LBL-23299
c.2

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Accelerator & Fusion Research Division

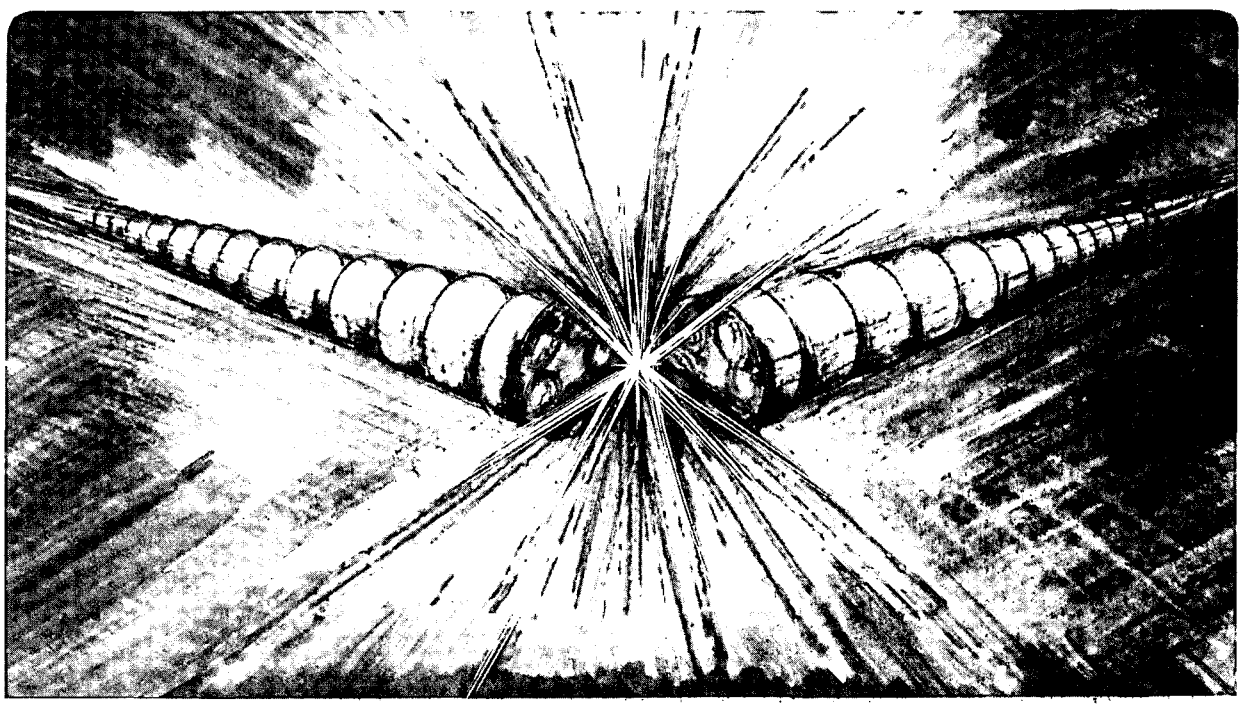
Presented at the Seventh APS Topical Conference on Applications of Radio-Frequency Power to Plasmas, Kissimmee, FL, May 4–6, 1987

CONGRUENT REDUCTION AND MODE CONVERSION IN 4-DIMENSIONAL PLASMAS

L. Friedland and A.N. Kaufman

April 1987

RECEIVED
LAWRENCE
BERKELEY LABORATORY
JUN 10 1987
LIBRARY AND
DOCUMENTS SECTION



LBL-23299
c.2

Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

CONGRUENT REDUCTION AND MODE CONVERSION IN 4-DIMENSIONAL PLASMAS

Lazar Friedland and Allan N. Kaufman
LBL and Physics Department, University of California, Berkeley, CA 94720

ABSTRACT

Standard eikonal theory reduces, to N=1, the order of the system of equations underlying wave propagation in inhomogeneous plasmas. The condition for this remarkable reducibility is that only one eigenvalue of the unreduced NxN dispersion matrix **D**(k,x) vanishes at a time. If, however, two or more eigenvalues of **D** become simultaneously small, the geometric optics reduction scheme becomes singular. These regions are associated with linear mode conversion, and are described by higher order systems. We develop a new reduction scheme based on congruent transformations of **D**, and show that, in "degenerate" plasma regions, a partial reduction of order is possible. The method comprises a constructive step-by-step procedure, which, in the most frequent (doubly) degenerate case, yields a second order system, describing the pairwise mode conversion problem, the solution of which in general geometry has been found recently.

INTRODUCTION

Small amplitude waves in inhomogeneous plasmas are described by systems of linear equations for such quantities as electromagnetic fields **E**, **B**, perturbed average velocities $\mathbf{V}_\alpha$ of various species in fluid models, etc. The complexity of the problem due to the multicomponent structure of the waves is typically resolved by using some sort of reduction (elimination) of several wave components from the problem. The conventional geometric optics theory,[1] for example, makes use of the reduced dispersion tensor **D**, a 3x3 matrix, describing the components of the electric field **E** alone. The geometric optics theory then uses a perturbation expansion technique to reduce the problem even further. In non-degenerate cases, when only one eigenvalue of **D** vanishes at a time, the theory reduces to the solution of a single first order equation for the scalar amplitude of the electric field. This perturbation scheme, however, becomes singular in cases when an additional eigenvalue of **D** becomes small.[2] More generally, the singular situations are characteristic of near degeneracies of the <u>unreduced</u> (N-th order) system and are associated with such phenomena as resonances and mode converison.

This paper summarizes our recent study of reduction in geometric optics,[3] and describes a constructive procedure which yields the optimally reduced system, describing the smallest possible number of "irreducible" wave components. The final system avoids singular coefficients, has the lowest possible order, and at the same time preserves important properties of the unreduced wave, such as the conservation of the wave-action flux. As an important application, in doubly degenerate plasmas, the theory yields a second order system, describing the pairwise mode conversion problem, the solution of which in general geometry has been obtained recently.[4]

TRANSPORT EQUATION AND CONGRUENCE TRANSFORMATION

Our starting point is the linear integral "evolution" equation

$$\int d^4x_2\ \hat{D}_{ij}(x_1, x_2)\, Z_j(x_2) = 0 \qquad (1)$$

for the (complex) N-component field $\mathbf{Z}(x)$ on space-time. The two point NxN kernel D_{ij} is assumed to be Hermitian and its local Fourier transform

$$\mathbf{D}(k,x) = \int d^4s\, \hat{\mathbf{D}}\,(x+\tfrac{1}{2}s, x-\tfrac{1}{2}s)\exp(-ik\cdot s) \qquad (2)$$

defines the dispersion matrix of the unreduced problem. The eikonal representation $\mathbf{Z}(x)=\mathbf{A}(x)\exp[i\psi(x)]$, with slowly varying amplitude $\mathbf{A}(x)$ and wave vector $k_\mu(x)=\partial\psi/\partial x^\mu$, leads to (Refs.1,3):

$$\mathbf{D}\bullet\mathbf{A} = i\left[\frac{\partial \mathbf{D}}{\partial k_\mu}\bullet\frac{\partial \mathbf{A}}{\partial x^\mu} + \frac{1}{2}\frac{d(\partial \mathbf{D}/\partial k_\mu)}{dx^\mu}\bullet\mathbf{A}\right] + O(\delta^2) \equiv \mathbf{L}(\mathbf{A}) \qquad (3)$$

where δ is the small parameter describing the slow variation of the background. Eq. (3) is the conventional transport equation, being the starting point of the geometric optics perturbation scheme. This equation also yields the action flux conservation

$$\partial J^\mu(x)/\partial x^\mu = 0, \quad J^\mu(x) \equiv A_i^*(\partial D_{ij}/\partial k_\mu)A_j + O(\delta) \qquad (4)$$

At this stage we make the transformation to a new amplitude $\mathbf{A}$ via

$$\mathbf{A}(x) = \mathbf{Q}(k(x),x)\bullet\bar{\mathbf{A}}(x) \qquad (5)$$

where the matrix $\mathbf{Q}(k,x)$ is assumed to be slowly varying and invertible. Then Eq.(3) can be written for $\mathbf{A}$:

$$\bar{\mathbf{D}}\bullet\bar{\mathbf{A}} = \mathbf{Q}^+\bullet\mathbf{L}(\mathbf{Q}\bullet\bar{\mathbf{A}}) \qquad (6)$$

where the <u>transformed</u> dispersion matrix $\mathbf{D}$ is given by the congruence transformation

$$\bar{\mathbf{D}} = \mathbf{Q}^+\bullet\ \mathbf{D}\bullet\mathbf{Q} \qquad (7)$$

The purpose of the transformation $\mathbf{Q}$ is to simplify the problem and in particular (see the next Section) to <u>annihilate</u>, if possible, some wave components, thus reducing the order of the system.

REDUCTION THEOREM

The main result of our studies is the following <u>Reduction Theorem</u>: If there exists at least one large (of O(1)) element in the matrix $\mathbf{D}(k,x)$, characterizing the unreduced N-component (N-th order) system, then one of the components of $\mathbf{A}$ can be eliminated from the problem in such a way that the remaining N-1 components of the wave are fully described by a reduced, Hermitian tensor $\mathbf{D}^r$ with the reduced transport equation of form (3), $\mathbf{D}$ being replaced by $\mathbf{D}^r$.

The proof of this theorem can be found in Ref.3. Here we shall briefly describe the reduction algorithm. The reduction proceeds differently in the following two cases :

Case 1. For general N, suppose that the element D_{kk} of **D** is O(1). Then the reduction to N-1 is achieved by $Q_{ij} = \delta_{ij} - \delta_{ik}D_{kj}/D_{kk} + \delta_{ik}\delta_{jk}$. This transformation annihilates the k-th component of **A**, leaving the remaining components of **A** unchanged, so that the reduced wave amplitude is $\mathbf{A}^r = \{A_i, i\neq k\}$. The reduced dispersion tensor, to lowest order, is

$$\mathbf{D}^r = D_{ij} - D_{ik}D_{kj}/D_{kk} \quad , \qquad i,j \neq k \tag{8}$$

Case 2. When all diagonal elements of **D** are O(δ), but D_{qr} ($r\neq q$) is of O(1), we use the transformation matrix $Q_{ij} = \delta_{ij} + \alpha\delta_{ir}\delta_{jq}$, where $\alpha = 1$ if Re D_{qr} = O(1) and $\alpha=i$ if Im D_{qr} = O(δ). After this transormation the q-th diagonal element of **D** is $D_{qq} = 2\mathrm{Re}(\alpha D_{qr}) + O(\delta) = O(1)$.Therefore the reduction to N-1 can be achieved as in Case 1. It can be also shown that the reduced matrix then has $D_{rr} \sim O(1)$, so that Case 2 actually allows reduction from N to N-2.

The Reduction Theorem provides a tool for reducing the order, step-by-step, for weakly varying plasmas of arbitrary geometry. The successive application of the algorithm yields a final reduced Hermitian matrix $\mathbf{D}^f$ of rank M≤N, such that all its elements are O(δ). A further attempt to reduce the system would yield singular coefficients, so that the system is "irreducible" within the geometric optics approximation. The simplest and most frequent case is M=1, when all but one of the components of **A** are eliminated from the problem. This is the non-degenerate case, when only one eigenvalue of **D** is small. The final transport equation in this case is a single first order PDE for the remaining wave component A^f.

NORMAL DEGENERACY AND MODE CONVERISON

Less frequent, but nevertheless important in applications, is the situation when M=2, in which case the final system comprises a set of two coupled PDE's. The final reduced matrix $\mathbf{D}^f$ in this case can be written as

$$\mathbf{D}^f = \begin{bmatrix} D_a & \eta \\ \eta^* & D_b \end{bmatrix} \tag{9}$$

where all elements are of O(δ), and let us denote $\mathbf{A}^f = (A_a, A_b)$.

Now we argue that since the degeneracy implies a simultaneous satisfaction of three smallness conditions ($D_a, D_b, \eta \sim O(\delta)$), the degeneracy is a rare phenonenon. The most probable scenario of the double degeneracy, therefore, is that due to some special physical conditions (existence of a global small parameter, for example) one of the elements of $\mathbf{D}^f$ is O(δ) in an extended region of the phase space. We also argue that since $\mathrm{Det}(\mathbf{D}^f) = D_aD_b - |\eta|^2$, only when $\eta \sim O(\delta)$ in an extended region of the phase space does there exist the possibility of having two distinct modes ($D_a \approx 0$, $D_b \approx 0$) away from the degenerate region. We shall refer to this situation as normal degenerancy.

The reduced transport equation (3) in the case of the normal degeneracy becomes[4]

$$V_a \cdot \partial A_a/\partial x - ix' \cdot R^a A_a = i\eta A_b$$
$$V_b \cdot \partial A_b/\partial x - ix' \cdot R^b A_b = i\eta^* A_a \qquad (10)$$

where $V^\mu_{a,b} = \partial D_{a,b}/\partial k_\mu$ and $R_\mu^{a,b} = \partial D_{a,b}/\partial x^\mu$ are the group four-velocity and spatial gradient (or four-refraction). These are constant vectors, evaluated at the crossing point x_o, k_o where $D_a(x_o,k_o) = D_b(x_o,k_o) = 0$. In (10), $x' \equiv x - x_o$. Solution of Eqs. (10) has been found in Ref. 4. It was shown that the action flux J_a associated with the dispersion relation $D_a = 0$ in the non-degenerate region is only partially transmitted through the neighborhood of the crossing point, and that the transmission coefficient is

$$T = \exp(-2\pi|\eta|^2/|B|) \qquad (11)$$

where B is the Poisson Bracket

$$B = (\partial D_a/\partial x^\mu)(\partial D_b/\partial k_\mu) - (\partial D_a/\partial k_\mu)(\partial D_b/\partial x^\mu) \qquad (12)$$

Summarizing, we have shown that pairwise mode conversion events are typically associated with normal degeneracy of the final 2x2 dispersion matrix (M=2), and that the reduction algorithm automatically provides the characteristic form of the dispersion tensor, describing two easily identifiable coupled modes asssociated with the diagonal elements of the matrix, while its non-diagonal element serves as a small coupling coefficient. These elements can then be used directly in the mode conversion theory[4] for calculating the transmission and mode conversion coefficients in cases of interest.

ACKNOWLEDGEMENT

This research was supported by the U.S. Department of Energy under contract No. DE-AC03-76SF00098.

REFERENCES

1. I. Bernstein and L. Friedland in "Handbook of Plasma Physics", M. Rosenbluth and R. Sagdeev, eds. (North-Holland, Amsterdam,1983) Vol.1, Ch. 2.5.
2. L. Friedland, Phys. Fluids 28, 3260 (1985).
3. L. Friedland, and A. Kaufman, Phys. Fluids (in press).
4. L. Friedland, G. Goldner, and A. Kaufman,Phys.Rev.Lett. 58,1392 (1987); A. Kaufman and L. Friedland, Phys. Lett. A (in press).

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

Reference to a company or product name does not imply approval or recommendation of the product by the University of California or the U.S. Department of Energy to the exclusion of others that may be suitable.

LAWRENCE BERKELEY LABORATORY
TECHNICAL INFORMATION DEPARTMENT
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720