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INTRODUCTION TO QUANTUM CHROMO TRANSPORT THEORY FOR QUARK-GLUON PLASMAS[1]

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ABSTRACT:

Upcoming heavy ion experiments at the AGS and SPS are aimed at producing and diagnosing a primordial form of matter, the quark-gluon plasma. In these lectures some recent developments on formulating a quantum transport theory for quark-gluon plasmas are introduced.

1. INTRODUCTION

At extreme nuclear densities, as in the cores of neutron stars, or at extreme temperatures, as before the first microsec of the Big Bang, matter may exist in a novel phase according to Quantum Chromodynamics (QCD) [1]-[9]. In that phase hadrons dissolve, strong interactions become weak, and an ideal color conducting plasma of quarks and gluons may be formed. Beginning late 1986 an ambitious new experimental program to discover this novel form of matter will commence at the AGS in the Brooghaven National Lab and at SPS at CERN. The goal is to simulate those extreme conditions by compressing and heating heavy nuclei via violent nuclear collisions at very high energy. At the AGS nuclei will be accelerated to $\sim$ 15 GeV/A (GeV per projectile nucleon). At the SPS nuclear collisions at energies up to 225 GeV/A will be studied. There are plans to build a Relativistic Heavy Ion Collider at BNL with energies up to 100 GeV/A in the center of mass.

Assuming that the phase transition point can be reached at sufficiently high energies, the hope is that the complex final debris of such collisions can provide information on the properties of the quark-gluon plasma. In that debris, there may be 1000 pions and 100 kaons together with 500 nucleons. Occasionally, a high energy photon or a very massive dilepton pair may also emerge. Unfortunately,

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the new phase we wish to study can only exist in nuclear collisions for $\sim 10^{-22}$ sec because the quarks and gluons become confined again into hadrons when the matter density gets below some critical density $\sim 3\rho_0$, where $\rho_0 \approx 0.145$ fm^{-3}. We are thus faced with the proverbial Humpty-Dumpty problem of trying to piece together all the shattered remnants of an elusive state. Whether "all the kings horses and all the kings men can put Humpty together again" depends in this case on how reliably we can formulate a Quantum Chromo Transport (QCT) theory to deal with the dynamical complications of very high energy nuclear collisions. After a decade of experience[10] with nuclear collisions at lower energies ~ 1 GeV/A, we have learned to appreciate the difficulties involved with trying to convert experimental differential cross sections into information about the thermodynamic properties of nuclear matter. The problem is that nuclei are small and nonequilibrium effects can camouflage signatures that would be expected if ideal chemical and thermal equilibrium were reached. To deal with that problem, nonequilibrium transport theories have recently been developed[11] with sufficient sophistication that real progress is finally being made toward the determination of the nuclear equation of state up to several times ρ_0.

Up to now most calculations of the signatures of quark-gluon plasmas have been made with models assuming local equilibration[1]-[4]. While those calculations have provided valuable guidance on what observables may provide the most information about the quark-gluon plasma, we should not be surprised to find discrepancies with the upcoming data. During the next several years, when only relatively light ions such as O^{16} and S^{32} can be accelarated, finite size nonequilibrium effects will be very important to consider. In the 1 GeV/A range we know that signatures of collective phenomena only become clear for nuclei with $A > 100$. Even in such heavy systems, 30-50% corrections due to finite surface effects are found. Thus, it is necessary to develop calculational methods that can deal with nonequilibrium conditions. Furthermore, in very high energy collisions the very rapid time scale of the collisions may make it necessary to incorporate quantum effects. Of course, we expect that QCD contains somehow all the answers. The problem is how to extract them from this subtle and still little understood theory. Unlike transport theory at lower energies based on known hadronic degrees of freedom and Yukawa interactions, the quark-gluon transport theory QCT is still in its infancy and many questions remain to be resolved. The aim of these lectures is to introduce some recent developments in QCT based on work in Refs. [12,13,14,15]. These lectures are not intended to provide a comprehensive review. For a broader perspective see the many excellent reviews in [1]-[4].

2. THE QUARK-GLUON PLASMA

Even without knowing about QCD, hadron phenomenology itself indicates that the properties of matter could change drastically at energy densities an order of

magnitude above that in ground state nuclei, $\epsilon_0 \approx m_N \rho_0 \approx 0.15 \mathrm{GeV/Fm}^3$. Consider zero baryon density matter. The energy density as a function of temperature T is given by

$$\epsilon(T) = \int dm \rho(m) e^{-m/T} \left(\frac{mT}{2\pi}\right)^{3/2} (m + \frac{3}{2}T + \cdots) \ , \tag{1}$$

where the density of hadronic states is approximately given by the Hagedorn form, $\rho(m) \sim m^{-a} e^{m/T_0}$ with $a \approx 5/2$ and $T_0 \approx m_\pi \approx 140$ MeV. Because of the exponential rise in the number of states ϵ and also the partition function $Z = \mathrm{tr}\ e^{-H/T}$ have an essential singularity at a finite critical temperature $T = T_0$. Taking the finite volume of hadrons into account as in the MIT Bag Model via the relation

$$V(m) = \frac{m}{4B} \ , \tag{2}$$

where $B \sim 200$ GeV/Fm^3 is the vacuum energy density, the energy density of matter in the excluded volume approximation was shown in Ref.[16] to be

$$\epsilon_B(T) = \frac{\epsilon(T)}{1 + \epsilon(T)/4B} \ , \tag{3}$$

where $\epsilon_H(T)$ is given by the point particle expression (1). With (3) the partition function has an essential singularity at the finite energy density

$$\epsilon_Q = \epsilon_B(T_0) = 4B \sim 1 \ \mathrm{GeV/Fm}^3 \ . \tag{4}$$

This suggests the end of the hadronic world and the breakdown of hadronic matter theory is not far from the domain of ordinary nuclei.

To discuss the properties of matter at energy densities beyond ϵ_Q, we must turn to QCD. In this standard model there are $N = 3$ colored quark and $N^2 - 1$ colored vector gluon fields that couple in a gauge invariant way under the color group, $SU(N)$. There are $N_f \approx 3$ flavors of quarks. In the absence of interactions, the energy density of this Stefan-Boltzmann gas of quarks and gluons is simply given by

$$\epsilon_{SB}(T,\mu) = \frac{\pi^2}{15}(N^2 - 1 + 7NN_f/4)T^4 + \frac{NN_f}{2}(T^2\mu^2 + \frac{1}{2\pi^2}\mu^4) \ . \tag{5}$$

with T, μ being the temperature and chemical potential. The pressure of this ideal relativistic gas is just $p_{SB}(T,\mu) = \frac{1}{3}\epsilon_{SB}(T,\rho)$. In perturbation theory the corrections to this ideal pressure have been computed to leading orders. Kapusta [17] calculated that for $\mu = 0$

$$p(T) = p_{SB}(T,0) - \frac{g^2T^4}{16}\frac{1}{9}(N^2-1)(N_c + \frac{5}{4}N_f) + \frac{g^3T^4}{12\pi}(N^2-1)(\frac{1}{3}(N+N_f/2))^{3/2} + \cdots \ . \tag{6}$$

The basic problem is to determine under what conditions *if any* perturbation theory could apply, i.e., when is g sufficiently small? The effective coupling strength in the plasma depends on the interplay between vacuum fluctuations that are responsible for asymptotic freedom at small distances and Debye screening that reduces the coupling at large distances. Recall that vacuum polarization and vertex corrections in QCD lead to the famous asymptotic freedom formula

$$\frac{g^2}{4\pi} = \alpha_{eff}(q) = \frac{4\pi}{(\frac{11}{3}N - \frac{2}{3}N_f)\ln(-q^2/\Lambda^2)} \quad . \tag{7}$$

where $\Lambda \sim 200$ MeV is a nonperturbative physical constant that appears in the theory on account of renormalization. The coupling depends on the momentum transfer, q, corresponding to the inverse of the characteristc interaction distance. For large q or small distances the coupling vanishes because of the impossibility of localizing a color charge. The gluons, so long as $N_f < \frac{11}{2}N$, smear the color charge of a quark over a scale $\sim \Lambda^{-1}$. So if we look very close to a heavy quark we will only find a small fraction of its original color charge.

On the other hand, at large distances the coupling becomes large. In fact, (7) has a singularity in perturbation theory at finite momentum transfers $t = -\Lambda^2$. Nonperturbative analysis suggests that α_{eff} should have a simple pole at $q^2 = 0$ instead corresponding to a linearly rising potential at large distances. In a many body medium not only by vacuum fluctuations but also by the polarizability of the medium modify the gluon propagator. In the one loop approximation [18]-[20] the the $\mu = \nu = 0$ part of the static ($\omega_0 = 0$) gluon propagator is modified such that the infrared singularity is screened away via

$$D^{00}(t) \approx (t - m_E^2)^{-1} \quad , \tag{8}$$

where the color electric mass is given by

$$m_E^2 \approx g^2 \left[(N_c + N_f/2)T^2/3 + (1/2\pi^2) \sum_{f=1}^{N_f} \mu_f^2 \right] \quad , \tag{9}$$

in terms of the termperature T and flavor chemical potentials μ_f. Of course, as T and μ tend to zero this many body correction tends to vanish. This modification of the gluon propagator implies from linear response theory that static color electric fields are screened on a length scale m_E^{-1}, called the Debye length. The physical origin of such screening is clear. Since the particles in the plasma carry color electric charge, a cloud of particles of opposite charge tend to accumulate around any external charge put into the plasma. On very small distance scales or large q the properties of the many body medium are of course irrelevant and the effective coupling just follows the asymptotic freedom form.

Thus in a plasma only we only need to consider the coupling on distance scales $\lesssim m_E^{-1}$. As the temperature or density increases that scale decreases and thus

the basic asymptotic freedom property causes the coupling to decrease. Thus, we see that perturbation theory might apply at extreme energy densities with an effective coupling $\alpha_{eff} \sim 1/\log(T/\Lambda)$. In perturbation theory, (5,6) the properties of matter at extreme energy densities would become very simple.

However, there is real trouble with QCD perturbation theory in higher orders associated with the lack of perturbative color magnetic screening as discussed in Refs.[21]. The thermodynamic potential diverges in the infrared limit for orders $n \geq 6$. Diagrams involving higher order interacting magnetic gluons, A^i, diverge because of the infinite range of color magnetic interactions in perturbative QCD. The problem is that there are no particles in the theory which carry color magnetic charge to screen such interactions. Color electric interactions are on the other hand easily Debye screened by the quarks and gluons which carry color electric charge. Screening of long range magnetic interactions requires nonperturbative effects in the plasma. A recent proposal[23] is that color magnetic monopoles could be spontaneously created to regulate the magnetic sector of QCD. Given such a "cure" of higher order infrared divergences, it may turn out in practice that lowest order perturbation theory could give good qualitative results.

Recent results using nonperturbative lattice techniques tend to confirm the qualitative picture of dense matter painted above. For details we refer to [1]-[9]. While the numerical results have still not converged to everyone's satisfaction, the following conclusions seem to be emerging. At some critical temperature, $T_c \sim \Lambda$, there is a rapid increase of the energy density with temperature until an energy density of a few GeV/Fm^3 is reached. Whether the transition is first or higher order remains unsettled. For $T > T_c$ the Stefan-Boltzmann law seems to hold remarkably accurately. There are some warnings, however, that this may be only accidental because there are indications the quasi-particles in the plasma are not the naive massless quarks and gluons[23,24]. In the absence of dynamical quarks, calculations[22] have now been performed on sufficiently finely grid lattices that the continuum limit of lattice theory is thought to be under control. The inclusion of dynamical quarks remains the greatest challenge of the lattice world and only calculations on very small lattices are practical at this time. It may take a new generation of computers to settle the issue with quarks (see Weingarten in [4]).

So where does this leave us from a phenomenological point of view? The most certain point is that qualitatively new phenomena will occur in hadronic matter at an energy density scale about one order of magnitude above ϵ_0. The exact nature of those phenomena depend on the quasi-particles and effective interactions that still remain to be determined. In constructing a quark-gluon plasma transport theory these uncertainties must always be kept in mind.

3. PLASMA INITIAL CONDITIONS

Unlike the controlled theoretical laboratory where the properties of the quark-

gluon plasma can be studied at leisure on a lattice, heavy ion experiments force us to consider plasmas in highly dynamical situation. In particular we need to get some idea of possible initial configurations the plasma may be produced in. Then we could go on to consider the dynamical evolution of that plasma to the detectors. We will consider here one particular model for the initial conditions based on ideas in string models [25]-[29]. We begin by reviewing the calculation[12] pair production in covariant constant SU(N) fields. We show that in the one loop approximation the quark gluon systems behaves as a simple multicomponent Abelian plasma.

In string or color rope models multiparticle production arises because multiple soft gluon exchange leaves both projectile and target in color non-singlet states. This leads to the creation of a color electric field, which due to confinement extends between the receeding projectile and target fragments in narrow color flux tubes of finite transverse area $A_\perp \sim 1\text{fm}^2$. The field is assumed to be a constant along the beam direction with a strength $E \sim gQ/A_\perp$, where Q is the net "charge" on the projectile. In *Abelian* string models [25,26,28,29], Q is treated simply as an integer.

The string field configuration is however unstable with respect to pair production at a rate per unit volume, $w_{1/2}(\sigma, m)$, for $q\bar{q}$ pairs of mass, m, is taken from the well known Schwinger formula [25].

$$w_{\frac{1}{2}}(\sigma, m) = \theta(\sigma)\frac{\sigma}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}\int_{m^2}^{\infty} dE_\perp^2 \exp(-n\pi E_\perp^2/\sigma) \approx \theta(\sigma)\frac{\sigma^2}{24\pi} \ , \tag{10}$$

with $\theta(x) = 0(1)$ for $x < (>)0$, and where the approximation holds for $\pi m^2/\sigma \ll 1$. In the above expression $\sigma = gE$ is the effective string tension, which phenomenologically is taken to be on the order of 1 GeV/fm. With (10) the rate with which the external "color" field is neutralized due to $q\bar{q}$ production can be estimated as in Ref.[29]. First we recognize that conservation of energy requires that in order to produce a pair, each with transverse energy $m_\perp$ and zero longitudinal momentum, that pair must separate by at least a distance $r_c \sim 2m_\perp/\sigma$. The minimum volume for pair production is thus $\delta V \sim A_\perp 2m_\perp/\sigma$. Once a pair has been produced in this finite volume element the field between them is reduced by $\delta E = g/A_\perp$ if we assume that all flux lines are confined to the original flux tube. With this crucial model assumption we can then estimate [29]

$$\frac{dE(t)}{dt} \approx -\frac{g}{A_\perp} w_{\frac{1}{2}} \delta V \propto \sigma\langle m_\perp\rangle = -aE(t)^{3/2} \ , \tag{11}$$

where a is a positive constant. The power 3/2 follows from dimensional considerations since locally $E(t)$ is the only dimensioned quantity that can set the time scale (also $\langle m_\perp\rangle \propto E^{1/2}$ for the same reason). Writing $E(t) = gQ(t)/A_\perp$, the solution of (11) is simply

$$Q(t) = Q_0/(1 + t/\tau_{1/4}(Q_0))^2 \ , \tag{12}$$

with $\tau_{1/4} = \tau_1/\sqrt{Q_0}$ in terms of the characteristic time $\tau_1 \sim 1$ fm/c required to neutralize 3/4 of the field strength in an elementary string with unit charge. The most important feature of the above solution is that it shows that the neutralization time of strong ($Q_0 \gg 1$) Abelian fields *decreases* as $Q_0^{-1/2}$. This is good news from the point of view of creating a quark-gluon plasma. The faster the fields neutralize the more time will there be for the quarks and gluons to come into chemical and thermal equilibrium and thus to produce interesting signatures [1]-[4] of this new state of matter.

Encouraged by the above Abelian analysis we turn next to investigate non-Abelian aspects of the problem. In particular, how do we treat the non-Abelian character of the charges of both quarks and gluons? How do quarks and gluons compete in the neutralization process? How close is the resulting quark-gluon plasma to local equilibrium conditions at time $\tau_{1/4}$? To answer these questions we compute the pair production rates for SU(N) in the one-loop $O(\hbar)$ approximation [12].

In conventional notation the generators, $t_a, a = 1, \cdots, N^2 - 1$, of SU(N) in the fundamental representation satisfy $[t_a, t_b] = if_{abc}t_c$, $tr(t_a t_b) = \delta_{a,b}/2$. The gluon field matrix is denoted by $A_\mu \equiv A_\mu^a t_a$, the covariant derivative matrix by $D_\mu = \partial_\mu + igA_\mu$, and the field tensor by $F_{\mu\nu} = [D_\mu, D_\nu]/(ig)$. The Heisenberg field equations for quarks and gluons are

$$(i\gamma^\mu D_\mu - m_f)\psi_f = 0 \ , \tag{13}$$

$$[D^\mu, F_{\mu\nu}] = gJ_\nu \ , \tag{14}$$

where $J_\nu = \sum_f \bar{\psi}_f \gamma_\nu t_a \psi_f t_a$ and f labels the quark flavors.

A covariant-constant field [30], which satisfies Eq.(14) in the source free region, is of the form

$$F_{\mu\nu} = \langle F_{\mu\nu}\rangle U(x) n_a t_a U^\dagger(x) \ , \tag{15}$$

where $\langle F_{\mu\nu}\rangle$ is independent of x_μ, and n_a is an N^2-1 dimensional color vector. The unitary matrix $U(x) = \exp(i\theta_a(x)t_a)$ implements arbitrary local gauge transformations. The covariant derivative is given by $D_\mu = U(x)(\partial_\mu - \frac{1}{2}ig\langle F_{\mu\nu}\rangle x^\nu n_a t_a)U^\dagger(x)$.

Since $n_a t_a$ is Hermitian, there exists an x_μ *independent* matrix $V \in$ SU(N) such that $V n_a t_a V^\dagger$ is diagonal. We can thus transform the external field into diagonal form by making a gauge transformation $G = V^\dagger U^\dagger(x)$ under which $D_\mu \to G D_\mu G^\dagger$. Since all real $N \times N$ traceless diagonal matrices can be expanded in terms of the $N-1$ diagonal matrices, h_i, representing the Cartan subgroup of SU(N), it is then most convenient to expand A_μ in that gauge in terms of the Cartan-Weyl basis of SU(N). That basis consists [32] of $N-1$ Abelian generators, h_i, and the $N(N-1)$ non-Abelian generators, $\{e_{ij},\ i,j = 1, \cdots, N;\ i \neq j\}$, which satisfy

$$[h_i, h_j] = 0 \ , \ [h_i, e_{jk}] = (\vec{\eta}_{jk})_i e_{jk} \ , \ [e_{ij}, e_{jk}] = \frac{1}{\sqrt{2}} e_{ik} \quad \text{for } i \neq j \neq k \ . \tag{16}$$

The h_j are the Gell-Mann matrices $h_j = (2j(j+1))^{-\frac{1}{2}} diag(1, \cdots, 1, -j, 0, \cdots, 0)$, with $-j$ appearing in the $j+1$ column. The $\vec{\eta}_{ij} = \vec{\epsilon}_i - \vec{\epsilon}_j$ are the root vectors of SU(N) as expressed in terms of the elementary weight vectors $\vec{\epsilon}_i = (\vec{h})_{ii} = ((h_1)_{ii}, \cdots, (h_{N-1})_{ii})$. (Note that $e_{ij} = \hat{e}_i \hat{e}_j^\dagger / \sqrt{2}$ in terms of the N orthonormal unit vectors $\hat{e}_i$.)

In the gauge which diagonalizes $\langle A_\mu \rangle$, the external field can be expressed in terms of $N-1$ Abelian components, $\vec{H}^\mu = (H_1^\mu, \cdots, H_{N-1}^\mu)$, as

$$\langle A^\mu \rangle = \sum_{i=1}^{N} H_i^\mu h_i \equiv \vec{H}^\mu \cdot \vec{h} \ , \tag{17}$$

where $\vec{h} \equiv (h_1, \cdots, h_{N-1})$. To take quantum fluctuations around this external field into account, the gluon field operator can then be written as

$$A_\mu = \langle A_\mu \rangle + B_\mu = \vec{H}_\mu \cdot \vec{h} + B_\mu \ , \tag{18}$$

where B_μ represents the quantum fluctuations.

The physical significance of $\vec{\epsilon}_i$ can be seen from Eq.(13) by considering the equation of motion for the tranformed quark field, $\psi' \equiv G^\dagger \psi$. Eq.(13) then reduces to the set of equations

$$(\gamma_\mu (i\partial^\mu - g\vec{\epsilon}_c \cdot \vec{H}^\mu) - m_f)\psi_c' = O(B\psi') \ . \tag{19}$$

The approximation of neglecting higher order quantum fluctuations beyond the one loop order is equivalent to neglecting the $O(B\psi')$ terms on the right hand side of (19). We therefore see that the equations for the N quarks (of each flavor) in the prime basis decouple and reduce to Abelian type equations in the one loop approximation with $\vec{H}^\mu$ playing the role of an effective electromagnetic potential that couples to quarks with effective "charges" $g\vec{\epsilon}_c$. Since we already know (10) the pair creation rate, $w_{\frac{1}{2}}(eF^{\mu\nu}; m)$, of fermions in an external Abelian field $F^{\mu\nu}$, we can immediately write down the pair creation rate per unit volume of ψ_c' quarks of flavor f as

$$w_{q_{c,f}} = w_{\frac{1}{2}}(g\vec{\epsilon}_c \cdot \vec{F}^{\mu\nu}; m_f) \ , \tag{20}$$

where $\vec{F}^{\mu\nu} = \partial^\mu \vec{H}^\nu - \partial^\nu \vec{H}^\mu$ is a constant $N-1$ dimensional vector.

Turning next to gluons, the equations of motion for B^μ in the one loop approximation are obtained by linearizing Eq.(14) in B^μ. It is most convenient to expand B^μ in the Cartan-Weyl basis as

$$B^\mu \equiv B_a^\mu t_a = \vec{C}^\mu \cdot \vec{h} + \sum_{i \neq j=1}^{N} W_{ij}^\mu e_{ij} \ . \tag{21}$$

Inserting (18) into Eq.(14) and using the Cartan-Weyl expansion (21) for B^μ together with the algebra (16) leads to the following equations of motion for the

fluctuations $\vec{C}^\mu$ and W^μ_{mn} in the (linearized) one loop approximation to

$$\partial_\mu(\partial^\mu \vec{C}^\nu - \partial^\nu \vec{C}^\mu) = 0 \ , \tag{22}$$

$$(D_{mn})_\mu(D^\mu_{mn}W^\nu_{mn} - D^\nu_{mn}W^\mu_{mn}) - (W_{mn})_\mu[D^\mu_{mn}, D^\nu_{mn}] = 0 \ , \tag{23}$$

where the effective covariant derivative D^μ_{mn} is given by

$$D^\mu_{mn} = \partial^\mu + ig\vec{\eta}_{mn} \cdot \vec{H}^\mu \ . \tag{24}$$

We therefore see that the Abelian fluctuations, $\vec{C}^\mu$, obey free field equations whereas the non-Abelian fluctuations, W^μ_{mn} obey Abelian vector field equations in the external field, $\vec{H}^\mu$, with an anomalous magnetic moment coupling [31]. Note that $[D^\mu_{mn}, D^\nu_{mn}] = ig\vec{\eta}_{mn} \cdot \vec{F}^{\mu\nu}$. Obviously these equations decouple in this approximation. The effective "charge" of the W^μ_{mn} gluon is given by $g\vec{\eta}_{mn}$. Pair production in SU(N) covariant constant fields is thus equivalent to $N(N-1)/2$ different SU(2) problems. Therefore, the pair creation rate per unit volume of $W_{mn}W_{nm}$ gluon pairs can be calculated from the known [30,31] rate, $w_1(gF^{\mu\nu})$, of vector mesons for SU(2) covariant constant fields as

$$w_{g_{mn}} = w_1(g\vec{\eta}_{mn} \cdot \vec{F}^{\mu\nu}) \ , \tag{25}$$

where $\vec{F}^{\mu\nu}$ is the same external covariant constant SU(N) field as in Eq.(20) and the spin 1 rate is given by [33]

$$w_1(\sigma) = \theta(\sigma)\frac{\sigma}{4\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^\infty dp_\perp^2 \exp(-n\pi p_\perp^2/\sigma) = \frac{1}{2}w_{\frac{1}{2}}(\sigma;0) \ . \tag{26}$$

The case of particular interest in phenomenological applications [25,27] corresponds to constant color electric fields created between interacting partons in high energy collisions. For that case $\vec{F}^{30} = -\vec{F}^{03} = \vec{E} = \vec{Q}E_0$, where $E_0 = g/A_\perp$ for a flux tube of transverse area $A_\perp$ and $\pm\vec{Q}$ are the effective color charges of the projectile and target. The elementary $q\bar{q}$ string corresponds in this picture to $\vec{Q} = \vec{\epsilon}_c$. The adjoint $g\bar{g}$ string corresponds to $\vec{Q} = \vec{\eta}_{ij}$. Note that $\vec{\eta}\cdot\vec{\eta}/(\vec{\epsilon}\cdot\vec{\epsilon}) = 2N/(N-1)$.

In high energy nuclear collisions [27] or very high energy hadronic collisions multiple soft gluon exchange may lead to large effective charges. If $\mathcal{N}$ gluons are exchanged with random charges, then $\langle Q^2\rangle = \mathcal{N}(1+N^{-1})^{-1}$ since only $N(N-1)$ of the N^2-1 gluons are "charged" with $\vec{\eta}_{ij}\cdot\vec{\eta}_{ij} = 1$. Of course such color electric fields are unstable against pair production as we saw above. The pair production rate per unit volume of massless quark-antiquark and gluon pairs is given from Eqs.(20,25) by

$$w_q = \frac{N_f g^2}{24\pi}\sum_{c=1}^{N}(\vec{\epsilon}_c\cdot\vec{E})^2 = \frac{Q^2}{2}N_f w_0 \ , \quad w_g = \frac{g^2}{48\pi}\sum_{i>j=1}^{N}(\vec{\eta}_{ij}\cdot\vec{E})^2 = \frac{Q^2}{4}N w_0 \ , \tag{27}$$

where $w_0 = (gE_0)^2/(24\pi)$ and N_f is the number of quark flavors such that $\pi m_f^2/\sigma \ll 1$.

We can now answer one of our initial questions. The ratio of quarks to gluons just after the color field is neutralized is $(q/g)_{neut} \approx 2N_f/2N = 1$ for $N = N_f = 3$. In comparison, local equilibrium at zero chemical potential is characterized by $(q/g)_{equil} = (\frac{3}{4}NN_f)/(N^2-1) = 0.84$. Thus, this ratio is surprisingly close to equilibrium in spite of the fact that non-equilibrium tunneling was involved. Note however that no "neutral" gluons are produced via pair production.

Generalizing the discussion in the Abelian case, we can estimate the rate of neutralization for strong covariant constant fields ($Q^2 \gg \eta^2 = 1$) by

$$\frac{d\langle\vec{E}\rangle}{dt} \approx -\sum_f \sum_{\vec{\epsilon}_i} \frac{g\vec{\epsilon}_i}{A_\perp} w_{\vec{\epsilon}_i} \delta V_{\vec{\epsilon}_i} - \sum_{\vec{\eta}_{ij}} \frac{g\vec{\eta}_{ij}}{A_\perp} w_{\vec{\eta}_{ij}} \delta V_{\vec{\eta}_{ij}} \ , \tag{28}$$

where the pair production rates per unit volume and effective volume elements, $\delta V_{\vec{q}}$, depend on the particular charges, $\vec{q}$, as well as on the mean external charge, $\langle\vec{Q}(t)\rangle = \langle\vec{E}(t)\rangle A_\perp/g$. The minimum volume elements for pair production are again constrained by energy conservation [29] with now $r_c = 2m_\perp/\sigma(\vec{q},\vec{Q})$ and where the effective string tension is given by

$$\sigma(\vec{q},\vec{Q}) = g\vec{q}\cdot\vec{E} = 2\vec{q}\cdot\vec{Q}\sigma_A \ . \tag{29}$$

Here $\sigma_A = g^2/(2A_\perp)$ is the effective string tension for adjoint strings. The factor of two arises due to interactions between the pairs as discussed in Refs.[12,26]. The average volume required for pair production is therefore

$$\delta V_{\vec{q}} \approx r_c A_\perp = \frac{2\langle m_\perp\rangle}{\sigma(\vec{q},\vec{Q})} A_\perp \ . \tag{30}$$

The neutralization rate is dominated by production of pairs with small mass ($\pi m^2/\sigma \ll 1$), for which $\langle m_\perp\rangle_{s=\frac{1}{2}} = (2-2^{-1/2})^{-1}\langle m_\perp\rangle_{s=1}$. Therefore, Eq.(28) reduces to

$$\frac{d\langle\vec{Q}(t)\rangle}{dt} \approx -\frac{2^{3/2}}{\tau_1}\left[N_f \sum_{\pm\vec{\epsilon}_i}{}' \vec{\epsilon}_i(\vec{\epsilon}_i\cdot\langle\vec{Q}\rangle)^{3/2} + (1-2^{-3/2})\sum_{\vec{\eta}_{ij}}{}' \vec{\eta}_{ij}(\vec{\eta}_{ij}\cdot\langle\vec{Q}\rangle)^{3/2}\right] \ , \tag{31}$$

where $\tau_1 \propto (g^2\sqrt{\sigma_A})^{-1}$ is independent of Q and where the sums are restricted to charges with $\vec{q}\cdot\vec{Q} > 0$.

Eq.(31) controls the rate of color neutralization of strong covariant constant SU(N) fields and is the natural generalization of the equation derived for the Abelian case in Ref.[29]. The SU(N $\geq$ 3) case is only complicated by the fact that Eq.(31) is a vector equation. The power 3/2 again follows from dimensional considerations under the assumption that the only dimensional quantity in the problem is $\vec{E}$.

For $\vec{Q}(0)$ pointing along one of the weight or root vectors Eq.(31) reduces to an Abelian equation

$$dQ/dt = -Q^{3/2}/\tau(\hat{Q}) \ , \tag{32}$$

where the relaxation time is given in the two special cases by

$$\begin{aligned} \tau(\hat{\epsilon}) &= \tau_1((N-1)/2N)^{1/4}\left[N_f((1-N^{-1})^{3/2}+N^{-3/2})+(1-2^{-3/2})N\right]^{-1} \\ \tau(\hat{\eta}) &= \tau_1\left[N_f+(1-2^{-3/2})(N+2^{3/2}-2)\right]^{-1} \end{aligned} \tag{33}$$

The terms proportional to N_f are those due to $q\bar{q}$ pair production. For SU(2) the two times are of course identical. Amazingly, for SU(3) they only differ in the fourth decimal place for $N_f = 3$.

The numerical coincidence of $\tau(\hat{\epsilon})$ and $\tau(\hat{\eta})$ for SU(3) has the pleasant consequence that $\tau(\hat{n}) \approx 0.18\tau_1$ is independent of the orientation of the color charge in the Cartan subspace to a very high accuracy. Therefore for SU(3), the vector nature of Eq.(31) is irrelevant, and the solution is accurately given by (12) with Q replaced by the $N-1$ dimensional Cartan charge vector, $\vec{Q}$ and with the characteristic time required to neutralize 3/4 of the initial color field replaced by

$$\tau_{1/4}(Q_0) = 0.36\ \tau_1\ Q_0^{-1/2} \ . \tag{34}$$

The factor 0.36 represents the combined effect of both $q\bar{q}$ and $g\bar{g}$ production and significantly shortens the neutralization time over the Abelian case where only $q\bar{q}$ contribute.

There are several interesting phenomenological consequences of the above results in connection with ultra-relativistic nuclear collisions. First, most of the quarks and gluons produced in the neutralization of strong color fields may appear at proper times an order of magnitude smaller than in elementary pp or e^+e^- collisions. Obviously shorter times imply that plasmas with higher initial energy densities can be produced [29] and the chances that local equilibrium can be reached are greater. On the other hand, the equilibration time due to ordinary kinetic phenomena could be long compared to the neutralization time [34]. Fortunately, the color neutralization mechanism leads to initial conditions that are not far from local equilibrium. First, as we noted, in strong fields the neutralization mechanism leads to production of quarks and gluons at comparable rates (for SU(3)). Furthermore, u,d,s quarks are produced with nearly the same abundance since their masses become irrelevant. Thus the chemical composition of the non-equilibrium plasma produced through neutralization is not far from that in local equilibrium. Second, the distributions of transverse momenta from (10,26) are approximately exponential as in local equilibrium although gluons materialize with ~30% larger transverse momentum than quarks due to their larger effective charge. Thus we conclude that the non-equilibrium quantum tunneling dynamics

and the specific non-Abelian features of gluon pair production may play an important role in creating plasma initial conditions close to local equilibrium at very early times in the collision.

4. QUANTUM CHROMO TRANSPORT THEORY

Once the quarks and gluons are produced, they are subject to accelerations in the neutralizing external field and scatter with each other as the plasma evolves closer toward equilibrium. To follow that evolution in detail we must turn to transport theory. First, we recall the familiar classical transport theory leading to the Vlasov- Boltzmann equation. Then the Wigner function and its equation of motion are derived in non relativistic quantum mechanics. The formulation of gauge covariant quantum transport theory is then introduced. A generalized quantum Vlasov and associated constraint equation are derived in connection with Abelian QED. Finally, features of non-Abelian transport theory are analyzed in the Abelian Dominance Approximation.

4.1 Classical Transport Theory

A classical plasma is characterized by the phase space density, $f(x,p,t) = \sum_i \delta(x - x_i(t))\delta(p - p_i(t))$, where the classical trajectories $(x_i(t), p_i(t))$ obey

$$\dot{x}_i = p_i/m \ , \ \dot{p}_i = F_{ext}(x_i) + F_2(x_i) \ , \tag{35}$$

where F_{ext} is the external force and $F_2(y) = \int d^3x d^3p \nabla_x V(y-x) f(x,p,t)$ is the self-consistent force that is a functional of the phase space density. From (35) we see that

$$\dot{f} = \sum_i (\dot{x}_i \nabla_{x_i} + \dot{p}_i \nabla_{p_i})\delta(x - x_i(t))\delta(p - p_i(t)) = -(\frac{p}{m} \cdot \nabla_x + F(x) \cdot \nabla_p) f(x,p,t) \ . \tag{36}$$

Taking the ensemble average of this (Klimontovich) equation leads to the Vlasov-Boltzmann equation:

$$(md/dt + p \cdot \nabla_x + m\langle F(x)\rangle \cdot \nabla_p)\langle f(x,p,t)\rangle = C(\langle f\rangle) \ , \tag{37}$$

where the mean force is given by $F_{ext} + F_2$ with f replaced by its ensemble average, $\langle f\rangle$, and where the collision term can be extracted [35] from the correlation term

$$C(f) = -m \int d^3x' d^3p' \nabla_{x'} V(x-x') \nabla_{p'} (\langle f(x,p,t) f(x',p',t)\rangle - \langle f(x,p,t)\rangle\langle f(x',p',t)\rangle) \ . \tag{38}$$

Neglecting two-body correlations in the plasma is equivalent to neglecting the collision term. In that case (37) reduces to the Vlasov equation which applies only to "collisionless" plasmas. What makes the collision term particularly difficult to calculate in QED is the infinite Coulomb cross section. Medium polarization effects that screen long range fields must then be taken into account. In addition a physical assumption must be made regarding different relaxation time scales for correlation functions (the BBGKY hierarchy). This procedure leads in the quasi-linear approximation to the Balescu-Lenard form for the collision term involving

the dielectric function.

4.2 Quantum Transport

The quantum mechanical analogue of the phase space density is the Wigner function [36]. We know that $\psi^\dagger(x)\psi(x)$ is the density of particles in coordinate space. Also $\psi^\dagger(p)\psi(p)$ is the density in momentum space. A natural candidate for the density in (x,p) phase space is then

$$W(x,p) = \psi^\dagger(x)\delta(p-\hat{p})\psi(x) \ , \tag{39}$$

where $\hat{p}$ is the momentum operator. The delta function acts as a projector onto momentum space and is defined by the Fourier transform as

$$W(x,p) = \int \frac{d^3y}{(2\pi\hbar)^3} e^{-ipy/\hbar}\psi^\dagger(x)e^{iy\hat{p}/\hbar}\psi(x) \ . \tag{40}$$

Recall that $\exp(iy\hat{p}/\hbar)$ generates a translation by y when acting to the right or $-y$ when acting to the left. In order to make W Hermitian we will make it act halfway to the left and halfway to the right by representing $\hat{p}$ as $\frac{1}{2}i\hbar(\partial_x - \partial_x^\dagger)$. In this way we recover the familiar expression for the Wigner function

$$W(x,p) = \int \frac{d^3y}{(2\pi\hbar)^3} e^{ipy/\hbar}\psi^\dagger(x+\tfrac{1}{2}y)\psi(x-\tfrac{1}{2}y) \ , \tag{41}$$

in terms of a mixed Fourier transform. The second quantized Wigner operator, $\hat{W}$, is then obtained by replacing $\psi(x)$ by the Heisenberg field operator $\hat{\psi}(x)$. The advantage of second quantization is of course that it allows us to pass from a single particle to many body quantum theory. The ensemble average of $\hat{W}$ is what corresponds to the classical phase space density.

The equation of motion for $\hat{W}$ obviously follows from the Schrödinger equation, $(i\hbar\partial_t - \hbar^2\nabla^2/2m + \hat{U}(x))\hat{\psi}(x) = 0$. To proceed we compute $p\cdot\nabla\hat{W}(x,p)$. Within the integration, p can be replaced by $-i\hbar\nabla_y$ after integrating by parts. Then, since $\nabla_y\cdot\nabla_x = \frac{1}{2}(\nabla^2_{x+y/2} - \nabla^2_{x-y/2})$, we obtain using the Schrödinger equation

$$p\cdot\nabla_x\hat{W} = m\int \frac{d^3y}{(2\pi\hbar)^3} e^{ipy/\hbar}(-\partial_t + \frac{i}{\hbar}[\hat{U}(x+\tfrac{1}{2}y) - \hat{U}(x-\tfrac{1}{2}y)])\hat{\psi}^\dagger(x+\tfrac{1}{2}y)\hat{\psi}(x-\tfrac{1}{2}y) \ . \tag{42}$$

Expanding in Taylor series, $U(x+\frac{1}{2}y) = \exp(\frac{1}{2}y\nabla_x)U(x)$, we can pull the exponential out of the integral by replacing y by $-i\hbar\nabla_p$ term by term in the series. With this trick, the right hand side of (42) can be written formally as

$$m(-\partial_t + \frac{2}{\hbar}[\sin(\tfrac{1}{2}\hbar\triangle)\hat{U}])\hat{W} \ ,$$

where we introduced [13] the "triangle" operator, $\triangle \equiv \nabla_x \cdot \nabla_p$ with ∇_x only acting on the potential and ∇_p only acting on the Wigner operator. In this way we have derived the non-relativistic quantum transport equation (see also [44])

$$(m\partial_t + p \cdot \nabla_x - \nabla_x \hat{U} \cdot \nabla_p)\hat{W}(x,p) = Q(\hbar\triangle)\hat{U}(x)\hat{W}(x,p) \ , \tag{43}$$

where the "quantum" operator is given by

$$Q(\hbar\triangle) = \frac{2}{\hbar}\sin(\tfrac{1}{2}\hbar\triangle) - \triangle = -\frac{\hbar^2}{24}\triangle^3 + O(\hbar^4) \ . \tag{44}$$

In the $\hbar \to 0$ limit we recover the Vlasov-Boltzmann equation. The Vlasov part follows from approximating the ensemble average of $\hat{U}\hat{W}$ on the l.h.s. by $\langle\hat{U}\rangle\langle\hat{W}\rangle$. The collision term is contained in the correlation function $C(x,p) = \triangle(\langle\hat{U}\hat{W}\rangle - \langle\hat{U}\rangle\langle\hat{W}\rangle)$ just as in the classical case. The terms on the right hand side correspond to genuine quantum corrections. It is however important to note that quantum corrections also occur in the collision term, C, since quantum fluctuations affect correlation functions [37].

The importance of quantum corrections in the collisionless domain depends on the magnitude of the dimensionless ratio $\hbar\triangle \sim \hbar/(\Delta R_U \Delta P_W)$, where ΔR_U is the characteristic *spatial* scale of variation of the potential and where ΔP_W is the characteristic *momentum* scale over which $W(x,p)$ varies appreciably. Classical Vlasov transport theory thus applies only for relatively slowly varying potentials and phase space densities. In Ref. [37] it was shown that quantum corrections modify the collision terms if the density is so high that the two body scattering rate approaches $\hbar/\Omega$, where Ω is the characteristic single-particle energy in the plasma.

4.3 Gauge Covariant Wigner Operator

Turning next to gauge theories we are faced with the new complication of formulating transport theory in a gauge invariant way. Recall that under a general non-Abelian gauge transformation, $G(x) = \exp(i\theta_a(x)t_a)$, the fields, $\psi(x)$ and $A_\mu(x)$, in the fundamental and adjoint representations transform as

$$\psi(x) \to G(x)\psi(x) \ , \ A_\mu(x) \to G(x)A_\mu(x)G^\dagger(x) + G(x)\partial_\mu G^\dagger(x)/(ig) \ . \tag{45}$$

Thus, the covariant derivative and field tensor transform covariantly, i.e., $D_\mu(x) \to G(x)D_\mu(x)G^\dagger(x)$. The relativistic generalization [38,39] of Wigner operator (39) as given by

$$\hat{W}_{\alpha\beta}(x,p) = \hat{\bar{\psi}}_\beta \delta^4(p - \hat{p})\hat{\psi}_\alpha(x) \tag{46}$$

is not satisfactory for gauge theories because under a gauge transformation:

$$\hat{W} \to \hat{\bar{\psi}}(x)G^\dagger(x) \otimes \delta^4(p - \hat{p})G(x)\hat{\psi}(x) \neq G(x)\hat{W}G^\dagger(x) \ .$$

We must demand that $\hat{W}$ transforms covariantly to insure for example that the momentum distribution, $\int d^4x \langle Tr\hat{W}(x,p)\rangle$, is gauge invariant. The problem is that ordinary derivatives do not commute with $G(x)$. Thus the ordinary derivative of a field, $\partial\psi(x) = \lim(\psi(x+\epsilon) - \psi(x))/\epsilon$ is not well defined in a gauge theory because under a gauge transformation it transforms inhomogeneously, $\partial\psi \to G\{\partial\psi + (G^{-1}\partial G)\psi\}$. The field at neighboring points can be compared only with the *covariant* derivative, $D_\mu(x)$. Under a gauge transformation $D_\mu(x)\psi(x) \to G(x)D_\mu(x)\psi(x)$. This is of course why covariant derivatives were invented in the first place. In addition, the covariant derivative is special because it represents the *kinetic* in contrast to the *canonical* momentum. Recall that in classical electrodynamics the kinetic momentum is given by $\pi^\mu = p^\mu - eA^\mu$, where p^μ is the *canonical* momentum conjugate to the coordinate x^μ. In quantum mechanics p^μ can be represented when sandwiched between $\hat{\psi}^\dagger$ and $\hat{\psi}$ by the operator $\hat{p}^\mu = \frac{1}{2}i(\partial^\mu - \partial^{\dagger\mu})$ where $\partial^\dagger$ acts to the left in bilinear forms. Thus, the operator representing the kinetic momentum can be represented by $\hat{\pi}_\mu = \hat{p}_\mu - eA_\mu(x) = \frac{1}{2}i(D - D^\dagger)$ in terms of the *gauge covariant* derivatives. Note that up to spin corrections it is π^μ and not p^μ which is on shell since $\hat{\pi}^2 = m^2 + O(\hbar)$ follows from the Dirac equation. We are thus led as emphasized in [13,14,15] to define the gauge covariant Wigner operator by substituting the gauge covariant derivative D^μ and its adjoint in place of ∂^μ and its adjoint in the naive gauge dependent definition. Applying this minimal substitution rule gives

$$\begin{aligned} \hat{W}(x,p) &= \bar{\psi}(x)\,\delta^4(p - \hat{\pi}(x))\;\psi(x)\;, \\ &= \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\;\bar{\psi}(x) e^{\frac{1}{2}y\cdot D^\dagger(x)} \otimes e^{-\frac{1}{2}y\cdot D(x)}\psi(x)\;. \end{aligned} \tag{47}$$

The symbol $\otimes$ indicates that $\hat{W}$ is a 4×4 matrix in spinor indices and a $N\times N$ matrix in color indices. Because $\hat{\pi}_\mu$ transforms covariantly, so does $\hat{W}$ Not only have we achieved gauge covariance but also it is now obvious that $\langle Tr\hat{W}(x,p)\rangle$ has the desired interpretation of being the Lorentz scalar density of particles at x with kinetic momentum p.

Next we need to know how $\exp(-y\cdot D(x))$ acts on $\psi(x)$. We note [13] that

$$\begin{aligned} \exp(-y\cdot D_x) &= \lim_{n\to\infty}\;(\,1 - \Delta y(\partial_x + igA(x))\,)^n \\ &= \lim_{n\to\infty}\;\Big((1 - ig\Delta y\cdot A(x))\;e^{-\Delta y\cdot\partial_x}\;(1 + O(\Delta y^2))\Big)^n \\ &= \lim_{n\to\infty}\;(1 - ig\Delta y\cdot A(x))\cdots(1 - ig\Delta y\cdot A(x - (n-1)\Delta y)\;\exp(-y\cdot\partial_x) \\ &= U(x, x-y)e^{-y\cdot\partial_x}\;, \end{aligned} \tag{48}$$

where $\Delta y \equiv y/n$. Thus the effect of the covariant translation operator is to translate the field and keep track of the phase via the path ordered phase factor

(or link operator)

$$U(b,a) = Pexp\left\{-ig\int_0^1 ds\frac{dz^\mu}{ds}A_\mu(z(s))\right\} = P\prod_{s=0}^{1}(1 - igA(z(s))\cdot dz(s)) \ , \quad (49)$$

Note that the path of integration is uniquely determined to be a *straight line* between the end points, $z(s) \equiv z(b,a,s) = a + (b-a)s$ with $0 \leq s \leq 1$.

Finally, we can express the gauge covariant Wigner operator in the form proposed in Ref.[40],

$$\hat{W}(x,p) = \int \frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\ \bar{\psi}(x+\tfrac{1}{2}y)U(x+\tfrac{1}{2}y,x)\otimes U(x,x-\tfrac{1}{2}y)\psi(x-\frac{1}{2}y) \ . \quad (50)$$

The advantage of our definition (47), is that there is no path ambiguity on account of the physical requirement that the momentum variable corresponds to the kinetic momentum. Of course, $U(b,a)$ depends on the path except in the trivial case, when $A^\mu \propto S\partial^\mu S^{-1}$ is a pure gauge field with $F^{\mu\nu} = 0$.

4.4 Gauge Covariant Operator Equation

To derive the equation of motion for $\hat{W}$ we need the following relation derived in [13] for how the link operator varies when the endpoints are varied:

$$\begin{aligned}\delta U(b,a) &= U(b+db,a+da) - U(b,a) \\ &= -igA(b)\cdot db\ U(b,a) + igU(b,a)A(a)\cdot da \\ &\quad +ig\int_0^1 ds\ U(b,z(s))F_{\mu\nu}(z(s))U(z(s),a)(b-a)^\mu(da+(db-da)s)^\nu \ ,\end{aligned} \quad (51)$$

with $z(s) \equiv a + (b-a)s$.

Consider first the equation obeyed by the kernel of the Wigner operator

$$\hat{\rho}(x_2,x_1) = \bar{\psi}(x_2)U(x_2,x)\otimes U(x,x_1)\psi(x_1) = 0 \ . \quad (52)$$

where $x_1 \equiv x - \frac{1}{2}y$ and $x_2 \equiv x + \frac{1}{2}y$ and thus $x = \frac{1}{2}(x_1+x_2)$. From the Dirac equation

$$\bar{\psi}(x_2)U(x_2,x)\otimes U(x,x_1)\,(i\gamma^\mu D_\mu(x_1) - m)\,\psi(x_1) = 0 \ . \quad (53)$$

Taking the derivative to the left we get

$$\begin{aligned}(i\gamma_\mu D^\mu(x_1) - m)\hat{\rho}(x_2,x_1) &= A^\mu(x_1)\psi(x_1) \\ &\quad +i\gamma_\mu\bar{\psi}(x_2)U(x_2,x)\otimes[\partial^\mu_{x_1}U(x,x_1)]\psi(x_1) \\ &\quad +i\gamma_\mu\bar{\psi}(x_2)[\partial^\mu_{x_1}U(x_2,x)]\otimes U(x,x_1)\psi(x_1) \ ,\end{aligned} \quad (54)$$

where we have defined the product of any operator $\hat{O}$ times $\hat{\rho}$ such that

$$\hat{O}\hat{\rho}(x_2,x_1) \equiv \bar{\psi}(x_2)U(x_2,x)\otimes\hat{O}U(x,x_1)\psi(x_1) \ ,$$

$$\hat{\rho}(x_2,x_1)\hat{O} \equiv \bar{\psi}(x_2)U(x_2,x)\hat{O}\otimes U(x,x_1)\psi(x_1) \ . \tag{55}$$

Using (51) we find that

$$\partial^\mu_{x_1}U(\tfrac{1}{2}(x_1+x_2),x_1) = -\tfrac{1}{2}igA^\mu(x)U(x,x_1)+igU(x,x_1)A^\mu(x_1)$$

$$+ig\int_0^1 ds\ U(x,z)F_\nu{}^\mu(x;z;x_1)U(z,x_1)\cdot\tfrac{1}{2}(x_2-x_1)^\nu\cdot(1-\tfrac{1}{2}s) \ , \tag{56}$$

and,

$$\partial^\mu_{x_1}U(x_2,\tfrac{1}{2}(x_1+x_2)) = +\tfrac{1}{2}igU(x_2,x)A^\mu(x)$$

$$+ig\int_0^1 ds\ U(x_2,z)F_\nu{}^\mu(x_2;z;x)U(z,x)\cdot\tfrac{1}{2}(x_2-x_1)^\nu\cdot\tfrac{1}{2}(1-s) \ , \tag{57}$$

where $F_{\mu\nu}(b;z;a)=F_{\mu\nu}(z)$ for z along the straight line path between a and b. The Wigner kernel thus obeys

$$i\gamma_\mu\left(\partial^\mu_{x_1}\hat{\rho}(x_2,x_1)+\tfrac{1}{2}ig[A^\mu(x),\hat{\rho}(x_2,x_1)]\right)-m\hat{\rho}(x_2,x_1)$$

$$= -\tfrac{1}{2}g(x_2-x_1)^\nu\gamma^\mu\left(\int_0^1 ds\ (1-\tfrac{1}{2}s)U(x,z)F_{\nu\mu}(x;z;x_1)U(z,x_1)U(x_1,x)\hat{\rho}(x_2,x_1)\right.$$

$$\left.+\hat{\rho}(x_2,x_1)\int_0^1 ds\ \tfrac{1}{2}(1-s)U(x,x_2)U(x_2,z)F_{\nu\mu}(x_2;z;x)U(z,x)\right) \ . \tag{58}$$

To simplify this expression further, we define the covariant derivative of a second-rank tensor $\mathcal{T}$ by

$$\mathcal{D}(x)\mathcal{T}(x)\equiv\partial_x\mathcal{T}(x)+ig[A(x),\mathcal{T}(x)] \ . \tag{59}$$

One may show by a similar proof as in eq.(48) that

$$e^{-y\cdot\mathcal{D}(x)}\ \mathcal{T}(x) = U(x,x-y)\mathcal{T}(x-y)U(x-y,x) \ . \tag{60}$$

Changing variables from x_2,x_1 to the midpoint and relative distance variables $x\equiv\frac{1}{2}(x_1+x_2)$ and $y\equiv x_2-x_1$, and using eqs.(59, 60), we can write (58) as

$$-i\gamma_\mu\left(\partial^\mu_y-\tfrac{1}{2}\mathcal{D}^\mu(x)\right)\hat{\rho}(x+\tfrac{1}{2}y,x-\tfrac{1}{2}y)-m\hat{\rho}(x+\tfrac{1}{2}y,x-\tfrac{1}{2}y)$$

$$= -\tfrac{1}{2}gy^\nu\gamma^\mu\left(\int_0^1 ds\ (1-\tfrac{1}{2}s)\ [e^{-\frac{1}{2}(1-s)y\cdot\mathcal{D}(x)}\ F_{\nu\mu}(x)]\ \hat{\rho}(x+\tfrac{1}{2}y,x-\tfrac{1}{2}y)\right.$$

$$\left.+\hat{\rho}(x+\tfrac{1}{2}y,x-\tfrac{1}{2}y)\int_0^1 ds\ \tfrac{1}{2}(1-s)\ [e^{\frac{1}{2}sy\cdot\mathcal{D}(x)}\ F_{\nu\mu}(x)]\right) \ . \tag{61}$$

Taking the relative Fourier transform of (61) and reinstating $\hbar$, we obtain finally the following equation for the Wigner operator:

$$\left(\gamma^\mu p_\mu - m + \tfrac{1}{2}i\hbar\gamma^\mu \mathcal{D}_\mu(x)\right)\hat{W}(x,p) \;=\; \tfrac{1}{4}\hbar g\gamma_\mu\left(\{Q_1(\tfrac{1}{2}\hbar\triangle)\,F_{\mu\nu}(x), \partial^p_\nu W(x,p)\}\right.$$
$$\left. + \,[Q_2(\tfrac{1}{2}\hbar\triangle)\,F_{\mu\nu}(x), \partial^p_\nu W(x,p)]\right)\;, \quad (62)$$

where

$$Q_1(x) = i\frac{\sin x}{x} + \frac{\sin x - x\cos x}{x^2}\;,\quad Q_2(x) = i\frac{x\sin x + \cos x - 1}{x^2} + \frac{1-\cos x}{x}\;. \quad (63)$$

and the *triangle operator*, $\triangle$, is given in this case by

$$\triangle = \partial_p \cdot \mathcal{D} = \partial_p \cdot \partial_x + \frac{1}{\hbar}ig\partial_p \cdot [A(x),\;\;]\;, \quad (64)$$

We emphasize that $\mathcal{D}(x)$ on the right hand side only acts on $F_{\nu\mu}$. To obtain this final expression we used the same trick as in the non-relativistic case of replacing y by $i\partial_p$ using $\int d^4y e^{-ip\cdot y} f(y)g(y) = f(i\partial_p)\int d^4y e^{-ip\cdot y} g(y)$ and performing the s integral explicitly. Because $\mathcal{D}_\mu$, $F_{\mu\nu}$, $\triangle$ and $\hat{W}(x,p)$ transform covariantly under a gauge transformation, eq.(62) is the sought after *gauge covariant operator equation* of motion. The ordering of the field operators follows the rules (55).

Equation (62) is completely equivalent to the original Dirac's equation for the Heisenberg field $\hat{\psi}$. However, it is in a form that allows us to extract the quantum transport equation. As we shall see in the next section it in fact contains two equations. One is the generalized quantum transport equation. The second is a generalized mass shell constraint equation that arises on account of the Heisenberg uncertainty principle.

4.5 Quantum Vlasov and Constraint Equations in the Abelian Limit

The physical content of (62) is especially clear in the Abelian limit when in addition $F_{\mu\nu}$ is replaced as the self-consistent c-number field [15]. In that case the operator ordering is irrelevant. Specifically we consider the Hartree approximation in which $F_{\mu\nu}$ is approximated by the mean (c-number) field $\bar{F}_{\mu\nu}$ satisfying Maxwell's equations

$$\partial_\mu \bar{F}^{\mu\nu}(x) = \langle J^\mu(x)\rangle = e\,tr\int d^4p\,\gamma^\mu W^H(x,p)\;. \quad (65)$$

Of course, an arbitrary external current could be added to the r.h.s. of (65). The trace is over spinor indices of $W^H(x,p) = \langle\hat{W}(x,p)\rangle$ corresponding to the ensemble average of the Wigner operator in the Hartree approximation. In this case the *triangle operator* is simply the mixed derivative

$$\triangle \equiv \partial^p \cdot \partial_x\;. \quad (66)$$

Notice that the triangle operator has the dimension of inverse action so that in units where $\hbar \neq 1$ it always appears multiplied by $\hbar$. Therefore, a power series expansion of the Bessel functions coincides with an expansion in terms of the ratio of $\hbar$ to a characteristic angular momentum, L, of the plasma. That characteristic angular momentum measures the product of the space-time scale, ΔR_F, over which the field tensor, $\bar{F}^{\mu\nu}(x)$, varies appreciably and the momentum scale, ΔP_W, over which the Wigner function varies appreciably. Therefore, a necessary condition for the validity of a power series expansion of the Bessel functions is that the mean field is slowly varying in the sense

$$\hbar \ll \Delta R_F \Delta P_W \ . \tag{67}$$

In the Abelian case the linear equation (62) can be written in the following suggestive form:

$$(\gamma \cdot K - m)\, W^H(x,p) = 0 \ , \tag{68}$$

in terms of the operator $K^\mu \equiv \Pi^\mu + \frac{1}{2} i\hbar \nabla^\mu$, where

$$\nabla^\mu \equiv \partial^\mu_x - e j_0(\tfrac{1}{2}\hbar\triangle)\, \bar{F}^{\mu\nu}\, \partial^p_\nu \ , \tag{69}$$

$$\Pi^\mu \equiv p^\mu - \tfrac{1}{2}\hbar e j_1(\tfrac{1}{2}\hbar\triangle)\, \bar{F}^{\mu\nu}\, \partial^p_\nu \ , \tag{70}$$

and where $j_i(x)$ are conventional spherical Bessel functions. We have reinstated $\hbar$ explicitly to show clearly the quantum character of this equation.

To second order in $\hbar$ these operators are given by

$$\begin{aligned} \nabla^\mu &= \partial^\mu_x - e\bar{F}^{\mu\nu}\partial^p_\nu + \tfrac{1}{24} e\hbar^2 \triangle^2 \bar{F}^{\mu\nu}\partial^p_\nu \ + \ O(\hbar^4) \ , \\ \Pi^\mu &= p^\mu - \tfrac{1}{12} e\hbar^2\, \triangle \bar{F}^{\mu\nu}\partial^p_\nu \ + \ O(\hbar^4) \ . \end{aligned} \tag{71}$$

The generalized constraint and transport equations can be extracted by first multiplying (68) by $(\gamma \cdot K + m)$. Noting that $\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}$, the following quadratic quantum equation of motion follows:

$$(K^2 - m^2 - \tfrac{1}{2} i\sigma^{\mu\nu}[K_\mu, K_\nu])\, W^H(x,p) = 0 \ . \tag{72}$$

Adding and subtracting the adjoint of this equation then leads to

$$(\Pi^2 - m^2 - \tfrac{1}{4}\hbar^2\nabla^2)\, W^H(x,p) \ = \ -\tfrac{1}{4}\hbar I^{\mu\nu}\{\sigma_{\mu\nu}, W^H(x,p)\} + \tfrac{1}{4} i\hbar R^{\mu\nu}[\sigma_{\mu\nu}, W^H(x,p)] \ , \tag{73}$$

$$\Pi^\mu \nabla_\mu\, W^H(x,p) \ = \ \tfrac{1}{4} i I^{\mu\nu}[\sigma_{\mu\nu}, W^H(x,p)] + \tfrac{1}{4} R^{\mu\nu}\{\sigma_{\mu\nu}, W^H(x,p)\} \ , \tag{74}$$

where the l.h.s. of the quantum transport equation (74) acquires the remarkably simple form on account of the relation $[\nabla_\mu, \Pi^\mu] = 0$ that holds to all orders in $\hbar$ because of the special form of Π^μ and ∇^μ and the relation $dj_0(z)/d_z = -j_1(z)$.

The real operators R and I associated with the spin terms on the r.h.s. are given by

$$\begin{aligned} R^{\mu\nu} + iI^{\mu\nu} &\equiv \hbar^{-1}[K^\mu, K^\nu] = -\tfrac{1}{2}\hbar\triangle j_0(\tfrac{1}{2}\hbar\triangle)\, e\bar{F}^{\mu\nu} - i[j_0(\tfrac{1}{2}\hbar\triangle) - \tfrac{1}{2}\hbar\triangle j_1(\tfrac{1}{2}\hbar\triangle)]\, e\bar{F}^{\mu\nu} \\ &= -ie\bar{F}^{\mu\nu} - \tfrac{1}{2}\hbar e\triangle\bar{F}^{\mu\nu} + O(\hbar^2 e\triangle^2\bar{F}^{\mu\nu}) \ . \end{aligned} \tag{75}$$

The generalized quantum constraint and quantum Vlasov equations [15] (73) and (74), specify the dynamics of Abelian plasmas in the "collisionless" regime. They generalize the non-relativistic quantum transport equation (43) to QED of spin-$\frac{1}{2}$ particles. Unlike previous work [42]-[45] based on gauge dependent formulations and/or with assumptions about "slowly varying" fields, the above Lorentz covariant, gauge invariant equations apply for arbitrarily strong and varying c-number fields. Quantum corrections in the colisionless domain may be computed systematically to any order in $\hbar$ by expanding the Bessel functions in powers of the triangle operator. Together with eq. (65) they form a coupled set of equations to determine simultaneously the fermion Wigner function and the field $\bar{F}^{\mu\nu}(x)$ in a self-consistent way. They form a closed set of equations because the expectation value of the current is directly related to the Wigner function in the Hartree approximation via (65)

The "collision" terms neglected in this approximation follow from the operator equation (62) only when correlations such as $\langle F_{\mu\nu}\hat{W}\rangle - \bar{F}^{\mu\nu}W$ are not neglected, i.e., only if the operator character of the gauge field is explicitly taken into account. Using the methods outlined in Ref.[38] for non-gauge theories, it is possible to compute explicitly such collisions terms. However, for gauge theories with long range forces, such as QED, the infrared divergences arising in perturbation theory need special care [43]. We note again that quantum corrections to the collision terms are small only in relatively dilute plasmas [37].

To second order in $\hbar$, the quantum Vlasov equation including spin corrections is thus given by

$$\begin{aligned} &(p\cdot\partial_x - ep_\mu\bar{F}^{\mu\nu}\partial^p_\nu)\, W^H(x,p) + \tfrac{1}{4}ie\bar{F}^{\mu\nu}\left[\sigma_{\mu\nu}, W^H(x,p)\right] \\ &= -\tfrac{1}{12}\hbar^2 e\triangle\bar{F}_{\mu\nu}[\partial^\nu_x - e\bar{F}^{\nu\lambda}\partial^p_\lambda]\partial^\mu_p\, W^H(x,p) - \tfrac{1}{8}\hbar e\triangle\bar{F}^{\mu\nu}\left\{\sigma_{\mu\nu}, W^H(x,p)\right\} \ . \end{aligned} \tag{76}$$

Equation (76) reduces to the equation derived in [41] for scalar QED when the spin terms are neglected.

It is important to appreciate that the above quantum Vlasov equation is not complete without the quantum constraint equation (73). As soon as quantum corrections become important in the transport equation, quantum corrections to the constraint equation must also be considered as emphasized in [15]. To second

order in $\hbar$, the constraint equation in the Hartree approximation is given by

$$\begin{aligned}(p^2 - m^2)\, W^H(x,p) &= \tfrac{1}{4}\hbar e \bar{F}^{\mu\nu}\{\sigma_{\mu\nu}, W^H(x,p)\} - \tfrac{1}{8} i\hbar^2 e \triangle \bar{F}^{\mu\nu}[\sigma_{\mu\nu}, W^H(x,p)] \\ &\quad + \hbar^2(\tfrac{1}{6} e p \cdot \triangle \bar{F} \cdot \partial_p + \tfrac{1}{12} e(\partial_x^\mu \bar{F}_{\mu\nu})\partial_\nu^p + \tfrac{1}{4}(\partial_x - e\bar{F}\cdot\partial_p)^2) W^H(x,p) \ .\end{aligned} \tag{77}$$

From eqs. (76) and (77) we see that quantum transport theory reduces to classical transport theory only if several conditions are satisfied simultaneously. In addition to the condition (67) necessary for the validity of the triangle operator expansion, the field strength must be small compared to the typical energy scale of particles in the plasma, i.e.,

$$\hbar e \|\bar{F}^{\mu\nu}\| \ll m^2 + |\vec{p}|^2 \ . \tag{78}$$

Thus the field strength has to be relatively weak and slowly varying for classical theory to hold. In addition, spatial and momentum gradients of the Wigner function need to be sufficiently small so that $\Delta m^2 = \frac{1}{4}\hbar^2(\partial_x - e\bar{F}\partial_p)^2 \ll m^2$. These off mass shell corrections obviously arise as a result of the uncertainty principle.

We remark that in the Hartree approximation the spinor structure of the above equations has been resolved in Ref.[15]. The scalar and axial vector components of W^H were shown to specify W^H completely to any order in $\hbar$. Furthermore the classical BMT equation for spinning particles was derived from (76) in the semiclassical limit.

4.6 Quark Transport and the Abelian Dominance Approximation

We turn next to the non Abelian features of (62). There is a very important difference between the triangle operator expansion in the Abelian and non-Abelian cases. The Abelian expansion is equivalent to the semiclassical expansion in powers of $\hbar$. In the non-Abelian case, there appears a commutator in (64) that is of *zeroth* order in $\hbar$!! We can still write the linear operator equation in the form (68). However, the operators Π^μ and ∇^μ are now given by

$$\begin{aligned}\nabla^\mu &\equiv \mathcal{D}_x^\mu - \tfrac{1}{2} g\{ImQ_1(\tfrac{1}{2}\hbar\triangle)\, F_{\mu\nu}, \}\, \partial_\nu^p - \tfrac{1}{2} g[ImQ_2(\tfrac{1}{2}\hbar\triangle)\, F_{\mu\nu},] \\ &= \frac{1}{\hbar} i g[A^\mu(x),] + \partial_x^\mu - \tfrac{1}{2} g\{F_{\mu\nu}, \}\partial_\nu^p - \tfrac{1}{24} g^3\{[A^\alpha, [A^\beta, F_{\mu\nu}]], \}\partial_\nu^p \partial_\alpha^p \partial_\beta^p + \cdots \ , \\ \Pi^\mu &\equiv p^\mu - \tfrac{1}{4}\hbar g \left(\{ReQ_1(\tfrac{1}{2}\hbar\triangle)\, F_{\mu\nu}, \} + [ReQ_2(\tfrac{1}{2}\hbar\triangle)\, F_{\mu\nu},]\right) \partial_\nu^p \ .\end{aligned} \tag{79}$$

We see that something terrible has happened. The non-Abelian commutator has changed the leading order in $\hbar$ for the operator ∇^μ to be $O(1/\hbar)$, and there now arise an infinite number of commutators and anti-commutators to zeroth order in

$\hbar$. Thus the simple classical limit we found in the Abelian case becomes hopelessly complicated in the non-Abelian case.

The only sensible semiclassical limit of a non-Abelian theory is one where the *covariant* derivative of $F_{\mu\nu}$ is small. Only for covariantly slowly varying fields, in the sense that

$$\langle F\hat{W}\rangle \gg \langle \mathcal{D}F\, \partial_p \hat{W}\rangle \ , \tag{80}$$

is it possible to carry out an expansion in powers of the $\triangle$ operator. The covariant constant field case considered in section 3 trivially satisfies this condition. In particular for covariant constant fields, the linear operator equation for $\hat{W}$ reduces to

$$\Big(\gamma_\mu(p^\mu + \tfrac{1}{2}i\hbar(\mathcal{D}^\mu - \tfrac{1}{2}g\{F_{\mu\nu},\ \}\partial^p_\nu - \tfrac{1}{4}g[F_{\mu\nu},\ ]\partial^p_\nu) - m \Big)\hat{W}(x,p) = 0 \ . \tag{81}$$

In the general case for strong or rapidly varying fields the full quantum equation, eq.(62), must be solved. This would be equivalent to solving the original field equations, i.e. hopeless at this time. Only under the rather restrictive conditions, (80), can we expect that the transport theory for quarks reduces to a simpler, more manageable form.

Proceeding as in the Abelian case, we can isolate the transport equation for (81) by converting it into a quadratic equation and isolating the anti-Hermitian part. This leads to the *semiclassical transport equation for the QCD Wigner operator* [13]:

$$\begin{aligned} & p\cdot\mathcal{D}(x)\,\hat{W}(x,p) \ + \ \tfrac{1}{2}gp^\mu\partial^\nu_p\Big\{\, F_{\nu\mu}(x),\hat{W}(x,p)\,\Big\} \ + \ \tfrac{1}{4}ig\Big[\,\sigma^{\mu\nu}F_{\mu\nu}(x),\hat{W}(x,p)\,\Big] \\ & = \ -\tfrac{1}{8}ig\partial^\mu_p\Big[\,F_{\mu\nu}(x),(\mathcal{D}^\nu(x)\,\hat{W}(x,p))\,\Big] \ - \ \tfrac{1}{16}ig^2\partial^\mu_p\partial^\nu_p\Big[\,F_\mu{}^\lambda(x)F_{\nu\lambda}(x),\hat{W}(x,p)\,\Big] \\ & \quad + O(\triangle F) \ . \end{aligned} \tag{82}$$

Equation (82) is the non Abelian generalization of Vlasov's equation in the following sense: i) it is still an operator equation; ii) the first and second term on the l.h.s. of eq.(82) present the usual combination of phase space variables and derivatives, however, modified by (anti)commutators; iii) explicit spin-dependent corrections arise; iv) quantum corrections can be systematically calculated via expansion in powers of the $\triangle$ operator. The general form for the transport equations valid to all orders in $\triangle$ was also derived in Ref.[13]. Neglecting all but the first two terms on the l.h.s. recovers the classical transport equation for spinless quarks first derived in Ref.[40]. Furthermore, we have shown that the $O(\triangle F)$ corrections not involving $\sigma^{\mu\nu}$ that follow from (62) reproduce the result for scalar quarks obtained in Ref.[46] modulo a sign error therein.

For covariant constant or slowly varying fields, the color structure of the above equation is particularly transparent. We consider essentially Abelian fields such

that all components of $F_{\mu\nu}$ can be simultaneously diagonalized in the same gauge:

$$\langle F^{\mu\nu}(x)\rangle \equiv S(x)\vec{F}^{\mu\nu}\cdot\vec{h}S^{-1}(x) \ , \tag{83}$$

where h_i the $N-1$ diagonal Gell-Mann matrices and $S(x)$ is a particular gauge transformation. We now make the bold model dependent *ansatz* that the ensemble of quarks is such that $\hat{W}$ is diagonal in the *same* gauge where $\langle F_{\mu\nu}(x)\rangle$ is diagonal. By assumption then we can express

$$\langle\hat{W}\rangle \equiv S(x)\left(\sum_{j=1}^{N-1} W^j h_j + W^0\mathbf{1}\right)S^{-1}(x) \ , \tag{84}$$

in terms of N Wigner *functions* depending on (x,p). The decomposition in spinor space proceeds as in the Abelian case [15]. Note that the property $\hat{W}^\dagger = \gamma^0\hat{W}\gamma^0$ insures that the N diagonal elements in color space $\langle\hat{W}\rangle^{ii}$ are real numbers after taking appropriate traces in spinor space. Equation (84) is a strong model assumption and we refer to it as the Abelian Dominance Approximation [32] for reasons that become clear below. There is no guaranty that such ensembles of quarks exist in nature or how well they approximate the conditions of the quark-gluon plasma formed in nuclear collisions. We proceed in the spirit of the MIT bag and string models and adopt this assumption for phenomenological purposes.

When (84) holds, it is most convenient to work in the $S(x)$ gauge, where

$$\langle\hat{W}\rangle_{ij} = (\vec{W}\cdot\vec{h} + W^0_{ij}\mathbf{1}) = \delta_{ij}(\vec{W}\cdot\vec{\epsilon}_j + W^0) \equiv \delta_{ij}f_j \ , \tag{85}$$

where the "charges" $\vec{\epsilon}_j$ are just the elementary weight vectors of SU(N) discussed in section 3.

The semiclassical transport equations for the $f_j(x,p)$ in this approximation are obtained by taking the expectation value of eq.(82) in this ensemble. We see that the color structure of those equations simplifies considerably:

$$(p\cdot\partial_x + g\vec{\epsilon}_j\cdot\vec{F}_{\mu\nu}p^\nu\partial_p^\mu)\ f_j(x,p) = -\tfrac{1}{4}ig\vec{\epsilon}_j\cdot\vec{F}_{\mu\nu}\,[\,\sigma^{\mu\nu}, f_j(x,p)\,] + \tilde{C}_j(x,p) \ , \tag{86}$$

where $\tilde{C}_j$ represents correlation terms of the form

$$\tilde{C}_j(x,p) = -\tfrac{1}{2}gp^\mu\partial_p^\nu\left(\langle\{F_{\nu\mu},\hat{W}(x,p)\}\rangle - 2\langle F_{\nu\mu}\rangle\langle\hat{W}(x,p)\rangle\right)_j + \cdots \ . \tag{87}$$

As usual, the collision terms are contained in such correlations.

Note that all the non-Abelian commutator terms dropped out for the model *ansatz* (84) and the transport theory for quarks has reduced to an effective multi-component Abelian plasma theory. The $f_i(x,p)$ just correspond to the phase space densities for quarks with "charge" $g\vec{\epsilon}_i$. Just as the pair production rate for $q_i\bar{q}_i$ in the semiclassical limit could be computed as if the ith colored quarks couple with an effective Abelian field with $e_iF^{\mu\nu}_{eff} = g\vec{\epsilon}_i\cdot\vec{F}^{\mu\nu}$, we see that those quarks obey an effective Abelian Vlasov equation with the same $e_iF^{\mu\nu}_{eff}$ appearing in the force term. Therefore, the quark plasma dynamics is very simple and intuitive in this Abelian Dominance Approximation.

4.7 Gluon Transport and the Abelian Dominance Approximation

To develop a gauge covariant transport theory for gluons we start with the following definition of the gauge covariant gluon Wigner operator proposed in Ref. [14].

$$\hat{\Gamma}_{\mu\nu}(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}[e^{\frac{1}{2}y\cdot\mathcal{D}(x)}F_\mu{}^\lambda(x)] \otimes [e^{-\frac{1}{2}y\cdot\mathcal{D}(x)}F_{\lambda\nu}(x)] \ , \tag{88}$$

where the covariant derivative of a second-rank tensor $\mathcal{T}$ is defined as in (59). The connection between gauge dependent and independent Wigner functions for Abelian photons is discussed in detail in Ref. [15]. In eq.(88) we suppressed four color indices of $\hat{\Gamma}$ but explicitly indicated its tensor structure. $\hat{\Gamma}$ is closely related to the energy-momentum tensor of the field,

$$\hat{T}_{\mu\nu}(x) \ \equiv \ Tr\left(F_\mu{}^\lambda F_{\lambda\nu} + \tfrac{1}{4}g_{\mu\nu}F_{\lambda\tau}F^{\lambda\tau}\right) \ = \ Tr\int d^4p\left(\hat{\Gamma}_{\mu\nu}(x,p) - \tfrac{1}{4}g_{\mu\nu}\hat{\Gamma}_\lambda{}^\lambda(x,p)\right) \ , \tag{89}$$

where the trace refers to color indices, $TrA\otimes B \equiv A_{ab}B_{ba}$. Eq.(89) provides the connection between $\hat{\Gamma}$ and observables of the gauge field. Formally,

$$Tr\hat{\Gamma}_{\mu\nu}(x,p) = F_\mu{}^\lambda(x)\delta^4(p - i\mathcal{D}(x))F_{\lambda\nu}(x) \ , \tag{90}$$

which shows that $Tr\hat{\Gamma}(x,p)$ measures the energy-momentum flux of gluons at x with kinetic momentum p.

The derivation of the quantum transport equation for $\hat{\Gamma}$ proceeds as for $\hat{W}$ [14]. We want to calculate $p\cdot\tilde{\mathcal{D}}\hat{\Gamma}_{\mu\nu}$, where the generalized covariant derivative acting on tensors of the structure of $\hat{\Gamma}$ is defined by

$$\tilde{\mathcal{D}}A\otimes B \equiv [\mathcal{D}A]\otimes B + A\otimes[\mathcal{D}B] \ . \tag{91}$$

Expressing the action of the covariant translation operators in (88) in terms of link operators, we carry $p\cdot\tilde{\mathcal{D}}$ into the integrand converting $p\to -i\partial_y$ (by partial integration) and $y\to i\partial_p$. We calculate all necessary derivatives of link operators which occur in the integrand by applying eq.(51). These straightforward though tedious manipulations lead to a very complex equation for $\hat{\Gamma}_{\mu\nu}(x,p)$ recorded in [14]. That equation becomes manageable only in the semiclassical limit obtained by expanding it to lowest order in powers of $\triangle \sim \hbar\partial_p\mathcal{D}$.

However, there is an additional complication for gluons not encountered for quarks. The $F_{\mu\nu}$ field contains in general *coherent* as well as *incoherent* parts. The coherent part corresponds to the external or self-consistent mean field $\bar{F}$. The incoherent part describes the interesting fluctuation part that most closely corresponds to a random gas of gluons for which $\langle F_{\mu\nu}\rangle = 0$. We separate these two parts by denoting the incoherent part of $\hat{\Gamma}$ by

$$\hat{G}_{\mu\nu} \ \equiv \ \hat{\Gamma}_{\mu\nu} - \bar{\Gamma}_{\mu\nu} \tag{92}$$

where $\bar{\Gamma}_{\mu\nu}$ is defined by eq.(88) with $F \to \bar{F} \equiv \langle F \rangle$. Barred quantities henceforth refer to $\bar{A}$. It is important to realize that $\langle \hat{\Gamma} \rangle \neq \bar{\Gamma}$ and therefore $G \equiv \langle \hat{G} \rangle \neq 0$. The classical mean field is determined from

$$[\bar{D}_\mu, \bar{F}^{\mu\nu}] = g(\langle J^\nu \rangle + \langle j^\nu \rangle) \ , \tag{93}$$

where $\langle J^\nu \rangle$ is the mean color current due to quarks and $\langle j^\nu \rangle$ is the mean induced color current due to the incoherent gas of gluons. We determine that induced current so as to simplify the transport equation for G.

We found that current source terms can be eliminated from in the transport equation by choosing the induced gluon current to satisfy

$$\bar{\bar{D}}_\mu(x)\langle j_\nu(x)\rangle + iTr' \int d^4p \, \{G_{\mu\nu}(x,p) - G_{\nu\mu}(x,p)\} \ = 0 \ , \tag{94}$$

with $(Tr'G)_{ac} = G_{abbc}$. In this case the gauge covariant analogue of Vlasov's equation for gluon fluctuations was found [14] to be

$$\begin{aligned} p \cdot \bar{\bar{D}} \, G_{\mu\nu} \ = \ & \tfrac{1}{2} g p^\sigma \partial_p^\tau \left\{ \left[\bar{F}_{\tau\sigma}, G_{\mu\nu} \right]_R \ + \ \left[G_{\mu\nu}, \bar{F}_{\tau\sigma} \right]_L \right\} \\ & + \tfrac{1}{8} i g \partial_p^\sigma \left\{ \left[\bar{F}^\tau_\sigma, \bar{\bar{D}}_\tau G_{\mu\nu} \right]_R \ - \ \left[\bar{\bar{D}}_\tau G_{\mu\nu}, \bar{F}^\tau_\sigma \right]_L \right\} \\ & + \tfrac{1}{16} i g^2 \partial_p^\sigma \partial_p^\eta \left\{ \left[\bar{F}_{\tau\sigma} \ , \ \left[\bar{F}^\tau_\eta, G_{\mu\nu} \right]_R \ + \ \left[G_{\mu\nu}, \bar{F}^\tau_\eta \right]_L \right]_R \right. \\ & \left. - \left[\left[\bar{F}^\tau_\eta, G_{\mu\nu} \right]_R \ + \ \left[G_{\mu\nu}, \bar{F}^\tau_\eta \right]_L \ , \ \bar{F}_{\tau\sigma} \right]_L \right\} \ , \end{aligned} \tag{95}$$

where the commutators are defined by

$$\left[\bar{F}, A \otimes B \right]_R \ \equiv \ \left[\bar{F}, A \right] \otimes B \ , \quad \left[A \otimes B, \bar{F} \right]_L \ \equiv \ A \otimes \left[\bar{F}, B \right] \ . \tag{96}$$

Eq.(95) describes a gluon plasma in the so-called collisionless regime to zeroth order in the triangle operator expansion. Eqs.(93,94,95) together form a consistent set of gauge covariant equations for the study of gluon fluctuations under the influence of a classical color field. Note that there is no need to solve for $\bar{\Gamma}$ since eq.(93) determines the classical field. Corrections to the above equations in powers of $\triangle \sim \hbar \partial_p \mathcal{D}$ can be calculated systematically from the exact transport equation in [14].

The color structure of the above equations simplifies greatly in the Abelian Dominance Approximation. If $\bar{F}^{\mu\nu}$ is to have the simple form in (83) then so must the mean currents. In particular $Tr'G$ must be diagonal in the same gauge as $\bar{F}$ is. This means that it must be possible to decompose G as

$$G_{\mu\nu}(x,p) \ \equiv \ G^{ij}_{\mu\nu}(x,p) e_{ij} \otimes e_{ji} + G^{i}_{\mu\nu}(x,p) h_i \otimes h_i \ \equiv \ G^{\vec{n}}_{\mu\nu}(x,p) + G^{h}_{\mu\nu}(x,p) \ , \tag{97}$$

with $G^{ij}_{\mu\nu} \equiv G^{ji}_{\nu\mu}$ *and* $G^{i}_{\mu\nu} \equiv G^{i}_{\nu\mu}$. In this case

$$\left[\vec{h}, G\right]_R = \vec{\eta}_{ij} G^{ij} e_{ij} \otimes e_{ji} = \left[G, \vec{h}\right]_L \ , \tag{98}$$

The induced gluon current satisfying (94) is

$$\langle j_\nu(x) \rangle = -\int d^4p \int d^4q \, e^{iq\cdot x} \frac{q^\mu}{q^2} \, diag(\ldots, \tfrac{1}{2} \sum_{j,j\neq i} \{G^{ij}_{\mu\nu}(q,p) - G^{ij}_{\nu\mu}(q,p)\}, \ldots) \ . \tag{99}$$

The $N-1$ Abelian mean fields, $F^{\mu\nu}_i$, related to $\bar{F}$ by $\bar{F}^{\mu\nu} = F^{\mu\nu}_i h_i$ satisfy in this approximation

$$\partial_\mu \vec{F}^{\mu\nu} = g(\vec{J}^\nu + \vec{j}^\nu) \ . \tag{100}$$

Inserting eq.(97) into the transport eq.(95), we can isolate the transport equations for the G^{ij}, G^i-components by multiplying with e_{ij}, h_i respectively and taking traces. Thus we find that the $N-1$ diagonal components of G defined in eq.(97) obey Abelian transport equations,

$$p \cdot \partial_x \, G^i_{\mu\nu}(x,p) = 0 \ . \tag{101}$$

Since G^i do not contribute to $\langle j \rangle$ in (99), and do not interact with the mean field $\vec{F}$, the $N-1$ ***neutral gluons*** described by its components G^i completely decouple from the dynamics of the system in this approximation. Using eq.(98), the Vlasov type equations for the $N(N-1)$ ***charged gluons*** described by the components G^{ij} are found to be

$$\left(p \cdot \partial_x - g p^\sigma \partial^\tau_p \vec{F}_{\tau\sigma} \cdot \vec{\eta}_{ij}\right) G^{ij}_{\mu\nu}(x,p) = 0 \ . \tag{102}$$

Notice the appearance of the effective coupling $g\vec{\eta}$ for charged gluons just as for quarks we found the effective coupling to be $g\vec{\epsilon}$.

In conclusion, we see that the transport theory for quark-gluon plasmas reduces in the Abelian Dominance Approximation to a multicomponent effectively Abelian plasma theory. In SU(N) there are N quarks with effective charges $g\vec{\epsilon}_i$, $N-1$ neutral gluons, and $N(N-1)$ charged gluons with effective charges $g\vec{\eta}_{ij}$. These particles are produced in pairs and accelerate in an effective $N-1$ component mean field, $\vec{F}^{\mu\nu}$. This suggests a simple extension of Abelian string models and transport theories that could be applied in the phenomenology of nuclear collisions [13]. Obviously, we have only scratched the surface of the topic of Quantum Chromo Transport theory and many fascinating theoretical challenges remain: the structure of collision terms with attention to color electric and magnetic screening, the inclusion of source terms in the phase space formulation due to color neutralization, and a deeper understanding of whether there exists a classical plasma transport limit for quark-gluon plasmas in the first place.

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