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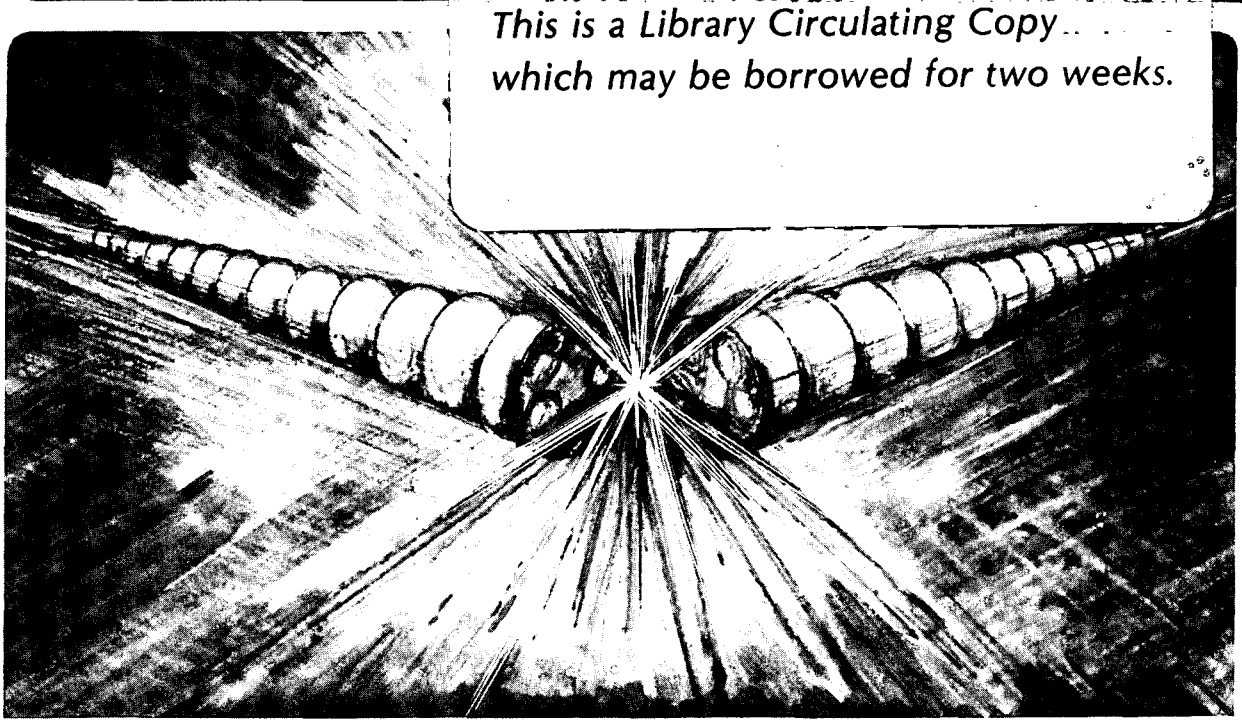

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Maslov Indices of Resonant Tori*

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Abstract

It is well known that the resonant tori of an integrable, classical Hamiltonian system break up under a small perturbation into new tori which wind around the remaining nonresonant tori. It turns out that the Maslov indices of one of these satellite tori depend on the Maslov indices of its unperturbed, resonant parent, and on the integers which enter into the resonance condition. They do not, however, depend on the functional form of the Hamiltonian or the perturbation. Our results are valid for any number of degrees of freedom.

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Semiclassical quantization schemes require a determination of the Maslov indices, integers which characterize the topology of the invariant tori of integrable and near-integrable classical Hamiltonian systems. According to the Einstein-Brillouin-Keller rule, the Maslov indices determine whether the quantized values of the classical actions are integral or half-integral:

$$I_k = (n_k + \mu_k/4)\hbar. \tag{1}$$

Here I_k is the k-th action, and n_k and μ_k represent the corresponding quantum number and Maslov index, respectively.

In recent years, many workers have applied semiclassical quantization techniques to regions of phase space surrounding a classical resonance.[1–3] Such regions are populated by satellite tori which wind around the resonant parent tori. The quantum mechanical states associated with the satellite tori are of physical interest, but the resonance complicates the computation of the Maslov indices. In particular, caustic counts become difficult because of the distorted appearance of the satellite tori in the standard $(\mathbf{q}, \mathbf{p})$ coordinates. (Caustic counts are difficult in any case in more than two degrees of freedom.)

In this Letter, we present a general formula for the Maslov indices of satellite tori surrounding an isolated, nonlinear resonance in a near-integrable system of any number N of degrees of freedom. These turn out to be simply related to the Maslov indices of the resonant parent torus. Moreover, the relation between the old and new indices depends only on the integers entering the resonance condition, and not on the functional form of the Hamiltonian or perturbation. Our results confirm formulas that have been proposed for systems of two degrees of freedom.[1,2]

Our calculation relies on a novel formula for the Maslov index, Eq. (2) below, which does not involve counting caustics. The ideas behind this formula originated in some topological observations of Arnold[4] and Voros,[5] which one of us (R.G.L.) has applied to wave packet evolution. The derivation of this formula is based on wave packet ideas and is discussed elsewhere[6]; here we merely explain what the formula means and how it may be used.

We let $(\boldsymbol{\theta}, \mathbf{J})$ be a set of proper action-angle variables in some region of interest in phase space. (The designation "proper" is due to Born,[7] and will be explained below.) We consider a closed contour $\Gamma(\lambda)$ in phase space, where λ is a parameter in the range $0 \leq \lambda < 2\pi$. The points of this contour have coordinates $\big(\boldsymbol{\theta}(\lambda), \mathbf{J}(\lambda)\big)$; we do not assume that $\Gamma(\lambda)$ necessarily lies on a torus (a surface of constant $\mathbf{J}$). For each point on $\Gamma(\lambda)$, we construct the $2N \times 2N$ matrix $\mathbf{S}(\lambda) = \partial(\mathbf{q}, \mathbf{p})/\partial(\boldsymbol{\theta}, \mathbf{J})$, which, being the Jacobian matrix of a canonical transformation, is a symplectic matrix. We only consider loops $\Gamma(\lambda)$ for which $\mathbf{S}(\lambda)$ is defined; this excludes loops which

cross separatrices (where some J_k is discontinuous) or the surfaces of vanishing action (where some θ_k is undefined). The set of matrices $\mathbf{S}(\lambda)$, $0 \leq \lambda < 2\pi$, can be thought of as a closed path in the space of symplectic matrices (i.e. the group manifold $Sp(2N)$). This group manifold has a "hole" in it (its fundamental group is $Z(1)$), so every closed path is characterized by a winding number, representing the number of times the path goes around the hole. Then the Maslov index μ corresponding to the matrix loop $\mathbf{S}(\lambda)$, and thereby to the loop in phase space $\Gamma(\lambda)$, is simply twice this winding number. We designate this relation by writing $\mu = 2\,\mathrm{wn}\big(\mathbf{S}(\lambda)\big)$.

In applications to quantization, the path $\Gamma(\lambda)$ is usually obtained by letting some θ_k vary between 0 and 2π, while holding all other θ's and all the J's fixed. (One can then set $\lambda = \theta_k$.) The Maslov index obtained thereby is the μ_k appearing in Eq. (1). We shall refer to this contour and Maslov index as being "generated by" the action J_k, since $\Gamma(\lambda)$ in this case is the orbit in phase space generated by treating J_k as a Hamiltonian. As we shall see, however, it is sometimes convenient to consider more general paths $\Gamma(\lambda)$, which are allowed by the definition above.

We will actually compute the winding numbers of symplectic matrix loops by the following method. Consider a symplectic matrix loop of the form $\mathbf{S}(\lambda) = \partial(\mathbf{q}_a, \mathbf{p}_a)/\partial(\mathbf{q}_b, \mathbf{p}_b)$, where $(\mathbf{q}_a, \mathbf{p}_a)$ and $(\mathbf{q}_b, \mathbf{p}_b)$ are any two sets of canonical variables (not just those used above). Define a complex $N \times N$ matrix $\mathbf{M}(\lambda)$ by

$$\mathbf{M}(\lambda) = \frac{\partial \mathbf{q}_a}{\partial \mathbf{q}_b}(\lambda) - i\,\frac{\partial \mathbf{p}_a}{\partial \mathbf{q}_b}(\lambda). \tag{2}$$

$\mathbf{M}(\lambda)$ is nonsingular whenever $\mathbf{S}(\lambda)$ is defined, so $\det \mathbf{M}(\lambda)$ is a nonzero complex number which executes a closed loop in the complex plane as λ varies between 0 and 2π. Then it turns out that the winding number of $\mathbf{S}(\lambda)$ is the same as the winding number of this loop about the origin in the complex plane. The latter can be written as an integral via Cauchy's theorem:

$$\mathrm{wn}\big(\mathbf{S}(\lambda)\big) = \mathrm{wn}\big(\det \mathbf{M}(\lambda)\big) = \frac{1}{2\pi i} \oint \frac{d}{d\lambda} \ln \det \mathbf{M}(\lambda)\; d\lambda. \tag{3}$$

We require our action-angle variables to be "proper," which means that as the angles vary independently between 0 and 2π, every point on the torus $\mathbf{J} = \text{const.}$ is

accessed precisely once. This is an important issue, because the Einstein-Brillouin-Keller quantization conditions are only valid for proper action-angle variables, and there are many instances in the literature where improper variables are used. However, as was already known to Born,[7] proper action-angle variables are not unique; they may always be subjected to a canonical transformation of the form $\mathbf{J}' = \mathbf{A}^{-1}\cdot\mathbf{J}$, $\boldsymbol{\theta}' = \tilde{\mathbf{A}}\cdot\boldsymbol{\theta}$, where $\mathbf{A}$ is an $N\times N$ matrix of integers with determinant ± 1, and the tilde represents the transpose. This is the same degree of indeterminacy in the choice of the basis contours on the tori, and therefore also in the Maslov indices.

The topological definition of the Maslov index, as given above, has two consequences of importance to us. First, since the Maslov index is a topological feature, it is invariant under continuous deformations of the path $\Gamma(\lambda)$ in phase space, so long as the conditions of the construction are met. This means that $\Gamma(\lambda)$ cannot be allowed to cross points of phase space where $\mathbf{S}(\lambda)$ is not defined. Second, if $\mathbf{S}_1(\lambda)$ and $\mathbf{S}_2(\lambda)$ are two loops of symplectic matrices, then the matrix product $\mathbf{S}(\lambda) = \mathbf{S}_1(\lambda)\mathbf{S}_2(\lambda)$ is another such loop, and the corresponding winding numbers are related by

$$\mathrm{wn}\big(\mathbf{S}(\lambda)\big) = \mathrm{wn}\big(\mathbf{S}_1(\lambda)\big) + \mathrm{wn}\big(\mathbf{S}_2(\lambda)\big). \tag{4}$$

That is, when we multiply symplectic matrix loops, the winding numbers simply add. This is a standard result from the homotopy theory of topological groups, as applied to the symplectic group, and is easy to prove. These two theorems are examples of the kinds of manipulations one can carry out with the topological definition of the Maslov index, which would be difficult to apply to caustic counts.

Let us now consider a near-integrable Hamiltonian $H = H_0(\mathbf{J}') + \epsilon H_1(\boldsymbol{\theta}', \mathbf{J}')$, expressed in terms of proper action-angle variables $(\boldsymbol{\theta}', \mathbf{J}')$, which has an isolated resonance at action values $\mathbf{J}' = \mathbf{J}'_r$. (The prime will be explained momentarily.) Thus, $\mathbf{n}'\cdot\boldsymbol{\omega}'_r = 0$, where $\mathbf{n}'$ is a relatively prime integer vector, and $\boldsymbol{\omega}'_r = \partial H_0(\mathbf{J}'_r)/\partial\mathbf{J}'$. (A relatively prime integer vector is one which collectively has no common divisor.) Under a canonical transformation to new proper action-angle variables, $\mathbf{J} = \mathbf{A}\cdot\mathbf{J}'$, $\boldsymbol{\theta} = \tilde{\mathbf{A}}^{-1}\cdot\boldsymbol{\theta}'$, the frequency and integer vectors transform according to $\boldsymbol{\omega}_r = \tilde{\mathbf{A}}^{-1}\cdot\boldsymbol{\omega}'_r$, $\mathbf{n} = \mathbf{A}\cdot\mathbf{n}'$. We claim it is always possible to find a unimodular matrix of integers $\mathbf{A}$ such that the new integer vector $\mathbf{n}$ has the form $(1, 0, \ldots, 0)$.

We prove this by induction on the number of degrees of freedom, N. The statement is trivial for $N = 1$. We assume it is true for $N - 1$. Let g be the greatest common divisor of $(n'_2, \ldots, n'_N)$. Then $(1/g)(n'_2, \ldots, n'_N)$ is a relatively prime $(N-1)$-vector of integers, so by the induction hypothesis, there exists an $(N-1)\times(N-1)$ unimodular matrix $\mathbf{B}$ which maps this $(N-1)$-vector of integers into the $(N-1)$-vector $(1, 0, \ldots, 0)$. Now define the $N\times N$ matrix $\mathbf{A}_2$ by its partition according to $N = 1 + (N-1)$:

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{B} \end{pmatrix}.$$

Then $\mathbf{A}_2 \cdot \mathbf{n}' = (n'_1, g, 0, \ldots, 0)$. But n'_1 and g are relatively prime, since $\mathbf{n}'$ overall has no common divisors. It follows by an elementary theorem of number theory[8] that there exist integers r and s (which can be calculated by Euclid's algorithm) such that $rn'_1 + sg = 1$. We use these integers to define another $N\times N$ matrix $\mathbf{A}_1$ by its partition according to $N = 2 + (N-2)$,

$$\mathbf{A}_1 = \begin{pmatrix} \begin{pmatrix} r & s \\ -g & n'_1 \end{pmatrix} & 0 \\ 0 & \mathbf{I} \end{pmatrix},$$

and we set $\mathbf{A} = \mathbf{A}_1\mathbf{A}_2$. Then $\mathbf{A}$ is unimodular and $\mathbf{A}\cdot\mathbf{n}' = (1, 0, \ldots, 0)$, as claimed.

Thus, if we begin with variables $(\boldsymbol{\theta}', \mathbf{J}')$, we can always switch to new variables $(\boldsymbol{\theta}, \mathbf{J})$ such that at the resonant values of the actions for the unperturbed Hamiltonian, $\mathbf{J} = \mathbf{J}_r$, the first component of $\boldsymbol{\omega}_r$ vanishes, $\omega_{r1} = 0$. Furthermore, the ω_{rk}, for $2 \le k \le N$, are rationally independent, because we assume the resonance is isolated. (Otherwise, we would have chaos, and no tori.) The invariant tori of the unperturbed Hamiltonian are given by either $\mathbf{J}' = \text{const.}$ or $\mathbf{J} = \text{const.}$; we denote the Maslov indices generated by these actions by $\boldsymbol{\mu}'$ and $\boldsymbol{\mu}$, respectively. As is apparent from Eq. (1), the Maslov indices transform like the actions, so we have $\boldsymbol{\mu} = \mathbf{A}\cdot\boldsymbol{\mu}'$

We now consider the effect of the perturbation. First, using standard resonant perturbation theory,[9] we average over the variables $(\theta_2, \ldots, \theta_N)$ to get a Hamiltonian of the form $H = H_0(\mathbf{J}) + \epsilon K_1(\mathbf{J}, \theta_1)$, in which only the angle θ_1 is not ignorable.

In the usual way in the analysis of resonances,[9] we treat $\mathbf{J} - \mathbf{J}_r$ as $O(\epsilon^{1/2})$ and expand H through $O(\epsilon)$, dropping constant terms, to get a Hamiltonian of the form

$$H = \omega_r \cdot (\mathbf{J} - \mathbf{J}_r) + \tfrac{1}{2}(\mathbf{J} - \mathbf{J}_r) \cdot \mathbf{D} \cdot (\mathbf{J} - \mathbf{J}_r) + \epsilon U(\theta_1),$$

where $U(\theta_1) = K_1(\mathbf{J}_r, \theta_1)$ and $D_{kl} = \partial^2 H_0(\mathbf{J}_r)/\partial J_k \partial J_l$. This is an integrable Hamiltonian, and its action-angle variables describe the satellite tori whose Maslov indices we seek. However, since the Maslov indices are constant inside separatrices, it suffices to examine only those tori very close to the stable fixed points of the (θ_1, J_1) motion, where $U'(\theta_1) = 0$. Usually there is only one such stable fixed point (since the resonance integers are $\mathbf{n} = (1, 0, \ldots, 0)$) and we take it to occur at $\theta_1 = 0$. We expand $U(\theta_1)$ about this point to get the harmonic oscillator potential, $\epsilon U''(0)\theta_1^2/2$.

It is then easy to transform this Hamiltonian to action-angle variables. We denote the canonical transformation by $(\boldsymbol{\theta}, \mathbf{J}) \to (\boldsymbol{\phi}, \mathbf{I})$; it is generated by

$$F(\boldsymbol{\theta}, \mathbf{I}) = \boldsymbol{\theta} \cdot (\mathbf{I} + \mathbf{J}_r) + \int_0^{\theta_1} d\alpha \left[\sqrt{2\Omega I_1 - \Omega^2 \alpha^2} - \mathbf{d} \cdot \mathbf{I}/D_{11}\right], \tag{5}$$

where $\Omega^2 = \epsilon U''(0)/D_{11}$ and $d_k = D_{1k}$. It is straightforward to show, by using Arnold's construction,[10] that the variables $(\boldsymbol{\phi}, \mathbf{I})$ are indeed proper. The satellite tori are given by $\mathbf{I} = \text{const.}$, and we denote the corresponding Maslov indices by $\boldsymbol{\nu}$. The Hamiltonian is a function only of $\mathbf{I}$ in the new coordinates, but, since we do not need it for subsequent work, we will not display it.

It turns out, for $k \neq 1$, that the new Maslov indices ν_k are the same as the old ones, μ_k. To see this, note first that the contours generated by the J_k are the same as those generated by the I_k, since Eq. (5) gives us $J_k = \partial F/\partial \theta_k = I_k + J_{rk}$ for $k \neq 1$. (That is, J_k and I_k, treated as Hamiltonians, have the same orbits in phase space.) This does not mean that the loops of symplectic matrices are the same, for we have $\mu_k = 2\,\mathrm{wn}\big(\partial(\mathbf{q}, \mathbf{p})/\partial(\boldsymbol{\theta}, \mathbf{J})\big)$ and $\nu_k = 2\,\mathrm{wn}\big(\partial(\mathbf{q}, \mathbf{p})/\partial(\boldsymbol{\phi}, \mathbf{I})\big)$. (Different sets of action-angle variables are used in the two cases, even if the phase space loops along which the Jacobians are evaluated are the same.) However, by using the chain rule and Eq. (4), we can write

$$\nu_k = 2\,\mathrm{wn}\left[\frac{\partial(\mathbf{q}, \mathbf{p})}{\partial(\boldsymbol{\theta}, \mathbf{J})} \cdot \frac{\partial(\boldsymbol{\theta}, \mathbf{J})}{\partial(\boldsymbol{\phi}, \mathbf{I})}\right] = 2\,\mathrm{wn}\left[\frac{\partial(\mathbf{q}, \mathbf{p})}{\partial(\boldsymbol{\theta}, \mathbf{J})}\right] + 2\,\mathrm{wn}\left[\frac{\partial(\boldsymbol{\theta}, \mathbf{J})}{\partial(\boldsymbol{\phi}, \mathbf{I})}\right]. \tag{6}$$

The first term on the right is simply μ_k. As for the second term, we apply Eqs. (2,3) to the canonical transformation generated by Eq. (5), and find, after a short calculation,

$$\det\left(\frac{\partial\boldsymbol{\theta}}{\partial\boldsymbol{\phi}} - i\,\frac{\partial\mathbf{J}}{\partial\boldsymbol{\phi}}\right) = \sqrt{\frac{2I_1}{\Omega}}\,(\cos\phi_1 + i\Omega\sin\phi_1). \tag{7}$$

But ϕ_1 and I_1 are constant along the orbit generated by I_k or J_k for $k \neq 1$, so the second winding number is zero. Thus we obtain $\nu_k = \mu_k$, as claimed.

As for ν_1, Eq. (6) is still valid with $k = 1$, but the loop in phase space along which the Jacobians are evaluated is now an orbit generated by I_1. This is no longer one of the basis contours on the parent torus (it does not coincide with the orbits generated by any of the J_k), so the first winding number on the right can no longer be related to any of the μ_k. In fact, it vanishes. To see this, express the orbit generated by I_1 in the $(\boldsymbol{\phi}, \mathbf{I})$ coordinates (which is easy, since only ϕ_1 is not constant), and use the canonical transformation generated by Eq. (5) to convert to the $(\boldsymbol{\theta}, \mathbf{J})$ coordinates. One can then easily show that this loop can be contracted to a point without crossing any point where $\partial(\mathbf{q},\mathbf{p})/\partial(\boldsymbol{\theta},\mathbf{J})$ is undefined. Therefore the first winding number is zero. The second winding number on the right in Eq. (6) can be computed, again, by appeal to Eqs. (2,3,7). Now, however, ϕ_1 is not constant, but rather ranges from 0 to 2π. As it does so, the indicated complex number encircles the origin once. Therefore the second term in Eq. (6) is 2, and, altogether, we have $\nu_1 = 2$.

To summarize, we have shown that in the appropriate coordinates, one of the Maslov indices of the satellite tori is 2, and the others are the same as those of the resonant parent. We have also shown explicitly how to find these special coordinates.

To illustrate these results, we consider the $1:1$ resonance in the water molecule, in the model which has been studied by Sibert *et al.*[11] and others. In this model, the water molecule consists of two coupled one-dimensional oscillators, which, if the coupling can be ignored, have Maslov indices $\boldsymbol{\mu}' = (2,2)$. Near the $1:1$ resonance, however, there are satellite tori. The integer vector for this resonance is $\mathbf{n}' = (1,-1)$, which is mapped into $\mathbf{n} = (1,0)$ by the unimodular matrix, $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, so that $\boldsymbol{\mu} = (2,4)$. Therefore the Maslov indices of the satellite tori are $\boldsymbol{\nu} = (2,4)$. More

generally, it is easy to show that an $n : m$ resonance in such a system gives satellite Maslov indices of $\nu = \big(2, 2(n+m)\big)$. This confirms the suggestions made by DeLeon, Davis, and Heller[1] and Ezra and Martens.[2]

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