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GEOMETRY AND PHYSICS

B. Zumino

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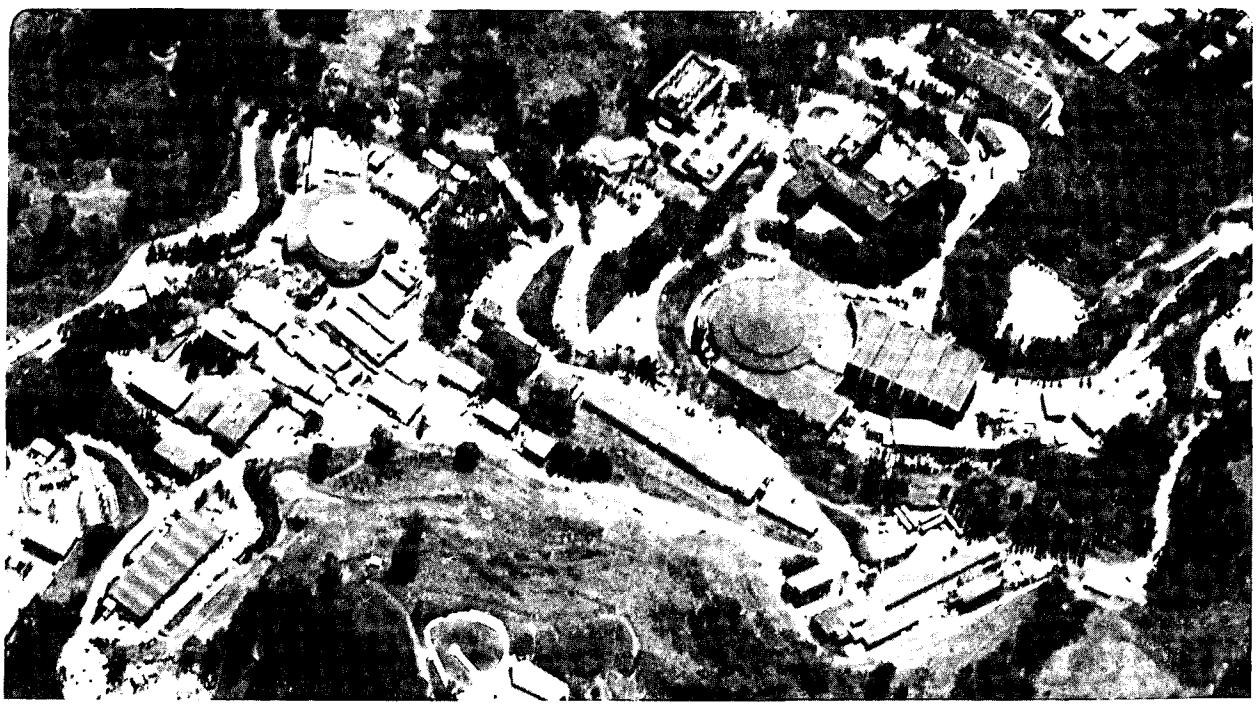

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GEOMETRY AND PHYSICS*

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Abstract

An introduction for physicists to the concept of line bundle and some of its physical applications.

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1. Introduction

I would like to describe in this lecture some aspects of geometry which have become relevant to physics during the past few years. The importance of differential geometry, topology and of concepts such as that of connection and curvature on fiber bundles has been recognized for some time. These concepts are important, for instance, in the study of classical solutions of Yang–Mills theory and relativity. More recently, a knowledge of the theory of line bundles and $U(1)$ bundles has become an essential tool for the understanding of a number of subjects. Among them are the anomalies in quantum field theory and string theories and various properties of Riemann surfaces and their moduli spaces, which are relevant for string theories. In this lecture I shall try to outline, by means of a simple example, some basic facts about line bundles for the inexperienced physicist. The best known physical example would be the Dirac point monopole. Instead, I shall describe the quantum mechanical phenomenon called Berry's phase, where the same mathematics emerges, without reference to physical monopoles. No new results will be presented. My aim is to be as clear and pedagogic as possible, even at the cost of mathematical rigor. Only some familiarity with exterior differentiation is assumed. I shall be satisfied if I have aroused enough curiosity to motivate the reader to look into the appropriate literature. Among the mathematical references [1–5], Refs. [2] and [3] are written for physicists and are very readable. Actually, [1], [4] and [5] are also quite readable, if one is willing to make a little effort. I quote only a few physics papers on anomalies [7–9], on strings [10–13] and on Berry's phase [14–16], chosen because they are representative of the use of the mathematical methods mentioned above. Reference [6] is a collection of papers and lecture notes and contains numerous references to earlier literature. A rather mathematical introduction to geometric quantization is given in [17].

This lecture is dedicated to T. D. Lee on the occasion of his 60th birthday. It is a small way to express my admiration for his qualities as a scientist and as a person and my gratitude for the friendship he has shown me on many occasions.

2. Non Integrable Quantum Phase

Let $H(\lambda)$ be a quantum mechanical Hamiltonian which depends upon a certain number of parameters $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_k)$ and let $|\lambda>$ be the normalized eigenvector corresponding to a given discrete isolated non–degenerate eigenvalue $E(\lambda)$

$$H(\lambda)|\lambda>= E(\lambda)|\lambda> . \tag{1}$$

The eigenvalue will remain non degenerate as λ varies in an open neighborhood in parameter space. The eigenvector $|\lambda>$ is determined only up to a phase, which can be different for different λ. We define the gauge potential

$$A = A_i(\lambda)d\lambda^i = i < \lambda|d|\lambda >= -i(d < \lambda|)|\lambda >, \tag{2}$$

where d denotes differentiation in λ, or more explicitly

$$A_i(\lambda) = i < \lambda|\frac{\partial}{\partial\lambda^i}|\lambda >= -i\left(\frac{\partial}{\partial\lambda^i} < \lambda|\right)|\lambda > . \tag{3}$$

A change of phase

$$|\lambda >\rightarrow |\lambda > e^{-i\beta(\lambda)} \tag{4}$$

induces a gauge transformation

$$A \rightarrow A + d\beta. \tag{5}$$

As defined, the gauge potential is real, since the eigevectors are normalized. The corresponding gauge invariant field strength is

$$F = dA = i(d < \lambda|)d|\lambda >, \tag{6}$$

or more explicitly

$$F_{ij} = i\left(\frac{\partial}{\partial\lambda^i} < \lambda|\right)\left(\frac{\partial}{\partial\lambda^j}|\lambda >\right) d\lambda^i \wedge d\lambda^j. \tag{7}$$

Remember that exterior differentials anticommute and that $d^2 = 0$. From now on we shall understand products of differentials to be exterior products and omit the wedge sign $\wedge$.

If the eigenvectors $|\lambda >$ are not normalized, (2) is replaced by

$$A = \frac{i}{2}\frac{< \lambda|d|\lambda > -(d < \lambda|)|\ \lambda >}{< \lambda|\lambda >} \tag{8}$$

and (6) becomes

$$F = dA = i\frac{1}{<\lambda|\lambda>}\left((d<\lambda|)d|\lambda> - \frac{(d<\lambda|)|\lambda><\lambda|d|\lambda>}{<\lambda|\lambda>}\right). \tag{9}$$

These expressions are sometimes more convenient for actual computations than (2) and (6). One can now consider more general gauge transformations

$$|\lambda> \to |\lambda>[\zeta(\lambda)]^{-1}, \tag{10}$$

$$A \to A - \frac{i}{2}d\ln\frac{\zeta}{\bar{\zeta}}, \tag{11}$$

where ζ may have modulus different from one. Here and in the following the bar denotes complex conjugation.

It is obvious, from (2) or (8), that, if the vectors $|\lambda>$ can be chosen to have real components in a finite neighborhood of parameter space, then A will vanish and consequently F will also vanish.

In quantum mechanics, a physical state is represented by a ray of non vanishing vectors differing by a complex proportionality factor. It is clear that the above potential cannot be defined on the space of rays, one needs the Hilbert space of vectors, or at least the space of normalized vectors. One can describe a ray by choosing a representative vector belonging to it, but this choice is obviously not unique. The arbitrariness gives rise to the gauge transformation (5) or (11), different choices of representative correspond to different gauges. On the other hand, the field strength F is gauge invariant. This means that it must be possible to define it on the space of rays itself.

Consider a vector $|>$ in the ray corresponding to a given eigenvalue $E(\lambda)$. It has the form

$$|> = \psi|\lambda>, \tag{12}$$

where ψ is a non vanishing complex number and $|\lambda>$ is a particular vector which we choose as representative for the ray. A different choice of representative

$$|\lambda>' = [\zeta(\lambda)]^{-1}|\lambda> \tag{13}$$

will determine the same ray. The same vector (12) will now be given by

$$|> = \psi'|\lambda>', \tag{14}$$

where

$$\psi' = \zeta(\lambda)\psi. \tag{15}$$

By letting the complex function $\zeta(\lambda)$ depend on λ we have explicitly allowed the possibility that, for different λ, we change the representative by a different factor. While λ labels the various rays, ψ is a coordinate which distinguishes different vectors in a particular ray. For given λ, (15) is a change of coordinates corresponding to the change of basis (13). The length of the vector (12) is given by

$$h(\psi,\bar{\psi},\lambda) = <|> = |\psi|^2 h(\lambda), \tag{16}$$

where

$$h(\lambda) = <\lambda|\lambda>. \tag{17}$$

Under the transformation (13), h changes into

$$h'(\lambda) = |\zeta(\lambda)|^{-2} h(\lambda). \tag{18}$$

We define the one-form

$$\sigma = \frac{<|d|>}{<|>} = d\ln\psi + \frac{<\lambda|d|\lambda>}{<\lambda|\lambda>} \tag{19}$$

and the complex conjugate

$$\bar{\sigma} = \frac{(d<|)|>}{<|>} = d\ln\bar{\psi} + \frac{(d<\lambda|)|\lambda>}{<\lambda|\lambda>}. \tag{20}$$

They are invariant under the gauge transformation (13), (15) and so is the real combination

$$B = \frac{i}{2}(\sigma - \bar{\sigma}) = \frac{i}{2}d\ln\frac{\psi}{\bar{\psi}} + A. \tag{21}$$

As we shall see in Section 3, in mathematical language we are dealing here with a line bundle. The rays corresponding to physical states are complex lines in Hilbert space. The set of all vectors is the collection of all points of all lines, a line bundle. The set of all rays is the base of the bundle, each ray is as fiber of the bundle. The eigenvector corresponding to a given eigenvalue, as λ varies in parameter space, spans a sub-bundle of the total bundle of all vectors. The one-form σ is a connection form on this sub-bundle and its vanishing defines parallel transport, i.e. it associates to a vector in a given ray a vector in a ray infinitesimally close to it. If we limit ourselves to normalized vectors, the coordinate ψ in the ray must have absolute value one. The vectors in a given ray differ just by a phase, which can be identified with an element of the group $U(1)$. One speaks then of a $U(1)$ bundle, rather than a line bundle.

To illustrate the above, we consider the standard example of a 2×2 Hermitian matrix (two level system, spin $\frac{1}{2}$ in magnetic field)

$$H(\vec{\lambda}) = \vec{\lambda} \cdot \vec{\sigma} = \begin{pmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{pmatrix}. \tag{22}$$

The eigenvalues are $\pm\lambda$, where $\lambda \equiv |\vec{\lambda}|$. Consider, for instance, the eigenvalue $+\lambda$. One can take as eigenvector

$$|\vec{\lambda}> = \begin{pmatrix} \lambda + \lambda_3 \\ \lambda_1 + i\lambda_2 \end{pmatrix}. \tag{23}$$

This eigenvector is not normalized

$$< \vec{\lambda}|\vec{\lambda} > = 2\lambda(\lambda + \lambda_3). \tag{24}$$

It is very easy to apply (8) and (9) with the result

$$A = -\frac{1}{2}\frac{\lambda_1 d\lambda_2 - \lambda_2 d\lambda_1}{\lambda(\lambda + \lambda_3)} \tag{25}$$

and

$$F = -\frac{1}{2}\frac{\lambda_1 d\lambda_2 d\lambda_3 + \lambda_2 d\lambda_3 d\lambda_1 + \lambda_3 d\lambda_1 d\lambda_2}{\lambda^3} \tag{26}$$

(remember that exterior differentials anticommute). In terms of the usual azymuthal angle ϕ of spherical coordinates

$$e^{i\phi} = \frac{\lambda_1 + i\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \tag{27}$$

and polar angle θ

$$\lambda_3 = \lambda \cos\theta, \tag{28}$$

the gauge potential is

$$A = -\sin^2\frac{\theta}{2}\, d\phi. \tag{29}$$

Equations (25) and (26) have the form of the gauge potential and the magnetic field of a Dirac point monopole of strength $-\frac{1}{2}$ situated at the origin of parameter space. The gauge potential is not defined globally, as made clear by the singularity (Dirac string) along the negative λ_3 axis. The normalized eigenvector is

$$\frac{1}{\sqrt{2\lambda(\lambda + \lambda_3)}} \begin{pmatrix} \lambda + \lambda_3 \\ \lambda_1 + i\lambda_2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}. \tag{30}$$

The eigenvector (30) has a phase ambiguity on the negative λ_3 axis, $\theta = \pi$. If one multiplies (30) by the λ dependent phase $e^{-i\phi}$, one obtains the new normalized eigenvector

$$\frac{1}{\sqrt{2\lambda(\lambda - \lambda_3)}} \begin{pmatrix} \lambda_1 - i\lambda_2 \\ \lambda - \lambda_3 \end{pmatrix} = \begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}. \tag{31}$$

Now the phase ambiguity is on the positive λ_3 axis, $\theta = 0$. The corresponding gauge potential has a singularity on the positive λ_3 axis and is given by

$$A + d\phi = \frac{1}{2}\frac{\lambda_1 d\lambda_2 - \lambda_2 d\lambda_1}{\lambda(\lambda - \lambda_3)}, \tag{32}$$

where, as one can easily see from (27),

$$d\phi = \frac{\lambda_1 d\lambda_2 - \lambda_2 d\lambda_1}{\lambda_1^2 + \lambda_2^2}. \tag{33}$$

The new potential is obtained from the old one by a singular gauge transformation. The gauge function ϕ is multivalued and undetermined along the entire λ_3 axis. The curvature F given by (26) is independent of gauge transformations.

The origin of the string-like singularity can be understood as follows. There is an exceptional point $\vec{\lambda} = 0$ in parameter space, where the two eigenvalues $\pm\lambda$ coincide. If we remove this point of degeneracy, parameter space acquires a non trivial topology. From the expressions (30) or (31) for the normalized eigenvector one can see that, once the phase is fixed, the eigenvector depends only on the direction of the vector $\vec{\lambda}$. Therefore we are really dealing with a $U(1)$ bundle having as base a sphere around the origin in $\vec{\lambda}$ space. The situation is exactly the same as for the Dirac monopole. Since the base manifold is topologically non trivial, the $U(1)$ bundle can also be non trivial.

If we move along a closed curve C in parameter space, starting say at the point $\vec{\lambda}_0$, a vector

$$|\vec{\lambda}_0 > e^{i\alpha_0} \tag{34}$$

of the fiber at λ_0 can be parallel transported in the $U(1)$ bundle and will come back as the vector

$$|\vec{\lambda}_0 > e^{i(\alpha_0 + \Delta\alpha)} \tag{35}$$

of the same fiber at λ_0, where

$$\Delta\alpha = \oint_C d\alpha = \oint_C A, \tag{36}$$

since $\psi = e^{i\alpha}$ and parallel transport means $B = 0$. By Stokes' theorem this can be tranformed into

$$\oint_C A = \int_S F, \tag{37}$$

where the surface S has the curve C as its boundary, $C = \partial S$. In the example described above, $\Delta\alpha$ equals the magnetic flux through the surface S of the magnetic field of a Dirac monopole of strength $-\frac{1}{2}$ at th origin. Equivalently, it equals $-\frac{1}{2}$ times the solid angle corresponding to the surface S as seen from the origin. Observe that the surface S is not uniquely determined by the curve C. If S is deformed smoothly the solid angle remains the same until the surface goes through the origin when the solid angle changes by $\pm 4\pi$. Then $\Delta\alpha$ changes by $\mp 2\pi$ so that the phase $e^{i\Delta\alpha}$ is unaffected. In general, the vector obtained from a given vector by parallel transport along a closed curve is unique: it differs from the given vector by the (non integrable) phase factor $e^{i\Delta\alpha}$, called Berry's phase, which is of completely geometric nature. Berry [14] has shown that this geometric phase factor arises in a variety of examples in addition to the well known dynamic phase factor

$$e^{-\frac{i}{\hbar}\int_0^T E(\lambda(t))dt} \tag{38}$$

when a quantum mechanical system is transported adiabatically along a closed curve in parameter space. The mathematical interpretation in terms of a $U(1)$ bundle is due to Barry Simon [15].

In the next section we shall see some general definitions and properties of line bundles. The example of this section can provide a concrete reference.

3. Line Bundles and $U(1)$ bundles

Consider a complex manifold M. In an open neighborhood we can use complex coordinates $t_1, t_2, \cdots t_n$. A line bundle over M associates a complex line as fiber to each point of the base M. Let the complex number ψ be a coordinate for the fiber. A connection is a one-form ω whose vanishing defines parallel transport of the (one dimensional) complex vectors of the fiber. In local coordinates one can write

$$\omega = d\psi + \Theta\psi, \tag{39}$$

where Θ is a one-form on M. Let there be a metric for the line bundle, given in local coordinates by

$$k = |\psi|^2 h(t, \bar{t}), \tag{40}$$

where h is a real non vanishing function. A metric compatible connection satisfies

$$dk = 0 \tag{41}$$

when the vector in the fiber is parallel transported

$$d\psi + \Theta\psi = 0, \tag{42}$$

$$d\bar{\psi} + \bar{\Theta}\bar{\psi} = 0. \tag{43}$$

Since ψ is arbitrary, this gives

$$dh - (\Theta + \bar{\Theta})h = 0 \tag{44}$$

as the condition for metric compatability. We require Θ to be of type (1,0), which means that it contains only the differentials $dt_1, \cdots dt_n$ and not $d\bar{t}_1, \cdots d\bar{t}_n$. Then the metric compatibility equation separates into the two equations

$$\partial h = \Theta h, \tag{45}$$

$$\bar{\partial} h = \bar{\Theta} h, \tag{46}$$

with solutions

$$\Theta = h^{-1}\partial h = \partial \ln h, \tag{47}$$

$$\bar{\Theta} = h^{-1}\bar{\partial} h = \bar{\partial} \ln h. \tag{48}$$

We recall here the well known definitions

$$d = \partial + \bar{\partial},$$

$$\partial = dt^i \frac{\partial}{\partial t^i}, \qquad \bar{\partial} = d\bar{t}^i \frac{\partial}{\partial \bar{t}^i} \tag{49}$$

and the relations

$$d^2 = \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0. \tag{50}$$

Observe that

$$\Theta + \bar{\Theta} = d \ln h. \tag{51}$$

The curvature two-form F can be defined from the equation

$$d\omega + \Theta\omega = -iF\psi, \tag{52}$$

where the left hand side is the covariant differential of ω. As defined, F is real and equals

$$F = id\Theta = -id\bar{\Theta} = dA, \tag{53}$$

where we have defined an equivalent real gauge potential

$$A = \frac{i}{2}(\Theta - \bar{\Theta}) = i\Theta - \frac{i}{2} d \ln h. \tag{54}$$

Using (53) we see that the curvature is given in terms of the metric by

$$F = i\bar{\partial}\partial \ln h = \frac{i}{h}\left(\bar{\partial}\partial h - \frac{\bar{\partial}h\partial h}{h}\right). \tag{55}$$

Clearly the connection form Θ is more convenient if one is interested in analiticity, A if one is interested in reality properties. A corresponds to the real gauge potential of Section 2.

By definition the curvature two-form satisfies the Bianchi identity

$$dF = 0, \tag{56}$$

i.e. it closed. Normalized by dividing it by -2π, the curvature is called the Chern form and its integral over a closed surface in M gives an integer, the Chern number.

A line bundle is a vector bundle, the fiber is a one-dimensional vector space, in our case complex. A change of basis in the vector space is a linear transformation, in our case it changes the coordinate ψ into

$$\psi' = \zeta\psi, \tag{57}$$

where $\zeta(t)$ is a non vanishing complex number, in general different for different fibers. Since the metric k must be invariant, h must change into

$$h' = |\zeta|^{-2} h. \tag{58}$$

Correspondingly Θ changes by a gauge transformation into

$$\Theta' = \partial \ln h' = \Theta - \partial \ln \zeta \tag{59}$$

and we have

$$\bar{\Theta}' = \bar{\Theta} - \partial \ln \bar{\zeta}, \tag{60}$$

$$A' = A - \frac{i}{2} d \ln \frac{\zeta}{\bar{\zeta}}, \tag{61}$$

$$\omega' = \zeta\omega, \quad \bar{\omega}' = \bar{\zeta}\bar{\omega}. \tag{62}$$

The null vector of the fiber remains null if we make the above change of coordinates. The null vector also remains null by parallel transport. The null vectors of all fibers form the null section, which is defined globally. For a nonvanishing vector ψ the metric k never vanishes. If ψ is nonzero we can define the one-form

$$\sigma = \frac{1}{\psi}\omega = d \ln \psi + \Theta = k^{-1}\partial k, \tag{63}$$

where differentiation operates now also on ψ. Under the transformation (57), (58), σ is invariant and so is the real combination

$$B = \frac{i}{2}(\sigma - \bar{\sigma}) = \frac{i}{2}(\partial - \bar{\partial}) \ln k. \tag{64}$$

The curvature (53) is also given by

$$F = dB = id\sigma = -id\bar{\sigma}. \tag{65}$$

The following example is relevant for Section 2 and illustrates also very clearly an important point.

Consider an $n+1$-dimensional Hilbert space whose vectors have complex components $z_i = z_0, z_1, \cdots z_n$. A physical state is described by a ray of non vanishing vectors differing from each other by a complex proportionality factor. The space of all rays is n-dimentional complex projective space $P_n(C)$. Since the vectors of a ray are non vanishing, for a given ray at least one of the coordinates will be different from zero, say z_j. By continuity z_j will be non vanishing for neighboring rays as well. The rays of $P_n(C)$ for which z_j does not vanish form an open neighborhood U_j of $P_n(C)$.

In U_j one can use as local coordinates for $P_n(C)$ the n rations $z_0/z_j, z_1/z_j \cdots z_n/z_j$, omitting z_j/z_j which equals 1. The entire $P_n(C)$ is covered by the $n+1$ neighborhoods $U_0, U_1, \cdots U_n$. The system of coordinates which is valid in each neighborhood cannot be extended to the entire $P_n(C)$, but the space can be described by using local coordinates in each neighborhood and giving the transformation functions in the overlap of two neighborhoods. Alternatively, one could use $z_0, z_1, \cdots z_n$ as homogeneous coordinates.

There is a natural line bundle over $P_n(C)$. It has $P_n(C)$ as base and the fiber over a point of $P_n(C)$ consists of all vectors belonging to the ray associated to that point. It is convenient to take out the null vector of each ray, then the natural line bundle can be identified with the original Hilbert space minus the null vector. The hermitean metric of the Hilbert space

$$k = |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 \tag{66}$$

induces a natural metric on the line bundle. For instance in U_0, using local coordinates

$$t_1 = \frac{z_1}{z_0}, t_2 = \frac{z_2}{z_0}, \cdots t_n = \frac{z_n}{z_0}, \tag{67}$$

we have

$$k = |z_0|^2 (1 + |t_1|^2 + \cdots + |t_n|^2). \tag{68}$$

Comparing with (40), we can say that, in a special gauge, $z_0 = \psi$ is the coordinate for the fiber and

$$h = 1 + |t_1|^2 + \cdots + |t_n|^2 . \tag{69}$$

Equations (54) and (55) now give, for the real gauge potential,

$$\begin{aligned} A &= \frac{i}{2}(\partial - \bar\partial) \ln h \\ &= \frac{i}{2} \frac{\bar t \cdot dt - t \cdot d\bar t}{1 + \bar t \cdot t} \end{aligned} \tag{70}$$

and for the curvature form

$$F = dA = i \frac{1}{1 + \bar t \cdot t} \left(d\bar t \cdot dt - \frac{d\bar t \cdot t \, \bar t \cdot dt}{1 + \bar t \cdot t} \right). \tag{71}$$

Up to a factor this is the Kähler form associated with the well known hermitean Fubini-Study metric on $P_n(C)$($P_n(C)$ is a Kähler manifold) [2]. We have used the abbreviation

$$\bar t \cdot dt = \sum_{i=1}^{n} t_i dt_i \tag{72}$$

and so on.

In a different gauge h is replaced by

$$\begin{aligned} h' &= |\zeta(t)|^{-2} (1 + \bar t \cdot t) \\ &= |u_0|^2 + |u_1|^2 + \cdots + |u_n|^2 , \end{aligned} \tag{73}$$

where we have defined the t dependent vector

$$u_0 = \frac{1}{\zeta(t)}, u_1 = \frac{t_1}{\zeta(t)} \cdots u_n = \frac{t_n}{\zeta(t)}. \tag{74}$$

Now

$$\begin{aligned} A' &= \frac{i}{2}(\partial - \bar\partial) \ln h' \\ &= \frac{i}{2} \frac{\bar u_0 du_0 + \bar u \cdot du - (u_0 d\bar u_0 + u \cdot d\bar u)}{\bar u_0 u_0 + \bar u \cdot u}. \end{aligned} \tag{75}$$

The gauge invariant field strength can therefore be written also as

$$F = dA' = \frac{i}{2} \frac{1}{\bar u \cdot u} \left(d\bar u \cdot du - \frac{\bar u \cdot du \; u \cdot d\bar u}{\bar u \cdot u} \right), \tag{76}$$

where, with a change in notation,

$$\bar u \cdot du = \sum_{i=0}^{n} \bar u_i du_i \tag{77}$$

etc.

We have introduced gauge transformations as operating on the fiber coordinate as a consequence of a change in the basis vector of the fiber. They are also essential in gluing together the local pieces of the bundle by providing the transition from one open neighborhood to another. For instance, in the neighborhood U_1 we can use local coordinates

$$s_1 = \frac{z_0}{z_1}, \; s_2 = \frac{z_2}{z_1}, \cdots s_n = \frac{z_n}{z_1} \tag{78}$$

and write

$$k = |z_1|^2 (1 + |s_1|^2 + \cdots |s_n|^2). \tag{79}$$

In the overlap of U_0 and U_1 both z_0 and z_1 are different from zero. The same vector $z_0, z_1 \cdots z_n$ can be said to have fiber coordinate z_0 (for U_0) and z_1 (for U_1). These two are obtained one from the other by the gauge transformation

$$z_1 = g_{10} z_0, \tag{80}$$

where the "transition function"

$$g_{10} = t_1 = \frac{1}{s_1} \tag{81}$$

can be expressed in terms of the local coordinates t of U_0 as well as in terms of the local coordinates s of U_1. In general, in the intersection of U_i and U_j one can use as fiber coordinates ψ_i or ψ_j, related by

$$\psi_i = g_{ij}\psi_j. \tag{82}$$

For consistency the transition functions g_{ij} must satisfy the cocycle conditions

$$g_{ij}g_{ji} = 1 \qquad \text{in} \quad U_i \cap U_j, \tag{83}$$

$$g_{ii} = 1 \tag{84}$$

and

$$g_{ij}g_{jk}g_{ki} = 1 \qquad \text{in} \quad U_i \cap U_j \cap U_k. \tag{85}$$

Finally, observe that the gauge invariant real form B can be written in terms of the homogeneous coordinates as

$$B = \frac{i}{2}\frac{\bar{z}\cdot dz - z\cdot d\bar{z}}{\bar{z}\cdot z} \tag{86}$$

and the curvature as

$$F = dB = i\frac{1}{\bar{z}\cdot z}\left(d\bar{z}\cdot dz - \frac{d\bar{z}\cdot z\;\bar{z}\cdot dz}{\bar{z}\cdot z}\right). \tag{87}$$

Here again, the summation index runs from 0 to n, as in (77).

The similarity of (75) and (76) with (8) and (9) is obvious. In the example of Section 2 the Hilbert space has two complex dimensions, the projective space of rays is $P_1(C)$ and has one complex dimensions ($P_1(C)$ is topologically equivalent to the two-dimensional real sphere S_2). The eigenvalue $E(\lambda)$ corresponding to a point λ in parameter space determines a ray of eigenvectors. Thus, as λ varies, the eigenvalue determines a map from parameter space into $P_1(C)$. More precisely, as we have seen above, this is a map m from a two sphere around the origin in parameter space into the space of rays $P_1(C)$. By this map, the natural bundle over $P_1(C)$ becomes a line bundle over the two-sphere in parameter space.

If we normalize the Hilbert space vectors we obtain a $U(1)$ bundle of phases. Notice that the set of normalized vectors of the two-dimensional Hilbert space is

topologically equivalent to a three-sphere S_3. If the map m can be extended to a map from the two-sphere in parameter space into this S_3, the $U(1)$ bundle will be trivial. This would require that it be possible to define globally for all λ the phase of the normalized vectors.

The generalization to a higher dimensional space of rays is obvious. Given an eigenvalue $E(\lambda)$, one has a map from parameter space into the projective space of rays. By this map, the natural line bundle over the space of rays becomes a line bundle over parameter space. If we normalize the Hilbert space vectors, we have a $U(1)$ bundle of phases. Criteria can be given [16] for the nontriviality of this $U(1)$ bundle.

4. Conclusion.

As we have seen in some detail, the non integrable quantum phase has a natural mathematical description in terms of a line bundle. As mentioned in the introduction, the concept of line bundle shows up in a number of other physics problems.

Consider, for instance, the bosonic string. Amplitudes are obtained by performing a functional integration over all world sheets which describe the possible space time evolution of the string. In the Euclidean formulation one must integrate over all Riemann surfaces of different genus. For a given genus g (number of handles) a Riemann surface is determined up to conformal equivalence by $3g-3$ complex parameters y^i, the coordinates of the moduli space. Because of the two-dimensional conformal invariance of the theory, the functional integral reduces to an integral over moduli space. The integrand $Z(y^i, \bar{y}^i)$, called the "partition function", has been the object of intensive study in recent time. In general, it is not really a function, rather it is a section of a potentially nontrivial line bundle having moduli space as base. Only when certain cancellations of anomalies occur, is Z truly a function that can be integrated to give physically acceptable results [12, 13].

A classic problem in the theory of Riemann surfaces, which comes up repeatedly in string theory, is the study of the functions on a given Riemann surface as charaterized by their zeros and singularities. This involves the theory of divisors for which the concept of line bundle is a natural tool. A related problem is that of the classification of different spin structures on a Riemann surface, important for the study of superstrings [11].

In a nonabelian gauge theory with chiral spinors, gauge invariance may be spoiled by the occurrence of anomalies. In the Hamiltonian formulation these anomalies manifest themselves as an anomalous Schwinger term in the commutator of the generators of gauge transformations. These Schwinger terms can be interpreted as the curvature two-form of a suitable line bundle [7, 8].

Finally, let me mention that line bundles over phase space occur as a first step (prequantization) towards the quantization of a dynamical system. If the phase space is topologically non trivial, the line bundle can also be non trivial. Quantization can then be performed consistently only if certain parameters upon which the Hamiltonian depends take quantized values (these parameters can be for instance coupling constants, masses etc.). Typical examples are the value of the magnetic charge of a Dirac monopole and the value of the spin of a particle [6, 17].

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