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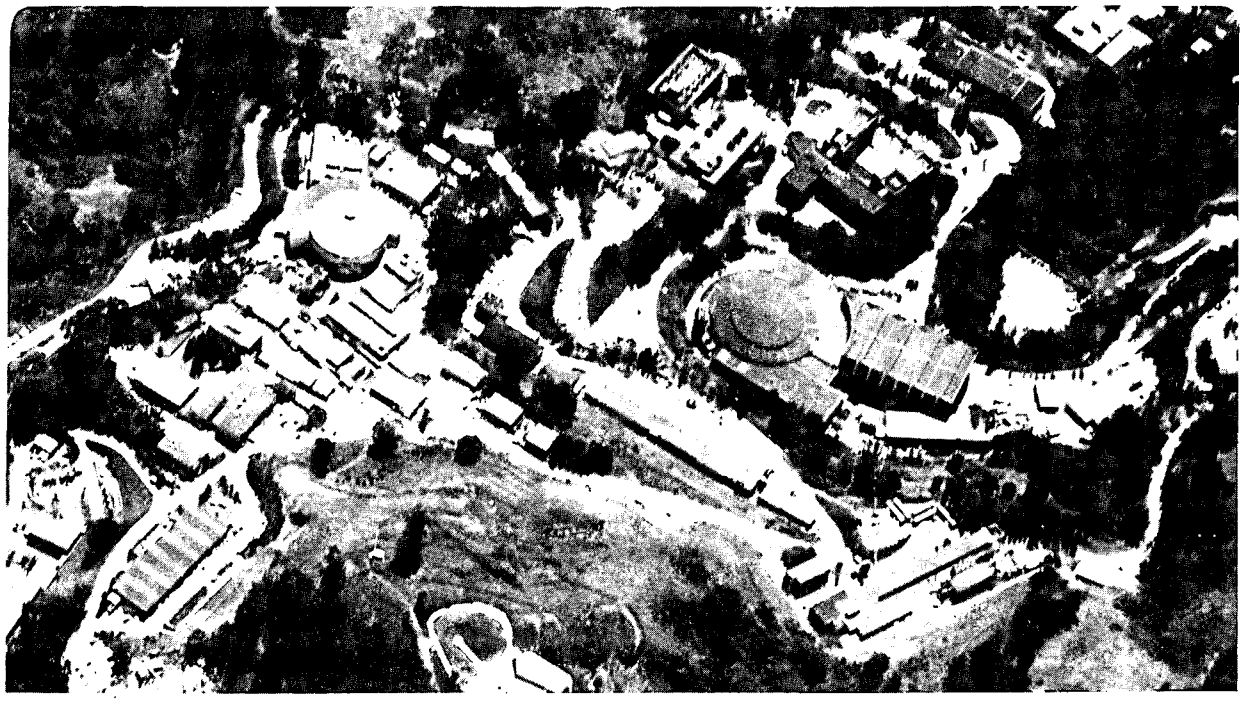

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THE GEOMETRY OF CONTINUUM REGULARIZATION*†

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ABSTRACT

This lecture is primarily an introduction to coordinate-invariant regularization, a recent advance in the continuum regularization program. In this context, the program is seen as fundamentally geometric, with all regularization contained in regularized DeWitt superstructures on field deformations.

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1. Introduction

The goal of the continuum regularization program[1-3] is uniform nonperturbative invariant continuum regularization across all quantum field theory. Successful applications have been given for the scalar prototype[2], gauge theory[1,3-7], gauge theory with fermions[8,9] and supersymmetric gauge theory[10]. An introductory discussion of the program may be found in Ref. 11. Recently, a general framework for coordinate-invariant regularization[12,13] has been given, with applications to the general non-linear sigma model[12,13] and quantum gravity[12-14] as examples.

I shall concern myself here primarily with this advance. The most general framework is in *phase-space*[13], and, with certain covariant spacetime Laplacians, this formulation should provide invariant regularization for any theory with Liouville measure. The development is geometric, with all regularization contained in regularized DeWitt[15] superstructures on field-coordinate deformations. Parallel development of invariant *coordinate-space* regularization follows with regularized functional integration[9] of the momenta, and here the previous regularizations of the program are seen as special cases in flat space and flat superspace.

For a given formal theory, defined as a functional integral with certain symmetries, the program always follows two basic steps:

1) Find a formally equivalent covariant stochastic[16,17] or Schwinger-Dyson (SD) formulation. Viability of the SD formulations is expected even when the formal action is unbounded, as in the case of Euclidean gravity discussed below.

2) Apply covariant-derivative regularization,

$$R(\Delta) = \begin{cases} \exp(\Delta/\Lambda^2), & \text{Euclidean} \\ (1-\Delta/\Lambda^2)^{-n}, & \text{Euclidean-Minkowski} \end{cases} \tag{1.1}$$

where Λ is the cutoff, R is the regulator and Δ is the relevant covariant spacetime Laplacian. The exponential or heat-kernel[5] regulator is guaranteed to regularize any Euclidean theory, and power-law regularization[1-4,6-7] is expected to succeed as well for n not less than some theory-dependent critical power.

2. Phase-space functional integrals

Introduce generic field-coordinates $\phi^M(\xi)$ and conjugate momenta $\pi_M(\xi)$ on a d-dimensional space time ξ^m, where the generic index M may include Einstein tensor indices. The formal theory of interest

$$Z = \int \mathcal{D}\omega\, e^{-H[\phi,\pi]} \tag{2.1a}$$

$$\mathcal{D}\omega = \prod_{\xi} d\pi_M(\xi) \wedge d\phi^M(\xi) \tag{2.1b}$$

$$\langle F \rangle = Z^{-1} \int \mathcal{D}\omega\, e^{-H} F[\phi,\pi] \tag{2.1c}$$

is taken with Liouville measure and an essentially arbitrary "Hamiltonian" H. I focus here on two possible coordinate-invariances of Z,

G_ϕ: field-coordinate diffeomorphisms $\bar{\phi}^M(\xi) = F^M(\phi(\xi))$

G_ξ: Einstein diffeomorphisms $\bar{\xi}^m = f^m(\xi)$

which, along with gauge invariance and other symmetries, should be respected by the regularization.

I also introduce DeWitt's[15] G_ϕ and G_ξ-invariant inner-product on field-deformations

$$\|\delta\phi\|^2 = \int (d\xi)\, \mathcal{G}_{MN}(\phi(\xi))\delta\phi^N(\xi)\delta\phi^M(\xi) \tag{2.2a}$$

$$\mathcal{G}_{MN}(\phi(\xi)) = \mathcal{E}_{MA}(\phi(\xi))\mathcal{E}_{NA}(\phi(\xi)) \tag{2.2b}$$

where $\mathcal{G}_{MN}$ and $\mathcal{E}_{MA}$ are the ultralocal supermetric and supervielbein respectively. For example, the most general supermetric for gravity $\phi^M \to g_{mn}$ is

$$\mathcal{G}^{mn;rs} = e\left[\frac{1}{2}(g^{mr}g^{ns} + g^{ms}g^{nr}) + \gamma g^{mn}g^{rs}\right] \tag{2.3}$$

with $e = \det^{1/2}[g_{mn}]$. The inverse superstructures are $\mathcal{G}^{MN}$ and $\mathcal{E}^{MA}$. I am going to employ these superstructures as *auxiliary* quantities or covariant kernels[16] in the construction of equivalent covariant stochastic and SD formulations below, independent of the specific form of H.

Some examples of particular interest are as follows. The Hamiltonian

$$H = \frac{1}{2}\int (d\xi)\, \pi_M \mathcal{G}^{MN}\pi_N + S[\phi] \tag{2.4}$$

corresponds to

$$Z = \int \mathcal{D}\phi\, \mathcal{E}[\phi] e^{-S[\phi]} \tag{2.5a}$$

$$\mathcal{E}[\phi] \equiv \prod_{\xi} \det{}^{1/2}\mathcal{G}_{MN}(\phi(\xi)) \tag{2.5b}$$

after a trivial momentum integration in (2.1), so this is the general theory with coordinate-space DeWitt measure. The class includes the general non-linear sigma model

$$S = \frac{1}{2}\int (d\xi)\, \mathcal{G}_{MN} g^{mn}\partial_n\phi^N\partial_m\phi^M + \cdots \tag{2.6a}$$

$$\mathcal{G}_{MN} = eG_{MN}(\phi) \tag{2.6b}$$

and DeWitt-measure Euclidean gravity.

Other cases of interest include real-time Hamiltonian and constrained Hamiltonian systems. As an example, consider canonical gravity[19] with $\phi^M \to (g_{ij}, \lambda_m)$, $\pi_M \to (\pi^{ij}, \pi^m)$,

$$H = -i\int (d\xi)[\pi^{ij}\partial_t g_{ij} - \lambda_m T^m(g_{ij}, \pi^{ij})] \tag{2.7a}$$

$$g_{mn}: \qquad g_{ij}, \quad g_{0i} = \lambda_i, \quad g_{00} = \lambda_i\lambda^i - \lambda_0^2 \tag{2.7b}$$

and the standard identification of the full metric g_{mn} is given in (2.7b). This unextended phase-space formulation is manifestly invariant only under $G_{\vec{\xi}}$, the group of spatial Einstein diffeomorphisms, but the full auxiliary supermetric is easily constructed from (2.3) and (2.7b).

Zwanziger's ghostless gauge-fixing[20] may be applied to unextended phase-space theories at the stochastic and SD levels below. A next step to Grassmann-extended phase-spaces, incorporating the developments of Refs. 8,9, will allow the regularization of more conventional ghost gauge-fixing[21].

3. Phase-space regularization

The original stochastic process studied by Langevin[22] was a phase-space process, and two recent studies in phase-space stochastic quantization[23] of flat-space scalar fields were helpful in constructing the general processes of this section.

I begin by stating the G_ϕ and G_ξ (or $G_{\tilde{\xi}}$) covariant[13] phase-space process

$$\dot{\pi}_M = -\frac{\delta H}{\delta \phi^M} - \beta \mathcal{G}_{MN}\frac{\delta H}{\delta \pi_N} + \sqrt{\beta}\mathcal{E}_{MA}\eta_A \tag{3.1a}$$

$$\dot{\phi}^M = \frac{\delta H}{\delta \pi_M} \tag{3.1b}$$

$$\langle \eta_A(\xi,t)\eta_B(\xi',t')\rangle = 2\delta_{AB}\delta^d(\xi-\xi')\delta(t-t') \tag{3.1c}$$

which I will show below is formally equivalent to the general phase-space functional integral (2.1). Here η_A is supertangent-space noise, t is a Markov time, overdot is Markov time derivative and β is a positive auxiliary parameter which generally controls the rate of equilibration. As promised above, the DeWitt superstructures $\mathcal{E}_{MA}$ and $\mathcal{G}_{MN}$ appear as covariant kernels[18] in the process, independent of the specific form of H. I also note the remarkable fact that such phase-space processes are *stochastically unambiguous*[13,24],

$$\overbracket{\mathcal{E}_{MA}\eta_A} = 0\,, \tag{3.2}$$

that is, insensitive to choice of stochastic calculus[25].

Invariant regularization of the phase-space process (3.1) is achieved by replacing the supervielbein in the noise term by a *regularized supervielbein* $\mathcal{E}^A_{M\xi;A\xi'}$

$$(\mathcal{E}_{MA}\eta_A)_\xi \to \int (d\xi')\,\mathcal{E}^A_{M\xi;A\xi'}\eta_A(\xi') \tag{3.3a}$$

$$\mathcal{E}^A_{M\xi;A\xi'} \equiv R(\tilde{\Delta})_{M\xi}{}^{iN\xi'}\mathcal{E}_{NA}(\phi(\xi')) \tag{3.3b}$$

where R is the covariant regulator. The correct matrix elements $(\tilde{\Delta})_{M\xi}{}^{iN\xi'}$ of the abstract spacetime Laplacian Δ are defined on objects $\tilde{V}_M \sim \pi_M$ which transform under G_ϕ and G_ξ like the momenta,

$$(\Delta\tilde{V}_M)_\xi \equiv (\tilde{\Delta}_\xi)_M{}^N V_N(\xi) \tag{3.4a}$$

$$(\tilde{\Delta})_{M\xi}{}^{iN\xi'} \equiv (\tilde{\Delta}_\xi)_M{}^N\delta^d(\xi-\xi') \tag{3.4b}$$

and the twiddle is a reminder that π_M is a G_ξ (or $G_{\tilde{\xi}}$)-tensor density when ϕ^M is a tensor. The regulator is constructed by matrix multiplication, and the unregularized process (3.1)

is regained in the formal large cutoff limit

$$R(\tilde{\Delta})_{M\xi}{}^{iN\xi'} \xrightarrow[\Lambda]{} \delta^N_M\delta^d(\xi-\xi') \tag{3.5a}$$

$$\mathcal{E}^A_{M\xi;A\xi'} \xrightarrow[\Lambda]{} \mathcal{E}_{MA}(\phi(\xi))\delta^d(\xi-\xi') \tag{3.5b}$$

of the regulator forms (1.1). The prescription (3.3) is a geometric generalization of the previous regularizations of the program.

The regularization is invariant[13] under all the covariances of Δ. The degree of difficulty in the explicit construction of Δ depends on an interplay of G_ϕ, G_ξ and other invariances. For example, if regularization of G_ξ only is of interest, the ordinary Laplacians $\Delta = g^{mn}D_mD_n$ of general relativity will suffice. The same Δ constructed with (2.7b) is adequate for regularization of canonical gravity. I call such Laplacians provisional, since it would be preferable to maintain manifest covariance under G_ϕ and G_ξ simultaneously. The original gauge-covariant Laplacian employed in the regularization of gauge theory[1,3] is provisional in the same sense.

In the case of the general non-linear sigma model (2.4) and (2.6), it is not difficult to obtain a spacetime Laplacian $\Delta = g^{mn}\Delta_m\Delta_n$ which is covariant under both G_ϕ and G_ξ. The covariant derivative Δ_m is simplest on $V_M \equiv e^{-1}\tilde{V}_M = G_\xi$-scalar,

$$\Delta_m V_M = \partial_m V_M - \tilde{\Gamma}^N_{mM}V_N \equiv V_{Mm} \tag{3.6a}$$

$$\Delta_n V_{Mm} = \partial_n V_{Mm} - \tilde{\Gamma}^r_{nm}(g)V_{Mr} - \tilde{\Gamma}^N_{nM}V_{Nm} \tag{3.6b}$$

$$\tilde{\Gamma}^N_{mM} \equiv \partial_m\phi^R\Gamma^N_{RM}(\mathcal{G}) \tag{3.6c}$$

where $\Gamma^r_{nm}(g)$ is Einstein connection, and

$$\Gamma^M_{NS}(\mathcal{G}) = \frac{1}{2}\mathcal{G}^{MT}\Big(\frac{\partial}{\partial\phi^N}\mathcal{G}_{TS} + \frac{\partial}{\partial\phi^S}\mathcal{G}_{TN} - \frac{\partial}{\partial\phi^T}\mathcal{G}_{NS}\Big) \tag{3.7}$$

is the *superconnection*.

Construction of Δ in more general cases deserves further study. For example, I have not yet constructed a G_ϕ and G_ξ-covariant Laplacian for gravity, and the regularization of gravity below is provisional in this sense.

When H is a non-singular quadratic form in the momenta, eq. (3.1b) may be used to eliminate π_M in favor of $\dot{\phi}^M$. The resulting regularized covariant *second-order* stochastic processes are discussed in Refs. 13,24.

4. Regularized phase-space Schwinger-Dyson systems

The invariant regularized SD system

$$0 = \Big\langle \{F,H\} + \beta\Big[\int(d\xi)\mathcal{G}_{MN}\frac{\delta H}{\delta\pi_N}\frac{\delta}{\delta\pi_M} + \Delta_\pi\Big]F\Big\rangle \tag{4.1a}$$

$$\{F,H\} \equiv \int(d\xi)\Big[\frac{\delta F}{\delta\phi^M}\frac{\delta H}{\delta\pi_M} - \frac{\delta H}{\delta\phi^M}\frac{\delta F}{\delta\pi_M}\Big] \tag{4.1b}$$

$$\Delta_\pi \equiv \int(d\xi)(d\xi')\mathcal{G}^A_{M\ell;N\ell'}\frac{\delta^2}{\delta\pi_N(\xi')\delta\pi_M(\xi)} \tag{4.1c}$$

is associated to the regularized process (3.1) and (3.3), under assumption of equilibration, and provides a d-dimensional formulation of the regularization. Here $F[\phi,\pi]$ is an arbitrary functional, the bracket is ordinary Poisson bracket, and Δ_π is the *regularized phase-space super-Laplacian*, in terms of the *regularized supermetric*

$$\mathcal{G}^A_{M\ell;N\ell'} = \int(d\xi'')\mathcal{E}^A_{M\ell;A\ell''}\mathcal{E}^A_{N\ell';A\ell''} \tag{4.2a}$$

$$= \int(d\xi'')\,R(\tilde\Delta)_{M\ell}{}^{;P\ell''}R(\tilde\Delta)_{N\ell'}{}^{;Q\ell''}\mathcal{G}_{PQ}(\phi(\xi''))\,. \tag{4.2b}$$

It is easy to check that the system (4.1) is a regularization of the original formal theory. In the formal large A limit,

$$\mathcal{G}^A_{M\ell;N\ell'} \xrightarrow[A]{} \mathcal{G}_{MN}(\phi(\xi))\delta^d(\xi-\xi') \tag{4.3}$$

and the SD eqs. (4.1a) are seen to correspond to the unregularized formal identities

$$0 = \int\mathcal{D}\omega\,\{e^{-H},F\} \tag{4.4a}$$

$$0 = \beta\int\mathcal{D}\omega\int(d\xi)\frac{\delta}{\delta\pi_M}\Big[e^{-H}\mathcal{G}_{MN}\frac{\delta}{\delta\pi_N}F\Big] \tag{4.4b}$$

at the level of the functional integral (2.1). The roles of β and $\mathcal{G}_{MN}$ as auxiliary quantities in the formulation are clearly seen in (4.4b). In fact, the systems (3.1) and (3.3) or (4.1) define a β-family[9,13,24] of invariant regularizations of the original theory. Such structure is analogous to latticeization ambiguities, and I will note below that large β is the simplest of the family.

5. Moment relations

In what follows, I study further structure in the special case of the general theory with DeWitt measure (2.4), for which (4.1) takes the form

$$0 = \Big\langle \{F,H\} + \beta\Big[\int(d\xi)\,\pi_M\frac{\delta}{\delta\pi_M} + \Delta_\pi\Big]F\Big\rangle\,. \tag{5.1}$$

In particular, I begin to study here a family of moment relations[9,13], which will provide the basis for the treatment both of Weyl anomalies and integration of the momenta below.

The moment expansion is in powers of the momenta. For example, the zeroth-moment choice $F_0[\phi,\pi] = F[\phi]$ in (5.1) gives the zeroth-moment relation

$$0 = \Big\langle\int(d\xi)\,\pi_M\mathcal{G}^{MN}\frac{\delta}{\delta\phi^N}F[\phi]\Big\rangle \tag{5.2}$$

which states the vanishing of translation-invariant averages linear in the momentum. The first-moment choice $F_1[\phi,\pi] = \int(d\xi)\,\pi_M\mathcal{G}^{MN}\delta/\delta\phi^N F[\phi]$ gives, after some algebra, the first-moment relation

$$0 = \Big\langle\Big[-\int(d\xi)\frac{\delta S}{\delta\phi^M}\mathcal{G}^{MN}\frac{\delta}{\delta\phi^N} + \Delta'\Big]F[\phi]\Big\rangle \tag{5.3}$$

where Δ' is a *mixed* phase/coordinate-space super-Laplacian

$$\Delta' = \int(d\xi)(d\xi')\,\pi_M(\xi)\pi_N(\xi')\mathcal{G}^{MP}(\phi(\xi))\mathcal{G}^{NQ}(\phi(\xi'))\frac{D}{D\phi^Q(\xi')}\frac{\delta}{\delta\phi^P(\xi)} \tag{5.4a}$$

$$\frac{D}{D\phi^Q(\xi')}\frac{\delta}{\delta\phi^P(\xi)} \equiv \Big[\frac{\delta}{\delta\phi^Q(\xi')}\delta^R_P - \delta^d(\xi-\xi')\Gamma^R_{QP}\Big]\frac{\delta}{\delta\phi^R(\xi)} \tag{5.4b}$$

and $D/D\phi^Q$ is *super-covariant derivative*. With regularized integration of the momenta, discussed below, averages involving the momenta are replaced with averages over the coordinates alone. Then, the relation (5.3) will provide a purely coordinate-space regularization.

A taste of the integration procedure is obtained with the mixed-moment choice $F_2' = H$, which gives

$$\Big\langle\int(d\xi)\,\pi_M\mathcal{G}^{MN}\pi_N\Big\rangle = \Big\langle\int(d\xi)\mathcal{G}^A_{M\ell}{}^{;M\ell}\Big\rangle \tag{5.5}$$

where

$$\mathcal{G}^A_{M\ell}{}^{;M\ell} = \mathcal{G}^A_{M\ell;N\ell}\mathcal{G}^{NM}(\phi(\xi)) \tag{5.6}$$

so the right side of (5.5) is the average of the invariant *trace* of the regularized supermetric. This relation is in momentum-integrated form, and will be seen to play a crucial role in the study of Weyl anomalies below.

In this connection, it will be useful to have the identity (5.5) in another form. It follows from (4.2) that

$$\mathcal{G}^{\Delta}_{\mathcal{M}\xi}{}^{;\mathcal{N}\xi} = \int (d\xi)\, R(\tilde{\Delta})_{\mathcal{M}\xi}{}^{;\mathcal{N}\xi'} R(\Delta)^{\mathcal{M}\xi}{}_{;\mathcal{N}\xi'} \tag{5.7}$$

where I have introduced another set of matrix elements of the spacetime Laplacian, now defined on objects $V^{\mathcal{M}} \equiv \mathcal{G}^{\mathcal{M}\mathcal{N}}\tilde{V}_{\mathcal{N}} \sim \delta\phi^{\mathcal{M}}$ which transform like field-coordinate deformations,

$$(\Delta V^{\mathcal{M}})_{\xi} \equiv (\Delta_{\xi})^{\mathcal{M}}{}_{\mathcal{N}} V^{\mathcal{N}}(\xi) \tag{5.8a}$$

$$(\Delta)^{\mathcal{M}\xi}{}_{;\mathcal{N}\xi'} = (\Delta_{\xi})^{\mathcal{M}}{}_{\mathcal{N}}\delta^{d}(\xi - \xi') \tag{5.8b}$$

$$= \mathcal{G}^{\mathcal{M}\mathcal{P}}(\phi(\xi))(\tilde{\Delta})_{\mathcal{P}\xi}{}^{;\mathcal{Q}\xi'}\mathcal{G}_{\mathcal{Q}\mathcal{N}}(\phi(\xi'))\,. \tag{5.8c}$$

Moreover, as seen below, these are the natural matrix elements for coordinate-space regularization.

6. Weyl anomaly in the general $d = 2$ non-linear sigma model

I define the phase-space stress-tensor θ^{mn} as the response of any matter Hamiltonian to a general deformation of the Einstein metric

$$\delta_g H_{\text{matter}} = -\frac{1}{2}\int (d\xi)\, e\theta^{mn}\delta g_{mn}\,. \tag{6.1}$$

This gives

$$e\theta \equiv e\theta^{mn}g_{mn} = -2g_{mn}\frac{\delta S}{\delta g_{mn}} + \frac{d}{2}\pi_{\mathcal{M}}\mathcal{G}^{\mathcal{M}\mathcal{N}}\pi_{\mathcal{N}} \tag{6.2}$$

for the general d-dimensional non-linear sigma model (2.4) and (2.6). The second term in (6.2) is the response of the DeWitt measure.

In $d = 2$ dimensions, the first term of (6.2) is the full response of the action to a Weyl deformation. For the classically Weyl-invariant S of (2.6), this term (and its regularized average) is zero. The relation

$$\left\langle \int (d\xi)\, e\theta \right\rangle = \left\langle \int (d\xi)\, \pi_{\mathcal{M}}\mathcal{G}^{\mathcal{M}\mathcal{N}}\pi_{\mathcal{N}} \right\rangle = \left\langle \int (d\xi)\, \mathcal{G}^{\Delta}_{\mathcal{M}\xi}{}^{;\mathcal{M}\xi} \right\rangle \tag{6.3a}$$

$$= \left\langle \int (d\xi)(d\xi')(e^{\tilde{\Delta}/\Lambda^2})_{\mathcal{M}\xi}{}^{;\mathcal{N}\xi'}(e^{\Delta/\Lambda^2})^{\mathcal{M}\xi}{}_{;\mathcal{N}\xi'} \right\rangle \tag{6.3b}$$

is then obtained with eqs. (6.2), (5.5) and (5.7) from the measure term alone. The result (6.3) is a nonperturbative geometric characterization of the general Weyl anomaly in the presence of the regulator.

As a comparison with known background-field (one-loop) results, consider the flat D-dimensional internal manifold $\mathcal{G}_{\mathcal{M}\mathcal{N}} = e\delta_{\mathcal{M}\mathcal{N}}$. Then the Δ of Section 3 reduces to ordinary G_{ξ}-covariant Laplacian on scalars and scalar densities,

$$(\Delta)^{\mathcal{M}\xi}{}_{;\mathcal{N}\xi'} = \delta^{\mathcal{M}}_{\mathcal{N}}(\Delta)_{\xi\xi'} \tag{6.4a}$$

$$(\tilde{\Delta})_{\mathcal{M}\xi}{}^{;\mathcal{N}\xi'} = \delta^{\mathcal{N}}_{\mathcal{M}}e(\xi)(\Delta)_{\xi\xi'}e^{-1}(\xi') \tag{6.4b}$$

$$= \delta^{\mathcal{N}}_{\mathcal{M}}(\Delta)_{\xi'\xi} \tag{6.4c}$$

$$(\Delta)_{\xi\xi'} \equiv (D^m\partial_m)_{\xi}\delta^2(\xi - \xi') \tag{6.4d}$$

and

$$\left\langle \int (d\xi)\, e\theta \right\rangle = D\int (d\xi)\,(e^{2\Delta/\Lambda^2})_{\xi\xi} \tag{6.5a}$$

$$= \int (d\xi)\, e\Big[\frac{\Lambda^2 D}{8\pi} + \frac{D}{24\pi}R(g) + O(\Lambda^{-1})\Big] \tag{6.5b}$$

is obtained by heat-kernel expansion, as expected[26], where $R(g)$ is Einstein curvature scalar. The general nonperturbative characterization (6.3) invites further analysis on non-trivial manifolds with heat-kernel and background-field methods.

7. Coordinate-space regularization

I now discuss regularized integration[9,13,24] of the momenta at large β to obtain the corresponding form of coordinate-space regularization in the case of the general theory with DeWitt measure. As in the case of regularized Grassmann integration[9], integration of the momenta at finite β seems prohibitively complex.

Since the formal unregularized theory is independent of β, no growth[9,26] of the averages of the theory is expected at large β. The large-β relation

$$0 = \left\langle \left[\int (d\xi)\, \pi_M \frac{\delta}{\delta \pi_M} + \mathbb{\Delta} \right] F \right\rangle_{(0)} \tag{7.1}$$

then follows from the SD eqs. (5.1), to prevent the growth of the other terms. The subscript zero denotes statements like (7.1) which are true only at large β. The relation (7.1) can be solved[13] by iteration with a general moment expansion, as in Section 5, and the results packaged as a generating functional for averages involving the momenta

$$\left\langle e^{\int (d\xi)\, J^M \pi_M} F[\phi] \right\rangle_{(0)} = \left\langle e^{\frac{1}{2} \int (d\xi)\, J^M(\xi) \mathcal{G}^{\Lambda}_{M\xi;N\xi'} J^N(\xi')} F[\phi] \right\rangle_{(0)} , \tag{7.2}$$

where $F[\phi]$ is arbitrary and J^M is a source. This states that the large-β momentum integration is still Gaussian, with the contraction rule

$$\overbrace{\pi_M(\xi)\pi_N(\xi')} = \mathcal{G}^{\Lambda}_{M\xi;N\xi'} \tag{7.3}$$

inside coordinate-space averages.

In particular, this result can be used to eliminate the momentum-dependence in the mixed phase/coordinate-space super-Laplacian $\mathbb{\Delta}'$ of eq. (5.3), which gives the regularized coordinate-space SD system

$$0 = \langle \mathbb{L} F[\phi] \rangle \tag{7.4a}$$

$$\mathbb{L} = -\int (d\xi) \frac{\delta S}{\delta \phi^M} \mathcal{G}^{MN} \frac{\delta}{\delta \phi^N} + \mathbb{\Delta} \tag{7.4b}$$

$$\mathbb{\Delta} \equiv \int (d\xi)(d\xi') \mathcal{G}^{M\xi;N\xi'}_{\Lambda} \frac{D}{D\phi^N(\xi')} \frac{\delta}{\delta \phi^M(\xi)} \tag{7.4c}$$

announced originally in Ref. 12. Here $\mathbb{\Delta}$ is the regularized *coordinate-space super-Laplacian*, in terms of the regularized *inverse* supermetric

$$\mathcal{G}^{M\xi;N\xi'}_{\Lambda} \equiv \int (d\xi'')\, R(\Delta)^{M\xi}{}_{;P\xi''} R(\Delta)^{N\xi'}{}_{;Q\xi''} \mathcal{G}^{PQ}(\phi(\xi'')) \tag{7.5}$$

which I have expressed in terms of the natural Laplacian $(\Delta)^{M\xi}{}_{;N\xi'}$ for coordinate-space regularization, given in (5.8).

As above at the phase-space level, it is easy to check that the SD system (7.4) is a coordinate-space regularization of the original unregularized theory. In the formal large Λ limit

$$R(\Delta)^{M\xi}{}_{;N\xi'} \xrightarrow[\Lambda]{} \delta^M_N \delta^d(\xi - \xi') \tag{7.6a}$$

$$\mathcal{G}^{M\xi;N\xi'}_{\Lambda} \xrightarrow[\Lambda]{} \mathcal{G}^{MN}(\phi(\xi)) \delta^d(\xi - \xi') \tag{7.6b}$$

and the SD eqs. (7.4) correspond to the unregularized formal identities

$$0 = \int \mathcal{D}\phi \int (d\xi) \frac{\delta}{\delta \phi^M(\xi)} \left[e^{-S} \mathcal{E}[\phi] \mathcal{G}^{MN}(\phi(\xi)) \frac{\delta}{\delta \phi^N(\xi)} F[\phi] \right] \tag{7.7}$$

at the coordinate-space action level (2.5).

Regularized Ito and Stratonovich coordinate-space processes equivalent to (7.4) on assumption of equilibration are given in Refs. 12,13.

The previous regularizations of the program are special cases of (7.4) in flat space and flat superspace. For example, the $\gamma = 0$ form[2,6] of regularized gauge theory is obtained from (7.4) with $\phi^M \to A^a_\mu$, $\mathcal{G}_{MN} \to \delta_{\mu\nu}\delta^{ab}$, $\Gamma^R_{MN}(\mathcal{G}) \to 0$ and

$$\mathcal{G}^{M\xi;N\xi'}_{\Lambda} \longrightarrow \delta_{\mu\nu} [R^2(\Delta)]^{ab} \tag{7.8}$$

where Δ is non-abelian gauge-covariant Laplacian. Like the G_ξ-covariant Laplacian of general relativity, this Laplacian is provisional, and a Laplacian covariant under both G_Λ and gauge-transformations is not yet known. The full G_ϕ-covariant Laplacian for the scalar prototype is, of course, a special case of the general sigma-model Laplacian given in Section 3.

8. Regularized Euclidean gravity

As an example, the general regularized coordinate-space SD eqs. (7.4) are easily written out for DeWitt-measure Euclidean gravity

$$0 = (\mathrm{L}F[g_{mn}]) \tag{8.1a}$$

$$\mathrm{L} \equiv \int(d\xi)\Big[\mathcal{L}_z g_{mn} - \mathcal{G}_{mn;rs}\frac{\delta S}{\delta g_{rs}}\Big]\frac{\delta}{\delta g_{mn}} + \Delta \tag{8.1b}$$

$$\Delta = \int(d\xi)(d\xi')\mathcal{G}^{A}_{mn\xi;rs\xi'}\frac{D}{Dg_{rs}(\xi')}\frac{\delta}{\delta g_{mn}(\xi)} \tag{8.1c}$$

where the supermetric for gravity was given in (2.3). I have also added a G_ξ-Zwanziger gauge-fixing term

$$\mathcal{L}_z g_{mn} = D_m Z_n + D_n Z_m \ , \tag{8.2}$$

whose generic form is always a gauge-transformation. The spacetime Laplacian is the provisional $\Delta = g^{mn}D_m D_n$ of general relativity on symmetric covariant tensors of rank two, and the regulator is taken as heat-kernel.

The differential SD formulation (8.1) bypasses the question of integration contour for the unstable conformal mode[27] of the Einstein action

$$S = \frac{1}{\kappa^2}\int(d\xi)\, eR \ , \tag{8.3}$$

giving directly the correct Euclidean result, at least in weak coupling. In particular, with

$$g_{mn} = \delta_{mn} + \kappa h_{mn} \ , \quad h \equiv h_{mm} \tag{8.4a}$$

$$Z_m = \frac{1}{2\kappa}(\partial_n h_{nm} - \frac{1}{2}\partial_m h) \tag{8.4b}$$

the free regularized graviton propagator

$$\begin{aligned}&\langle h_{mn}(\xi)h_{rs}(\xi')\rangle^{(0)}\\ &=\Big[\Big(1+\frac{\mathrm{T}}{2-d}\Big)_{mn;rs} - \frac{1+2\gamma}{3-d+2\gamma}\Big(\delta_{mn}\frac{\partial_r\partial_s}{\Box} + \delta_{rs}\frac{\partial_m\partial_n}{\Box}\Big)\Big]\Big(\frac{2e^{2\Box/\Lambda^2}}{-\Box}\Big)_{\xi\xi'}\end{aligned} \tag{8.5a}$$

$$\mathrm{1}_{mn;rs} \equiv \frac{1}{2}(\delta_{mr}\delta_{ns} + \delta_{ms}\delta_{nr}) \ , \quad \mathrm{T}_{mn;rs} \equiv \delta_{mn}\delta_{rs} \tag{8.5b}$$

is obtained directly, and the free regularized n-point functions are constructed according to Wick's theorem from (8.5). The result is gauge-equivalent for all γ to the correct Euclidean Feynman gauge (first term), and also indicates that the perturbation expansion is much simpler for the supermetric parameter choice $\gamma = -1/2$.

Chan and I[14] have generalized the SD system (8.1) to regularize arbitrary power-law measure $\prod_\xi e(\xi)^\nu$, where $\nu_{DW} = (d+1)(d-4)/4$ is the DeWitt power in d-dimensions,

and studied the weak-coupling expansion. The expansion provides a geometrization of previous regularized SD rules, whose structure will be generally applicable in coordinate-invariant regularization. We have also completed a non-trivial one-loop explicit check of Einstein invariance of the regularization. Our conclusion is that a one-loop cosmological counterterm

$$\lambda_O = \frac{\kappa^2\Lambda^d}{(8\pi)^{d/2}}\Big[\nu - \frac{d^2-7d-2}{4}\Big] \tag{8.6}$$

is necessary to stabilize the flat vacuum, and that, with this counterterm, the graviton mass is zero for all (d,ν), as it should be.

References

1) Z. Bern, M.B. Halpern, L. Sadun and C. Taubes, Phys. Lett. 165B, (1985) 151.

2) Z. Bern, M.B. Halpern, L. Sadun and C. Taubes, Nucl. Phys. B284 (1987) 1.

3) Z. Bern, M.B. Halpern, L. Sadun and C. Taubes, Nucl. Phys. B284 (1987) 35.

4) Z. Bern, M.B. Halpern, and L. Sadun, Nucl. Phys. B284 (1987) 92.

5) Z. Bern, M.B. Halpern and N.G. Kalivas, Phys. Rev. D35 (1987) 753.

6) Z. Bern, M.B. Halpern, and L. Sadun, "Continuum Regularization of Quantum Field Theory IV. Langevin Renormalization", LBL-21960, UCB-PTH 86/26.

7) L. Sadun, "Continuum Regularization of Quantum Field Theory V. Schwinger-Dyson Renormalization", Harvard University preprint, Dec 1986.

8) Z. Bern, H.S. Chan, and M.B. Halpern, Z. Phys. C33 (1986) 77.

9) Z. Bern, H.S. Chan, and M.B. Halpern, "Non-Grassmann Formulation of Regularized Gauge Theory with Fermions", LBL-21786 UCB-PTH-86/17 (To appear in Z. Phys. C).

10) N. Kalivas, "Continuum Regularization of Superfield Supersymmetry", Oxford preprint, Nov 1986.

11) M.B. Halpern, in: "Proceedings of the Symposium on Topological and Geometric Methods in Field Theory, Espoo, 1986. (World Scientific 1986).

12) M.B. Halpern, "Schwinger-Dyson Formulation of Coordinate-Invariant Regularization", LBL-22088, UCB-PTH-86/28 (To appear in Phys. Lett. B).

13) M.B. Halpern, "Coordinate-Invariant Regularization", LBL-22847, UCB-PTH-87/2.

14) H.S. Chan and M.B. Halpern, "Continuum-Regularized Quantum Gravity", LBL-22986, UCB-PTH-87/8.

15) B.S. De Witt, J. Math. Phys. 3, (1962) 1073;
B.S. De Witt, in: "General Relativity", edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge 1979);
B.S. De Witt, in: "Recent Developments in Gravitation", edited by M. Lévy and S. Deser (Plenum Press, New York 1979).

16) N.G. van Kampen, "Stochastic Processes in Physics and Chemistry" (North-Holland, 1981).

17) G. Parisi and Wu Yong-Shi, Sci. Sin. 24, (1981) 483.

18) B. Sakita, in: "The Seventh Johns Hopkins Workshop", edited by G. Domokos and S. Kovesi-Domokos (World Scientific, 1983).

19) R. Arnowitt, S. Deser and C.W. Misner, in: "Gravitation, an Introduction to Current Research", edited by L. Witten (John Wiley, New York, 1962);
K. Kuchar, in: "Quantum Gravity 2, A Second Oxford Symposium", edited by C.J. Isham, R. Penrose and D.W. Sciama (Clarendon Press, 1981).

20) D. Zwanziger, Nucl. Phys. B192, (1981) 259;
E.G. Floratos and J. Iliopoulos, Nucl. Phys. B214, (1983) 392;
E.G. Floratos, J. Iliopoulos and D. Zwanziger, Nucl. Phys. B241, (1984) 221;
H.S. Chan and M.B. Halpern, Phys. Rev. D33, (1986) 540.

21) C. Becchi, A. Rouet and R. Stora, Phys. Lett. 52B (1974) 344;
E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. 55B (1975) 224, and preprint CERN-TH-2332 (1977); Lettere al Nuovo Cimento 13 (1975) 187;

I.A. Batalin and G.A. Vilkovisky, Phys. Lett. 69B (1977) 309;
E.S. Fradkin and T.E. Fradkina, Phys. Lett. 72B (1978) 343;
M. Henneaux, Phys. Rep. 126 (1985) 1.

22) P. Langevin, C.R. Acad. Sci. Paris 146, (1908) 530.

23) S. Ryang, T. Saito and K. Shigemoto, Prog. Theor. Phys. 73 (1985) 1295;
A.M. Horowitz, Phys. Lett. 156B (1985) 89.

24) Z. Bern and M.B. Halpern, "Phase-Space versus Coordinate-Space Regularization", in preparation.

25) R.E. Mortensen, Jour. Stat. Phys. 1. No. 2, 271 (1969).

26) O. Alvarez, Nucl Phys. B216 (1983) 125.

27) G.W. Gibbons, S.W. Hawking and M.J. Perry, Nucl. Phys. B138 (1978) 141;
S.W. Hawking, in: "General Relativity", edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge 1979).

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