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July 1986

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LATTICE VORTEX MODELS AND TURBULENCE THEORY[1]

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LATTICE VORTEX MODELS AND TURBULENCE THEORY*

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Abstract

Vortex lattice models for use in turbulence theory are explained, a simple one dimensional version is presented and is seen to provide reasonable qualitative information about intermittency and vortex stretching, and results obtained with three dimensional models are summarized.

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Introduction

The lattice vortex models that we shall discuss are designed to provide qualitative understanding and quantitative predictions in turbulence theory.

The prospects for an accurate and detailed numerical solution of the Navier-Stokes or Euler equations in a fully turbulent situation are dim. Simple estimates of the number of degrees of freedom that must be resolved (see e.g. [2], [12]) show that the computation would be overwhelmingly large. The complexity of the phenomena observed in simplified problems (see e.g. [7], [8]) and estimates of the rate of growth of errors (see e.g. [4]) show that the problem is worse even than the simple counting argument indicates. Even if the computational problem were somehow solved, the problem of analyzing and understanding the results of the calculation would remain open. After all, nature presents us with detailed analogue calculations of turbulent flows and these have not yet led to a solution of the problem of turbulence.

There is a vast literature on the modeling of turbulence by averaged or phenomenological equations, usually involving empirical parameters. Such modeling is very useful as an engineering tool (see e.g. [2], [11]) but does not lead to general equations or to qualitative understanding [5]. Some of the reasons will be discussed below.

A way of tackling the turbulence problem is suggested by recent calculations ([7], [8]). The turbulent flow field is determined by its vorticity. The evolution of vorticity is typically dominated by a small number of coherent structures that interact through vortex / vortex interactions, through the exchange of vorticity (for a visualization, see [19]), and through processes of vortex stretching and creation. The details of these interactions are amazingly complex, but the calculations suggest that the overall statistics are independent of the details as long as certain constraints are satisfied. In many practical problems one is interested in scales of motion that are comparable with the scales of the coherent structures, and it is not desirable to carry out scaling or averaging operations that eliminate the spottiness (= "intermittency") of the solution. It is therefore

natural to turn to discrete models of the vorticity field; the lattice models are convenient because they allow one to separate scales in a natural fashion and keep the calculations relatively small.

What the lattice vortex models are not

At this point a disclaimer is appropriate. There are a number of mathematical constructions that use words similar to the ones we shall be using, and confusion is to be expected.

There is a substantial literature on lattice models and their hydrodynamical limits (see e.g. [18]) which is totally unrelated to the present work. As far as the present work is concerned, the Navier-Stokes and Euler equations have been derived, and the goal is to construct discrete models that mimic their behavior. If the models we introduce have a hydrodynamical limit it would presumably be, depending on the scaling, either the Navier-Stokes equations or some averaged model of turbulence.

Similarly, there have been in recent years a number of attempts ([13], [17]) to model continua by discrete models; even if these models can be made consistent with the equations, and even if they can be made more efficient than more standard methods (at present, a rather doubtful proposition), the problem of obtaining information about turbulence from solutions of the Navier-Stokes equations would remain as open as described in the introduction. The lattice models offered here and the lattice models proposed for example in the context of "cellular automata" are on opposite sides of the Euler or Navier-Stokes equations.

Vortex methods (see e.g. [8]) provide genuine approximations to solutions of the equations of motions and are thus quite distinct from the models discussed here. However, some of the features of the models were suggested by calculations with vortex methods. The closest analogues to the present models that I know of are the coarse grained collective variable models [1], [6], but the analogy is not close, in particular because the coarse-grained models do not include a vortex model of energy dissipation.

Euler's equations

The equations we shall be starting from are Euler's equations:

$$\partial_t \underline{\xi} + (\underline{u}\cdot\underline{\nabla})\underline{\xi} - (\underline{\xi}\cdot\underline{\nabla})\underline{u} = 0, \tag{1a}$$

$$\underline{\xi} = \text{curl } \underline{u}, \quad \text{div } \underline{u} = 0, \tag{1b,1c}$$

where $\underline{u}$ is the velocity, $\underline{\xi}$ is the vorticity, t is the time, and $\underline{\nabla}$ is the differentiation vector. These equations are appropriate for the analysis of scales larger than the dissipation scales, as discussed for example in [8]. For a derivation of these equations, see [10]; for a recent review of their theory, see [16]. The kinetic energy of a fluid whose motion obeys equations (1) is

$$T = \frac{1}{2}\int |\underline{u}|^2 d\underline{x} = \frac{1}{8\pi}\int d\underline{x}\int d\underline{x}'\frac{\underline{\xi}(\underline{x})\cdot\underline{\xi}(\underline{x}')}{|\underline{x}-\underline{x}'|}, \tag{2}$$

where $\underline{x}$ is the position vector (see [15]). The mean squared vorticity is

$$Z = \int |\underline{\xi}|^2 d\underline{x}. \tag{3}$$

Equations (1) allow vortex tubes to stretch (for definitions, see [10]). Experiments, both numerical and physical, show that the stretching can be very substantial, and that as a result Z can become very large, possibly infinitely large in a finite time ([7], [16]).

Note that the integrand

$$B = B(\underline{x},\underline{x}') = \underline{\xi}(\underline{x})\cdot\underline{\xi}(\underline{x}') \;/\; (8\pi|x-x'|)$$

in equation (2) is not positive definite. To see the significance of this fact, write

$$T = \int B d\underline{x} d\underline{x}' = \int_{|\underline{x}-\underline{x}'|<\epsilon} B d\underline{x} d\underline{x}' + \int_{|\underline{x}-\underline{x}'|\geq\epsilon} B d\underline{x} d\underline{x}' = T_< + T_> .$$

If ϵ is small and $\underline{\xi}$ is reasonably smooth, $T_< \geq 0$ and $T_>$ can change sign. As vortex tubes stretch and thin out, the vorticity associated with them increases and if ϵ is of the order of a tube diameter $T_<$ should increase. Since the sum T is non-increasing $T_>$ must decrease and possibly become negative. If $B < 0$, $\underline{\xi}(\underline{x})\cdot\underline{\xi}(\underline{x}') < 0$, and the vortex tubes must fold. In picturesque language, $T_>$ provides the screen behind which $T_<$ can grow and vortex tubes can stretch.

Assumptions in the vortex lattice models

We now list the assumptions we shall be making in the lattice models. These assumptions are suggested by numerical experiment. The first four are merely global properties of Euler's equations:

(a) the energy T is conserved as long as Z is finite. If Z is not finite, energy is non-increasing.

(b) specific volume is conserved.

(c) circulation ($\int_C \underline{u}\cdot d\underline{s}$, where C is any closed curve), is conserved.

(d) the connectivity of vortex lines is invariant in three space dimensions; connectivity is important because it bestows some semblance of realism on the cascade models that lead to Kolmogorov's spectrum (see e.g. [9]).

(e) vortex lines stretch rapidly and irreversibly.

Assumption (e) hides interesting mathematical difficulties, analogous to the problems in deriving the irreversible Boltzmann or Navier-Stokes

equations from the reversible Liouville equation. The solutions of Euler's equations are reversible at least as long as they remain smooth, yet the tendency towards stretching is very marked. Similarly, the solutions of Burgers' equation exhibit a tendency to form steep gradients for "most" initial data. Somehow the space of solutions and the "natural" space of initial data are different, but attempts to formulate this or similar statements precisely have not yet been successful. (For an interesting attempt, see [14].) Vortex stretching, subject to the conservation properties above, drives the flow towards turbulence. The details of the flow are hard to determine and can be viewed as random, with an unknown distribution. We shall pick that distribution at our convenience, in the hope that the choice does not unduly affect the gross statistical features of the flow.

These assumptions are similar to assumptions made by Childress [3] his Γ-model.

A one-dimensional model of vortex stretching

We now present a one dimensional "cartoon" of vortex stretching subject to energy conservation. In three dimensional space vortex tubes can move, stretch and fold; we mimic these processes on the line.

Consider the lattice $x_i = ih$, $i = 1,\ldots,N$, $Nh = 1$, with periodic boundary conditions, where h is the length of the lattice bond. At time $t = 0$, place "vortices" of lengths $\ell_i = 1$ at the sites where i is odd and no vortices elsewhere. Endow the system with the energy

$$T = \sum_i \sum_{j \neq i} \mathrm{sgn}(\ell_j) \cdot \mathrm{sgn}(\ell_i) \frac{1}{r_{ij}} + \sum_i |\ell_i|, \tag{4}$$

where $\mathrm{sgn}(z) = 1$ if $z > 0$, -1 if $z < 0$, 0 if $z = 0$, and $r_{ij} = \left|(i-j)h\right| (\mathrm{mod}\ \frac{1}{2})$. The double sum mimics $T_>$ with $\epsilon = h/2$. The

second term should mimic $T_<$ and depends on the unknown details of the vorticity distribution within each "vortex"; it is reasonable to set $|\ell_i|$ to be the contribution of each site to $T_<$ (and this statement may serve as a definition of ℓ_i). Let $T(t=0) = T_0$.

We now set up rules of motion for the vortices. Pick i at random, $1 < i \leqslant N$, and perform with equal probability one of the following operations:

(a) Translation. Consider the possibility of translating the "vortex" at $x_i = ih$ either to x_{i+1} or to x_{i-1}, if the new site is empty. If as a result $T > T_0$, do nothing; if $T \leqslant T_0$, perform the translation.

(b) "Stretching". When a real vortex is stretched its contribution to $T_<$ increases. In the one-dimensional model there is no realistic way to evaluate this increase, so we arbitrarily try to set $\ell_{i+1} = 2\ell_i$; if the resulting $T > T_0$ we do nothing, if $T \leqslant T_0$ the "stretching" is made. (A discussion of the scaling of $T_<$ in three dimension is given in [9]).

(c) "Folding" Consider setting $\ell_i = -\ell_i$; if the resulting $T \leqslant T_0$ do it, if $T > T_0$ do nothing.

T is thus allowed to decrease; it is plotted as a function of the number of moves in figure 1. T decreases when the "vortices" "fold" (i.e., one of a pair changes sign) and approach each other. T increases when "stretching" occurs. On a finite lattice the stretching eventually brings T back to T_0. When T reaches T_0 a new "vortex" configuration is obtained, stretched but with the same T as the original configuration. In figure 2, we plot part of the original configuration and the final configuration with the same T for N = 40. A negative ℓ_i is marked with an arrow pointing downward. We note that the new configuration is "intermittent", i.e., energy is not distributed uniformly in space. Intermittency is an important property of turbulence. Clearly without intermittency "vortex stretching" would be very limited. The formation of pairs of "vortices" of opposing sign corresponds to the formation of "hairpins" in real flow [8], [20].

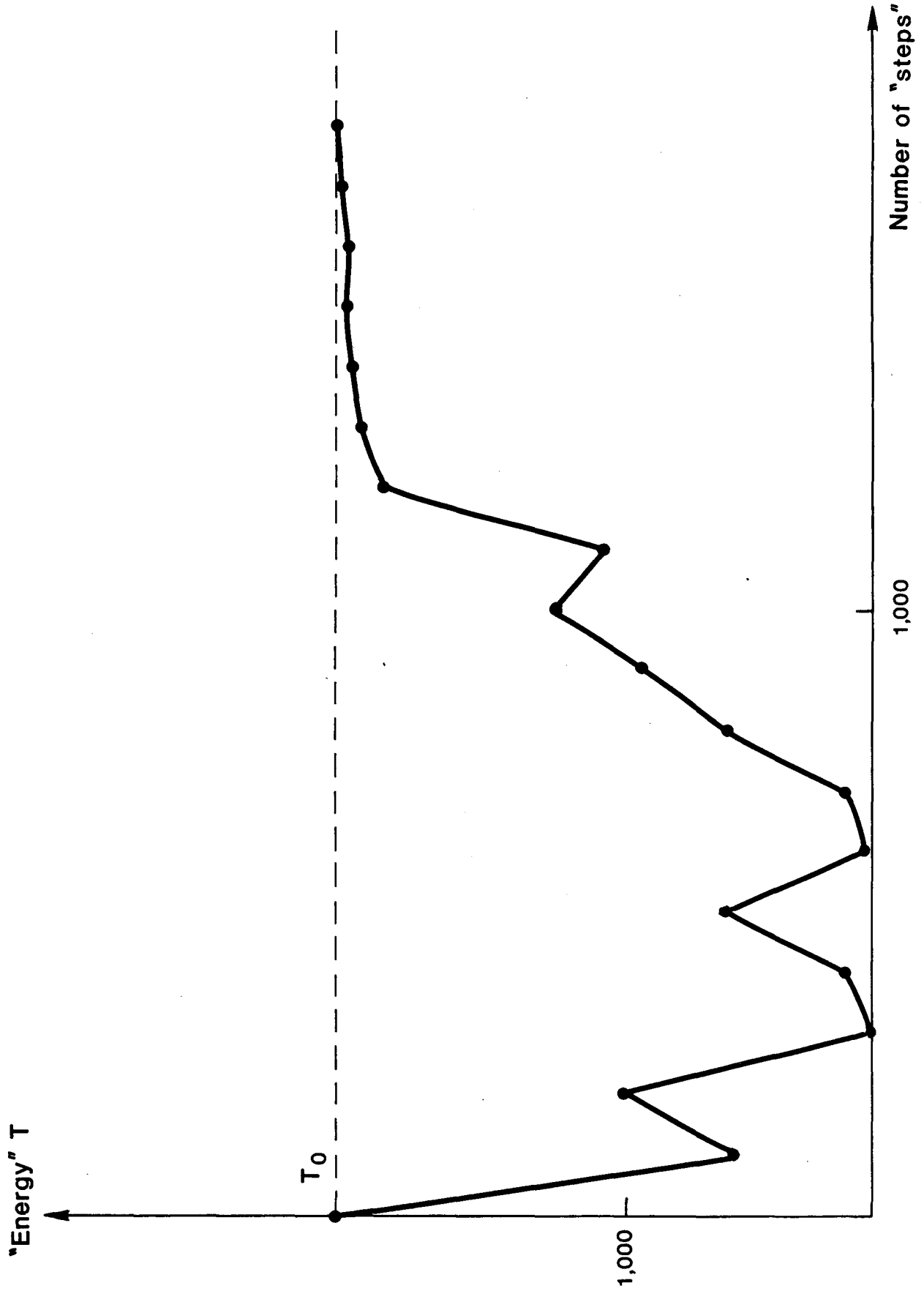


figure 1

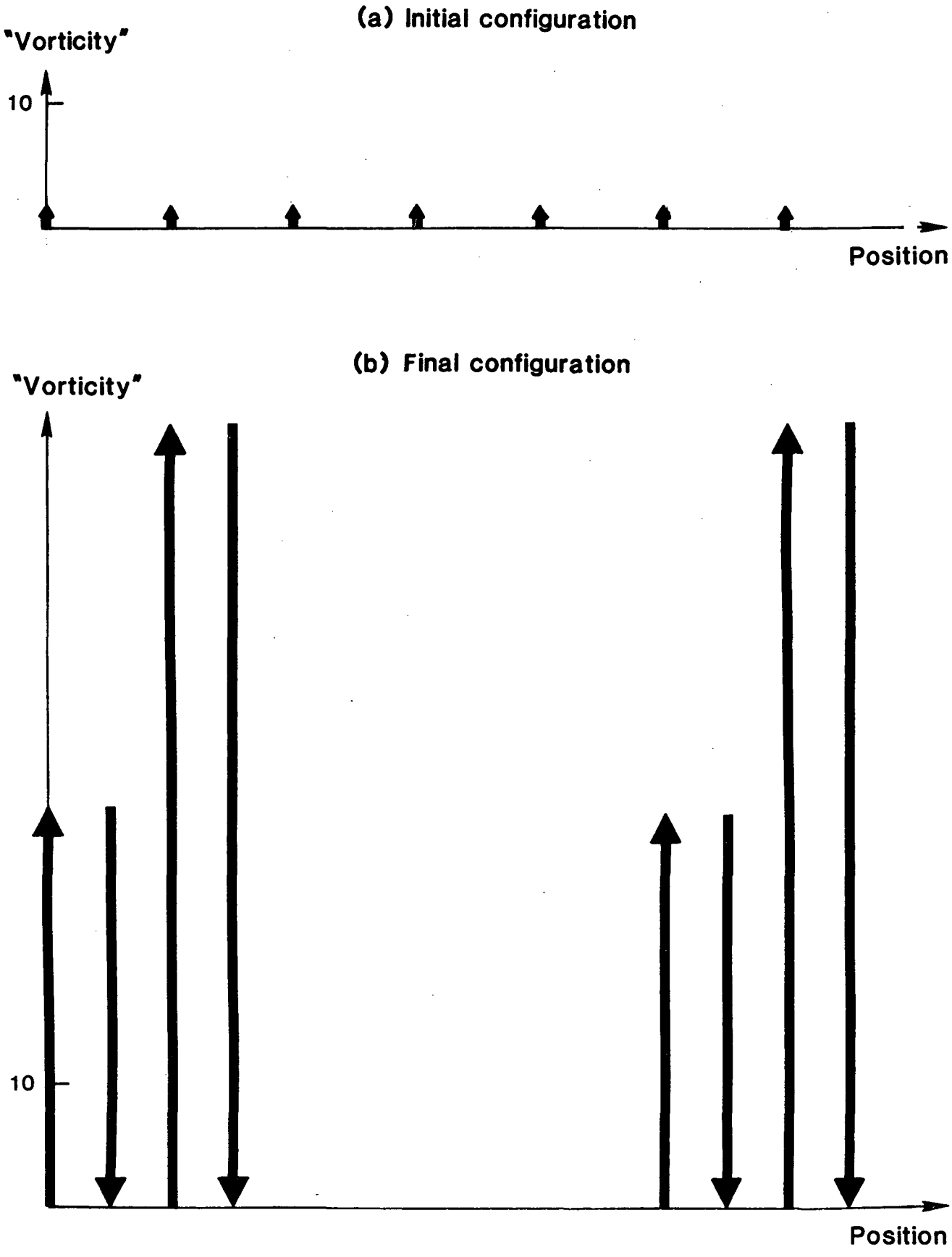

(a) Initial configuration
"Vorticity"
10
Position
(b) Final configuration
"Vorticity"
10
Position

figure 2

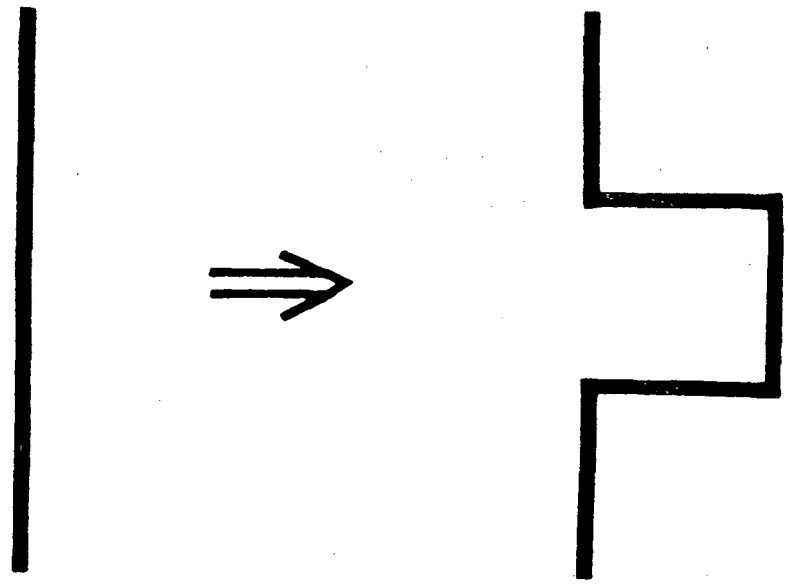
The length of the new "vortices" depends on h. Having gone as far as figure 2, we could refine the mesh and stretch some more. We could extract a part of the lattice, claim that its interactions with the rest of lattice are less significant than its internal dynamics, throw away the rest of lattice and concentrate on the part that remains. All these operations will be carried out in the three-dimensional case. Finally, if dissipation is important, on some scale one should allow cancellations between "vortices" of opposing sign. It is a reasonable conjecture that such cancellations are the major mechanism of enhanced energy dissipation in turbulent flow.

The simple model of this section is sufficient to exhibit intermittency, its relation to vortex stretching, hairpin formation and a mechanism of turbulent energy dissipation.

Summary of results in a three dimensional calculations

A three-dimensional calculation, of which the one-dimensional calculation of the preceeding section is a very simplified version, is presented in [9]. The vorticity field is assumed to consist of vortex tubes whose axes coincide with the bonds in a three-dimensional cubic lattice. An energy $T = T_> + T_<$ is defined, analogous to formula (2). The tubes are subjected to a random sequence of stretchings of which one is drawn in figure 3. The energy connected with the tubes before and after stretching is calculated in a manner consistent with equation (2), and with the constraints of conservation of specific volume and of circulation. Vortex tubes are not allowed to intersect nor to break. The energy decreases and then increases, in a manner similar to figure 1. When the energy returns to its initial value we view the outcome as an eddy that has broken down to smaller scales. Once an eddy has broken down, an eigth of the calculation is extracted, the mesh is refined and further breakdown becomes possible. This extraction/refinement process can be repeated ad infinitum. At each step in the process, the mean squared vorticity Z (equation 3) can be estimated.

figure 3



Basic stretching in 3D

It is observed and proved that if the volume occupied by the "active" vorticity, i.e., vorticity still available for stretching, remains equal to the original volume, then the stretching will come to a halt, or else conservation of energy will be violated (as in the one dimensional model problem). The remedy is to force the stretching vorticity into ever smaller volumes (as in fact happens in solution of Euler's equation, [6], [7]). The sequence of decreasing volumes defines a similarity dimension D for the "active" portion of the flow. $T_>$ becomes negative and vorticity "tangles" appear. The amount of vortex stretching that occurs is a function of D. In homogeneous turbulence, the relation

$$Z(k) = k^2 E(k), \tag{5}$$

where k is the wave number and E, Z are respectively the energy and vorticity spectra (see e.g. [5]), determines the right increase in Z and leads to an equation for D that can be solved numerically, yielding an estimate $D \sim 2.4$. Equation (5) is a consequence of the definition of vorticity (1b) and of homogeneity. The calculation also yields an estimate of the inertial range exponent close to the Kolmogorov value $\gamma = 5/3$, a fact that is probably of little significance, since Kolmogorov's theory is dimensionally correct and any reasonably self-consistent model should reproduce it. It is significant however

that the Kolmogorov spectrum is not incompatible with intermittency, and that intermittency appears as a necessary consequence of vortex stretching and energy conservation.

There are two distinct length scales connected with the shrinking "active eddies". One is the cube root of the typical volume in which the active eddies are confined and is comparable with the radius of curvature of the vortex tubes; the other is the diameter of the vortex tubes. The second length scale decreases much more slowly than the first, as one can already see from figure 3 where the basic stretching is shown. The first length scale decreases as a result of this stretching by a factor $1/h \geqslant 2$, while the second decreases only by a factor $\sqrt{3} \sim 1.732$. The conclusion must be that as the energy cascade proceeds vortex tubes must divide into distinct subtubes that curve independently. A similar conclusion is reached in [3] by other arguments. This conclusion also constitutes a warning for straightforward vortex calculations in three dimensions: they must have a fine enough resolution to allow for the bursting of vortex tubes.

The time it takes an eddy to break down can be assumed to be comparable with its characteristic time $L/\sqrt{T}$, where L is its linear dimension. The sum of the characteristic times of the sequence of shrinking eddies converges, and thus, within the framework of the model, it takes a finite time t_* for Z to become infinite (see [7] for the details of the argument). To continue the calculation beyond t_* one has to allow for the interaction of several vorticity regions and for energy dissipation; these more elaborate models will be presented elsewhere.

Conclusion

On the basis of the results obtained so far, it is reasonable to predict that lattice vortex models will become a useful tool for investigating the structure of turbulence and a bridge between engineering models and the Euler and Navier-Stokes equations.

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