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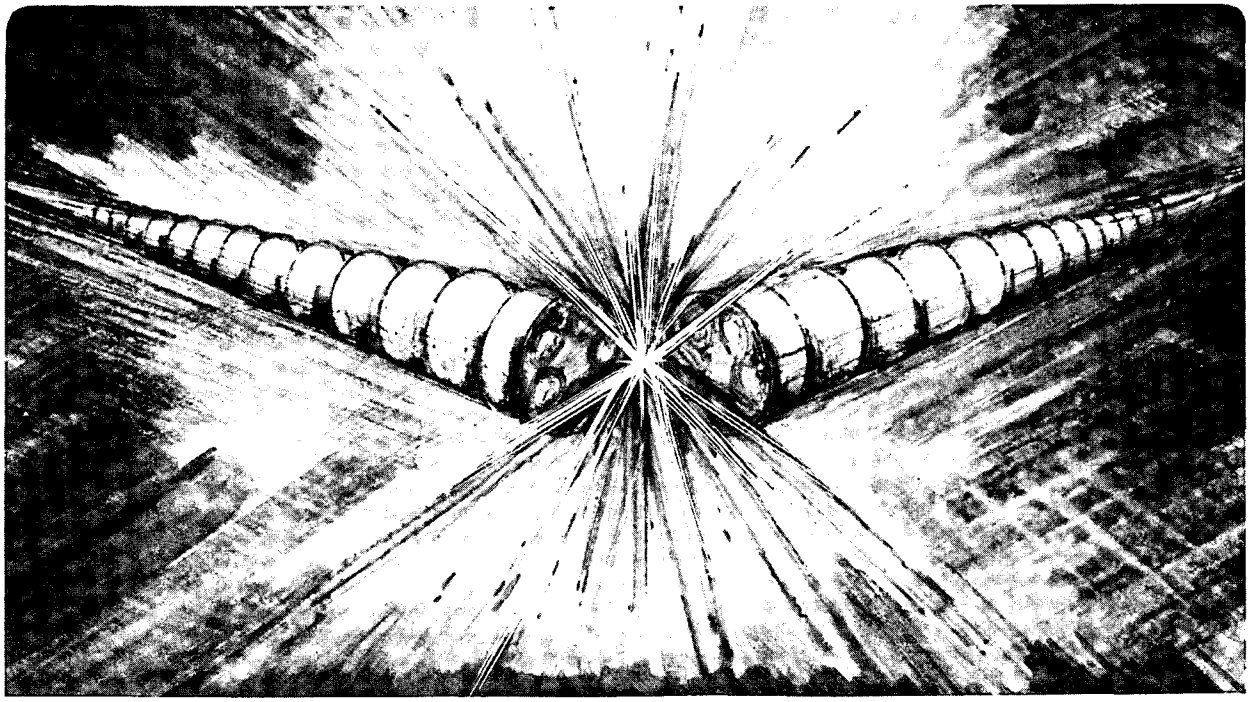

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# VARIATIONAL FORMULATION OF EIKONAL THEORY FOR VECTOR WAVES

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## ABSTRACT

The eikonal theory of wave propagation is developed by means of a Lorentz-covariant variational principle, involving functions defined on the natural eight-dimensional phase space of rays. The wave field is a four-vector representing the electromagnetic potential, while the medium is represented by an anisotropic, dispersive nonuniform dielectric tensor $D^{\mu\nu}(k,x)$. The eikonal expansion yields, to lowest order, the Hamiltonian ray equations, which define the Lagrangian manifold $k(x)$, and the wave action conservation law, which determines the wave amplitude transport along the rays. The first-order contribution to the variational principle yields a concise expression for the transport of the polarization phase. The symmetry between k-space and x-space allows for a simple implementation of the Maslov transform, which avoids the difficulties of caustic singularities.

Wave propagation in various media is often well represented by the eikonal theory, also known as ray optics, geometrical optics, and WKB [1]. In this paper, we present a variational approach, leading to some new results, as well as concise expressions for old results. While our immediate motivation is electromagnetic wave propagation in magnetized plasma [2], we expect that our methods are applicable to other media as well, such as elastic waves [3].

Our emphasis here is on the <u>vector</u> character of the wave field, which propagates through an anisotropic medium. We also emphasize the <u>phase-space</u> concept, as a way of avoiding the singularities of caustics.

In addition, it is convenient to utilize a covariant formulation [4], treating space and time on an equal footing; this is the key to conciseness. We represent the electromagnetic wave field by the four-potential $A_\mu(x)$, with $A_\mu = (\underline{A},-\phi)$, and $x = (\underline{x},t)$. (We set $c = 1$).

The wave equation for <u>linear</u> <u>dissipationless</u> propagation can be expressed as an integral equation of the form

$$\int d^4x''\, D^{\mu\nu}(x',x'')A_\nu(x'') = 0. \qquad (1)$$

Here $D^{\mu\nu}(x',x'')$ represents the (in general) anisotropic dispersion tensor of the medium, as a two-point kernal. We consider it as given; methods for its deirvation are discussed in our previous work. The variational equation below requires that it be symmetric:

$$D^{\mu\nu}(x',x'') = D^{\nu\mu}(x'',x'); \tag{2}$$

asymmetry represents dissipation, which must be treated by other methods. Nonlinearity can also be included, by allowing the medium to respond "ponderomotively" to the wave.

We introduce the quadratic action functional:

$$S(A) = \tfrac{1}{2}\int d^4x' \int d^4x''\ A_\mu(x')D^{\mu\nu}(x',x'')A_\nu(x''). \tag{3}$$

The requirement that S be stationary with respect to arbitrary variations of the four-potential field is equivalent to Eq. (1), since the functional derivative $\delta S/\delta A_\mu(x')$ is just the left side of that equation.

It is advantageous to introduce the phase-space concept as early as possible. We define the dispersion tensor as a function on phase space (k,x):

$$D^{\mu\nu}(k,x) = \int d^4s\ D^{\mu\nu}(x'=x+s/2,\ x''=x-s/2)\exp(-ik\cdot s) \tag{4}$$

It is thus the Fourier transform of $D^{\mu\nu}(x',x'')$ with respect to the two-point separation $s=x'-x''$. The wave four vector $k = (\underline{k},-\omega)$, together with $x = (\underline{x},t)$, coordinatizes the eight-dimensional phase space (k,x) of the rays.

Analogously, we introduce the bilinear Wigner tensor:

$$A^2_{\mu\nu}(k,x) = \int d^4s\ A_\mu(x+s/2)A_\nu(x-s/2)\exp(-ik\cdot s), \tag{5}$$

which likewise lives on the phase space. In terms of these, the action functional reads

$$S(A) = \tfrac{1}{2}\int\!\!\int D^{\mu\nu}(k,x)\ A_{\nu\mu}(k,x), \tag{6}$$

where $\int\!\!\int = \int\!\!\int d^4x d^4k/(2\pi)^4$.

We see that k-space and x-space enter (6) on an equal footing. At each point (k,x) in this space, we represent the tensor $D^{\mu\nu}$ in terms of its local (orthonormal) <u>eigenvectors</u> $e_\alpha$ and (real) <u>eigenvalues</u> $D_\alpha$ :

$$D^{\mu\nu}(k,x) = \Sigma_\alpha D_\alpha(k,x)e^\mu_\alpha(k,x)e^{\nu *}_\alpha(k,x), \tag{7}$$

where

$$D^{\mu\nu}(k,x)e^\alpha_\nu(k,x) = D_\alpha(k,x)e^\mu_\alpha(k,x) \tag{8}$$

and

$$e^{\mu *}_\alpha(k,x)e^\beta_\mu(k,x) = \delta^\beta_\alpha. \tag{9}$$

(We assume that the eigenvalues are non-degenerate.) Substituting (7) into (6), we obtain

$$S(A) = \tfrac{1}{2}\sum_\alpha \int\!\!\int D_\alpha(k,x)A^2_\alpha(k,x), \tag{10}$$

where

$$A^2_\alpha(k,x) = e^{\nu *}_\alpha(k,x)A^2_{\nu\mu}(k,x)e^\mu_\alpha(k,x) \tag{11}$$

is the projection of the Wigner tensor on the local polarization $e_\alpha(k,x)$, and $D_\alpha(k,x)$ is the <u>scalar</u> dispersion function.

We now assume that the wave field $A_\mu(x)$ can be expressed in <u>eikonal</u> <u>form</u>:

$$A_\mu(x) = a_\mu(x)\ \exp\ i\theta(x) + \text{c.c.}, \tag{12}$$

where the phase $\theta(x)$ is real and has slowly varying first-derivative

$k_\mu(x) = \partial\theta(x)/\partial x^\mu$, while the amplitude $a_\mu(x)$ is complex and is slowly varying. The eikonal assumption introduces an implicit small parameter, representing $\partial a_\mu/\partial x^\nu$ and $\partial k_\mu/\partial x^\nu$.

When we substitute (12) into (5), we obtain, to <u>lowest</u> order in the eikonal parameter,

$$A^2_{\mu\nu}(k,x) = (2\pi)^4[\delta^4(k-\partial\theta/\partial x)a^*_\mu(x)a_\nu(x) + \delta^4(k+\partial\theta/\partial x)a_\mu(x)a^*_\nu(x)], \tag{13}$$

where we have discarded rapidly oscillating terms ~ $\exp 2i\theta(x)$, which vanish upon x-integration in (6).

Substituting (13) into (11), and performing the k-integration of (10), we obtain the action functional

$$S(A) = \int d^4x\, L(x) \tag{14}$$

in terms of a <u>Lagrangian density</u>:

$$L(x) = \sum_\alpha D_\alpha(k=\partial\theta/\partial x,\ x)|a_\alpha|^2(x), \tag{15}$$

where

$$a_\alpha(x) = e^\mu_\alpha(k=\partial\theta/\partial x,\ x)\ a_\mu(x) \tag{16}$$

is the scalar projection of the amplitude on the polarization direction $e_\alpha$.

The variation of the action (14) is now taken with respect to the amplitude $a_\mu(x)$ and the phase $\theta(x)$. The former variation yields

$$D_\alpha(k=\partial\theta/\partial x,\ x)a_\alpha(x) = 0, \tag{17}$$

for each $\alpha$. For a non-trivial solution, we require $D_\alpha=0$ for <u>one</u> polarization, denoted I, and allow $a_\alpha = 0$ for the <u>other</u> polarizations $\alpha \neq I$.

Thus we obtain the eikonal equation

$$D_I(k=\partial\theta/\partial x,\ x) = 0, \tag{18}$$

with the amplitude expressed as

$$a_\mu(x) = e^I_\mu(x)a_I(x). \tag{19}$$

The polarization

$$e^I_\mu(x) = e^I_\mu(k=\partial\theta/\partial x,\ x) \tag{20}$$

can absorb the complex phase of the amplitude vector field $a_\mu(x)$, allowing the scalar amplitude $a_I(x)$ to be real and positive.

The eikonal equation (18) is solved by Hamilton's method. In eight-dimensional phase space, the ray equations are

$$\frac{dx^\mu}{d\sigma} = -\frac{\partial D_I}{\partial k_\mu}, \qquad \frac{dk_\mu}{d\sigma} = +\frac{\partial D_I}{\partial x^\mu} \quad . \tag{21}$$

These Hamiltonian equations yield the family of ray orbits $[k(\sigma),\ x(\sigma)]$. With appropriate initial conditions, the rays generate a four-dimensional "Lagrangian submanifold". $k(x)$. The phase $\theta(x)$ is then obtained by integrating along a ray:

$$\theta(x(\sigma)) = \int_{\sigma_0}^{\sigma} d\sigma'\ k_\mu(\sigma')dx^\mu(\sigma')/d\sigma'. \tag{22}$$

On varying S with respect to the phase $\theta(x)$, we obtain a conservation law [5]:

$$\partial J^{\mu}(x)/\partial x^{\mu} = 0, \tag{23}$$

for the wave-action four-flux $J^{\mu}(x)$, defined as

$$\begin{aligned} J^{\mu}(x) &= -\ \partial L/\partial k_{\mu} \\ &= -\ [\partial D_I(k,x)/\partial k_{\mu}](x)\ a_I^2(x). \\ &= (dx^{\mu}/d\sigma)\ a_I^2(x). \end{aligned} \tag{24}$$

By (21), the temporal component of $J^{\mu}(x)$ is the familiar wave-action density:

$$J(\underline{x},t) = (\partial D_I/\partial\omega)a_I^2, \tag{25}$$

while the spatial part of $J^{\mu}(x)$ is the wave-action flux density $(\partial\omega/\partial\underline{k})J$. Thus (23) reads

$$\partial J(\underline{x},t)/\partial t = -\ \nabla \cdot (J\ \partial\omega/\partial\underline{k}); \tag{26}$$

this conservation law reflects the invariance of the action functional under a uniform phase shift $\theta(x) \to \theta(x) + C$.

On substituting the expression (24) for $J^{\mu}(x)$ into the conservation law (23), we obtain the amplitude transport equation:

$$\frac{d}{d\sigma}\ \ln a_I^2 = \frac{\partial^2 D_I(k,x)}{\partial k_{\mu}\partial x^{\mu}} + \frac{\partial^2 D_I}{\partial k_{\mu}\partial k_{\nu}}\ \frac{\partial^2\theta(x)}{\partial x^{\mu}\partial x^{\nu}}\ . \tag{27}$$

The first term on the right represents the medium nonuniformity, while the second term represents the divergence (or convergence) of a ray bundle.

The latter quantity is determined, in turn, by <u>its</u> transport relation:

$$\begin{aligned} \frac{d}{d\sigma}\ \frac{\partial^2\theta}{\partial x^{\mu}\partial x^{\nu}} &= \frac{\partial^2 D_I}{\partial x^{\mu}\partial x^{\nu}} + \frac{\partial^2 D_I}{\partial x^{\mu}\partial k_{\rho}}\ \frac{\partial^2\theta}{\partial x^{\rho}\partial x^{\nu}} + \frac{\partial^2 D_I}{\partial x^{\nu}\partial k_{\rho}}\ \frac{\partial^2\theta}{\partial x^{\rho}\partial x^{\mu}} \\ &+ \frac{\partial^2 D_I}{\partial k_{\rho}\partial k_{\lambda}}\ \frac{\partial^2\theta}{\partial x^{\lambda}\partial x^{\mu}}\ \frac{\partial^2\theta}{\partial x^{\rho}\partial x^{\nu}}\ , \end{aligned} \tag{28}$$

obtained by differentiating (18) twice with respect to x. This nonlinear equation for the ray divergence leads in general to a singularity in a finite distance, i.e., to a caustic. To avoid that singularity, we may use the Maslov transform.

Inserting (19) into (12), we now have, for the field,

$$A_{\mu}(x) = e_{\mu}^{I}(x)\ a_I(x)\ \exp\ i\theta(x) + c.c., \tag{29}$$

with $\theta(x)$ determined by (22) and (21), $a_I(x)$ determined by (27), and $e_{\mu}(x)$ determined by (20) up to a complex phase factor. That is, the replacement $e_{\alpha}^{\mu}(k,x) \to e_{\alpha}^{\mu}(k,x)\exp\ i\psi(x)$ leaves (7) invariant, so that the polarization phase is not as yet determined.

For the polarization phase, it is necessary to expand the action functional to first order in the eikonal parameter. After some algebra, and using the zero-order equations, we find the <u>first order</u> Lagrangian to be

$$L'(x) = i\, a_I^2(x)[e_I \cdot de_I / d\sigma + \tfrac{1}{2} D^{\mu\nu}\{e_\nu^I, e_\mu^{I*}\}], \tag{30}$$

where {,} represents the canonical Poisson bracket:

$$\{f,g\} = \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial k_\mu} - \frac{\partial f}{\partial k_\mu}\frac{\partial g}{\partial x^\mu} \quad . \tag{31}$$

On taking the variation of the action functional with respect to the amplitude, we now obtain the additional equation $L' = 0$, which yields the desired polarization phase transport:

$$e_I^* \cdot \frac{de_I}{d\sigma} = -\tfrac{1}{2} D^{\mu\nu}\{e_\nu^I, e_\mu^{I*}\} \ . \tag{32}$$

It is interesting to note that this transport relation lives in phase space; i.e., it is not necessary to project onto x-space. On the other hand, the ray divergence equation (28) definitely refers to x-space.

We now proceed to the elimination of caustic singularities. Let us again substitute the Wigner tensor (13) into (11), obtaining the scalar field:

$$A_I^2(k,x) = (2\pi)^4 a_I^2(x)[\delta^4(k - \partial\theta/\partial x) + \delta^4(k + \partial\theta/\partial x)] \tag{33}$$

Integrating over an element of phase space volume [see (10)], we have

$$d^4x\, d^4k\, A_I^2(k,x)/(2\pi)^4 = a_I^2(x) d^4x. \tag{34}$$

We see that $A_I^2(k,x)$ is the wave density in <u>phase space</u>, while $a_I^2(x)$ is the density in <u>x-space</u>. From (33), we see that $A_I^2$ is supported on the Lagrangian manifold $k = k(x) = \partial\theta(x)/\partial x$, i.e., $k_1 = \partial\theta(x)/\partial x^1$, and so on. At a caustic, $k(x)$ becomes double-valued, so that the x-representation of the eikonal breaks down.

The Maslov procedure [6] is a simple way to avoid the caustic singularity. One Fourier transforms the field $A_\mu(x)$ from $(x^1, x^2, x^3, x^4)$- space, to $(k_1, x^2, x^3, x^4)$ - space, where the $(x^1, k_1)$ pair is chosen to represent the singular direction in $\partial^2\theta/\partial x^\mu \partial x^\nu$. One now can make the eikonal assumption (12) in the <u>new</u> space, and the whole procedure outlined above carries through in the same way. Space does not permit a fuller discussion here.

The Maslov transform is a special case of a more sophisticated approach, recently developed by Littlejohn [7-9]. His method involves a <u>continuous</u> transformation of coordinates in phase space, and is based on the wave packet approximation.

The variational method discussed here allows for a treatment of wave angular momentum [10] transport, and its consequences. This work is in progress.

Applications of the present formalism are now under way.

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