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A NUMERICAL STUDY OF VORTEX MOTION
IN SUPERFLUID HELIUM

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September 1986

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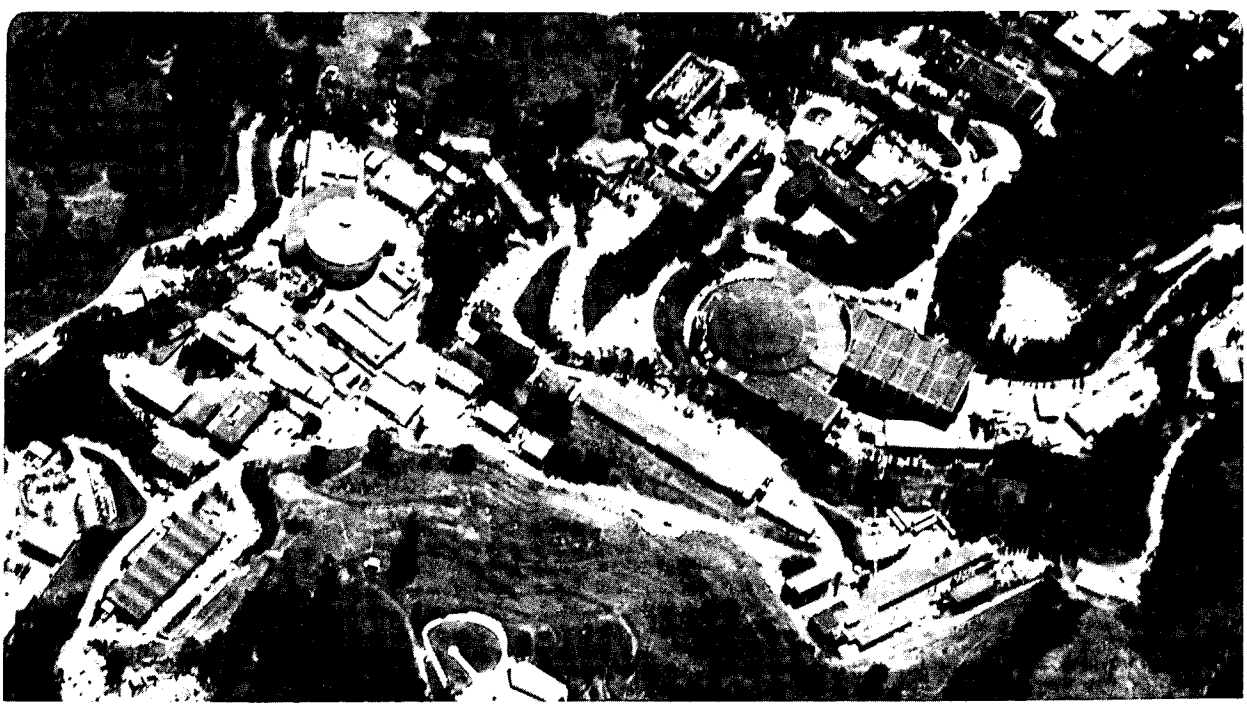

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A NUMERICAL STUDY OF VORTEX MOTION IN SUPERFLUID HELIUM[1]

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A Numerical Study of Vortex Motion in Superfluid Helium

Thomas F. Buttke

ABSTRACT

We carry out a careful numerical solution of the equations of motion of a superfluid helium vortex in the self-induction approximation where reconnection is allowed. Contrary to earlier results we find that the turbulence produced is not homogeneous and that critical velocities are not present. We conclude that the model equation used is inadequate to describe superfluid turbulence. We explain the source of error in earlier results.

It is generally postulated that turbulence in superfluid ^{4}He is caused by the creation of a tangle of quantized vortex filaments [5]. It is not fully understood how the vortices are created, but it is observed experimentally that the turbulence appears when the characteristic velocities of the superfluid system become large enough. It is generally assumed that superfluid ^{4}He vortices obey Euler's equations and thus the exact motion of a superfluid vortex is difficult to determine. For a rotating fluid the experiments of Hall and Vinen [6,7] indicate that a superfluid vortex undergoes much less stretching and folding than the corresponding turbulent vortex in an ordinary non-rotating fluid [3,4,14,15]. Schwarz [11] incorporates the experimental observations of Hall and Vinen into a model where the motion of a superfluid vortex depends only on the local geometry of the vortex and on the local average velocity fields. Schwarz then combines this local model of vortex dynamics with the reconnection ansatz of Feynman [5] to produce a description of superfluid turbulence [12]. (We recall that the reconnection ansatz states that whenever two vortices cross each other they reconnect.)

We have developed the first stable numerical method to solve for the motion of a vortex in the self-induction approximation; we show how this method can be generalized to include perturbations which cause a vortex to stretch. We use our numerical method to solve the local model of Schwarz. We carefully analyze the truncation error in the numerical method and the resolution needed near the kinks in the vortices introduced by the reconnections; we find that this leads to restrictions on the numerical scheme. Our results disagree with earlier work; we find that the turbulence produced is not homogeneous; there is no critical velocity independent of initial conditions. We conclude that the model used is inadequate to describe turbulence in superfluid helium.

Assume that the location of a superfluid vortex is given by the function $\mathbf{r}(\xi,\tau)$, where ξ is a Lagrangian variable labeling the fluid particles along the vortex, and τ is dimensionless time. Then we have that the velocity of the vortex is given by [12]

$$\partial \mathbf{r}/\partial \tau = \mathbf{r}' \times \mathbf{r}'' + \alpha \mathbf{r}' \times \mathbf{v}_0 + \alpha \mathbf{r}'' \ , \tag{1}$$

where $'$ indicates differentiation with respect to arclength, $\mathbf{v}_0$ is the average counterflow velocity, and α is proportional to the Hall-Vinen coefficient B [6]. The first term on the right side of equation (1) is the self-induction approximation [8,9] of the motion of a vortex; the second and third terms result from the interaction of the superfluid vortex with the normal fluid. Equation (1) is written in dimensionless form, and it is assumed that $\mathbf{v}_0$ is a constant. Vortices evolving according to equation (1) stretch and contract but, contrary to the evolution of a turbulent vortex in an ordinary fluid, the stretching is linear in time and tends to smooth kinks in the vortex.

If we differentiate the terms of equation (1) with respect to ξ, we can rewrite equation (1) in terms of the tangent vector $\mathbf{t}$ as

$$\partial \mathbf{g}/\partial \tau = g\Big(\mathbf{t}\times\mathbf{t}'' + \alpha \mathbf{t}' \times \mathbf{v}_0 + \alpha \mathbf{t}''\Big) \ , \tag{2}$$

where $\mathbf{g} \equiv \partial \mathbf{r}/\partial \xi$ and $g \equiv |\,\mathbf{g}\,|$. From equation (2) we see

$$\partial g\,/\partial \tau = \alpha g\, \kappa(\mathbf{b}\cdot\mathbf{v}_0 - \kappa) \ , \tag{3}$$

where $\mathbf{b}$ is the binormal vector to the vortex and κ is the curvature of the vortex. The change in g indicates whether the vortex is stretching or contracting. From equation (3) we see that the vortex contracts if $\kappa > \mathbf{b}\cdot\mathbf{v}_0$, and stretches otherwise; the vortex contracts in regions of large curvature. Even though equation (2) is dimensionless, equation (3) indicates that there is still a characteristic length present in equation (2). Define $\gamma \equiv |\,\mathbf{v}_0\,|$, then if we measure arclength in units of $1/\gamma$ we find that equation (2) becomes

$$\frac{1}{\gamma^2}\frac{\partial \mathbf{g}}{\partial \tau} = g\Big(\mathbf{t}\times\mathbf{t}'' + \alpha \mathbf{t}' \times \hat{\mathbf{v}}_0 + \alpha \mathbf{t}''\Big) \ , \tag{4}$$

where $\hat{\mathbf{v}}_0$ is a unit vector pointing in the $\mathbf{v}_0$ direction. We see that if we redefine time, equations (2) and (4) are identical except for the fact that equation (4) no longer depends on $\gamma = |\,\mathbf{v}_0\,|$. If we consider solving equation (2) with given initial conditions for several different values of γ, then the analagous solutions could be found by solving equation (4), but the initial conditions of equation (4) depend on γ whereas for equation (2) they would be

fixed.

Consider equation (1) in the case $\alpha = 0$. In this case equation (1) becomes

$$\partial \mathbf{r}/\partial \tau = \mathbf{r}' \times \mathbf{r}'' \quad . \tag{5}$$

Equation (5) is the self-induction equation [8]. It can be shown that a vortex which evolves according the self-induction equation neither contracts nor stretches [1]; thus it is convenient to use arclength measured along the vortex to parameterize the vortex. If we differentiate the terms of equation (5) with respect to arclength s we find

$$\partial \mathbf{t}/\partial \tau = \mathbf{t} \times \mathbf{t}'' \quad , \tag{6}$$

where $' \equiv \partial/\partial s$. We denote $\mathbf{t}_j^n$ to be an approximation to $\mathbf{t}(j\,\Delta s\,, n\,\Delta \tau)$, where j and n are integers, Δs is the distance between approximation points along the vortex, and $\Delta \tau$ is the time between approximations. We define $\mathbf{t}_j^n$ by the finite difference equation:

$$\mathbf{t}_j^{n+1} - \mathbf{t}_j^n = \frac{\Delta \tau}{\Delta s^2} \mathbf{t}_j \times (\mathbf{t}_{j-1} + \mathbf{t}_{j+1}) \,, \tag{7}$$

where $\mathbf{t}_j \equiv (\mathbf{t}_j^n + \mathbf{t}_j^{n+1})/2$. Hasimoto [10] has shown that equation (6) is equivalent to a non-linear Schrödinger equation which has an infinite number of invariants; the numerical scheme is stable as a result of the fact that it preserves three invariants [1,2].

We have carried out a careful comparison of exact smooth solutions of equation (6) with the approxmate solutions obtained from equation (7) and conclude that the approximate solutions are second order accurate in space and time for smooth solutions. Furthermore we have found a new exact self-similar solution of equation (6) which has the same singularity that is present in a vortex just after it has undergone a reconnection. The initial conditions for the self-similar problem are

$$\mathbf{t}(s\,,0) = \begin{cases} \mathbf{t}_- & \text{if } s < 0 \\ \mathbf{t}_+ & \text{if } s > 0 \end{cases} ,$$

where $\mathbf{t}_-$ and $\mathbf{t}_+$ are constant unit vectors. The solution to the self-similar problem is the curve which has curvature equal to $\kappa_0/\sqrt{\tau}$ and torsion equal to $s\,/2\tau$, where κ_0 is a constant determined by the angle between $\mathbf{t}_-$ and $\mathbf{t}_+$ [1]. We have carried out a careful comparison of

the exact self-similar solution with the approximate solution and we find that the approximate solutions converge to the exact solution.

We generalize equation (7) so that we can add perturbations to the self-induction equation which cause the vortices to stretch. The generalized scheme preserves two invariants. We write down the generalized finite difference scheme used to approximate equation (2):

$$\begin{aligned} \mathbf{g}_j^{n+1} - \mathbf{g}_j^n = & \frac{\Delta\tau}{4\Delta\xi^2}\,\mathbf{t}_j \times \left(h_{j+1}\mathbf{t}_{j+2} + h_{j-1}\mathbf{t}_{j-2}\right) \\ & + \alpha\frac{\Delta\tau}{2\Delta\xi}\left(\mathbf{t}_{j+1} - \mathbf{t}_{j-1}\right) \times \mathbf{v}_0 \\ & + \alpha\frac{\Delta\tau}{4\Delta\xi^2}\left(h_{j+1}\mathbf{t}_{j+2} - (h_{j+1} + h_{j-1})\mathbf{t}_j + h_{j-1}\mathbf{t}_{j-2}\right), \end{aligned} \tag{8}$$

where $\mathbf{g}_j^n$ is an approximation to $\mathbf{g}(j\,\Delta\xi, n\,\Delta\tau)$, $\mathbf{g}_j \equiv (\mathbf{g}_j^n + \mathbf{g}_j^{n+1})/2$, $h_j \equiv 1/\,|\,\mathbf{g}_j\,|$, and $\mathbf{t}_j \equiv h_j\,\mathbf{g}_j$. We have carefully compared the known exact solutions of equation (2) with the approximate solutions and find that the approximate solutions converge to the exact solutions for smooth solutions. We have also carefully examined the approximate solutions in the case in which a reconnection is allowed; we find that in this case there is a precipitous loss of accuracy if the distance between approximation points $\Delta\xi$ is chosen too large. As a result of the scaling arguments introduced in equation (4) and numerical evidence [1], we find that in order for the approximate solutions to be close to the exact solutions the condition

$$\gamma\Delta\xi \leq 0.5 \tag{9}$$

must be satisfied.

We wish to calculate the line length densities of vortex tangles for different values of γ, a calculation attempted by Schwarz [12]. We calculate the line length inside of a unit cube. We impose periodic boundary conditions and choose $\hat{\mathbf{v}}_0$ perpendicular to one of the boundary faces. The natural way to carry out the numerical experiments using equation (8), before one is aware of condition (9), is to fix $\Delta\xi$ and vary the magnitude of the counterflow velocity γ. In figures 1 and 2 we show the results of the numerical experiments when we fix $\Delta\xi$ and vary γ.

For the first set of experiments we choose simple initial conditions consisting of four circles and we fix $\Delta\xi$ independent of γ. As shown in figure 1, we find that a vortex tangle is established provided $\gamma > 20$; the data suggests a critical velocity below which no vortex tangle can be self-sustaining. In order to verify that this is a genuine critical velocity we perform a second set of experiments for which we choose 'turbulent' initial conditions. We again leave $\Delta\xi$ fixed, but take for initial conditions the vortex system obtained from the simple initial conditions with $\gamma = 45.0$ and time $= 0.8$. From figure 2 we see that there is no critical velocity present. The apparent critical velocity present in figure 1 is caused by two things. The first cause is that we chose our initial conditions too simple to cause a self-sustaining vortex tangle. The second cause of the apparent critical velocity is the fact that most of the data shown in figure 1 is calculated with values of $\Delta\xi$ and γ such that condition (9) is *not* satisfied. As we see from figure 3 the correct line lengths, calculated when condition (9) is satisfied, are much smaller than the incorrect values calculated in figure 1. Thus we see that the two effects taken together produce what appears to be a critical velocity; for these reasons we conclude that the critical velocities reported earlier [13] were a result of choosing an approximation which was too coarse and initial conditions which were too simple to cause a vortex tangle.

In the calculations used to obtain the values of line length shown in figures 1 and 2, $\Delta\xi$ is fixed; thus as γ is increased there comes a point at which condition (9) is no longer satisfied. The $\Delta\xi$ used to calculate the line lengths in figure 2 satisfies the inequality $\gamma\Delta\xi \leq 0.5$ for $\gamma \leq 25.0$; thus we recalculate the line lengths for $\gamma \geq 30.0$ using a $\Delta\xi$ which scales with γ so that $\gamma\Delta\xi \leq 0.5$. The results of the calculations are shown in figure 3. We see that an incorrect line length is obtained if condition (9) is not satisfied.

If the turbulence is homogeneous a dimensional argument shows that the line length density should be proportional to γ^2 [12]. From figure 3 we see that the line length density is definitely not proportional to γ^2. If $\Delta\xi$ is chosen too large, however, we obtain results that

erroneously indicate that the line length density is proportional to γ^2 as claimed in previous calculations [12]. The results in figures 1,2 and 3 are obtained for $\alpha = 0.1$. In figure 4 we show the results of line length calculations where we have set $\alpha = 0.03$. From figures 3 and 4 we see that line lengths calculated when condition (9) is *not* satisfied give inaccurate results which agree with earlier calculations. We have done calculations for values of α ranging from $\alpha = 0.01$ to $\alpha = 1.0$ and we find that homogeneous turbulence is not produced for any of these values. We conclude that earlier calculations were too crude.

The results of our calculations indicate that, contrary to earlier results, the model equation (1) is inadequate for a description of turbulence in superfluid helium. The model equation does not produce enough folding and stretching of the superfluid vortices. Our results indicate that models used to describe superfluid helium turbulence will have to maintain more of the character of Euler's equation than self-induction does. The self-induction term alone is inadequate to model the evolution of a turbulent vortex even with the reconnection ansatz included. These observations are consistent with what is known in the theory of vortex motion in classical fluids [4],[14].

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Figure Captions

Figure 1 : Average line length as a function of γ, the magnitude of the counterflow velocity. The initial condition consists of four circles. The spacing between approximation points $\Delta\xi$ =0.02 and $\alpha = 0.1$.

Figure 2 : $\Delta\xi = 0.02$, $\alpha = 0.1$ The line lengths represented by the squares were calculated from the simple initial conditions and the line lengths represented by the octagons were calculated from 'turbulent' initial conditions.

Figure 3 : $\alpha = 0.1$, The line lengths represented by the squares were calculated with $\gamma\Delta\xi \leq 0.5$ and the line lengths represented by the octagons were calculated with $\gamma\Delta\xi \geq 0.6$. The straight line corresponds to the results given in reference [12].

Figure 4 : $\alpha = 0.03$, The line length represented by the square was calculated with $\gamma\Delta\xi \leq 0.5$ and the line lengths represented by the octagons were calculated with $\gamma\Delta\xi \geq 0.6$. The upper straight line corresponds to the results given in reference [12]. The lower straight line is a least squares fit to the data given by the octagons.

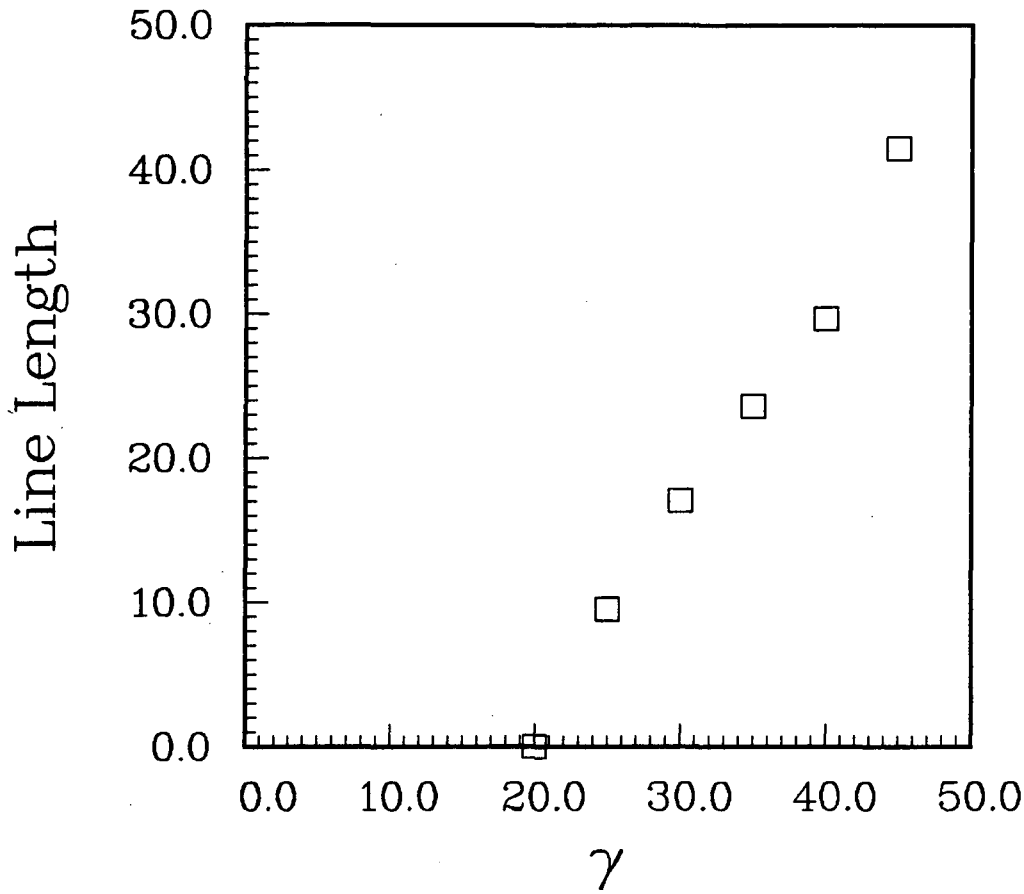

figure 1

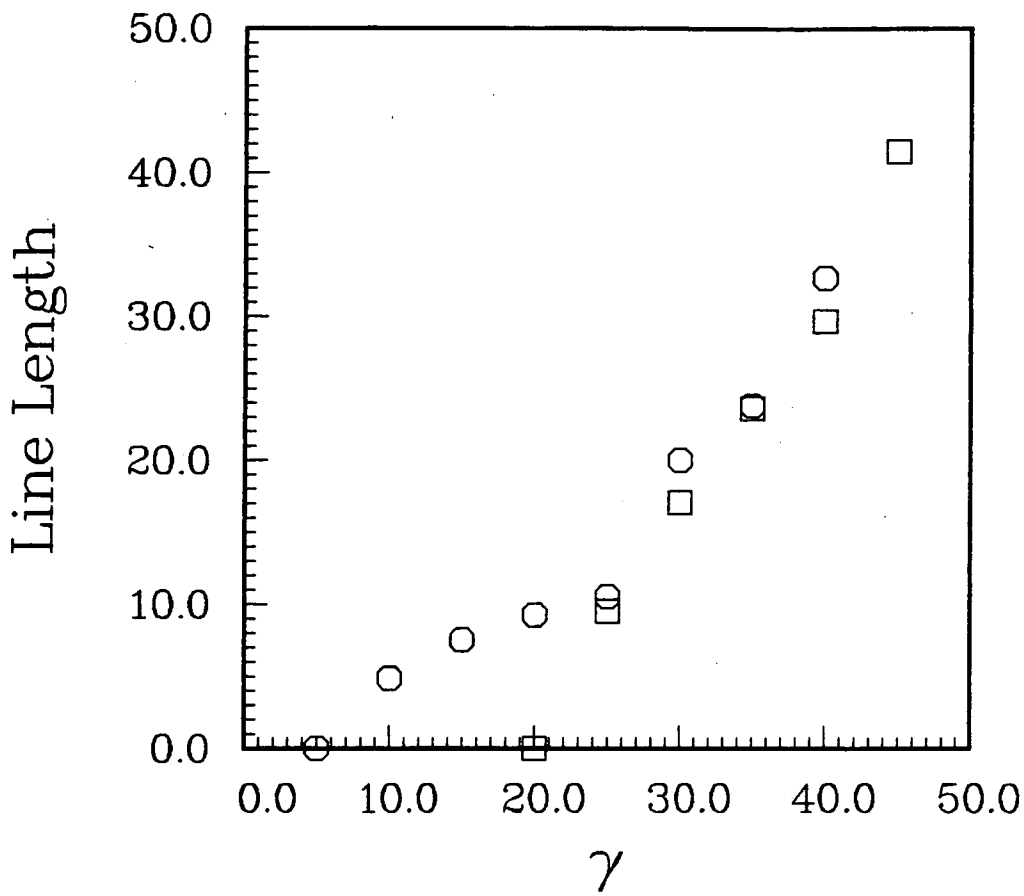

figure 2

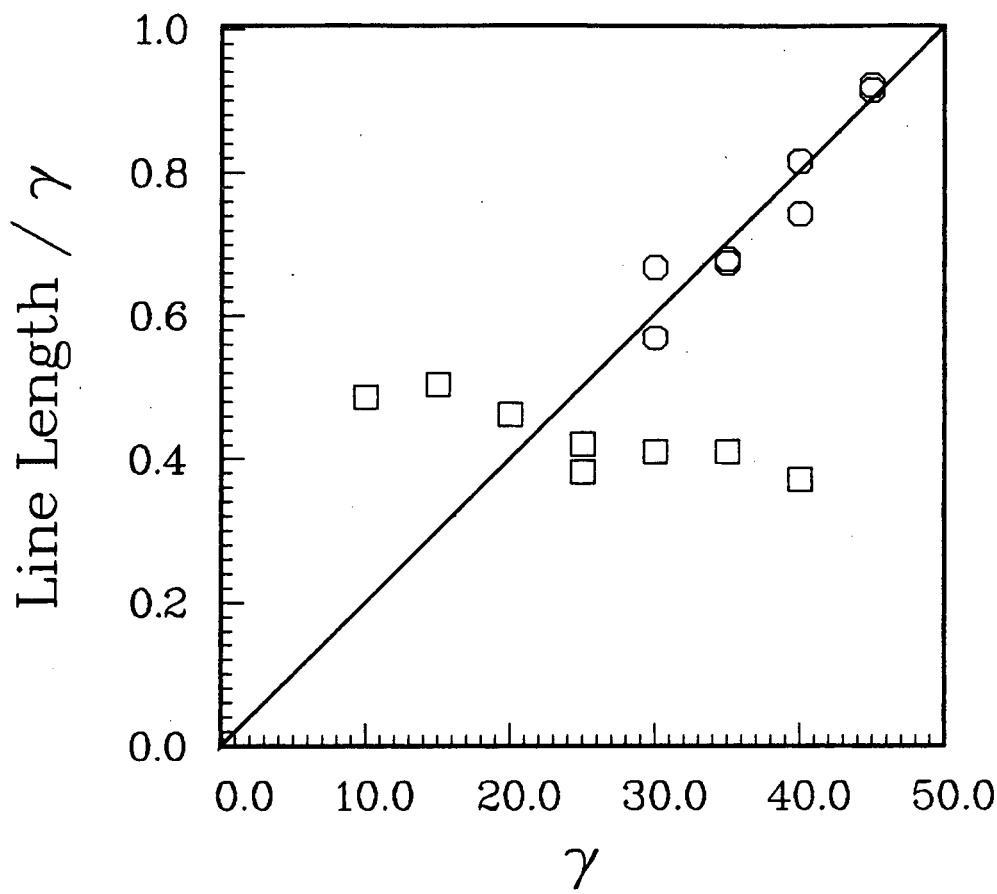

figure 3

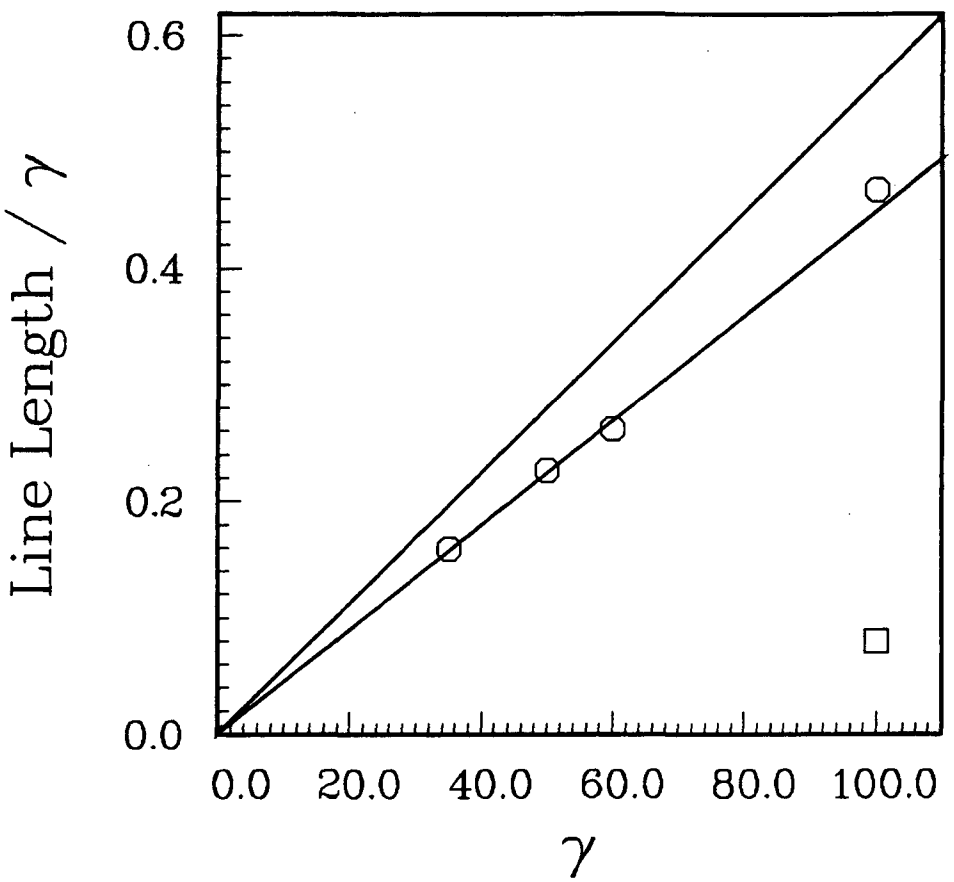

figure 4

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