LBL-22897 c.2
Preprint

Lawrence Berkeley Laboratory
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Accelerator & Fusion Research Division

Submitted to Physical Review Letters

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MAR 1 0 1987

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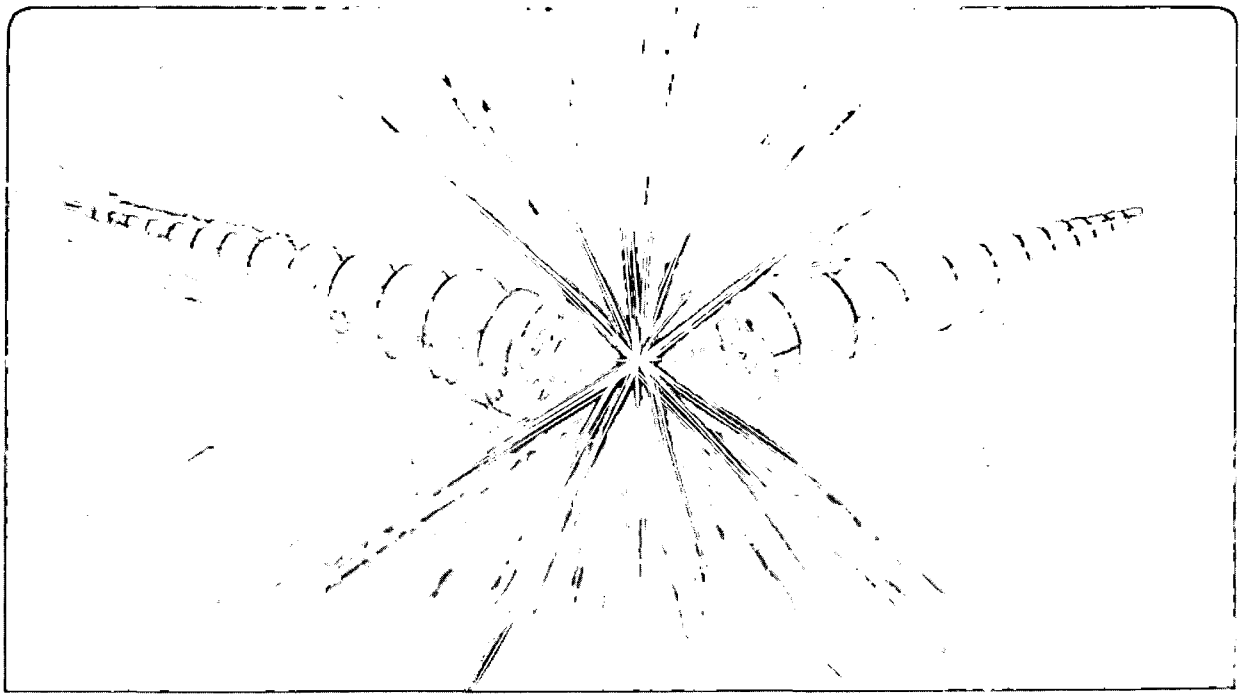
January 1987



Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

LBL-22897 c.2

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FOUR-DIMENSIONAL THEORY OF LINEAR MODE CONVERSION*

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January 1987

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*This research was supported by the U.S. Department of Energy under Contract No. DE-AC03-76SF00098, and by a grant from the U.S.-Israel Binational Science Foundation.

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Abstract

The problem of linear mode conversion is solved for general geometry, in that the dispersion functions $D_a(k,x)$, $D_b(k,x)$ of the two, orthogonally polarized, coupled modes *a,b* have spatial gradients $\partial D_a/\partial x^\mu$, $\partial D_b/\partial x^\mu$ which need not be parallel. The transmission ratio is $\exp(-2\pi |\eta|^2 / |B|)$, where η is the coupling coefficient, and B is the Poisson Bracket $\{D_a,D_b\} = (\partial D_a/\partial x^\mu)(\partial D_b/\partial k_\mu) - (\partial D_a/\partial k_\mu)(\partial D_a/\partial x^\mu)$. The further generalization to weak dissipation and ray divergence is included.

The propagation of waves in an anisotropic weakly-nonuniform medium has been successfully treated by eikonal theory.[1a] The medium supports a set of N modes, with mutually orthogonal polarizations $\mathbf{e}_a$, each mode having its own dispersion function $D_a(k,x)$, which is one of the N local eigenvalues of the dispersion matrix $D_{ij}(k,x)$. It is assumed that the variation of **D** and of D_a with respect to $x^\mu = (\mathbf{x},t)$ is slow, and with respect to $k_\mu = (\mathbf{k}, -\omega)$ is smooth. The rays of each mode are generated independently in the 8-dimensional phase space (k,x), by the respective ray Hamiltonians $D_a(k,x)$: $dx^\mu/d\sigma_a = -\partial D_a/\partial k_\mu$, $dk_\mu/d\sigma_a = \partial D_a/\partial x^\mu$; the ray-orbit parameter σ_a is related to time by $dt/d\sigma_a = \partial D_a/\partial\omega$. For a coherent wave of mode *a*, its rays form a 4-dimensional surface, imbedded in the 7-dimensional dispersion surface $D_a(k,x) = 0$.

This conventional eikonal analysis breaks down when two of the eigenvalues (D_+ and D_-, say) are near-degenerate near zero ($D_+ \approx D_- \approx 0$) in some (k,x) region, causing these eigenvalues and their eigenvectors to become rapidly varying; wave-action can then be effectively transferred between the two corresponding modes. The geometrical picture is explained in Fig. 1.

Almost all studies of this linear mode conversion process have been limited to the one-dimensional case; i.e., it has been assumed that the components of the dispersion matrix vary in only one direction locally.[1b,2,3] In general, however, the two interacting modes, with polarizations $\mathbf{e}_a$, $\mathbf{e}_b$, have non-parallel dispersion gradients $\partial D_a/\partial x^\mu$, $\partial D_b/\partial x^\mu$ in the conversion region. In that case, we conclude that an effectively two-dimensional treatment is required, which is presented in this letter. Our approach generalizes previous work[4] by one of us (LF) on the one-dimensional case; that work was inspired by the unifying local analyses of Cairns and Lashmore-Davies[2] and of Fuchs, Ko, and Bers.[3]

Before presenting our analysis, let us summarize our results, for the case that the

dispersion matrix is Hermitian. In the mode-conversion region, centered at (k_o,x_o), its two relevant eigenvalues $D_\pm(k,x)$ can be expressed as

$$D_\pm(k,x) = \tfrac{1}{2}(D_a + D_b) \pm \tfrac{1}{2}[(D_a - D_b)^2 + 4|\eta|^2]^{\frac{1}{2}}, \qquad (1)$$

where D_a, D_b are locally *linear* functions on phase space:

$$D_a(k,x) = (k - k_o)_\mu V_a^\mu + (x - x_o)^\mu R_\mu^a, \qquad (2)$$

and η is a small constant, representing one-half the minimum eigenvalue separation $|D_+ - D_-|$. The $\pm$ signs in (1) are to be chosen so that [see Fig. 1] $D_+ \to D_a$ on the incident ray, $D_+ \to D_b$ on the converted ray, and $D_- \to D_a$ on the transmitted ray. Thus D_a, D_b are eigenvalues of **D** in the asymptotic region $|D_+ - D_-| >> |\eta|$. The vector coefficients in (2) are the group 4-velocity $V_a{}^\mu = \partial D_a/\partial k_\mu$ and spatial gradient (or 4-refraction) $R_\mu{}^a = \partial D_a/\partial x^\mu$, evaluated at (k_o,x_o); similarly for mode **b**.

The transmission ratio T is defined as the absolute square of the ratio of the amplitudes of the transmitted and incident rays, at equal distances from x_o, in the asymptotic region. We derive the result

$$T = \exp[-2\pi\, |\eta|^2/|B|], \qquad (3)$$

where B is the Poisson Bracket

$$B = \{D_a, D_b\} = R^a \cdot V_b - R^b \cdot V_a, \qquad (4)$$

of the dispersion functions of the two modes. In the one-dimensional case, (3) reduces

to the results of Refs. 2 and 4.

Our starting point is the linear integral equation

$$\int d^4x_2 \; \hat{D}_{ij}(x_1,x_2)\, Z_j(x_2) = 0 \tag{5}$$

for the N-component field $\mathbf{Z}(x)$ on space-time. The two-point N x N kernel matrix $\hat{D}_{ij}(x_1,x_2)$ is considered given; its local Fourier transform

$$\mathbf{D}(k,x) = \int d^4s\, \hat{\mathbf{D}}(x+\tfrac{1}{2}s,\, x-\tfrac{1}{2}s)\, \exp(-ik\cdot s) \tag{6}$$

was referred to above. We allow $\mathbf{D} = \mathbf{D}^H + \mathbf{D}^A$ to have a small anti-Hermitian part $\mathbf{D}^A$, representing dissipation. The eikonal form $\mathbf{Z}(x) = \mathbf{A}(x) \exp i\Psi(x)$, with *slowly-varying* amplitude $\mathbf{A}(x)$ and wave-vector $k_\mu(x) = \partial\Psi/\partial x^\mu$, leads, to *first* order, to

$$\mathbf{D}^H \circ \mathbf{A} = i[(\partial\mathbf{D}^H/\partial k_\mu)\circ(\partial\mathbf{A}/\partial x^\mu) + \tfrac{1}{2}d(\partial\mathbf{D}^H/\partial k_\mu)/dx^\mu \circ \mathbf{A}] - \mathbf{D}^A\circ\mathbf{A} \tag{7}$$

Expanding $\mathbf{A}(x) = \mathbf{A}^{(o)}(x) + \mathbf{A}^{(1)}(x) + \ldots$, we have, to zero order,

$$\mathbf{D}^H(k,x)\circ\mathbf{A}^{(o)}(x) = 0\,. \tag{8}$$

In the non-degenerate region, we would conclude that the N-vector $\mathbf{A}^{(o)}(x)$ is a scalar multiple of a local eigenvector $\mathbf{e}_a(k(x),x)$ of $\mathbf{D}^H(k,x)$ with zero eigenvalue $D_a(k,x)$. However, in the mode-conversion region, the eigenvectors vary rapidly, so we use the method of Ref. (4). At a point (x_*,k_*) where an incident ray *enters* the conversion

region, we construct the *constant* basis pair $\mathbf{m}_a = \mathbf{e}_a(x_*)$, $\mathbf{m}_b = \mathbf{e}_b(x_*)$. Setting $\mathbf{A}(x) \doteq A_a(x)\mathbf{m}_a + A_b(x)\mathbf{m}_b$, we obtain

$$
\begin{aligned}
D_a A_a + \eta A_b &= i[(\partial D_a/\partial k)\cdot(\partial A_a/\partial x) + \tfrac{1}{2}(d/dx)\cdot(\partial D_a/\partial k)A_a] - \mathbf{m}_a\cdot\mathbf{D}^A\cdot\mathbf{A}\,, \\
\eta^* A_a + D_b A_b &= i[\partial D_a/\partial k)\cdot(\partial A_a/\partial x) + \tfrac{1}{2}(d/dx)\cdot(\partial D_b/\partial k)A_b] - \mathbf{m}_b\cdot\mathbf{D}^A\cdot\mathbf{A}\,,
\end{aligned}
\tag{9}
$$

where $D_a(k,x) = \mathbf{m}_a^* \circ \mathbf{D}^H(k,x) \circ \mathbf{m}_a$, similarly for *b*, and $\eta(k,x) = \mathbf{m}_a^* \circ \mathbf{D}^H(k,x) \circ \mathbf{m}_b$. We note that at (x_*,k_*), $D_a = 0 = \eta$ and $D_b \neq 0$. We now continue on the ray *a* generated by $D_a(k,x)$ [see Fig. 1], to locate the point (x_o,k_o) where $D_b = 0$. (The distance along ray *a* is found to be $\Delta\sigma_a = - D_b(k_*, x_*)/B$.) We set $\eta = \eta(k_o,x_o)$ (to lowest order). The left side of (9) now yields Eq. (1). Further, we use the underdeterminacy of our system to set $k = k_o$ throughout the conversion region. [4] After some algebra, we obtain

$$
\begin{cases}
V_a\cdot\partial A_a/\partial x - (ix'\cdot R^a + C_a)A_a = i\eta_{ab}A_b\,, \\[2ex]
V_b\cdot\partial A_b/\partial x - (ix'\cdot R^b + C_b)A_b = i\eta_{ba}A_a\,.
\end{cases}
\tag{10}
$$

where $x' = x-x_o$, $C_a = i\,\mathbf{m}_a^* \circ \mathbf{D}^A \circ \mathbf{m}_a - \tfrac{1}{2}\,dV_a^\mu/dx^\mu$, $\eta_{ab} = \mathbf{m}_a^* \circ \mathbf{D}(k_o,x_o) \circ \mathbf{m}_b$. The (real) coefficients C incorporate dissipation and divergence due to nonuniform group velocity, and are evaluated at (k_o, x_o). These equations yield the conservation law for total wave-action:

$$(\partial/\partial x^\mu)[V_a^\mu(x)|A_a|^2(x) + V_b^\mu(x)|A_b|^2(x)] = i\mathbf{A}^*\circ\mathbf{D}^A\circ\mathbf{A};$$

for each mode, $V_n^\mu(x)|A_n|^2(x)$ is the action flux four-vector.

With the coefficients V, R, C, η in (10) all constants, we proceed to the solution. The change of dependent variable: $Y_n(x) = A_n(x) \exp\{-[i(x'\cdot R^n)^2/2 + C_n\, x'\cdot R^n]/(V_n\cdot R^n]\}$, and of coordinates $x \to \sigma$ in the conversion plane: $x'^\mu = V_a^\mu\, \sigma_a + V_b^\mu\, \sigma_b$, yields

$$\begin{cases} \partial Y_a/\partial\sigma_a = i\eta_{ab}\, Y_b \exp[-i\phi(\sigma_a,\sigma_b)]\,, \\ \\ \partial Y_b/\partial\sigma_b = i\eta_{ba}\, Y_a \exp[+i\phi(\sigma_a,\sigma_b)]\,, \end{cases} \tag{11}$$

With $\phi = c_a\sigma_a^2/2 - c_b\sigma_b^2/2 - B\,\sigma_a\sigma_b + id_b\sigma_a - id_a\sigma_b$, $\quad c_a = (V_a\cdot R^a) - (V_a\cdot R^b)^2/(V_b\cdot R^b)$, $d_a = C_b - C_a(V_b\cdot R^a)/(V_a\cdot R^a)$. Eliminating Y_b from (11) and substituting

$g(\sigma_a,\sigma_b) = Y_a(\sigma_a,\sigma_b) \exp(-i\, c_b\, \sigma_b^2/2)$ yields

$$\partial^2 g/\partial\sigma_a\partial\sigma_b = -\eta_{ab}\eta_{ba}\, g + (iB\sigma_a - d_a)\, \partial g/\partial\sigma_a\,. \tag{12}$$

By separation of variables, the solution of (12) is

$$g(\sigma_a,\sigma_b) = \int d\beta\, f(\beta) \exp(i\beta\sigma_b)\, [\sigma_a - (\beta - id_a)/B]^{-i\rho}\,, \tag{13}$$

with $\rho = \eta_{ab}\eta_{ba}/B$, and $f(\beta)$ arbitrary. To determine $f(\beta)$, we return to (10), and

consider the initial state ($\sigma_a \to -\infty$), with no excitation of mode *b*; we obtain

$$g \to \alpha\, \sigma_a^{-i\rho} \exp\left(-i\, V_b \cdot R^b\, \sigma_b^2/2\right), \qquad (14)$$

with α a constant. Obtaining $f(\beta)$ as the Fourier transform of (14) with respect to σ_b: $f(\beta) = \alpha' \exp\left[i\beta^2/(2V_b \cdot R^b)\right]$, we pass to the transmission ratio:

$$T = \lim_{\sigma_a \to \infty} \left| A_a(+\sigma_a,\sigma_b)/A_a(-\sigma_a,\sigma_b)\right|^2 = \left|(-1)^{-i\rho}\right|^2 \exp(-4C_a \Delta\sigma_a). \qquad (15)$$

With $(-1) = \exp(\pm \pi i)$, we choose the causal sign, and obtain

$$T = \exp\left[(-2\pi \left|\mathrm{Re}(\eta_{ab}\eta_{ba})/B\right| - 4\, C_a\, \Delta\sigma_a)\right]. \qquad (16)$$

In the case of zero dissipation and no divergence ($\eta_{ab} = \eta_{ba}^*$, $C_a = 0$), this is the result (3).

In summary, we have derived an expression for the transmission ratio in linear mode conversion, for general geometry, on the basis of a local analysis. We plan to report on applications of this result in future publications.

This research was supported by the U.S. Department of Energy under Contract No. DE-AC03-76SF00098, and by a grant from the U.S. - Israel Binational Science Foundation.

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Figure Caption

Figure 1:

Geometry of the linear mode conversion process in 4-dimensional space-time. The incident and transmitted modes, with polarization $\mathbf{e}_a$, have group 4-velocity V_a^μ and dispersion gradient $R_\mu^a = \partial D_a(k,x)/\partial x^\mu$. The converted mode, with polarization $\mathbf{e}_b$ (orthogonal to $\mathbf{e}_a$) has group velocity V_b^μ and dispersion gradient $R_\mu^b = \partial D_b(k,x)/\partial x^\mu$. Note that the four 4-vectors are not co-planar, in general. The curved ray is generated by the eigenvalue $D_+(k,x)$ of the dispersion matrix $\mathbf{D}(k,x)$. The two dispersion functions D_a, D_b are the asymptotic limits of D_+.

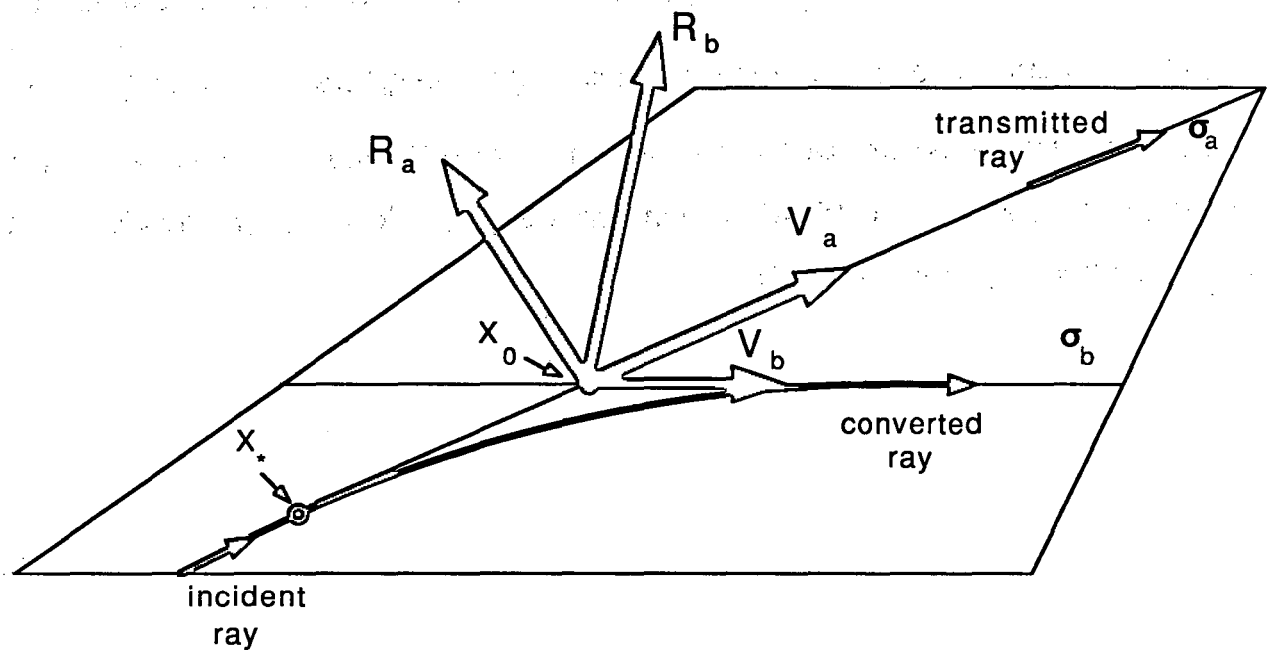
R_b
R_a
transmitted ray
σ_a
V_a
X_0
V_b
σ_b
converted ray
X_*
incident ray

Fig. 1

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

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