude, $e^{-i\alpha_n(t)}$ is a rapidly varying phase, and $\psi_n(\underset{\sim}{x})$ the spatial function of the $n^{\underline{th}}$ mode. Both A_n and α_n are real-valued functions of time; ψ_n may be complex-valued. We assume here for simplicity that the waves are normal modes of the plasma; in Appendix A we discuss how to treat external sources.

When ψ_n is a normal mode, its frequency

$$\dot{\alpha}_n \equiv \omega_n = \omega_n^{(0)} + \Delta\omega_n(t) \quad , \tag{9}$$

has a constant term $\omega_n^{(0)}$ from linear theory, with nonlinear corrections $\Delta\omega_n(t)$ which vary on the slow time scale.

The beats between primary waves, which depend on the phases, are represented by $\phi_b(\underset{\sim}{x}, t)$. It suffices to consider second order beats of the form:

$$\phi_b(\underset{\sim}{x}, t) = \sum_{nm} A_n A_m \left[e^{-i(\alpha_n + \alpha_m)} \psi_{nm}(\underset{\sim}{x}) + e^{-i(\alpha_n - \alpha_m)} \psi_{n\bar{m}}(\underset{\sim}{x}) + \text{c.c.} \right] . \tag{10}$$

Here $\psi_{nm} = \psi_{mn}$, $\psi_{n\bar{m}} = \psi^*_{m\bar{n}}$, and we let $\psi_{n\bar{n}} \equiv 0$ since this term represents a slowly varying correction (i.e. no phase) to the unperturbed potential ϕ_e.

With this form for Φ, the Hamiltonian (2) is the sum of a zero order piece,

$$h_e = H_{kin} + \int d^3x \; \rho \, \phi_e(x) , \tag{11}$$

a first order perturbing term,

$$h_m = \int d^3x \; \rho \, \phi_m , \tag{12}$$

and a second order perturbing term,

$$h_b = \int d^3x \; \rho \; \phi_b \quad . \tag{13}$$

Inserting (7) into (1), we can effect the variation with respect to $\Phi(x, t)$ by independently varying each of the functions ϕ_e, A_n, α_n, ψ_n, ψ_{nm}, $\psi_{n\bar{m}}$. Since the amplitudes A_n are assumed small, stationarity ($\delta S = 0$) must be satisfied at each order in A_n.

By transforming (1) to oscillation-center variables, the variation $\delta S/\delta\alpha_n = 0$ yields the desired beat-resonant coupling. We discuss this in the following sections. The evaluation of $\delta S = 0$ <u>before</u> transforming to oscillation-center variables is presented in Appendix B; we describe two of those results here to indicate how the dielectric response enters.

To state these results concisely requires some additional notation. Let $\hat{\chi}(\omega)$ denote the susceptibility operator defined by

$$\hat{\chi}(\omega)\,\Lambda(\underset{\sim}{x}) = -(2\pi)^3 \int d^3x' \left[\int d^3I \sum_{\underset{\sim}{j}} (\partial_{\underset{\sim}{j}} f_e)(\omega_{\underset{\sim}{j}} - \omega - i0)^{-1}\, \rho_{\underset{\sim}{j}}(\underset{\sim}{x}';\underset{\sim}{I})^* \rho_{\underset{\sim}{j}}(\underset{\sim}{x}';\underset{\sim}{I}) \right] \Lambda(\underset{\sim}{x}') \tag{14}$$

where $f_e(\underset{\sim}{I})$ is the unperturbed distribution function, $\partial_{\underset{\sim}{j}}$ denotes the differential operator $\underset{\sim}{j} \cdot \frac{\partial}{\partial \underset{\sim}{I}}$, $\omega_{\underset{\sim}{j}} \equiv \partial_{\underset{\sim}{j}} h_e$, and $\rho_{\underset{\sim}{j}}(\underset{\sim}{x};\underset{\sim}{I})$ is the Fourier coefficient of $\rho(\underset{\sim}{x};\underset{\sim}{I},\underset{\sim}{\theta}) = \sum_{\underset{\sim}{j}} e^{i\underset{\sim}{j}\cdot\underset{\sim}{\theta}} \rho_{\underset{\sim}{j}}(\underset{\sim}{x};\underset{\sim}{I})$. We may write $\hat{\chi}(\omega) = \hat{\chi}'(\omega) + i\hat{\chi}''(\omega)$ using the Plemelj formula[35] to separate $\hat{\chi}$ into Hermitian ($\hat{\chi}'$) and anti-Hermitian ($i\hat{\chi}''$) terms.

Now combining (7) and (1), the variation $\delta S/\delta\psi_n(\underset{\sim}{x})^* = 0$ yields (B9), the linear mode equation:

$$\nabla^2\psi_n(\underset{\sim}{x}) - 4\pi\,\hat{\chi}'(\omega_n^{(0)})\psi_n(\underset{\sim}{x}) = 0 \tag{15}$$

Since the damping of the primary modes is neglected $\hat{\chi}'(\omega_n^{(0)})$, rather than $\hat{\chi}(\omega_n^{(0)})$, appears in (15).

The variation $\delta S/\delta\psi_{n\bar{m}}(\underset{\sim}{x})^* = 0$ yields a similar result (B7) for the shielded beat potential

$$\nabla^2 \psi_{n\bar{m}}(x) - 4\pi\, \hat{\chi}(\omega_{n\bar{m}})\psi_{n\bar{m}} = -i(2\pi)^4 \int d^3I \sum_{\underset{\sim}{j}} \rho_{\underset{\sim}{j}}(\underset{\sim}{x}, \underset{\sim}{I})^* C_{n\bar{m},\underset{\sim}{j}}(\underset{\sim}{I})(\omega_{\underset{\sim}{j}} - \omega_{n\bar{m}} - i0)^{-1} \tag{16}$$

where $\omega_{n\bar{m}} \equiv \omega_n^{(0)} - \omega_m^{(0)}$ and $C_{n\bar{m},j}$ is the Fourier coefficient of $C_{n\bar{m}}$ (B6). From Johnston's earlier discussion[7] we know that physically the right-hand side of (16) represents the sum of the "polarization charge density" and the charge density due to the beat pseudo-potential which acts on the oscillation centers.

Introducing the Green function[36] associated with (16),

$$[\nabla^2 - 4\pi\, \hat{\chi}(\omega)]\, G(\underset{\sim}{x}, \underset{\sim}{x}', \omega) = \delta(\underset{\sim}{x} - \underset{\sim}{x}') \tag{17}$$

we have

$$\psi_{n\bar{m}}(x) = -i(2\pi)^4 \int d^3x'\, G(\underset{\sim}{x}, \underset{\sim}{x}'; \omega_{n\bar{m}}) \int d^3I \sum_{\underset{\sim}{j}} \rho_{\underset{\sim}{j}}(\underset{\sim}{x}', \underset{\sim}{I})^* C_{n\bar{m},\underset{\sim}{j}}(\underset{\sim}{I})(\omega_{\underset{\sim}{j}} - \omega_{n\bar{m}} - i0)^{-1} \tag{18}$$

The shielding of $\psi_{n\bar{m}}$ is seen in the eigenfunction representation of G,

$$G(\underset{\sim}{x}, \underset{\sim}{x}', \omega) = -\sum_{\mu} \frac{\Lambda_\mu(\underset{\sim}{x}, \omega)\, \tilde{\Lambda}_\mu(\underset{\sim}{x}', \omega)^*}{\epsilon_\mu(\omega)} \tag{19}$$

where $\{\Lambda_\mu\}$ are the eigenfunctions (assumed complete),[37]

$$(\nabla^2 - 4\pi\, \hat{\chi}(\omega))\, \Lambda_\mu(\underset{\sim}{x}, \omega) = -\epsilon_\mu(\omega)\, \Lambda_\mu(\underset{\sim}{x}, \omega) \tag{20}$$

and $\tilde{\Lambda}_\mu$ are the adjoint eigenfunctions

$$(\nabla^2 - 4\pi\, \hat{\chi}(\omega)^+)\, \tilde{\Lambda}_\mu(\underset{\sim}{x}, \omega) = -\epsilon_\mu(\omega)^*\, \tilde{\Lambda}_\mu(\underset{\sim}{x}, \omega)\,. \tag{21}$$

The adjoint susceptibility is defined by $\langle \hat{\chi}(\omega)^{\dagger} \phi_1, \phi_2 \rangle = \langle \phi_1, \hat{\chi}(\omega) \phi_2 \rangle$ where $\langle \phi_1, \phi_2 \rangle = \int d^3x \, \phi_1^* \phi_2$. We assume $\int d^3x \, |\Lambda_\mu|^2 = 1$ and an adjoint normalization such that $\langle \tilde{\Lambda}_\mu, \Lambda_\mu \rangle = 1$

Physically $\epsilon_\mu(\omega)$ may be interpreted as the dielectric response at frequency ω to the $\mu^{\underline{th}}$ component of an electrostatic perturbation. This interpretation is motivated by noting that (15) implies $\epsilon_n'(\omega_n^{(0)}) = 0$, where $\epsilon_n(\omega) = \epsilon_n'(\omega) + i\epsilon_n''(\omega)$ is given by

$$\epsilon_n(\omega) = \int d^3x \, |\nabla \psi_n|^2 + 4\pi \chi_n(\omega) \quad . \tag{22}$$

$$\chi_n(\omega) = \langle \psi_n, \hat{\chi}(\omega) \psi_n \rangle \quad . \tag{23}$$

Thus undamped normal modes correspond to eigenfunctions with eigenvalue zero, and the normal mode frequencies are given by the zeros of the real part of $\epsilon_n(\omega)$.

IV. Oscillation Center Variables

For a particle oscillating in the field of an electrostatic wave, the oscillation center description[7, 18, 24-28] is analogous to the guiding center description of a gyrating particle. We shall use a Lie transform to construct the oscillation center representation. The efficiency of Lie methods for constructing canonical transformations is well known, the technique having been described by various authors.[18-22] We only outline the construction here.

The oscillation center transformation is a canonical, time-dependent map $\Lambda_t(\underset{\sim}{\theta}, \underset{\sim}{I}) = (\underset{\sim}{\theta}', \underset{\sim}{I}')$ to oscillation center variables $(\underset{\sim}{\theta}', \underset{\sim}{I}')$. Λ_t is constructed to ensure that the dynamics, re-expressed in $(\underset{\sim}{\theta}', \underset{\sim}{I}')$ variables, is

free of the rapid, first order oscillations due to the waves in ϕ_m. Because Λ_t is canonical, we may regard it as generated by the flow of an auxiliary Hamiltonian vector field with Hamiltonian $W(\underset{\sim}{\theta}, \underset{\sim}{I}, t)$.

In practice, the calculations associated with the transformation may be expressed directly in terms of $W(\underset{\sim}{\theta}, \underset{\sim}{I}, t)$, derived as a perturbation series in the wave amplitudes: $W = W^{(0)} + W^{(1)} + W^{(2)} + \cdots$. We requires only the first order term; the zeroth term is set to zero to assure that the new variables reduce to the original particle variables in the absence of the waves. Thus $W = W^{(1)}$; we henceforth omit the superscript.

The generator W is determined from[20]

$$\frac{\partial W}{\partial t} + \{W, h_e\} = -h_m \quad . \tag{24}$$

By assuming that h_e is integrable (i.e. independent of $\underset{\sim}{\theta}$), and neglecting the time dependence of $A_n(t)$ and $\dot{\alpha}_n(t)$, this is easily solved. Let

$$W = \sum_n A_n e^{-i\alpha_n} W_n(\underset{\sim}{\theta}, \underset{\sim}{I}) + \text{cc.}, \text{ and } W_n = \sum_{\underset{\sim}{j}} e^{i\underset{\sim}{j}\cdot\underset{\sim}{\theta}} W_{n,\underset{\sim}{j}}(\underset{\sim}{I})$$

then

$$W_{n,\underset{\sim}{j}} = i(\omega_{\underset{\sim}{j}}(I) - \omega_n)^{-1} \int d^3x \; \rho_{\underset{\sim}{j}}(\underset{\sim}{x};\underset{\sim}{I})\, \psi_n(\underset{\sim}{x}) \, . \tag{25}$$

Since we are neglecting the linear damping of the primary waves, only the principal value contribution $(\omega_{\underset{\sim}{j}} - \omega_n)^{-1}$ of the primary resonance is kept. The problem of incorporating primary resonant particles into the oscillation center representation has been discussed exhaustively by Dewar.[18, 24, 38]

Given W, standard perturbation theory[20] provides an expression for the new Hamiltonian $K(\underset{\sim}{\theta}, \underset{\sim}{I}, t)$. The terms in K needed to describe the beat-resonance interaction are:

$$K(\underset{\sim}{\theta}, \underset{\sim}{I}, t) = h_e(\underset{\sim}{I}) + \int d^3x \; \phi_b(\underset{\sim}{x}, t)\, \hat{\rho}(\underset{\sim}{x}; \underset{\sim}{\theta}, \underset{\sim}{I}, t) + \frac{1}{2}\{W, h_m\} \tag{26}$$

where

$$\hat{\rho}(\underset{\sim}{x}; \underset{\sim}{\theta}, \underset{\sim}{I}, t) \equiv \rho(\underset{\sim}{x}; \underset{\sim}{\theta}, \underset{\sim}{I}) + \{W, \rho\} + \frac{1}{2}\{W, \{W, \rho\}\} + \cdots \tag{27}$$

is charge density kernel for oscillation centers. The expression for $\hat{\rho}$ is a "multipole" expansion of the physical charge density about the oscillation center. The first term in (27) is the charge density if the particle is located at its oscillation center. The next term $\{W, \rho\}$ is a first order correction due to the displacement of the particle from its oscillation center; $\{W, \{W, \rho\}\}$ is a second order correction and so on.

Finally the Vlasov distribution function transforms as

$$f(\underset{\sim}{I}, \underset{\sim}{\theta}, t) = F(\Lambda_t(\underset{\sim}{I}, \underset{\sim}{\theta}), t) = F(\underset{\sim}{I}, \underset{\sim}{\theta}, t) - \{W, F\} + \frac{1}{2}\{W, \{W, F\}\} + \cdots \tag{28}$$

where $F(\underset{\sim}{I}, \underset{\sim}{\theta}, t)$ is the oscillation center distribution function. Note that at $t = 0$, $f(\underset{\sim}{I}, \underset{\sim}{\theta}, t = 0) = f_o(\underset{\sim}{I}, \underset{\sim}{\theta}) = F_o(\Lambda_{t=0}(\underset{\sim}{I}, \underset{\sim}{\theta}))$ where $F_o(\underset{\sim}{I}, \underset{\sim}{\theta}) = F(\underset{\sim}{I}, \underset{\sim}{\theta}, t = 0)$.

To rewrite the variational principle in the oscillation center description requires only that the particle Lagrangian be transformed. The Lagrangian fields for particles $(\underset{\sim}{I}, \underset{\sim}{\theta})$ and oscillation centers $(\underset{\sim}{I}', \underset{\sim}{\theta}')$ are related by

$$\Lambda_t(\underset{\sim}{\theta}(z_o,t), \underset{\sim}{I}(z_o,t)) = (\underset{\sim}{\theta}'(z_o',t), \underset{\sim}{I}'(z_o',t)) \tag{29}$$

where $z_o' = \Lambda_{t=0}(z_o)$. In words, (29) equates the new variables $\underset{\sim}{\theta}'$, $\underset{\sim}{I}'$ evaluated at the new initial condition z_o' to the particle variables $\underset{\sim}{\theta}$, $\underset{\sim}{I}$ evaluated at z_o and then mapped by Λ_t, the oscillation center transformation. The <u>form</u> of the phase space Lagrangian is invariant under a canonical transformation,[15,17] and the Jacobian of the transformation is unity. Thus inserting (29) into (1) yields an oscillation center variational principle S' satisfying $S'(\underset{\sim}{I}', \underset{\sim}{\theta}', \Phi) = S(\underset{\sim}{I}, \underset{\sim}{\theta}, \Phi)$ where

$$S'(\underset{\sim}{I}', \underset{\sim}{\theta}', \Phi) = \int_{t_1}^{t_2} dt \left\{ \int d^6 z_o F_o(z_o) \left[\underset{\sim}{I}'(z_o,t) \cdot \dot{\underset{\sim}{\theta}}'(z_o,t) - K(\underset{\sim}{I}', \underset{\sim}{\theta}', \Phi) \right] + \frac{1}{8\pi} \int d^3x |\nabla\Phi|^2 \right\} \tag{30}$$

From this point, we shall drop the primes on oscillation center quantities.

V. The Oscillation Center Variational Principle

The qualitatively new feature of the transformed variational principle is its explicit dependence on $\omega_n = \dot{\alpha}_n$ through the resonant denominator in $W_n(\underset{\sim}{\theta}, \underset{\sim}{I})$, which enters $S(\underset{\sim}{I}, \underset{\sim}{\theta}, \Phi)$ through the Hamiltonian K. Our central concern therefore is the variation $\delta S/\delta\alpha_n = 0$ which will yield the amplitude dynamics. We shall also require the oscillation center dynamics and the Vlasov equation for F.

As before, $\delta S/\delta\underset{\sim}{\theta} = 0$ and $\delta S/\delta\underset{\sim}{I} = 0$ give Hamilton's equations, but now for the oscillation centers,

$$\dot{\underset{\sim}{I}} = -\partial K/\partial \underset{\sim}{\theta}$$
$$\dot{\underset{\sim}{\theta}} = \partial K/\partial \underset{\sim}{I} \tag{31}$$

In terms of the solution to (31), $F(\underset{\sim}{\theta}, \underset{\sim}{I}, t)$ is related to $F_o(\underset{\sim}{\theta}, \underset{\sim}{I})$ by

$$F(\underset{\sim}{\theta}', \underset{\sim}{I}', t) = \int d^2 z_o \, F_o(z_o) \, \delta^3(\underset{\sim}{\theta}' - \underset{\sim}{\theta}(z_o, t)) \, \delta^3(\underset{\sim}{I}' - \underset{\sim}{I}(z_o, t)) \tag{32}$$

which is equivalent to

$$\frac{\partial F}{\partial t} + \{F, K\} = 0 , \quad F(\underset{\sim}{\theta}, \underset{\sim}{I}, t=0) = F_o(\underset{\sim}{\theta}, \underset{\sim}{I}) . \tag{33}$$

It can be shown that the variations of S with respect to ϕ_o, ψ_n^*, and $\psi_{nm}^{(\pm)*}$ reproduce the results of Appendix B. The variation with respect to A_n is described at the end of this section.

We now perform the phase variation $\delta S/\delta \alpha_n$. Let $\mathcal{L}$ denote the Lagrangian of (30)

$$\mathcal{L} = \int d^6 z_o \, F_o(z_o) [\underset{\sim}{I} \cdot \dot{\underset{\sim}{\theta}} - K] + \int d^3 x \, |\nabla \Phi|^2 / 8\pi , \tag{34}$$

then $\partial \mathcal{L}/\partial \dot{\alpha}_n \equiv J_n$ defines a canonical momentum, conjugate to the phase, which we shall refer to as the <u>action</u> of mode n. The phase variation $\delta S/\delta \alpha_n = 0$ leads to the Euler-Lagrange equation for J_n,

$$\dot{J}_n = \partial \mathcal{L}/\partial \alpha_n . \tag{35}$$

To focus on the slow time scale, we implicitly assume that (35) has been time averaged to remove rapidly oscillating phases.[39] This could be formally achieved by a coarse graining in time.

In evaluating $J_n = \partial \mathcal{L}/\partial \omega_n$ for the left hand side of (35), we simply note that ω_n appears only in K,

$$J_n = -\int d^3I_o\, d^3\theta_o\, F_o(\underset{\sim}{I}_o, \underset{\sim}{\theta}_o)\, \partial K(\underset{\sim}{I}, \underset{\sim}{\theta}, t)/\partial \omega_n$$

$$= -\int d^3I\, d^3\theta\, F(\underset{\sim}{I}, \underset{\sim}{\theta}, t)\, \partial K/\partial \omega_n$$

$$= -\frac{1}{2}\int d^3I\, d^3\theta\, F_e(\underset{\sim}{I})\, \frac{\partial}{\partial \omega_n}\, \{W, h_m\} + \text{h.o.t.} \tag{36}$$

Here F_e is the zeroth order piece of F, given by $F_e(\underset{\sim}{I}) = f_e(\underset{\sim}{I})$ from (28). In (36) we require only the slowly varying terms of $\{W, h_m\}$ to obtain the lowest order contribution to J_n (time averaged). From our definitions of W (25) and h_m (12),

$$\frac{1}{2}\{W, h_m\} = \sum_{n,m} A_n A_m \left[e^{-i(\alpha_n + \alpha_m)} K_{nm} + e^{-i(\alpha_n - \alpha_m)} K_{n\bar{m}} + \text{c.c.}\right] \tag{37}$$

with coefficients,

$$K_{nm} \equiv \frac{1}{2}\left\{W_n, \int d^3x\, \rho\psi_m\right\} \tag{38}$$

$$K_{n\bar{m}} = \frac{1}{2}\left\{W_n, \int d^3x\, \rho\psi_m^*\right\} . \tag{39}$$

Thus (36) becomes,

$$J_n = -A_n^2 \int d^3\theta\, d^3I\, f_e(\underset{\sim}{I})\, \partial K_n/\partial \omega_n \tag{40}$$

where

$$K_n = (K_{n\bar{n}} + K_{n\bar{n}}^*) . \tag{41}$$

The <u>K-χ theorem</u>[25-29] which relates $K_n(\underset{\sim}{\theta}, \underset{\sim}{I})$ to the real linear susceptibility χ_n' (23) provides a simple physical interpretation for (40). In our notation, the theorem states

$$\chi_n'(\omega_n^{(0)}) = - \int d^3\theta \; d^3I \; f_e(\underset{\sim}{I}) \, K_n(\omega_n^{(0)}) \tag{42}$$

With (42) and (22), our result for J_n becomes

$$J_n = \frac{A_n^2}{4\pi} \frac{\partial \epsilon_n'}{\partial \omega_n} (\omega_n^{(0)}) + \text{h.o.t} \quad . \tag{43}$$

Thus the magnitude of the lowest order contribution to J_n is the "plasmon density."[3,10] (To check this interpretation one can easily show that the second order mode energy is $\omega_n^{(0)} J_n$ with J_n given by (43).[40])

We now evaluate $\partial \mathcal{L}/\partial \alpha_n$, the right side of (35), keeping those terms which contribute to beat-resonant coupling. The electrostatic term, $d^3x \, |\nabla \Phi|^2$, contributes only rapidly varying terms to $\partial \mathcal{L}/\partial \alpha_n$ and may be neglected. This leaves

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \alpha_n} &= - \int d^3I \; d^3\theta \; F(\underset{\sim}{I}, \underset{\sim}{\theta}, t) \, \partial K/\partial \alpha_n \\ &= - \int d^3I \; d^3\theta \; F(\underset{\sim}{I}, \underset{\sim}{\theta}, t) \, \frac{\partial}{\partial \alpha_n} \left[\frac{1}{2} \{W, h_m\} + \int d^3x \; \hat{\rho} \, \phi_b \right] + \text{h.o.t.} \end{aligned} \tag{44}$$

For the beat-resonant coupling, the relevant terms in K carry the beat phase $e^{-i(\alpha_n \pm \alpha_m)}$. At second order, these terms may be written:

$$\frac{1}{2}\{W,h_m\} + \int d^3x\, \rho\, \phi_b = \sum_{nm}{}' A_n A_m \left[e^{-i(\alpha_n - \alpha_m)} T_{n\bar{m}} + e^{-(\alpha_n + \alpha_m)} T_{nm} + c.c. \right] \quad (45)$$

where

$$T_{nm} = K_{nm} + \int d^3x\, \rho\, \psi_{nm}(\underset{\sim}{x}) \quad (46)$$

$$T_{n\bar{m}} = K_{n\bar{m}} + \int d^3x\, \rho\, \psi_{n\bar{m}}(\underset{\sim}{x}) \quad (47)$$

and the summation Σ'_{nm} is primed to indicate that when $n=m$ the terms with phase $e^{\pm i(\alpha_n - \alpha_m)}$ are neglected.

To obtain a slowly varying contribution to (44) we combine (45) with the appropriate second order, driven response in F given by

$$\frac{\partial F}{\partial t} + \{F,h_e\} = - \left\{ F_e \,,\, \left[\frac{1}{2}\{W,h_m\} + \int d^3x\, \rho\, \phi_b \right] \right\} \quad . \quad (48)$$

We solve (48) using the approximation that A_n and ω_n are time-independent, and select the causal response by assuming that ω_n has a small imaginary part which damps the wave as $t \to -\infty$. For the Fourier coefficients of F, $F = \sum_{\underset{\sim}{j}} e^{i\underset{\sim}{j}\cdot\underset{\sim}{\theta}} F_{\underset{\sim}{j}}(\underset{\sim}{I}, t)$, this gives

$$F_{\underset{\sim}{j}} = (\partial_{\underset{\sim}{j}} f_e) \sum_{nm}{}' A_n A_m \left[\frac{T_{nm,\underset{\sim}{j}}\, e^{-i(\alpha_n + \alpha_m)}}{\omega_{\underset{\sim}{j}} - (\omega_{nm} + i0)} + \frac{T_{n\bar{m},\underset{\sim}{j}}\, e^{-i(\alpha_n - \alpha_m)}}{\omega_{\underset{\sim}{j}} - (\omega_{n\bar{m}} + i0)} \right.$$

$$\left. + \frac{(T_{nm,-\underset{\sim}{j}})^*\, e^{i(\alpha_n + \alpha_m)}}{\omega_{\underset{\sim}{j}} + (\omega_{nm} - i0)} + \frac{(T_{n\bar{m},-\underset{\sim}{j}})^*\, e^{i(\alpha_n - \alpha_m)}}{\omega_{\underset{\sim}{j}} + (\omega_{n\bar{m}} - i0)} \right] \quad (49)$$

where $T_{nm,\underset{\sim}{j}}$ is the $\underset{\sim}{j}^{th}$ Fourier coefficient of T_{nm}, $\omega_{nm} = \omega_n^{(0)} + \omega_m^{(0)}$, and $\omega_{n\bar{m}} = \omega_n^{(0)} - \omega_m^{(0)}$.

With (45) and (49), we find from (44)

$$\frac{\partial \mathcal{L}}{\partial \alpha_n} = -iA_n^2 \sum_m{}' A_m^2 (2\pi)^3 \int d^3I \sum_{\underset{\sim}{j}} (\partial_{\underset{\sim}{j}} f_e) \left[|T_{mn,\underset{\sim}{j}} + T_{nm,\underset{\sim}{j}}|^2 \left(\frac{1}{\omega_{\underset{\sim}{j}} - (\omega_{nm} + i0)} - \frac{1}{\omega_{\underset{\sim}{j}} - (\omega_{nm} - i0)} \right) \right.$$

$$\left. + |T_{n\bar{m},\underset{\sim}{j}} + (T_{m\bar{n},-\underset{\sim}{j}})^*|^2 \left(\frac{1}{\omega_{\underset{\sim}{j}} - (\omega_{n\bar{m}} + i0)} - \frac{1}{\omega_{\underset{\sim}{j}} - (\omega_{n\bar{m}} - i0)} \right) \right]$$

$$= A_n^2 \sum_m{}' A_m^2 (\Gamma_{nm} + \Gamma_{n\bar{m}}) \tag{50}$$

where the resonance denominators have been evaluated by the Plemelj formula,[35] and we have used the identities

$$\delta(\omega_{\underset{\sim}{j}} - \omega_{n\bar{m}})(T_{n\bar{m},\underset{\sim}{j}})^* = \delta(\omega_{\underset{\sim}{j}} - \omega_{n\bar{m}}) T_{m\bar{n},-\underset{\sim}{j}} \tag{51}$$

$$\delta(\omega_{\underset{\sim}{j}} - \omega_{nm}) T_{nm,\underset{\sim}{j}} = \delta(\omega_{\underset{\sim}{j}} - \omega_{nm}) T_{mn,\underset{\sim}{j}} \tag{52}$$

to obtain the couplings,

$$\Gamma_{nm} = (2\pi)^4 \sum_{\underset{\sim}{j}} \int d^3I (\partial_{\underset{\sim}{j}} f_e)\ \delta(\omega_{\underset{\sim}{j}} - \omega_{nm})\ |2T_{nm,\underset{\sim}{j}}|^2 \tag{53a}$$

$$\Gamma_{n\bar{m}} = (2\pi)^4 \sum_{\underset{\sim}{j}} \int d^3I (\partial_{\underset{\sim}{j}} f_e)\ \delta(\omega_{\underset{\sim}{j}} - \omega_{n\bar{m}})\ |2T_{n\bar{m},\underset{\sim}{j}}|^2 . \tag{53b}$$

Combining (35) and (50) gives the <u>action-transfer equation</u>:

$$\dot{J}_n = A_n^2 \sum_m{}' A_m^2 (\Gamma_{nm} + \Gamma_{n\bar{m}}) \tag{54}$$

which describes the amplitude dynamics.

The form of the coupling is very simple. Consider (53b): for each beat-resonance $\omega_{\underset{\sim}{j}} - (\omega_n^{(0)} - \omega_m^{(0)}) = 0$, we integrate the action gradient of the equilibrium $\partial_{\underset{\sim}{j}} f_e$ over the resonance surface with a weight $|2T_{n\bar{m},\underset{\sim}{j}}|^2$ which reflects the strength of the $\underset{\sim}{j}^{\underline{th}}$ Fourier component of the beat disturbance produced by primary modes n and m. Then all such resonances are summed.

There are two distinct contributions to the weighting factor $|2T_{n\bar{m},\underset{\sim}{j}}|^2$. The unshielded contribution arises from $\{W, h_m\}$ in the oscillation center Hamiltonian, while the other contribution arises from the shielded, self-consistent beat potential (18) $\psi_{n\bar{m}}$. We shall evaluate these contributions for an example in the following section.

Our derivation shows that the modulus-squared form of the weight arises quite generally. Early studies[4,5,8] of beat-resonant interaction which did not use the oscillation center representation, were able to demonstrate this form for only the simplest models and even then with very much greater effort.

There are conservation laws associated with (54). From the relations (51), (52), $\omega_{-\underset{\sim}{j}} = -\omega_{\underset{\sim}{j}}$, $\omega_{nm} = \omega_{mn}$, and $\omega_{n\bar{m}} = -\omega_{m\bar{n}}$, we find the symmetries

$$\Gamma_{nm} = \Gamma_{mn} \quad \text{and} \quad \Gamma_{n\bar{m}} = -\Gamma_{m\bar{n}} \ . \tag{55}$$

This implies a simple result for the action transfer of two modes interacting via these couplings:

$$\frac{d}{dt}(J_n \pm J_m) = \begin{cases} \Gamma_{nm} A_n^2 A_m^2 \\ \Gamma_{n\bar{m}} A_n^2 A_m^2 \end{cases} \tag{56}$$

Thus, for example, $J_n + J_m$ is conserved by the coupling $\Gamma_{n\bar{m}}$. That (56) should also apply to many modes in the random phase approximation is clear from the preceding derivation.

By combining (43) and (56), we can recover the usual Manley-Rowe relations.[3,7] Of course, whether the process described by Γ_{nm} (or $\Gamma_{n\bar{m}}$) is interpreted physically as induced scattering or wave emission/absorption depends on whether $\left(\left|\frac{\partial \epsilon_n'}{\partial \omega_n}\right| A_n^2 + \left|\frac{\partial \epsilon_m'}{\partial \omega_m}\right| A_m^2\right)$ is conserved by the interaction.[3,7]

Before turning to the example, we briefly discuss the variation $\delta S/\delta A_n = 0$, which determines $\Delta\omega_n$ in (9). From (30) we have

$$\delta S/\delta A_n = -\int d^6 z_o F_o(z_o)\, \partial K/\partial A_n + \frac{1}{8\pi}\int d^3x\, \partial|\nabla\Phi|^2/\partial A_n \tag{57}$$

The electrostatic contribution is (after time averaging),

$$\int d^3x\, \frac{\partial}{\partial A_n}|\nabla\Phi|^2 = 4A_n \int d^3x\, |\nabla\psi_n|^2 + \text{h.o.t.} \tag{58}$$

From (26), (37), and (45) the oscillation center contribution may be written,

$$\int d^6 z_o F_o(z_o)\, \partial K/\partial A_n = 2A_n \int d^3I\, d^3\theta\, f_e(\underset{\sim}{I}) K_n$$
$$+ \sum_m{}' A_m \sum_{\underset{\sim}{j}} (2\pi)^3 \int d^3I (F_{\underset{\sim}{j}}(I,t))^* \Big[e^{-i(\alpha_n - \alpha_m)} (T_{n\bar{m},\underset{\sim}{j}} + (T_{m\bar{n},-\underset{\sim}{j}})^*)$$
$$+ e^{-i(\alpha_n + \alpha_m)} (T_{nm,\underset{\sim}{j}} + T_{mn,\underset{\sim}{j}}) + \text{c.c.}\Big] \tag{59}$$

where, as before, the prime on the summation $\sum_m'$ indicates that for $m = n$ the term $(T_{n\bar{n},\underset{\sim}{j}} + (T_{n\bar{n},-\underset{\sim}{j}})^*)$ is omitted. Inserting $(F_{\underset{\sim}{j}})^*$ from (49), and time averaging we find that only the principal value contribution of the beat resonance contributes,

$$\int d^6 z_o F_o(z_o) \partial K/\partial A_n = 2A_n \int d^3 I\, d^3\theta\, f_e K_n + A_n \sum_m' A_m^2 (\Omega_{nm} + \Omega_{n\bar{m}}) \tag{60}$$

where

$$\Omega_{nm} \equiv 2(2\pi)^3 \sum_{\underset{\sim}{j}} \int d^3 I (\partial_{\underset{\sim}{j}} f_e)\, |T_{nm,\underset{\sim}{j}} + T_{mn,\underset{\sim}{j}}|^2 / (\omega_j - \omega_{nm}) \tag{61}$$

$$\Omega_{n\bar{m}} \equiv 2(2\pi)^3 \sum_{\underset{\sim}{j}} \int d^3 I (\partial_{\underset{\sim}{j}} f_e)\, |T_{n\bar{m},\underset{\sim}{j}} + (T_{m\bar{n},-\underset{\sim}{j}})^*|^2 / (\omega_{\underset{\sim}{j}} - \omega_{n\bar{m}}) \quad . \tag{62}$$

From (22) and the K-χ theorem (42) we note that

$$\int d^3 x\, |\nabla \psi_n|^2 - 4\pi \int d^3\theta\, d^3 I\, f_e K_n(\omega_n) = \epsilon_n'(\omega_n^{(0)}) + \frac{\partial \epsilon_n'}{\partial \omega_n} \Delta\omega_n + \text{h.o.t.}$$

$$= \frac{\partial \epsilon_n'}{\partial \omega_n} \Delta\omega_n \tag{63}$$

since $\epsilon_n'(\omega_n^{(0)}) = 0$. Thus adding (58) and (54) gives, from $\delta S/\delta A_n = 0$, the second order frequency shift due to the beat-resonant interactions:

$$\frac{\partial \epsilon_n'}{\partial \omega_n} \Delta\omega_n = 2\pi \sum_m' A_m^2 (\Omega_{nm} + \Omega_{n\bar{m}}) \quad . \tag{64}$$

Since our derivations are based on the variational principle (30), it is interesting to note that our results for the wave action-transfer (54) and non-linear frequency shift (64) are consistent with the canonical relations for

action-transfer and frequency shift described by Sturrock using a Hamiltonian description of wave dynamics.[41] In particular, (J_n, α_n) serve as action-angle variables for mode n and we may rewrite (54) and (64) in the form[42]

$$\dot{J}_n = -\frac{\partial H_I}{\partial \alpha_n} \tag{65}$$

$$\dot{\alpha}_n - \omega_n^{(0)} = \Delta\omega_n = \frac{\partial H_I}{\partial J_n} \tag{66}$$

if we assume the beat-resonance interaction Hamiltonian H_I is,

$$H_I = \int d^6 z_o F_o(z_o) \left[\frac{1}{2}\{W, h_m\} + \int d^3x\, \hat{\rho}\, \phi_b \right] \tag{67}$$

with $\frac{1}{2}\{W, h_m\} + \int d^3x\, \hat{\rho}\, \phi_b$ given by (45).

That (65) reproduces (54) is clear by inspection; to recover (64) from (66) we write $\partial H_I/\partial J_n = (\partial H_I/\partial A_n)(\partial A_n/\partial J_n)$ and use $\partial A_n/2J_n = 2\pi(A_n(\partial\epsilon_n'/\partial\omega_n)^{-1}$ from (43) to obtain

$$\frac{A_n}{2\pi}\frac{\partial \epsilon_n'}{\partial \omega_n}\Delta\omega_n = \partial H_I/\partial A_n \quad . \tag{68}$$

(64) follows immediately.

VI. Application to a Bounded, Single Species, Guiding Center Plasma

Recent experiments[43,44] have explored the confinement of a cylindrical electron plasma in a field geometry which is nominally axisymmetric. A strong, uniform axial magnetic field provides the radial confinement and the axial confinement is provided by electrostatic fields which are due to negatively biased end cylinders. Since the plasma is nonneutral, there is a radial electric field which produces an azimuthal $\underset{\sim}{E} \times \underset{\sim}{B}$ drift rotation of the plasma. For typical experimental parameters the cyclotron frequency, plasma frequency, and rotation frequency satisfy $\omega_c \gg \omega_p \gg \omega_r$. The plasma dynamics can be modeled by a drift kinetic description of the electron guiding centers.[33,45]

The radial confinement of such a plasma is intimately related to the cylindrical symmetry of the system. To understand this link, it is useful to introduce the canonical angular momentum (conjugate to the azimuthal angle Ψ) of the electron guiding centers,[17]

$$P_\Psi = -\frac{1}{2}\frac{eB}{c}\sum_i r_i^2 \tag{69}$$

where r_i is the radial position of the $i^{\underline{th}}$ electron guiding center and $-e$ is the electron charge. We have taken the vector potential $A_\Psi(r) = \frac{1}{2}Br$, corresponding to a uniform axial magnetic field. If the system is cylindrically symmetric, P_Ψ is conserved and $\sum_i r_i^2$ is fixed. The mean square radius of the guiding center plasma can increase only if the angular momentum decreases. An investigation of external sources of angular momentum is therefore of

utmost importance for an understanding of the radial transport and confinement.

It is believed that small asymmetries in the confining electric and magnetic fields cause a significant torque ($dP_\psi/dt \neq 0$) in the experiments[43] and mechanisms allowing such asymmetries to transfer angular momentum to the plasma have been proposed.[33,46] Among the nonlinear mechanisms which have been conjectured[33] is the coupling of a field asymmetry to a collective mode via the beat-resonant interaction. If this interaction were strong enough to destabilize the mode, then the angular momentum of the plasma could be significantly changed by the angular momentum of the growing mode.

A quantitative analysis of this particular process has been hindered by the complexity of the beat-resonant coupling when derived using non-Hamiltonian perturbation theory.[3,8] In this section, we evaluate our general expression (53) for the drift kinetic model used in previous studies.[33,45] The angular momentum transferred to the plasma when a mode is driven unstable by a field asymmetry has been recently studied[47] in detail using the general form of the coupling in (53).

In the model mentioned above, we take the plasma equilibrium to be a cylindrically symmetric, quiescent electron column of length L with periodic boundary conditions. There is a uniform magnetic field in the axial direction $\underset{\sim}{B} = B\hat{e}_z$ and a radial equilibrium electron field $\underset{\sim}{E}_e = (-\partial\varphi_e/\partial r)\hat{e}_r$. The electron guiding centers stream freely along the magnetic field and $\underset{\sim}{E} \times \underset{\sim}{B}$ drift azimuthally. There is a perfectly conducting wall at radius R_o. The

justification of this model is discussed in references 33 and 45.

Since cylindrical coordinates will be used for both the guiding center phase space and three dimensional physical space, we shall write guiding center variables. e.g. (R, Ψ, Z), in upper case and let (r, ψ, z) denote a point in physical space. The guiding center Hamiltonian is

$$H(\underset{\sim}{I}, \underset{\sim}{\Theta}, \Phi) = P_z^2/2m + \int d^3x \; \rho(\underset{\sim}{x}; \underset{\sim}{I}, \underset{\sim}{\Theta}) \; \Phi(\underset{\sim}{x}, t) \tag{70}$$

where $\underset{\sim}{\Theta} = (\Psi, Z)$ and $\underset{\sim}{I} = (P_\Psi, P_Z)$ are canonical coordinates. (We assume for simplicity that $L = 2\pi$ so Z is an angle.) The charge density $\rho(\underset{\sim}{x}; \underset{\sim}{I}, \underset{\sim}{\Theta}) = -e\,\delta(\psi - \Psi)\,\delta(z - Z)\,\delta(r - R)/r$ has the Fourier expansion

$$\rho(\underset{\sim}{x}; \underset{\sim}{I}, \underset{\sim}{\Theta}) = \sum_{\underset{\sim}{j}} e^{i\underset{\sim}{j} \cdot \underset{\sim}{\Theta}} \; \rho(\underset{\sim}{x}; \underset{\sim}{I}) \tag{71}$$

with

$$\rho_j(\underset{\sim}{x}; \underset{\sim}{I}) = -e\, e^{-i\underset{\sim}{j} \cdot \underset{\sim}{\theta}}\, \delta(r - R)/(2\pi)^2 r \quad . \tag{72}$$

In (72), $\underset{\sim}{\theta} = (\psi, z)$ are the azimuthal and axial coordinates (lower case) in physical space. Finally, let ℓ and k denote the integer components of $\underset{\sim}{j}$ so that $\underset{\sim}{j} \cdot \underset{\sim}{\theta} = \ell\psi + kz$.

We will consider only two modes and the beat-resonance associated with the difference frequency $\omega_1^{(0)} - \omega_2^{(0)}$. For the coupling $\Gamma_{1\bar{2}}$ in (53b) we require $T_{1\bar{2}}$ from (47): $T_{1\bar{2}} = K_{1\bar{2}} + \int d^3x \; \rho\psi_{1\bar{2}}$, where $K_{1\bar{2}} = \frac{1}{2}\left\{W_1, \int d^3x \; \rho\, \psi_2^*\right\}$.

For this model, the susceptibility operator (14) may be rewritten using (72) so that for an arbitrary function,

$$\Lambda(\underset{\sim}{x}) = \sum_{\underset{\sim}{j}} e^{i\underset{\sim}{j}\cdot\underset{\sim}{\theta}} \Lambda_{\underset{\sim}{j}}(r) \ ,$$

$$\hat{\chi}(\omega)\,\Lambda(\underset{\sim}{x}) = -\sum_{\underset{\sim}{j}} e^{i\underset{\sim}{j}\cdot\underset{\sim}{\theta}} \left[4\pi e^2 m\omega_c \int dP_z (\partial_{\underset{\sim}{j}} f_e)(\omega_{\underset{\sim}{j}} - \omega - i0)^{-1}\right] \Lambda_{\underset{\sim}{j}}(r) \ . \qquad (73)$$

Note that the resonance denominator is the familiar drift kinetic resonance, with

$$\omega_{\underset{\sim}{j}} = \underset{\sim}{j}\cdot\frac{\partial h_e}{\partial \underset{\sim}{I}} = \ell\,\frac{\partial h_e}{\partial P_\psi} + k\,\frac{\partial h_e}{\partial P_z} = \ell\omega_r + kv$$

where $v = \dot{z}$ is the parallel velocity; the momentum gradient is

$$\partial_{\underset{\sim}{j}} f_e = k\frac{\partial f_e}{\partial P_z} + \ell\,\frac{\partial f_e}{\partial P_\psi} = \frac{1}{m}\left(k\frac{\partial f_e}{\partial v} - \frac{\ell}{r\omega_c}\frac{\partial f_e}{\partial r}\right) \ .$$

Therefore the eigenfunctions introduced in (20) have the form $\Lambda_\mu = e^{+i\underset{\sim}{j}\cdot\underset{\sim}{\theta}}\eta_\mu(r;\omega)$ where $\mu = (\underset{\sim}{j}, p)$ and $p = 1, 2, 3, \ldots$ indexes the radial eigenfunction, η_μ, which satisfies

$$\left[\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{\ell^2}{r^2} - k^2 + 4\pi e^2 m\omega_c \int dP_z(\partial_{\underset{\sim}{j}} f_e)(\omega_{\underset{\sim}{j}} - \omega - i0)^{-1}\right]\eta_\mu(r,\omega) = -\epsilon_\mu(\omega)\,\eta_\mu(r,\omega) \qquad (74)$$

with boundary conditions $\eta_\mu\ (r = 0)$ finite and $\eta_\mu = 0$ at the wall $r = R_o$. Standard theorems assert that the set of solutions, to such a non-self-adjoint boundary value problem, is complete.[48]

The primary modes in (8) which satisfy (15) have the form $\psi_n(\underset{\sim}{x}) = e^{+i\underset{\sim}{j}_n \cdot \underset{\sim}{\theta}} \eta_n(r, \omega_n^{(0)})$ where $\eta_n(r, \omega_n^{(0)})$ satisfies[49] (74) with $(\omega_{\underset{\sim}{j}} - \omega - i0)^{-1}$ replaced by $(\omega_{\underset{\sim}{j}_n} - \omega_n^{(0)})^{-1}$ and $\epsilon_\mu = 0$. We assume here for simplicity that even if ψ_n represents an externally driven mode (see Appendix A) it still has a unique $\underset{\sim}{j}_n$ value. For such modes, $\int d^3x\, \rho\psi_n = -e\, e^{i\underset{\sim}{j}_n \cdot \underset{\sim}{\Theta}} \eta_n(R; \omega_n^{(0)})$ and the Lie generator (25) is

$$W_n = -ie(\omega_{\underset{\sim}{j}_n} - \omega_n)^{-1} e^{i\underset{\sim}{j}_n \cdot \underset{\sim}{\Theta}} \eta_n(R; \omega_n^{(0)}) \ . \tag{75}$$

Hence the unshielded term in $T_{1\bar{2}}$ is

$$K_{1\bar{2}} = -\frac{e^2}{2} e^{i(\underset{\sim}{j}_1 - \underset{\sim}{j}_2)\cdot \underset{\sim}{\Theta}} Q_{1\bar{2}} \tag{76}$$

where

$$Q_{1\bar{2}} = (\omega_{\underset{\sim}{j}_1} - \omega_1)^{-1} \eta_1 (\partial_{\underset{\sim}{j}_1} \eta_2) + \eta_2\, \partial_{\underset{\sim}{j}_2} [(\omega_{j_1} - \omega_1)^{-1} \eta_1] \tag{77}$$

The remaining contribution to $T_{1\bar{2}}$ is the shielded beat potential $\psi_{1\bar{2}}$ given by (18). From (B6) we obtain the charge density source $C_{1\bar{2}}$ for the beat potential,

$$C_{1\bar{2}} = ie^2 e^{i(\underset{\sim}{j}_1 - \underset{\sim}{j}_2)\cdot \underset{\sim}{\Theta}} (N_{1\bar{2}} - (N_{2\bar{1}})^*) \tag{78}$$

where

$$N_{1\bar{2}} = \eta_2(R)\, \partial_{\underset{\sim}{j}_2} [(\partial_{\underset{\sim}{j}_1} f_e)\, \eta_1(R)/(\omega_{\underset{\sim}{j}_1} - \omega_1^{(0)})]$$

$$+ (\partial_{\underset{\sim}{j}_1} \eta_2(R))(\partial_{\underset{\sim}{j}_1} f_e)\, \eta_1(R)/(\omega_{\underset{\sim}{j}_1} - \omega_1^{(0)}) \quad . \tag{79}$$

From (73) one easily sees that the adjoint eigenvalue problem (21) has solutions $\tilde{\Lambda}_\mu(\underset{\sim}{x}, \omega) = e^{i\underset{\sim}{j}\cdot\underset{\sim}{\theta}} \eta_\mu(r, \omega)^*$ with normalization $(2\pi)^2 \int_0^{R_o} \eta_\mu(r, \omega)^2 r dr = 1$. With the adjoint solutions, we construct the Green function (19) and then evaluate the beat potential from (18),

$$\psi_{1\bar{2}}(\underset{\sim}{x}) = (-2e/m^2) \sum_\mu \delta_{\underset{\sim}{j}, \underset{\sim}{j}_1 - \underset{\sim}{j}_2} A_{\mu, \mu_1, \mu_2} \Lambda_\mu(\underset{\sim}{x}, \omega_{1\bar{2}})/\epsilon_\mu(\omega_{1\bar{2}}) \tag{80}$$

where $\mu = (\underset{\sim}{j}, p)$, $\mu_i = (\underset{\sim}{j}_i, p_i)$, and

$$A_{\mu,\mu_1,\mu_2} \equiv \pi\omega_c m^3 \left\langle \tilde{\Lambda}_\mu, e^{i\underset{\sim}{j}\cdot\underset{\sim}{\theta}} \int dP_z (N^*_{2\bar{1}} - N_{21})(\omega_{\underset{\sim}{j}} - \omega_{1\bar{2}} - i0)^{-1} \right\rangle . \tag{81}$$

Finally, the shielded contribution to $T_{1\bar{2}}$ is ,

$$\int d^3x\, \rho\, \psi_{1\bar{2}} = (2e^4/m^2)\, e^{i(\underset{\sim}{j}_1 - \underset{\sim}{j}_2)\cdot\underset{\sim}{\Theta}} \sum_p A_{\mu, \mu_1, \mu_2}\, \eta_\mu(R; \omega_{1\bar{2}})/\epsilon_\mu(\omega_{1\bar{2}}) \tag{82}$$

where $\mu = (\underset{\sim}{j}_1 - \underset{\sim}{j}_2, p)$.

Thus the full coupling, obtained by summing (76) and (82), is

$$2T_{1\bar{2}} = e^2 \left[-Q_{1\bar{2}} + (4e^2/m^2) \sum_p A_{\mu, \mu_1, \mu_2}\, \eta_\mu(R)/\epsilon_\mu \right] e^{i(\underset{\sim}{j}_1 - \underset{\sim}{j}_2)\cdot\underset{\sim}{\Theta}} \quad . \tag{83}$$

There are several interesting features to (83). First, the motivation for the definition of A_{μ,μ_1,μ_2} in (81) is the resulting simple correspondence with the three-wave coupling for this model.[33] Taking into account notational differences,[50] one finds $A_{\mu,\mu_1,\mu_2} = M_{\underset{\sim}{j}p,\underset{\sim}{j}_1p_1,-\underset{\sim}{j}_2p_2}$ where $M_{\underset{\sim}{j}p,\underset{\sim}{j}_1p_1-\underset{\sim}{j}_2p_2}$ is the three-wave matrix element defined in reference 33. Thus $\psi_{1\bar{2}}$ is the result of a three-wave interaction in which the third wave $(\psi_{1\bar{2}})$ is not a normal mode, consequently the effective coupling is smaller due to shielding $\epsilon_\mu(\omega_{1\bar{2}})^{-1}$. This interpretation is familiar from simpler homogeneous plasma models.[4]

The radial summation $\sum_p$ in the shielded beat potential is not familiar from homogeneous plasma theory; it arises from the radial inhomogeneity of the equilibrium $f_e(P_\psi, P_z)$. The equilibrium is homogeneous in the azimuthal and axial directions which permits modes of plane wave form $e^{i\underset{\sim}{j}\cdot\underset{\sim}{\theta}}$ and implies the beat wave (80) will also be of plane wave form $e^{i(\underset{\sim}{j}_1-\underset{\sim}{j}_2)\cdot\underset{\sim}{\theta}}$ in the ψ and z coordinates. However, the radial dependence of f_e requires a more complicated radial dependence in the primary modes, and their beat excites <u>all</u> the radial modes $\eta_\mu(r, \omega_{1\bar{2}})$. This is reflected in the summation $\sum_p$ in (83).

VII. Conclusions

We have derived the beat-resonant coupling for small amplitude electrostatic waves in a Vlasov plasma. Our key assumption, besides weak nonlinearity, is the requirement of integrable particle dynamics in the unperturbed plasma.

The derivation reformulates and generalizes Johnston's oscillation center theory of induced scattering by utilizing the variational principle (30) for an oscillation center plasma. We obtain the action-transfer equation and nonlinear frequency shift in canonical form.

The coupling has a simple structure; in particular, we easily obtain the weighting factor $|T_{n\bar{m},\underset{\sim}{j}} + T^*_{m\bar{n},\underset{\sim}{j}}|^2$ in modulus-squared form. This form has previously been obtained only for homogeneous plasma models, and it allows us to readily derive the Manley-Rowe relations satisfied by the beat-resonant interaction.

The formalism has been applied to a drift kinetic model of an electron plasma. This is a specific example of an inhomogeneous Vlasov equilibrium for which previous non-Hamiltonian derivations of the beat-resonant coupling have required considerable labor and yielded results of great algebraic complexity.

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Appendix A

In conjunction with our ansatz (7) for $\Phi(\underset{\sim}{x}, t)$ we may adopt various boundary conditions. Let D denote the region containing the plasma and ∂D denote the region boundary.

The simplest possibility is a time-independent boundary condition $V_e(\underset{\sim}{x})(\underset{\sim}{x} \in \partial D)$ appropriate to the equilibrium $\phi_e(\underset{\sim}{x})$. For this choice,

$$\left.\begin{aligned} \phi_e(\underset{\sim}{x}) &= V_e(\underset{\sim}{x}) \\ \phi_m(\underset{\sim}{x}, t) &= 0 \\ \phi_b(\underset{\sim}{x}, t) &= 0 \end{aligned}\right\} \text{ for } \underset{\sim}{x} \in \partial D \tag{A1}$$

Here the primary waves $\psi_n(\underset{\sim}{x})$ vanish on ∂D and correspond to normal modes of the plasma.

A second type of boundary condition, relevant to recent experiments on electron plasmas,[34] arises when an external voltage is applied at the boundary to excite waves. When this applied signal is peaked at a single frequency ω_o we have a boundary condition of the form $V(\underset{\sim}{x}, t) = V_e(\underset{\sim}{x}) + (G(t)\, e^{-i\omega_o t}\, V_o(\underset{\sim}{x}) + \text{c.c.})$ which is satisfied by assuming, for $\underset{\sim}{x} \in \partial D$,

$$\phi_e(\underset{\sim}{x}) = V_e(\underset{\sim}{x})$$

$$\phi_m = \begin{cases} A_1(t)\, e^{-i\alpha_1(t)}\, \psi_1(\underset{\sim}{x}) = G(t)\, e^{-i\omega_o t}\, V_o(\underset{\sim}{x}) \\ \\ \underset{n \neq 1}{A_n\, e^{-i\alpha_n}\, \psi_n(\underset{\sim}{x})} = 0 \end{cases} \tag{A2}$$

$$\phi_b(\underset{\sim}{x}, t) = 0 \ .$$

Thus one of the waves in ϕ_m is driven by the external signal and all others are normal modes. Note that for the driven mode we have $A_1 = G$ and $\alpha_1 = \omega_o t$ so that these fields are completely determined by the boundary conditions and accordingly are not varied in calculating $\delta S = 0$.

Evidently more complicated external sources can be treated in a similar way.

Appendix B

In this Appendix, we discuss the variation $\delta S = 0$ using the form for Φ given by (7) but without transforming to oscillation center variables. This is instructive because the results obtained from the amplitude and phase variation depend on whether one uses particle variables or oscillation center variables. We also wish to provide a derivation of the equations for the primary mode potentials (15) and the beat mode potentials (16). For these purposes it is sufficient to evaluate the stationarity condition for the slow-time scale (i.e. neglecting terms with rapidly varying phases) and at lowest order in the wave amplitude.

We first consider the variation of S (1) with respect to $\psi^*_{n\bar{m}}(\underset{\sim}{x})$, and find from $\delta S/\delta\psi^*_{n\bar{m}} = 0$,

$$A_n^2 A_m^2 \nabla^2 \psi_{n\bar{m}}/2\pi + A_n A_m e^{i(\alpha_n - \alpha_m)} \int d^3I\, d^3\theta\, \rho\, f(\underset{\sim}{I}, \underset{\sim}{\theta}, t) = 0 \ , \qquad \text{(B1)}$$

In obtaining (B1) we took into account the identity $\psi_{n\bar{m}} = \psi^*_{m\bar{n}}$.

For the second term in (B1), we require the driven response in $f(\underset{\sim}{I}, \underset{\sim}{\theta}, t)$ proportional to $A_n A_m e^{-i(\alpha_n - \alpha_m)}$. This response is found from the second order Vlasov equation (6),

$$\partial f^{(2)}/\partial t + \{f^{(2)}, h_e\} = -\{f^{(1)}, h_m\} - \{f_e, h_b\} \qquad \text{(B2)}$$

where h_e, h_m and h_b are given by (11) - (13) and $f^{(1)}$ satisfies the first order equation,

$$\partial f^{(1)}/\partial t + \{f^{(1)}, h_e\} = -\{f_e, h_m\}. \qquad \text{(B3)}$$

We solve (B2) and (B3) neglecting the time dependence of the amplitudes A_n and the frequencies $\dot{\alpha}_n$. This gives for $f^{(1)}$,

$$f^{(1)} = -\{W, f_e\} \tag{B4}$$

in terms of W (25), and for the Fourier coefficients of $f^{(2)}$,

$$f^{(2)}_{\underset{\sim}{j}} = A_n A_m e^{-i(\alpha_n - \alpha_m)} \left[2(\partial_{\underset{\sim}{j}} f_e) \int d^3x\, \rho_{\underset{\sim}{j}} \psi_{n\bar{m}} + i C_{n\bar{m},\underset{\sim}{j}} \right] (\omega_{\underset{\sim}{j}} - (\omega_{n\bar{m}} + i0))^{-1}. \tag{B5}$$

In (B5) we retain only the term with the beat phase $e^{-i(\alpha_n - \alpha_m)}$. The second term in square brackets, $C_{n\bar{m},j}$, is the Fourier coefficient of $C_{n\bar{m}}$,

$$C_{n\bar{m}} \equiv \left\{ \{f_e, W_n\}, \int d^3x\, \rho \psi^*_m \right\} + \left\{ \{f_e, W^*_m\}, \int d^3x\, \rho \psi_n \right\}, \tag{B6}$$

which arises from the inhomogeneous term $\{f^{(1)}, h_m\}$ in (B2).

With f given by (B5), we may rewrite (B1) as

$$\nabla^2 \psi_{n\bar{m}} - 4\pi \hat{\chi}(\omega_{n\bar{m}}) \psi_{n\bar{m}} = -(2\pi)^4 i \int d^3I \sum_{\underset{\sim}{j}} \rho_{\underset{\sim}{j}}(\underset{\sim}{x}; \underset{\sim}{I})^* C_{n\bar{m},\underset{\sim}{j}} (\omega_{\underset{\sim}{j}} - (\omega_{n\bar{m}} + i0))^{-1} \tag{B7}$$

where $\hat{\chi}(\omega)$ is the susceptibility operator (14). This is the desired equation for the beat potential; the variation with respect to $\psi^*_{nm}(\underset{\sim}{x})$ leads to a similar result.

The primary mode potentials are determined by $\delta S/\delta\psi^*_n = 0$ in a similar calculation. We first obtain

$$A_n^2 \nabla^2 \psi_n / 4\pi + A_n e^{i\alpha_n} \int d^3I\, d^3\theta\, \rho f(\underset{\sim}{I}, \underset{\sim}{\theta}, t) = 0 , \tag{B8}$$

then substituting for f the first order response (B4) with phase $e^{-i\alpha_n}$ gives,

$$\nabla^2 \psi_n - 4\pi \hat{\chi}'(\omega_n^{(0)}) \psi_n = 0 . \tag{B9}$$

For normal modes we may also vary S with respect to A_n and α_n. The amplitude variation, $\delta S/\delta A_n = 0$, gives the dispersion relation $\epsilon_n'(\omega_n^{(0)}) = 0$ for the linear frequency.

For S expressed in particle variables, there is no explicit dependence on the frequencies $\dot{\alpha}_n$. The variation with respect to α_n gives the Euler-Lagrange equation $d/dt(\partial \mathcal{L}/\partial \dot{\alpha}_n) = \partial \mathcal{L}/\partial \alpha_n$ where $\mathcal{L}$ denotes the Lagrangian of (1). Unlike the oscillation center Lagrangian (34), we now have $\partial \mathcal{L}/\partial \dot{\alpha}_n \equiv 0$, and consequently the phase variation in particle variables implies $\partial \mathcal{L}/\partial \alpha_n = 0$. If we evaluate $\partial \mathcal{L}/\partial \alpha_n$, <u>without</u> neglecting the linear damping of the primary modes a-priori, then

$$\begin{aligned} \partial \mathcal{L}/\partial \alpha_n &= - \int d^6 z_o f_o(z_o)\, \partial H/\partial \alpha_n \\ &= -A_n^2\, \epsilon_n''(\omega_n^{(0)})/2\pi \end{aligned} \tag{B10}$$

where ϵ_n'' is the imaginary part of the dielectric (22). Thus stationarity requires the linear damping to be negligible. Stated differently, by ignoring the time dependence of A_n when integrating (B3) we have assumed the linear damping is negligible on the fast time scale. Our evaluation of $\delta S/\delta \alpha_n = 0$ in (B10) reflects this assumption.

References

1. W. E. Drummond and D. Pines, Nucl. Fusion Suppl. Pt. III, 1049 (1962).
2. B. B. Kadomtsev and V. I. Petviashvilli, Zh. Eksp. Teor. Fiz. 43 2234 (1962) [Sov. Phys. J.E.T.P. 16 1578 (1963)].
3. R. Z. Sagdeev and A. A. Galeev, Nonlinear Plasma Theory (Benjamin, New York, 1969), Chap. 3.
4. M. L. Sloan and W. E. Drummond, Phys. Fluids 13 2554 (1970).
5. E. Ott and C. T. Dum, Phys. Fluids 14 959 (1971).
6. R. L. Dewar, J. Plasma Phys. 7 267 (1972).
7. S. Johnston, Phys. Fluids 19 93 (1976).
8. M. Porkolab and R. P. H. Chang, Rev. Mod. Phys. 50 745 (1978).
9. A. Kaufman, Physica Scripta T2/2, 517 (1982).
10. R. Davidson, Methods in Nonlinear Plasma Theory (Academic Press, New York, 1972).
11. F. E. Low, Proc. Roy. Soc. (London) A248 282 (1958).
12. H. R. Lewis and K. Symon, J. Math. Phys. 20 413 (1979) (Appendix).
13. A. N. Kaufman, XIII Int'l. Colloq. on Group Theoretical Methods in Physics, College Park, MD (World Scientific Publishing, 1984) 26.
14. A. N. Kaufman and B. Boghosian, Contemp. Math. 28 169 (1984).
15. V. I. Arnold, Mathematical Methods of Classical Mechanics (Springer, New York, 1978) Chapter 9.
16. R. G. Littlejohn, J. Math. Phys. 23 742 (1982).

17. R. G. Littlejohn, J. Plasma Phys. 29 111 (1983).

18. R. Dewar, J. Phys. A9 2043 (1976).

19. A. N. Kaufman in Topics in Nonlinear Dynamics, S. Jorna (ed.), (A.I.P. Conference Proceedings, 46, A.I.P., New York, 1978).

20. J. R. Cary, Phys. Rep. 79 131 (1981).

21. S. N. Rasband, Dynamics (J. Wiley, New York, 1983).

22. A. J. Lichtenberg and M. A. Lieberman, Regular and Stochastic Motion (Springer, New York, 1983).

23. S. Johnston and A. N. Kaufman in Proc. of the Sherwood Theory Meeting, Gatlinburg, TN, April, 1978, Poster D23 (unpublished).

24. R. Dewar, Phys. Fluids 16 1102 (1973).

25. J. R. Cary and A. N. Kaufman, Phys. Rev. Lett. 39 402 (1977).

26. J. R. Cary and A. N. Kaufman, Phys. Fluids 24 1238 (1981).

27. S. Johnston and A. N. Kaufman, Phys. Rev. Lett. 40 1266 (1978).

28. S. Johnston and A. N. Kaufman, J. Plasma Phys. 22 105 (1979).

29. A. N. Kaufman and D. Holm, Phys. Lett. 105A 277 (1984).

30. P. Similon, Phys. Lett. 112A 33 (1985).

31. J. S. Kim and J. R. Cary, Phys. Fluids 26 2167 (1983).

32. J. S. Kim, Ph.D. thesis, University of California, Berkeley (1984).

33. J. D. Crawford, T. M. O'Neil, and J. H. Malmberg, Phys. Rev. Lett. 54 697 (1985).

34. J. R. Cary and R. G. Littlejohn, Ann. Phys. 151 1 (1983).

35. B. W. Roos, Analytic Functions and Distributions in Physics and Engineering (J. Wiley and Sons, New York, 1969).

36. A. N. Kaufman and T. Nakayama, Phys. Fluids 13 956 (1970).

37. In conjunction with equation (20), we require the eigenfunctions to satisfy appropriate boundary conditions. Completeness can sometimes be established from the theory of non-self-adjoint boundary value problems; see reference 48.

38. R. Dewar, Physica D, 17 37 (1985).

39. An alternative procedure would be to restrict a-priori the allowed variation $\delta\alpha_n(t)$ to functions varying slowly in time, then the rapidly varying terms would be averaged out by the $\int dt$ in the action functional

40. To calculate the mode energy in oscillation center theory we simply subtract the oscillation center energy, $\int d^6z\, FK$, from the total plasma energy $\int d^6z\, fH$. This gives $\sum_n \omega_n^{(0)} J_n$.

41. P. A. Sturrock, Ann. Phys. 9 422 (1960).

42. Compare these expressions to (3.1) in reference 41.

43. J. H. Malmberg and C. F. Driscoll, Phys. Rev. Lett. 44 654 (1980); C. F. Driscoll and J. H. Malmberg, Phys. Rev. Lett. 50 167 (1983).

44. D. L. Eggleston, T. M. O'Neil, and J. H. Malmberg, Phys. Rev. Lett. 53 982 (1984).

45. S. A. Prasad and T. M. O'Neil, Phys. Fluids 27 206 (1984).

46. R. Keinigs, Phys. Fluids 24 860 (1981); Phys. Fluids 27 1427 (1984).

47. J. D. Crawford and T. M. O'Neil, to be published.

48. E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations (McGraw-Hill, New York, 1955) Chapter 12.

49. When there is no ambiguity, we use the abbreviated notation $\eta_n(r\,;\,\omega_n^{(0)})$ to denote $\eta_{\mu_n}(r\,;\,\omega_n^{(0)})$, the radial eigenfunction associated with Λ_{μ_n}.

50. In reference 33, the radial modes were denoted by $\psi_{\ell kn}$ and normalized by $\int \psi_{\ell kn}^2\, r dr = 1$; the equilibrium distribution function F_o in ref. 33 is related to f_e in this paper by $F_o = m^2 \omega_c f_e$.

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