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R.G. Littlejohn, A.N. Kaufman, and G.L. Johnston

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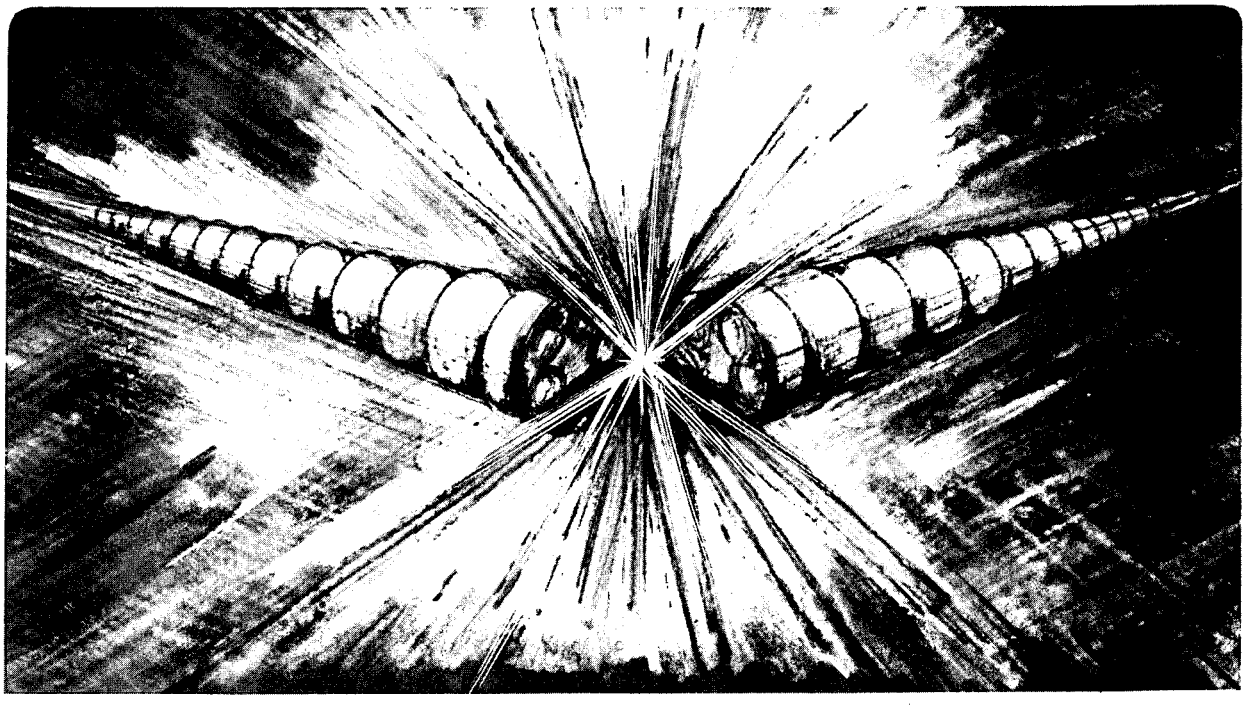

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Hamiltonian Structure of Particle Motion in an Ideal Helical Wiggler with Guide Field*

Robert G. Littlejohn and Allan N. Kaufman

Lawrence Berkeley Laboratory and Physics Department
University of California, Berkeley, CA 94720

and

George L. Johnston

Plasma Fusion Center, Massachusetts Institute of Technology
Cambridge, MA 02139

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Abstract

The ideal helical wiggler with guide field is shown to possess an integrable Hamiltonian. Explicit generating functions are presented for the canonical transformation to action/angle variables.

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The following is a reduction of the motion of a particle in the magnetic field:

$$\underset{\sim}{B} = B_o\hat{z} + B_w(\hat{x}\cos kz + \hat{y}\sin kz),$$

to integrable form, including a partial reduction to action/angle variables. Essentially the same problem was solved by Roberts and Buchsbaum [1], but without revealing the Hamiltonian structure. The practical use of the calculation presented here is as a basis for a perturbative treatment of more difficult and/or realistic magnetic field models [2].

Choose units so that e=m=c=1, and the following gauge:

$$\underset{\sim}{A} = B_o x\hat{y} + \frac{B_w}{k}(-\hat{x}\cos kz - \hat{y}\sin kz) .$$

Then the relativistic Hamiltonian (squared) is

$$H^2(\underset{\sim}{x},\underset{\sim}{p}) = (p_x + \frac{B_w}{k}\cos kz)^2 + (p_y - B_o x + \frac{B_w}{k}\sin kz)^2 + p_z^2 + 1.$$

Now introduce the following canonical transformation:
$(\underset{\sim}{x},\underset{\sim}{p}) \rightarrow (Q_1,Q_2,z=Q_3;P_1,P_2,P_3)$, given by the generating function:

$$F(x,y,z;P_1,P_2,P_3) = (P_1 + B_o y)(x - \frac{B_w}{B_o k}\sin kz) + P_2(\frac{P_1}{B_o} + y + \frac{B_w}{B_o k}\cos kz)$$

$$+ P_3 z - \frac{B_w^2}{4B_o k^2}\sin 2kz - \frac{B_w^2}{2B_o k} z.$$

This yields the following transformation equations:

$$p_x = P_1 + B_o y \ ,$$

$$p_y = P_2 + B_o x - \frac{B_w}{k} \sin kz \ ,$$

$$p_z = P_3 - \frac{B_w}{B_o} (\cos kz)\ (P_1 + B_o y) - \frac{B_w}{B_o} (\sin kz) P_2 - \frac{B_w^2}{B_o k} \cos^2 kz \ ,$$

$$Q_1 = P_2/B_o + x - \frac{B_w}{B_o k} \sin kz \ ,$$

$$Q_2 = P_1/B_o + y - \frac{B_w}{B_o k} \cos kz \ ,$$

$$Q_3 = z.$$

Untangling this, we obtain

$$x = Q_1 - P_2/B_o + \frac{B_w}{B_o k} \sin kz \ ,$$

$$y = Q_2 - P_1/B_o - \frac{B_w}{B_o k} \cos kz \ ,$$

$$z = Q_3,$$

$$p_x = B_o Q_2 - \frac{B_w}{k} \cos kz \ ,$$

$$p_y = B_o Q_1,$$

$$p_z = P_3 - \frac{B_w}{B_o} (B_o Q_2 \cos kz + P_2 \sin kz).$$

In the new variables, we have

$$H^2 = P_2^2 + B_o^2\ Q_2^2 + [P_3 - \frac{B_w}{B_o} (B_o Q_2 \cos kz + P_2 \sin kz)]^2 + 1.$$

Note that this H is ignorable in <u>both</u> Q_1 and P_1, so both of these are invariants. In fact, Q_1 and P_1 are closely related to a kind of generalized guiding center position, denoted by X and Y:

$$X = Q_1, \quad Y = -P_1/B_o$$

The Hamiltonian now has only 2 degrees of freedom. It is still not clear that it is integrable, since it depends on all 4 canonical variables: Q_2, P_2; z, P_3. Notice that it has a harmonic oscillator term in Q_2 and P_2, with frequency B_o. The physical significance of the variables Q_2 and P_2 is that they are essentially the x and y components of the four-velocity:

$$\dot{x} = B_o Q_2, \qquad \dot{y} = P_2$$

However, the harmonic oscillator term is coupled to the longitudinal motion through the term $[P_3 - \ldots]^2$.

Introduce action/angle variables for the Q_2, P_2 variables, i.e. set

$$Q_2 = (2J/B_o)^{1/2}\sin\theta,$$
$$P_2 = (2JB_o)^{1/2}\cos\theta,$$

so that the new canonical variables are $(\theta,J;\ z,P_3)$. The Hamiltonian becomes

$$H^2 = 2B_oJ + [P_3 - B_w\,(2J/B_o)^{1/2}\sin(\theta+kz)]^2 + 1.$$

Now introduce a new canonical transformation: $(\theta,J;\ z,P_3) \rightarrow (\phi,J;\ z,P_3')$, generated by

$$F'(\theta,z;\ J,P_3') = J\,(\theta + kz) + z\,P_3',$$

yielding

$$\phi = \theta + kz, \qquad P_3 = P_3' + kJ.$$

Then the Hamiltonian becomes

$$H^2 = 2B_oJ + [P_3' + kJ - B_w(2J/B_o)^{1/2}\sin\phi]^2 + 1.$$

This is now ignorable in z, so that P_3' is an invariant.

The Hamiltonian now has only one essential degree of freedom, and so is integrable. The integration can be carried out via elliptic functions [3].

Acknowledgments

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References

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2. H. Freund and A. Ganguly, IEEE J. Qu. Electronics 21 (1985) 1073.
3. H. Freund and A. Drobot, Phys. Fluids 25 (1982) 736.

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