LBL-23146 c.2
Preprint

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Accelerator & Fusion Research Division

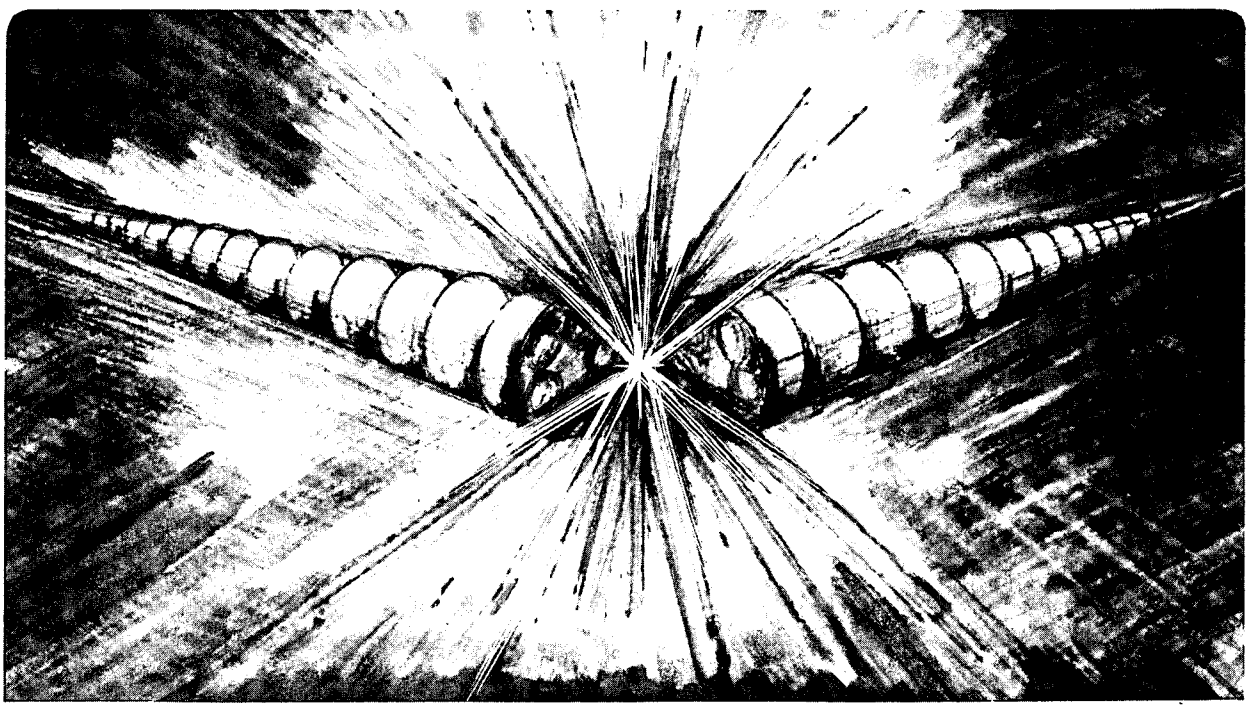
RECEIVED
JUN 3 1987
DOCUMENTS SECTION

Submitted to Physics Letters A

PHASE-SPACE SOLUTION OF THE LINEAR MODE-CONVERSION PROBLEM

A.N. Kaufman and L. Friedland

March 1987



Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

LBL-23146 c.2

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

LBL-23146

Phase-Space Solution of the Linear Mode-Conversion Problem*

Allan N. Kaufman and Lazar Friedland

Lawrence Berkeley Laboratory & Physics Department
University of California
Berkeley, CA 94720

ABSTRACT

The problem of linear mode conversion in a weakly inhomogeneous medium is posed and solved by phase-space methods. The PDEs for the two coupled modes are transformed to a simple first-order ordinary differential equation by a canonical transformation, wherein the two dispersion functions become essentially a locally conjugate pair of coordinates.

*This work was supported by the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

Wave propagation problems have traditionally been solved either in physical 4-dimensional space-time (x-space) or in terms of Fourier transforms (k-space). Yet, because the Hamiltonian ray equations describe orbits in 8-dimensional phase space, one would expect that more general coordinates would sometimes be of use.

Such is the case in the problem of linear mode conversion, when the components $D_{ij}(k,x)$ of the wave dispersion tensor have spatial variation in different directions [1]. To introduce notation, let us first use reduction techniques [2] to make **D** a 2x2 matrix field, assumed to be Hermitian:

$$\mathbf{D}(k,x) = \begin{pmatrix} D_a(k,x) & \eta(k,x) \\ \eta^*(k,x) & D_b(k,x) \end{pmatrix} . \qquad (1)$$

Here D_a and D_b are the respective dispersion functions for two modes, **a** and **b**, while η is the (small) coupling. Typically, for a ray of mode **a**, whose orbit is generated by $D_a(k,x)$:

$$\frac{dx^\mu}{d\sigma_a} = -\frac{\partial D_a}{\partial k_\mu}, \qquad \frac{dk_\mu}{d\sigma_a} = +\frac{\partial D_a}{\partial x^\mu} \qquad (2)$$

and lies on the 7-dimensional manifold $D_a(k,x) = 0$, the other dispersion function D_b is irrelevant, so long as $D_b \neq 0$.

However, if a ray of mode **a** pierces [at the point (k_c, x_c)] the 7-dimensional manifold $D_b(k,x) = 0$, it will transfer wave-action to mode **b**, mediated by the coupling parameter η. Using x-space eikonal methods, we have shown [2] that the transmission ratio (transmitted intensity/incident intensity) is given by

$$T = \exp[-2\pi|\eta|^2/|\{D_a,D_b\}|] , \tag{3}$$

in terms of the Poisson Bracket

$$\{D_a, D_b\} = \frac{\partial D_a}{\partial x^\mu}\frac{\partial D_b}{\partial k_\mu} - \frac{\partial D_a}{\partial k_\mu}\frac{\partial D_b}{\partial x^\mu} , \tag{4}$$

and the coupling η, both evaluated at the conversion point (k_c, x_c).

The appearance of the Poisson Bracket (4) in the result (3) suggests that this problem could benefit from phase-space methods. We shall demonstrate that this is indeed the case, allowing for an extremely simple derivation of the result (2).

We first localize the problem about the conversion point, by expanding D_a, D_b, and η in a Taylor series, and keep *only* the leading terms. Thus D_a and D_b are linear in the first derivatives, evaluated at (k_c, x_c):

$$(i=a,b) \qquad D_i(k,x) = (\partial D_i/\partial x^\mu)\,(x-x_c)^\mu + (\partial D_i/\partial k_\mu)\,(k-k_c)_\mu , \tag{5}$$

while the leading term in η is its value at (k_c,x_c).

The wave field $\mathbf{Z}(x) = (Z_a(x), Z_b(x))$ satisfies [3] the field equation:

$$\mathbf{D}(k \to -i\frac{\partial}{\partial x}, x) \bullet \mathbf{Z}(x) = 0 , \tag{6}$$

which is a set of two coupled linear partial differential equations. The operator $\hat{k} \equiv -i\frac{\partial}{\partial x}$ satisfies

$$[x^\mu, \hat{k}_\nu] = i\,\delta^\mu_\nu , \tag{7a}$$

corresponding to the Poisson Bracket

$$\{x^\mu, k_\nu\} = \delta^\mu_\nu . \tag{7b}$$

A *general* canonical change of coordinates has the form:

$$(i = 1,2,3,4) \qquad q_i = q_i(k,x), \qquad p_i = p_i(k,x) , \tag{8a}$$

with

$$\{q_i, p_j\} = \delta_{i,j} . \tag{8b}$$

Then the Hamiltonian equations are form invariant, and we have, for mode **a**,

$$\frac{dq_i}{d\sigma_a} = - \frac{\partial \overline{D}_a(q,p)}{\partial p_i} , \qquad \frac{dp_i}{d\sigma_a} = + \frac{\partial \overline{D}_a(q,p)}{\partial q_i} , \tag{9a}$$

where $\overline{D}_a(q,p)$ is the original dispersion function in terms of the new variables:

$$\overline{D}_a(q,p) = D_a(k,x) . \tag{9b}$$

Let us now choose

$$p_1(k,x) \equiv -D_a(k,x) ; \tag{10}$$

i.e., $\overline{D}_a(q,p) = -p_1$. Then (9a) yields

$$dq_1/d\sigma_a = 1, \quad dq_i/d\sigma_a = 0 \ (i \neq 1), \quad dp_i/d\sigma_1 = 0 ; \tag{11}$$

in other words, q_1 is the orbit parameter of mode **a**.

Next we choose

$$q_1(k,x) = \alpha D_b(k,x) , \tag{12}$$

with α a constant determined by (8b): $\{q_1, p_1\} = 1$. This yields $\alpha = B^{-1}$, where $B = \{D_a,D_b\}$ evaluated at (x_c,k_c). Then, similarly to (11), we have

$$dp_1/d\sigma_b = B, \quad dp_i/d\sigma_b = 0 \ (i \neq 1), \quad dq_i/d\sigma_b = 0 ; \tag{13}$$

i.e., p_1 is (within the factor B) the orbit parameter of the converted ray of mode **b**. Thus q_1 and p_1 are not only conjugate, but are the natural phase-space coordinates based on the two rays.

The dispersion matrix now reads

$$\overline{\mathbf{D}}\,(p,q) = \begin{pmatrix} -p_1 & \eta \\ \eta^* & Bq_1 \end{pmatrix} \tag{14}$$

The field equation (6) now reads

$$\overline{\mathbf{D}}\,(p \rightarrow -i\frac{\partial}{\partial q},\ q) \bullet \overline{\mathbf{Z}}\,(q) = 0\ , \tag{15}$$

where $\overline{Z}_i(q)$ is related to $Z_i(x)$ by a metaplectic transformation [4], which need not concern us here.

Using (14) in (15), we have

$$\begin{cases} i\,\partial\overline{Z}_a(q)/\partial q_1 + \eta\,\overline{Z}_b(q) = 0\ , & (16a) \\ \eta^*\,\overline{Z}_a(q) + Bq_1\,\overline{Z}_b(q) = 0 & (16b) \end{cases}$$

Eliminating $\overline{Z}_b$, we have the first-order ordinary differential equation:

$$\partial\,\overline{Z}_a(q)/\partial q_1 = (i|\eta|^2/Bq_1)\overline{Z}_a(q)\ , \tag{17}$$

whose solution is

$$\overline{Z}_a(q) = f(q_2, q_3, q_4)q_1^{i|\eta|^2/B}\ , \tag{18}$$

with f arbitrary.

The *amplitude* transmission ratio R is defined to be

$$R \equiv \overline{Z}_a(+q_1, q_2, q_3, q_4)/\overline{Z}_a(-q_1, q_2, q_3, q_4)\,; \tag{19}$$

i.e., the amplitude of mode **a** is compared at equal $|q_1|$ distances from the conversion point $q_1 = 0$. From (18) we find

$$R = (-1)^{i|\eta|^2/B} = e^{\pm\pi|\eta|^2/B} \tag{20}$$

Choosing the proper sign for causality, we finally obtain

$$T \equiv |R|^2 = \exp[-2\pi\,|\eta|^2/|B|]\,. \tag{21}$$

Comparison of this derivation with our previous x-space approach shows how remarkably simple the phase-space method is.

We are indebted to Robert Littlejohn for introducing us to metaplectic techniques.

References

[1] L. Friedland, G. Goldner, and A. N. Kaufman, Phys. Rev. Letters 58 (1987) 1392.

[2] L. Friedland and A. N. Kaufman, submitted to Physics of Fluids (1987).

[3] V. Maslov and M. Fedoriuk, Semi-Classical Approximation in Quantum Mechanics (Reidel, Boston), 1981.

[4] R. G. Littlejohn, Physics Reports 138 (1986) 193.

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

Reference to a company or product name does not imply approval or recommendation of the product by the University of California or the U.S. Department of Energy to the exclusion of others that may be suitable.

LAWRENCE BERKELEY LABORATORY
TECHNICAL INFORMATION DEPARTMENT
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720