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EXACT TREATMENT OF THE DWBA BY ANALYTIC MEANS FOR PARTICLE TRANSFER BETWEEN HEAVY IONS*

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Abstract

The Fourier-Bessel expansion is utilized for the exact evaluation of the DWBA amplitude in transfer reactions. The partial wave radial components of the elastic scattering wave function in the outgoing channel and one of the bound-state wave functions of the transferred particle are expanded in terms of a complete set of spherical Bessel functions over a finite interval. For a radial wave function of a given angular momentum, the basis is chosen to be a complete set of spherical Bessel functions of the same angular momentum, so that the addition theorem for the product of spherical Bessel function and spherical Harmonic can be utilized. It is shown that the method proposed has the distinct advantage that the recoil angular momentum appears as a natural expansion parameter, and that the various approximate methods can be retrieved as simple limits of the exact expression. The method is applied to the transfer of a single nucleon or a cluster, and to the transfer of several nucleons.

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1. Introduction

Reactions between complex nuclei in which one or several nucleons are transferred to form discrete states in the product nuclei are thought to be direct reactions, analogous to light ion reactions such as the (d,p) reaction[1]). In this case, in lowest order, the reaction may be computed by the distorted wave Born-approximation (DWBA).

The DWBA involves the evaluation of multidimensional integrals, which is difficult because the integrands consist of products of a number of functions which depend on different vector coordinates, with, however, only several of them being independent. In the case of light nuclide reactions, such as (d,p), the evaluation is simplified by neglecting the effect of the finite size of the light nuclide (d in the case of (d,p) reaction). For reactions between complex nuclei, the neglect of their finite size is not justified, however. In addition to this, the small de Broglie wavelength of typical heavy-ion reactions requires that the particular dependence of the functions in the integrand on their vector coordinates be retained with an accuracy commensurate with the wavelength. Evaluation of the integrals which approximate the vector dependences of the integrands while retaining the finite range[2]) are referred to as no-recoil approximations. (The zero-range approximation of the integrals automatically neglects recoil.)

There appear to be three distinct approaches to evaluating the DWBA integral without making either the zero-range or recoilless approximation. These are usually referred to as "exact", the sense of the word being that any errors are due to the finite accuracy of the numerical methods, and not

to the neglect of physical effects. One approach evaluates directly the multidimensional integral, as in the work of Bayman[3]). A second approach employs a Legendre expansion in two vectors by which the angle integrals can then be done in closed form. There remains a double-radial integral. This is the formulation of Austern, et al.[4]), originally developed for (d,p) reactions. It is the method employed in the work of Yoshida et al.[5]), of Devries[6]), and of Low and Tamura[7]). The third approach expands the functions of the dependent coordinates in terms of functions for which a separation into the independent coordinates is possible. The most transparent way of achieving this is the expansion of the distorted waves on a plane wave basis. Then the coordinates separate trivially. This method was formulated by Charlton and Robson[8]). Sawaguri and Tobocman[9]) had earlier expanded the distorted waves on an oscillator function basis which also permits a separation of coordinates.

So far the direct evaluation of the multidimensional integrals for heavy ion reactions has been done only for angular momentum transfer equal to zero, perhaps because the method takes much computer time. Computer programs based on the approach of Austern et al. also have turned out to be costly to execute. In this connection, Low and Tamura have discussed how to choose the integration regions judiciously so as to save computer time, apparently at the cost of accuracy in absolute, though not in relative, cross sections. Of the third approach, the expansions of Sawaguri and Tobocman converge slowly and have consequently been applied only to light systems (N_1+ O). The expansion in plane waves used by Charlton has been thus far applied only to very light ion reactions.

In view of the above critique there is evidently a need for a fast and accurate method of evaluating the DWBA integrals. We propose the method discussed in detail in the next section, which appears to us to be a fast method, and one in which the convergence with respect to recoil angular momentum can be exploited. The method is based first on the existence of an addition theorem for the product of a spherical Bessel function and spherical harmonic, analogous to that for a Hankel function. This addition theorem which is the vehicle for the separation of coordinates, can be exploited by representing the functions whose coordinates are to be separated, by a Fourier-Bessel series. Our approach belongs to the third category mentioned above. A virtue of our particular formulation is that the recoil angular momentum appears as the natural expansion parameter. This fact can be exploited in numerical calculations by truncating the series when satisfactory convergence is achieved.

In Section 2 the method is formulated. Then the recoil angular momentum and selection rules are discussed in Section 3, while in Section 4 we retrieve earlier approximate solutions from our general result. The Fourier-Bessel series is discussed with several examples in Section 5. The general case of multinucleon transfer is formulated in the Appendix.

2. Formulation of the Problem

We shall use a notation that is suggestive of a (d,p) reaction though, of course, our interest is in heavy ions. We label the reaction as

$$D + A \rightarrow P + B$$

where

$$D = P + N \quad , \quad B = A + N$$

and show the coordinates in fig. 1. We write the wavefunctions for D and B as

$$\Phi_{J_D}^{M_D}(D) = \sum_{J_P L_1 S_1 J_1} B_{L_1 S_1 J_1}(P,D) \left[\Phi_{J_P}(P)\ \phi_{L_1 S_1 J_1}(\underset{\sim}{R}_{PN})\right]_{J_D}^{M_D} \tag{1}$$

$$\Phi_{J_B}^{M_B}(B) = \sum_{J_A L_2 S_2 J_2} B_{L_2 S_2 J_2}(A,B) \left[\Phi_{J_A}(A)\ \phi_{L_2 S_2 J_2}(\underset{\sim}{R}_{AN})\right]_{J_B}^{M_B} \tag{2}$$

in which the B's are parentage amplitudes. As usual in DWBA, the nuclei A and P are regarded as inert cores of D and B.

The transferred object N may be either a single nucleon or a cluster of several nucleons whose intrinsic angular momentum is S. In the latter case $\underset{\sim}{R}_{PN}$ and $\underset{\sim}{R}_{AN}$ are to be interpreted as the centers of mass of the group of transferred nucleons and L_1 and L_2 the orbital angular momenta

of their center of mass about P and A, respectively. Also ϕ depends in this case on the internal coordinates of the cluster. These details are, however, irrelevant to our method so we shall proceed as though N were a nucleon. The more general case of multinucleon transfer in which the cluster condition is relaxed is treated in appendix A.

The DWBA amplitude for the reaction is

$$T = \iint \psi_{k_p}^{(-)*}(\underset{\sim}{R}_P) \langle \Phi_{J_B}^{M_B} \Phi_{J_P}^{M_P} | V(R_{PN}) | \Phi_{J_D}^{M_D} \Phi_{J_A}^{M_A} \rangle \psi_{k_D}^{(+)}(\underset{\sim}{R}_D) \, d\underset{\sim}{R}_D \, d\underset{\sim}{R}_{PN} \tag{3}$$

which upon introducing the parentage expansions, appears more explicitly as

$$T = \sum_{L_1 S_1 J_1} \sum_{L_2 S_2 J_2} B_{L_1 S_1 J_1}(P,D) \, B_{L_2 S_2 J_2}(A,B)$$

$$C \begin{matrix} J_P & J_1 & J_D \\ M_P & M_1 & M_d \end{matrix} \quad C \begin{matrix} J_A & J_2 & J_B \\ M_A & M_2 & M_B \end{matrix} \quad t \tag{4}$$

$$t \equiv \iint \psi_{k_p}^{(-)*}(\underset{\sim}{R}_P) \, \phi^*_{L_2 S_2 J_2 M_2}(\underset{\sim}{R}_{AN}) \, V(R_{PN}) \, \phi_{L_1 S_1 J_1 M_1}(R_{PN}) \, \psi_{k_D}^{(+)}(\underset{\sim}{R}_D) \, d\underset{\sim}{R}_D \, d\underset{\sim}{R}_{PN} \, . \tag{5}$$

The difficulty mentioned above of evaluating this six-dimensional integral is now apparent. The coordinates of integration, $\underset{\sim}{R}_D$ and $\underset{\sim}{R}_{PN}$, appear

as arguments of only three of the functions, but appear in the other two in the combinations

$$\underset{\sim}{R}_P = \frac{A}{B}\underset{\sim}{R}_D - \frac{N}{B}\frac{M}{D}\underset{\sim}{R}_{PN}$$

$$\underset{\sim}{R}_{AN} = \underset{\sim}{R}_D + \frac{P}{D}\underset{\sim}{R}_{PN} \tag{6}$$

$$M = B + P = A + D \quad .$$

(In this context we use A to denote the mass of nucleus A.) The recoilless approximation consists in neglecting the second term on the right side of $\underset{\sim}{R}_P$. Aside from the trivial scaling factor A/B, the arguments of the two distorted waves are then the same. This simplifies the integration very much. However, this neglect would be justified only when $\psi_{k_p}(R_p)$ varies slowly over distances

$$d = \frac{N}{D}\frac{M}{B} r_D \tag{7}$$

where we use r_D to denote the radius of D which is the approximate bound on $\underset{\sim}{R}_{PN}$ imposed by the presence of $V(R_{PN})$ in (5). Hence recoil effects will be small only if the de Broglie wavelength $\lambda\!\!\!^{-} = 1/k_p$ is large compared to d, i.e.

$$x \equiv \frac{N}{D}\frac{M}{B} k_P r_D \ll 1 \quad . \tag{8}$$

This inequality is rarely satisfied (see Table 1).

To evaluate the integral (5) without making the zero-range or recoilless approximation, we seek to express the functions $\psi(R_P)$ and $\phi(R_{AN})$ each as the product of a function of R_D and of R_{PN}. To this end we first introduce, as usual, the partial wave expansions

$$\psi^{(+)}_{k_D}(\underset{\sim}{R}_D) = 4\pi \sum_{L_D \mu_D} e^{i\sigma_D} i^{L_D} F_{L_D}(k_D, R_D)\, Y^{\mu_D^*}_{L_D}(\hat{k}_D)\, Y^{\mu_D}_{L_D}(\hat{R}_D)$$

$$\psi^{(-)*}_{k_P}(\underset{\sim}{R}_P) = 4\pi \sum_{L_P \mu_P} e^{i\sigma_P} i^{-L_P} F_{L_P}(k_P, R_P)\, Y^{\mu_P}_{L_P}(\hat{k}_P)\, Y^{\mu_P^*}_{L_P}(\hat{R}_P) \quad . \tag{9}$$

If F_{L_P} were a spherical Bessel function, then the addition theorem mentioned in the introduction could be used. This we arrange by use of the Fourier-Bessel expansion

$$F_{L_P}(k_P, R_P) = \sum_n a_n^{L_P} j_{L_P}(\alpha_n^{L_P} R_P) \qquad (0 \leqslant R_P \leqslant R_m) \quad . \tag{10}$$

In section 5 we discuss this expansion in more detail. The important fact, shown there, is that only a few terms, about 10 generally, are needed to represent F_{L_P} in the necessary range to good accuracy.

The range, R_m, over which the representation is made is arbitrary in principle. The convergence is faster the smaller the R_m. On the other hand, R_m must be chosen large enough to obtain all significant contributions to the integral (5). In practice this is achieved if R_m is a few fermi's larger than the sum of the radii of D and A.

Similarly, if we denote the radial part of $\phi_{L_2 S_2 J_2}(\underset{\sim}{R}_{AN})$ by $u_{L_2 J_2}$

$$\phi_{L_2 S_2 J_2}(\underset{\sim}{R}_{AN}) = u_{L_2 J_2}(R_{AN}) \; [Y_{L_2}(\hat{R}_{AN}), S_2]_{J_2}$$

we can expand it

$$u_{L_2 J_2}(R_{AN}) = \sum_m b_m^{L_2} \, j_{L_2}(\beta_m^{L_2} R_{AN}) \quad . \tag{11}$$

The addition theorem is written for these as (see appendix B)

$$\begin{aligned} j_{L_P}(\alpha R_P) \; Y^*_{L_P \mu_P}(\hat{R}_P) = \sum_{\ell_P \ell_R} & A_{\ell_P \ell_R L_P} (-)^{\ell_P} \, j_{\ell_P}(\alpha \tfrac{A}{B} R_D) \\ & \times j_{\ell_R}(\alpha \tfrac{N}{D} \tfrac{M}{B} R_{PN}) \left[Y_{\ell_P}(\hat{R}_D) \; Y_{\ell_R}(-\hat{R}_{PN}) \right]^*_{L_P \mu_P} \end{aligned} \tag{12}$$

and

$$j_{L_2}(\beta R_{AN})\, Y_{L_2M_2}(\hat{R}_{AN}) = \sum_{\ell\Lambda} A_{\Lambda\ell L_2} (-)^{\Lambda}\, j_{\Lambda}(\beta R_D) \times j_{\ell}\left(\beta \frac{P}{D} R_{PN}\right) \left[Y_{\Lambda}(\hat{R}_D)\, Y_{\ell}(\hat{R}_{PN})\right]_{L_2M_2} . \tag{13}$$

Now the integral (5) can be written

$$\begin{aligned} t = (4\pi)^2 \sum_{L_p\mu_p} \sum_{L_D\mu_D} & e^{i(\sigma_p+\sigma_D)}\, i^{L_D-L_P}\, Y^*_{L_D\mu_D}(\hat{k}_D)\, Y_{L_p\mu_p}(\hat{k}_p) \\ & \times \sum_{\ell_p\ell_R} A_{\ell_p\ell_R L_p} (-)^{\ell_p+\ell_R} \sum_{\Lambda\ell} A_{\Lambda\ell L_2} (-)^{\Lambda} \\ & \langle \left[Y_{\ell_p}(\hat{R})\, Y_{\ell_R}(\hat{r})\right]_{L_p\mu_p} , \left[\left[Y_{\Lambda}(\hat{R})\, Y_{\ell}(\hat{r})\right]_{L_2} , S_2\right]_{J_2M_2} | \\ & \times | Y_{L_D\mu_D}(\hat{R}), \left[Y_{L_1}(r), S_1\right]_{J_1M_1} \rangle \\ & \times \sum_{nm} a_n^{L_p}\, b_m^{L_2} \int_0^{R_m} j_{\ell_R}\left(\alpha_n^{L_p} \frac{N}{D}\frac{M}{B} r\right) j_{\ell}\left(\beta_m^{L_2} \frac{P}{D} r\right) V(r)\, u_{L_1J_1}(r)\, r^2\, dr \\ & \times \int_0^{R_m} j_{\ell_p}\left(\alpha_n^{L_p} \frac{A}{B} R\right) j_{\Lambda}\left(\beta_m^{L_2} R\right) F_{L_D}(k_D, R)\, R^2\, dR \end{aligned} \tag{14}$$

The main result of our paper is expressed in eq. (14). The six-fold integral representing the t-matrix has been reduced to products of two <u>one</u>-dimensional integrals.

The angle integrals appearing in eq. (14) can be evaluated in closed form as

$$\langle \text{angle integral} \rangle = \delta_{S_1 S}\, \delta_{S_2 S_1} \sum_{\mathcal{L}\mu} \begin{pmatrix} L_P & L_D & \mathcal{L} \\ \mu_P & -\mu_D & \mu \end{pmatrix} \begin{pmatrix} J_1 & J_2 & \mathcal{L} \\ M_1 & -M_2 & \mu \end{pmatrix}$$

$$(-)^{L_D+\mu_D+J_2+M_2}\, G(J_1 J_2 L_1 L_2 L_P L_D \ell \Lambda \ell_R \ell_P\, S\mathcal{L}) \tag{15}$$

where G is a geometric factor

$$G = \frac{1}{4\pi} (-)^{L_D+L_1+J_1+S+\mathcal{L}} \begin{pmatrix} \ell_P & \Lambda & L_D \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_R & \ell & L_1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{Bmatrix} L_2 & \mathcal{L} & L_1 \\ J_1 & S & J_2 \end{Bmatrix} \begin{Bmatrix} L_2 & \mathcal{L} & L_1 \\ \ell_R & \ell & \Lambda \end{Bmatrix} \begin{Bmatrix} L_P & \mathcal{L} & L_D \\ \Lambda & \ell_P & \ell_R \end{Bmatrix} (\hat{\ell}_P \hat{\Lambda} \hat{\ell}_R \hat{\ell} \hat{L}_1 \hat{J}_1 \hat{L}_2 \hat{J}_2 \hat{L}_P \hat{L}_D)^{1/2} \tag{16}$$

For convenience let us define

$$\mathcal{F}^{(L_1 S J_1 L_2 S J_2)\mathcal{L}}_{L_P L_D}(R) = \sum_{\ell_P \ell_R} \sum_{\Lambda \ell} A_{\ell_P \ell_R L_P} (-)^{L_P}$$

$$\times A_{\Lambda \ell L_2} (-)^{\Lambda}\, G(J_1 J_2 L_1 L_2 L_P L_D \ell \Lambda \ell_R \ell_P S \mathcal{L}) \tag{17}$$

$$\times \sum_{nm} a_n^{L_P}\, b_m^{L_2}\, I_{nm}(L_1 L_2 L_P \ell_R \ell)\; j_{\ell_P}(\alpha_n^{L_P} \tfrac{A}{B} R)\; j_{\Lambda}(\beta_m^{L_2} R)$$

where I_{nm} denotes the r integral in eq. (14). Then the result for t is

$$t = (4\pi)^2 \sum_{L_p\mu_p L_D\mu_D} e^{i(\sigma_p+\sigma_D)}\, i^{L_D-L_p}\, Y^*_{L_D\mu_D}(\hat{k}_D)\, Y_{L_p\mu_p}(\hat{k}_p)$$
$$\sum_{\mathcal{L}\mu} \begin{pmatrix} L_p & L_D & \mathcal{L} \\ \mu_p & -\mu_D & \mu \end{pmatrix} \begin{pmatrix} J_1 & J_2 & \mathcal{L} \\ M_1 & -M_2 & \mu \end{pmatrix} \sqrt{\hat{\mathcal{L}}}\; K^{(n_1 n_2)\mathcal{L}}_{L_p L_D} \tag{18}$$

where

$$\hat{\mathcal{L}} = 2\mathcal{L} + 1 \;, \qquad n_1 \equiv L_1 S J_1 \;, \qquad n_2 = L_2 S J_2 \tag{19}$$

and

$$K^{(n_1 n_2)\mathcal{L}}_{L_p L_d} = \frac{1}{\sqrt{\hat{\mathcal{L}}}} \int \mathcal{F}^{(n_1 n_2)\mathcal{L}}_{L_p L_D}(R)\; F_{L_D}(k_D, R)\; R^2\, dR \;. \tag{20}$$

This is the distorted wave integral as it appears in this formulation. In the appropriate limit, the usual zero-range expression is retrieved, as discussed in section 4.

Since the scattered functions $F_L(k,R)$ are complex, the expansion (10) is done for both real and imaginary parts although we have not carried this through in our notation, for the sake of simplicity of appearance.

Usually in transfer reaction calculations, the asymptotic region of all components of a given state are chosen to be equal (to the separation energy for example). Therefore, the same expansion parameters b_m and β_m in (11) apply generally to all components independent of L_2.

The differential cross section is related to the transition amplitude, Eq. (4), through the expression

$$\frac{d\sigma}{d\Omega} = \frac{\mu_{DA}\ \mu_{PB}}{(2\pi\hbar^2)^2}\ \frac{k_P}{k_D}\ \frac{1}{(2J_D+1)\ (2J_A+1)} \sum_{\substack{M_A M_B \\ M_P M_D}} |T|^2 \tag{21}$$

where μ_{DA} and μ_{PB} are the reduced masses in the initial and final channels of the reaction. Utilizing Eqs. (4) and (18), we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\mu_{DA}\ \mu_{PB}}{(2\pi\hbar^2)^2}\ \frac{k_P}{k_D}\ \frac{2J_B+1}{2J_A+1} \sum_{J_1 J_2 \mathcal{L} \mu} \frac{1}{(2J_1+1)\ (2J_2+1)} \left| \beta_{J_1 J_2 \mathcal{L} \mu} \right|^2 \tag{22a}$$

where

$$\begin{aligned} \beta_{J_1 J_2 \mathcal{L} \mu} = {} & (4\pi)^2 \sum_{L_1 S_1 L_2 S_2} \sum_{L_P \mu_P L_D \mu_D} B_{L_1 S_1 J_1}(P,D)\ B_{L_2 S_2 J_2}(A,B) \\ & \times\ i^{L_D - L_P}\ e^{i(\sigma_P+\sigma_D)}\ Y^*_{L_D \mu_D}(k_D)\, Y_{L_P \mu_P}(k_P) \begin{pmatrix} L_P & L_D & \mathcal{L} \\ \mu_P & -\mu_D & \mu \end{pmatrix} \\ & \times\ K^{(n_1 n_2)\mathcal{L}}_{L_P L_D}\ \delta_{S_1 S}\ \delta_{S_2 S} \end{aligned} \tag{22b}$$

The contributions from different J_1, J_2 and $\mathcal{L}$ add incoherently.

3. Recoil Angular Momentum and Selection Rules

The angular momentum couplings are exhibited in fig. 2. It is observed that $\mathcal{L}$ is the angular momentum transfer. It is a particular virtue of this approach that the recoil angular momentum ℓ_R appears explicitly. We give it this name because in the recoilless limit $N/D \to 0$

$$j_{\ell_R}(\alpha_n \frac{N}{D}\frac{M}{B} r) \longrightarrow \delta_{\ell_R 0} \quad \text{as} \quad \frac{N}{D} \to 0 \quad . \tag{23}$$

Thus ℓ_R is the angular momentum associated with the recoil.

From an examination of the geometrical factors we can derive the parity selection rules.

$$(-)^{L_p+L_D} = (-)^{L_1+L_2} = (-)^{\Lambda+\ell_R} \quad . \tag{24}$$

The first two factors express the conservation of parity and the third indicates that the angular momentum transfer $\mathcal{L}$, which is subject to the vector selection rules,

$$\underset{\sim}{\mathcal{L}} = \underset{\sim}{L}_1+\underset{\sim}{L}_2 = \underset{\sim}{J}_1+\underset{\sim}{J}_2 \tag{25}$$

is <u>not</u> restricted by the parity selection rule. Only when $\ell_R = 0$ does $\Lambda \to \mathcal{L}$, in which case the recoilless selection rules hold. In general

$$\underset{\sim}{\mathcal{L}} = \underset{\sim}{\Lambda} + \underset{\sim}{\ell}_R \quad . \tag{26}$$

That the angular momentum transfer $\mathcal{L}$ is not generally restricted by the parity change has very important consequences as observed in earlier work[10]).

It is a convenient feature of this formalism that the recoil angular momentum appears explicitly because we anticipate that this quantity is restricted to small values. Thus, the convergence in this variable can be exploited in numerical calculations. The convergence in ℓ_R depends on the magnitude of the r integrals in eq. (14) for various ℓ_R. This in turn depends on the size of the argument for j_{ℓ_R}. The smaller this argument the faster the convergence. It will be clearer in the next section that, as concerns the sum on n, the most important terms are $\alpha_n \simeq k_p$. Because of the appearance of $V(r)u_{L_1J_1}(r)$ in the r integral, the main contribution comes from $r < r_D$, the "size" of D. Thus, the argument of j_{ℓ_R} is typically bounded by

$$X = \frac{N}{D}\frac{M}{B} k_p \, r_D \quad . \tag{27}$$

Since a Bessel function $j_\ell(x)$ has the property that its first maximum occurs at $x_{max} > \sqrt{\ell(\ell+1)}$, we conclude that the recoil angular momentum ℓ_R is restricted to values

$$\sqrt{\ell_R(\ell_R+1)} \lesssim \frac{N}{D}\frac{M}{B} k_p \, r_D \tag{28}$$

This limit is consistent with the one already discussed following eq. (6) from which we learned that x must be much smaller than unity before recoil can, ā priori, be neglected. Now we learn from eq. (28) what range of recoil angular momenta will be significant. For a few typical reactions we show x, and the maximum recoil angular momentum expected in Table 1.

4. Retrieval of Earlier Results

Here we briefly show how several special cases can be retrieved from the exact result given above. These can be used as necessary conditions of a computer programs correctness.

4.1. ZERO-RANGE

The zero-range approximation used in (d,p) calculations can be obtained by choosing

$$V(r)\ u_{L_1 J_1}(r) = D_0 \delta(r)\ \delta_{L_1 0} \quad . \tag{29}$$

Then

$$I_{nm}(L_1 L_2 L_p \ell_R \ell) \longrightarrow D_0\ \delta_{\ell_R 0}\ \delta_{\ell 0}\ \delta_{L_1 0} \quad . \tag{30}$$

The n and m sums now yield back the original functions so that we obtain

$$\mathcal{J}_{L_p L_D}(R) \longrightarrow \delta_{\mathcal{L} L_2}\ G_0\ F_{L_p}(k_p, \frac{A}{B} R)\ u_{L_2 J_2}(R) \tag{31}$$

Thus, $K_{L_p L_D}$ becomes the usual DWBA integral. The geometrical factor is denoted in this limit by G_0.

We note that for convenience in checking a computer program in this limit, it can also be achieved by taking $\frac{N}{D} \to 0$, $\frac{P}{D} \to 0$. Referring to the r integral, eq. (14), we see this has the same effect as written above, since in these limits $j_{\ell_R} \to \delta_{\ell_R 0}\delta(r)$, and $j_\ell \to \delta_{\ell 0}\ \delta(r)$, respectively.

4.2. BUTTLE-GOLDFARB

These authors treated the finite range in the problem but neglected recoil[2]). They also used Hankel function tails to represent the bound state wavefunctions, which we do not. We obtain the recoilless approximation by taking $N/D \to 0$ in eq. (14). This yields

$$j_{\ell_R} \longrightarrow \delta_{\ell_R 0}\, \delta(r) \quad \text{as} \quad \frac{N}{D} \to 0 \quad . \tag{32}$$

Referring to fig. 2, we see that as a consequence of $\ell_R = 0$, then $\ell \to L_1$, $\Lambda \to \mathcal{L}$ and $\ell_p \to L_p$. This is the finite range recoilless approximation in which the correct bound state wavefunction, u_{L_2} is still employed. If instead it is replaced by a Hankel function,

$$u_{L_2} \longrightarrow \mathcal{N}_{L_2}\, i^{-L_2}\, h^{(1)}_{L_2}\, (i\, K_2\, r) \tag{33}$$

then the nm sum of the two integrals in eq. (14) becomes

$$\begin{aligned} &\mathcal{N}_{L_2}\, i^{-L_2} \int j^*_{L_1}\, (i\, K_2\, \tfrac{P}{D}\, r)\; V(r)\; u_{L_1 J_1}(r)\; r^2\, dr \\ &\times \int F_{L_p}(k_p, \tfrac{A}{B} R)\; h^{(1)^*}\, (i\, K_2\, R)\; F_{L_D}(k_D, R)\; R^2\, dR \end{aligned} \tag{34}$$

which is recognized as the result of Buttle and Goldfarb.

4.3. RECOIL CORRECTIONS BY TAYLOR EXPANSION

The first order treatments discussed by Nagarajan[10]) and by Baltz and Kahana[11]) involve expansions in the mass ratio N/D. The first order approximation thus treats $\ell_R = 0$ and 1 approximately, that is to first order in N/D. They could be tested against the $\ell_R' = 0$ and 1 terms of our formulation which compute these contributions to all orders in N/D. From eq. (14) their approximation would be retrieved by replacing

$$j_{\ell_R}(\alpha_n \frac{N}{D}\frac{M}{B} r) \longrightarrow \begin{cases} 1 & , \ell_R = 0 \\ \alpha_n \frac{N}{D}\frac{M}{B} r & , \ell_R = 1 \\ 0 & , \ell_R \geqslant 2 \end{cases} . \tag{35}$$

The second order recoil correction[12]) can be obtained by keeping ℓ_R up to 2 and approximating the Bessel function (35) by its series up to $(N/D)^2$.

5. Fourier-Bessel Expansions

This expansion is especially useful for reaction calculations because it allows the representation of a function in a specified interval[13]). The appropriate interval is of course the one extending to the maximum distance at which the transfer reactions can occur with appreciable probability. This is typically 15 to 20 F. This expansion allows the representation of an arbitrary function F(r) in the interval

$$0 \leqslant r \leqslant R \tag{36}$$

by

$$F(r) = \sum_{n=1}^{\infty} a_n j_\ell(\alpha_n r) \tag{37}$$

where the α_n are determined by the condition

$$A\, j_\ell(\alpha_n R) + B \frac{d}{dR} j_\ell(\alpha_n R) = 0 \tag{38}$$

and

$$\frac{A}{B} = -\left(\frac{1}{F}\frac{dF}{dr}\right)_{r=R} . \tag{39}$$

We see that the series has the property that it has the same logarithmic derivative at the boundary as the function itself. In particular, if R is chosen to be a zero of F(r) then we choose B = 0, and every term in the series vanishes at the boundary R.

The order ℓ of the Bessel functions is arbitrary, but clearly since we want to use the addition theorem (12), if F(r) is associated with angular momentum L, we should choose $\ell = L$.

The coefficients in the expansion are given by

$$a_n = \frac{2}{R^3[j_\ell^2(\alpha_n R) - j_{\ell-1}(\alpha_n R) j_{\ell+1}(\alpha_n R)]} \int_0^R F(r)\, j_\ell(\alpha_n r)\, r^2\, dr \quad . \qquad (40)$$

For the purpose of exhibiting the accuracy of the method for a scattering function, we display in Fig. 3 a Coulomb function corresponding to

$$\ell = 85 \quad , \qquad k = 10\ F^{-1} \ , \qquad \eta = 21$$

(typical of O + Sn, E = 160 MeV). In this case we choose to represent the function up to its zero at

$$R = 15.408\ F$$

and choose therefore the boundary condition B = 0 in (38). Since a plot of its Fourier-Bessel representation with 10 or more terms would lie generally within a line width on the graph, we plot instead the percentage error corresponding to 10 terms. This is seen to be small except where the function itself is small, and there even 100% error is immaterial. Where the function is large, the error is seen to be less than several percent. A 15 term series represents the function at its maxima to an accuracy of better than 1/10%.

For a bound state, where the function does not have a zero in the asymptotic region, the logarithmic derivative of the representation is matched to the function in accord with (39). For a 2d function shown in fig. 4, R

is chosen to be 15 F. Again the percentage error corresponding to 10 terms is seen to be small except when the function itself is small. Even so, with 15 terms the error is less than several percent at its worst point, on the boundary R.

It is worth noting, that for a scattered function, the best choice for the boundary is a zero of the function, for then, by construction, the basis functions vanish at the same point.

APPENDIX A

We wish to generalize the formulation to include multinucleon transfer. For simplicity, we assume that the interaction causing the transfer depends only upon the separation between the center of mass of the nucleons being transferred and the residual core. Suppose a group of N nucleons is transferred.

We have to replace eq. (1) by the following parentage expression

$$\Phi_{J_D}^{M_D}(D) = \sum_{J_p L_1 S_1 J_1} B_{L_1 S_1 J_1}(P,D) \left[\Phi_{J_P}(P)\ \phi_{L_1 S_1 J_1}(\underset{\sim}{R}_{PN}, \underset{\sim}{\rho}_1 \cdots \underset{\sim}{\rho}_N\right]_{J_D}^{M_D} \tag{A1}$$

Where $\underset{\sim}{R}_{PN}$ represents the separation of the centers of mass of the N transferred nucleons and the residual core, and $\underset{\sim}{\rho}_1, \underset{\sim}{\rho}_2 \cdots$ represents the rest of the internal co-ordinates of the group. One separates the internal motion of the cluster and the motion of its center-of-mass by a transformation of the form

$$\phi_{L_1 S_1 J_1}(\underset{\sim}{R}_{PN}, \underset{\sim}{\rho}_1 \cdots \underset{\sim}{\rho}_N) = \sum_{\lambda_1 \Lambda_1} C_{\lambda_1 \Lambda_1}^{L_1} \tag{A2}$$

$$\times \left[\left\{\phi_{\Lambda_1}(\underset{\sim}{R}_{PN})\ \phi_{\lambda_1}(\underset{\sim}{\rho}_1 \cdots \underset{\sim}{\rho}_N)\right\} L_1, S_1\right]_{J_1}^{M_1}$$

A similar representation is sought for the other nucleus B;

$$\Phi_{J_B}^{M_B}(B) = \sum_{J_A L_2 S_2 J_2} B_{L_2 S_2 J_2}(A,B) \left[\Phi_{J_A}(A)\, \phi_{L_2 S_2 J_2}(\underset{\sim}{R}_{AN}, \underset{\sim}{\rho}_1 \cdots \underset{\sim}{\rho}_N)\right]_{J_B}^{M_B} \tag{A3}$$

$$\phi_{L_2S_2J_2}^{M_2}(\underset{\sim}{R}_{AN},\underset{\sim}{\rho}_1\cdots\underset{\sim}{\rho}_N) = \sum_{\lambda_2\Lambda_2} C_{\lambda_2\Lambda_2}^{L_2}\left[\left\{\phi_{\Lambda_2}(\underset{\sim}{R}_{AN})\ \phi_{\lambda_2}(\underset{\sim}{\rho}_1\cdots\underset{\sim}{\rho}_N)\right\}L_2,S_2\right]_{J_2}^{M_2} \tag{A4}$$

Again we have an equation like (5) with the ϕ_{LSJ} given by the above (A2) and (A4). Just as before the function of $\underset{\sim}{R}_{AN}$, namely ϕ_{Λ_2} can be expressed in terms of $\underset{\sim}{R}_D$ and $\underset{\sim}{R}_{PN}$ by use of a Fourier-Bessel series as in (11) and (13). Proceeding as before, we obtain, analogous to (14) the equation

$$t = (4\pi)^2 \sum_{L_P\mu_P}\sum_{L_D\mu_D} e^{i(\sigma_P+\sigma_D)}\, i^{L_D-L_P}\, Y^*_{L_D\mu_D}(\hat{k}_D)\, Y_{L_P\mu_P}(\hat{k}_P)$$

$$\times \sum_{\ell_P\ell_R} A_{\ell_P\ell_R L_D}\,(-)^{\ell_P+\ell_R} \sum_{\lambda\Lambda_1\Lambda_2} C_{\lambda\Lambda_1}^{L_1}\, C_{\lambda\Lambda_2}^{L_3} \sum_{\Lambda\ell} A_{\Lambda\ell\Lambda_2}(-)^{\Lambda}$$

$$\times \left\langle \left[Y_{\ell_P}(\hat{R})Y_{\ell_R}(\hat{r})\right]_{L_P\mu_P},\ \left[\left\{\left[Y_\Lambda(\hat{R})Y_\ell(\hat{r})\right]_{\Lambda_2}\ \phi_\lambda(\underset{\sim}{\rho}_1\cdots\underset{\sim}{\rho}_N)\right\}L_2,S_2\right]_{J_2M_2}\right.$$

$$\left.\left|Y_{L_D\mu_D}(\hat{R}),\ \left[\left\{Y_{\Lambda_1}(\hat{r})\ \phi_\lambda(\underset{\sim}{\rho}_1\cdots\underset{\sim}{\rho}_N)\right\}L_1,S_1\right]_{J_1M_1}\right.\right\rangle \tag{A5}$$

$$\times \sum_n\sum_m a_n^{L_P}\, b_m^{\Lambda_2} \int_0^{R_m} j_{\ell_R}\left(\alpha_n^{L_P}\frac{N}{D}\frac{M}{B}r\right) j_\ell\left(\beta_m^{\Lambda_2}\frac{P}{D}r\right) V(r)\,u_{L_1J_1}(r)\,r^2dr$$

$$\times \int_0^{R_m} j_{\ell_P}\left(a_n^{L_P}\frac{A}{B}R\right) j_\Lambda\left(\beta_m^{\Lambda_2}R\right) F_{L_D}(k_D,R)\,R^2dR$$

The angle bracket can be evaluated to yield

$$\langle \text{angle integral} \rangle = \delta_{S_1 S}\, \delta_{S_1 S_2} \sum_{\mathcal{L}\mu} \begin{pmatrix} L_P & L_D & \mathcal{L} \\ \mu_P & -\mu_D & \mu \end{pmatrix} \begin{pmatrix} J_1 & J_2 & \mathcal{L} \\ \mu_1 & -\mu_2 & \mu \end{pmatrix} \tag{A6}$$

$$\times\ (-)^{L_D+\mu_D+J_2+M_2}\ G(J_1 J_2 L_1 L_2 \Lambda_1 \Lambda_2 \lambda L_P L_D \ell \Lambda \ell_R \ell_P S \mathcal{L})$$

with

$$G \equiv \frac{\Omega_\lambda}{4\pi} (-)^{L_1+\Lambda_1+\Lambda_2-S-\lambda+J_1+L_D} \begin{pmatrix} \ell_P & \mathcal{L} & L_D \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_R & \ell & \Lambda_1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\times \begin{Bmatrix} L_2 & \mathcal{L} & L_1 \\ J_1 & S & J_2 \end{Bmatrix} \begin{Bmatrix} \Lambda_2 & \mathcal{L} & \Lambda_1 \\ \ell_R & \ell & \Lambda \end{Bmatrix} \begin{Bmatrix} \ell_P & \mathcal{L} & L_D \\ \Lambda & \ell_P & \ell_R \end{Bmatrix} \begin{Bmatrix} \Lambda_1 & \mathcal{L} & \Lambda_2 \\ L_2 & \lambda & L_1 \end{Bmatrix} \tag{A7}$$

$$\times\ (\hat{\ell}_P \hat{\Lambda} \hat{\ell} \hat{\ell}_R \hat{L}_1 \hat{J}_1 \hat{L}_2 \hat{J}_2 \hat{L}_P \hat{L}_D \hat{\Lambda}_1 \hat{\Lambda}_2)^{1/2}$$

and

$$\Omega_\lambda(B,D) = \int d\rho_1 \cdots d\rho_N\ \phi^*_{B\lambda}\ (\underset{\sim}{\rho}_1 \cdots \underset{\sim}{\rho}_N)\ \phi_{D\lambda}\ (\underset{\sim}{\rho}_1 \cdots \underset{\sim}{\rho}_N) \tag{A8}$$

is the overlap integral for the internal state of the group of N as they appear in the composite nuclei B and D, respectively.

One can obtain the earlier results (14), (15), and (16) for transfer of a cluster from the above general result by setting

$$\lambda \to 0 \quad , \qquad \Omega_\lambda \to 1 \quad , \qquad C^L_{\lambda\Lambda} \to 1 \quad . \tag{A9}$$

We shall illustrate how the representations (A1) and (A2) may be typically obtained by considering a two-nucleon transfer reaction. It is usual to express shell model wavefunctions on a jj coupled basis. The nuclear state of D for example can be expressed as

$$\Phi_{J_D} = \sum B_{jj'J}(P,D) \left[\Phi_{J_P}, \phi_{jj'J}\right]_{J_D} \tag{A10}$$

where $B_{jj'J}$ will be, in general, a product of a two particle coefficient of fractional parentage and a configuration mixing coefficient. The jj coupled two particle symmetrized state $\phi_{jj'J}$ is of the form

$$\phi_{jj'J} = \frac{1}{\sqrt{2(1+\Delta_{jj'})}} \left[\phi_{jj'J}(1,2) + (-1)^T \phi_{jj'J}(2,1)\right] . \tag{A11}$$

Where we have used the shorthand notation

$$\Delta_{jj'} = \delta_{nn'} \delta_{\ell\ell'} \delta_{jj'} .$$

For like nucleons, we must choose T = 1.

We obtain the representation (A1) by use of the LS-jj transformation coefficient,

$$\phi_{jj'J} = \frac{1}{\sqrt{2(1+\Delta_{jj'}}} \sum_{LS} \begin{bmatrix} \ell & 1/2 & j \\ \ell' & 1/2 & j' \\ L & S & J \end{bmatrix} \left[\phi_{\ell\ell'LSJ}(1,2) + (-)^T \phi_{\ell\ell'LSJ}(2,1)\right]. \tag{A12}$$

Combining eqs. (A12) and (A10), we obtain

$$B_{LSJ} = \sum_{jj'} \begin{bmatrix} \ell & 1/2 & j \\ \ell' & 1/2 & j' \\ L & S & J \end{bmatrix} B_{jj'J} \tag{A13}$$

If now we use the Moshinsky transformation to separate the center-of-mass and relative motion, we obtain

$$\phi_{jj'J} = \frac{1}{\sqrt{2(1+\Delta_{jj'})}} \sum_{LS} \begin{bmatrix} \ell & 1/2 & j \\ \ell' & 1/2 & j' \\ L & S & J \end{bmatrix} \langle \nu\lambda N\Lambda;L|n\ell n'\ell';L \rangle$$

$$\times \quad (1-(-)^{\lambda+S+T}) \,|\, [\phi_{\nu\lambda}(\vec{r})\phi_{N\Lambda}(\vec{R})],\ S;J \rangle \quad . \tag{A14}$$

Comparing this with (A4), we obtain

$$C^{L}_{\lambda\Lambda} = \frac{(-1)^{\lambda+\Lambda-L}}{\sqrt{2(1+\Delta_{jj'})}} \left(1 - (-1)^{\lambda+S+T}\right) \langle \nu\lambda N\Lambda;L|n\ell n'\ell';L \rangle \tag{A15}$$

APPENDIX B

The addition theorem (Eq. 12) is easily obtained by using the relation

$$j_L(qR)\, Y^*_{LM}(\hat{R}) = \frac{i^{-L}}{4\pi} \int d\hat{q}\; e^{i\vec{q}\cdot\vec{R}}\; Y^*_{LM}(q) \tag{B1}$$

where the integral is over the direction of the vector $\vec{q}$. If

$$\vec{R} = \alpha\vec{r}_1 + \beta\vec{r}_2 \tag{B2}$$

one obtains (see Eqn. 12)

$$j_L(qR)\, Y^*_{LM}(\hat{R}) = \sum_{L_1L_2} A_{L_1L_2L}(-)^{L_1}\, j_{L_1}(\alpha q r_1) \times j_{L_2}(\beta q r_2) \left[Y_{L_1}(\hat{r}_1)\, Y_{L_2}(\hat{r}_2)\right]^*_{LM} \tag{B3}$$

where

$$A_{L_1L_2L} = \left[\frac{4\pi\,(2L_1+1)\,(2L_2+1)}{(2L+1)}\right]^{\frac{1}{2}} i^{L_2-L_1-L}\, \langle L_1 0 L_2 0 | L 0 \rangle\,. \tag{B4}$$

For completeness we add that the addition theorem given by Buttle and Goldfarb for the Hankel function is

$$h^{(1)}_L(iqR)\, Y_{LM}(\hat{R}) = \sum_{L_1L_2} A_{L_1L_2L}\,(-)^{L_1}\, h^{(1)}_{L_1}(iq\alpha r_1)\, j_{L_2}(iq\beta r_2) \left[Y_{L_1}(\hat{r}_1)\, Y_{L_2}(\hat{r}_2)\right]_{LM}. \tag{B5}$$

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Table I. For a few typical reactions the parameter x which gives a measure of the importance of recoil (Eq. 8) is listed. The largest recoil angular momentum ℓ_R is given by $\sqrt{\ell_R(\ell_R+1)} = x$ which for such a rough estimate we solve as $\ell_R \simeq x$.

Reaction	E_{Lab} (MeV)	$x \simeq \ell_R$
$^{13}C(^{12}C,^{13}C)^{12}C$	87	2
$^{208}Pb(^{16}O,^{15}N)^{209}Bi$	104	2
$^{120}Sn(^{18}O,^{16}O)^{122}Sn$	100	4
$^{12}C(^{20}Ne,^{16}O)^{16}O$	78	5

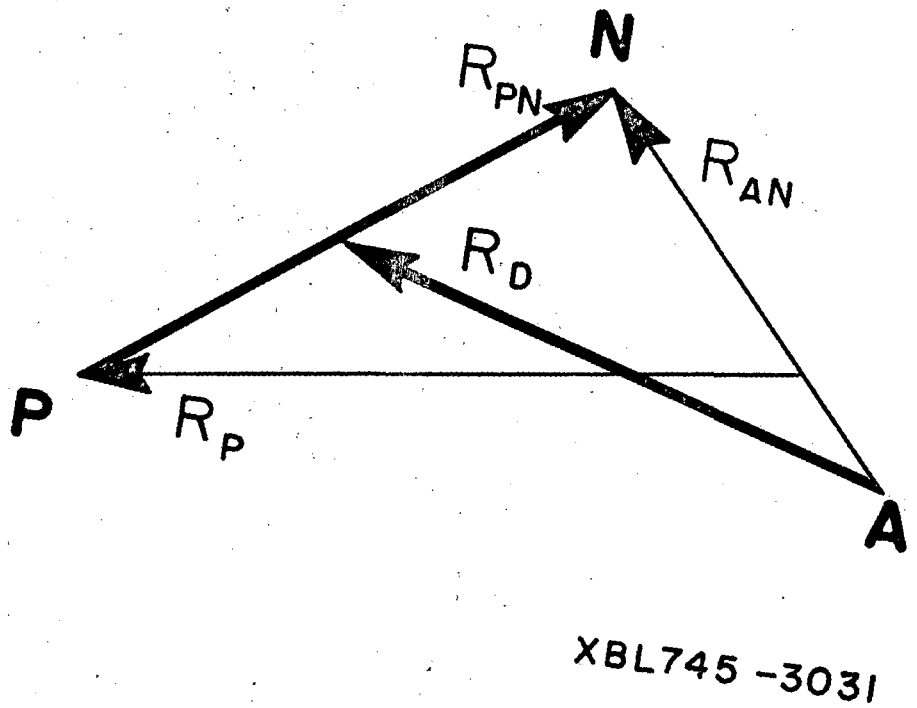
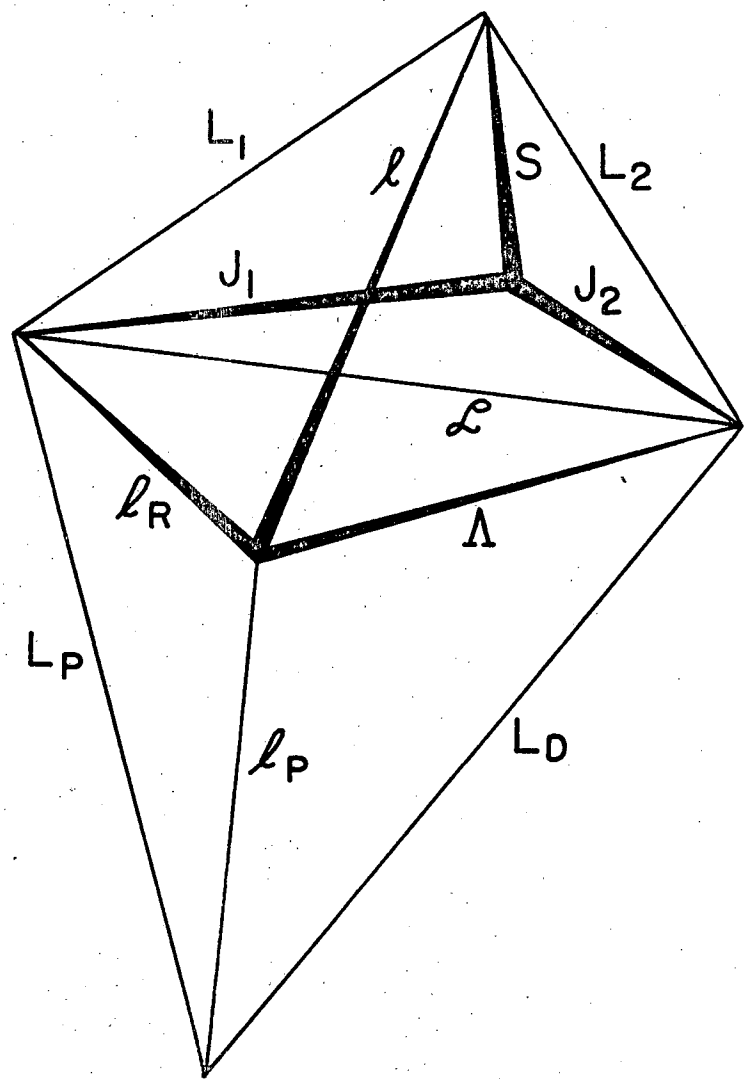


Fig. 1

The coordinates for the reaction A + (N+P) → (A+N) + P, with D = N+P, B = A+N. We choose to represent all vectors in terms of the two shown by heavy lines.



XBL745-3030

Fig. 2.

The angular momentum coupling is displayed by this figure. The relative angular momenta in the entrance and exit channels, D and P are labelled L_D and L_P. The angular momentum of the center-of-mass of N in D = P+N is $\underset{\sim}{J}_1 = \underset{\sim}{L}_1 + \underset{\sim}{S}$ and it is transferred to $\underset{\sim}{J}_2 = \underset{\sim}{L}_2 + \underset{\sim}{S}$ in B = A+N. The total angular momentum transfer is $\mathcal{L}$ which is the sum of the recoil angular momentum ℓ_R and Λ.

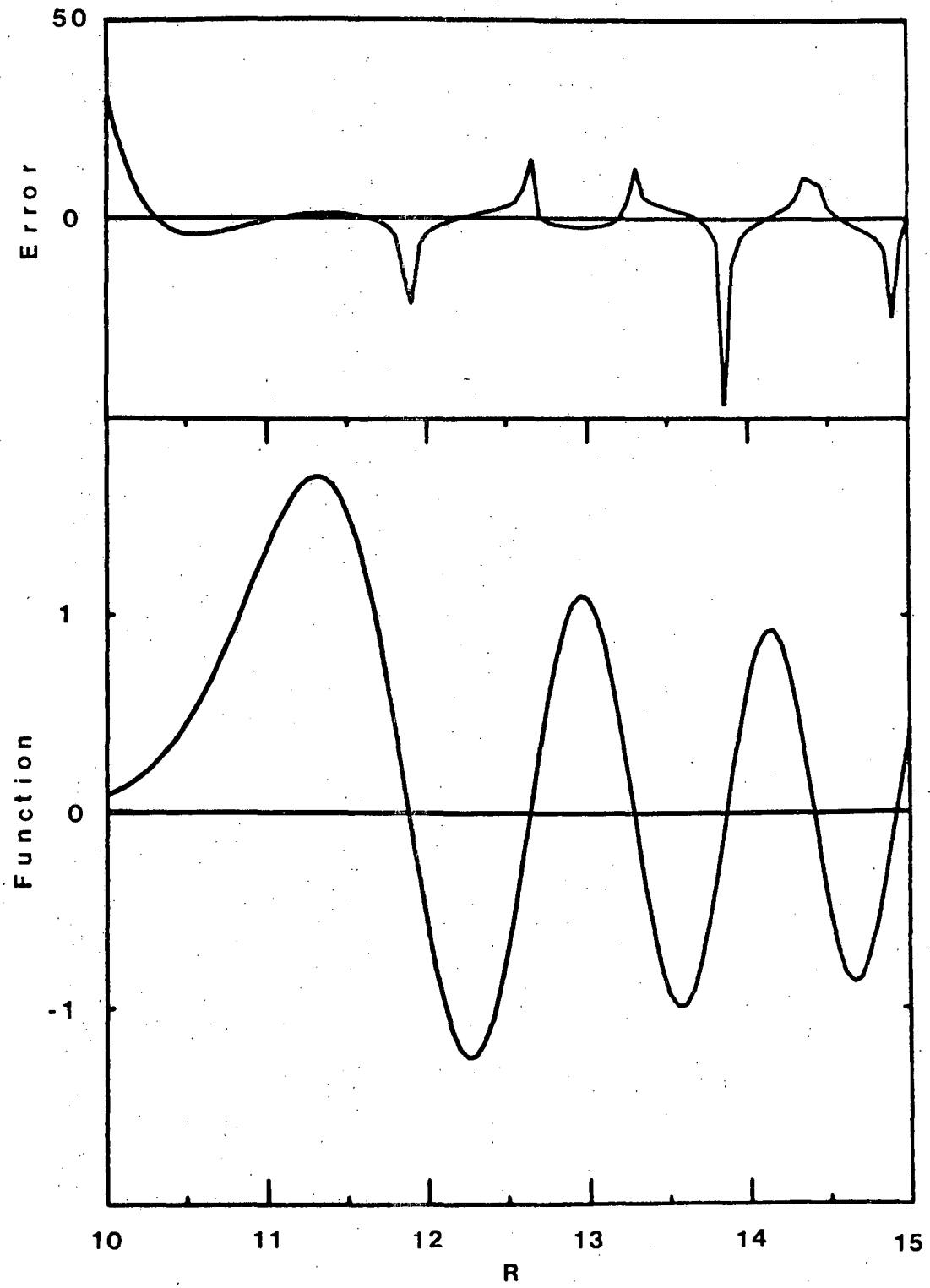


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Fig. 3

A Coulomb function for L = 85, k = 10 F^{-1}, η = 21 is shown together with the error with which it is represented by a 10 term Fourier-Bessel series. Except where the function itself is small, the error is small.

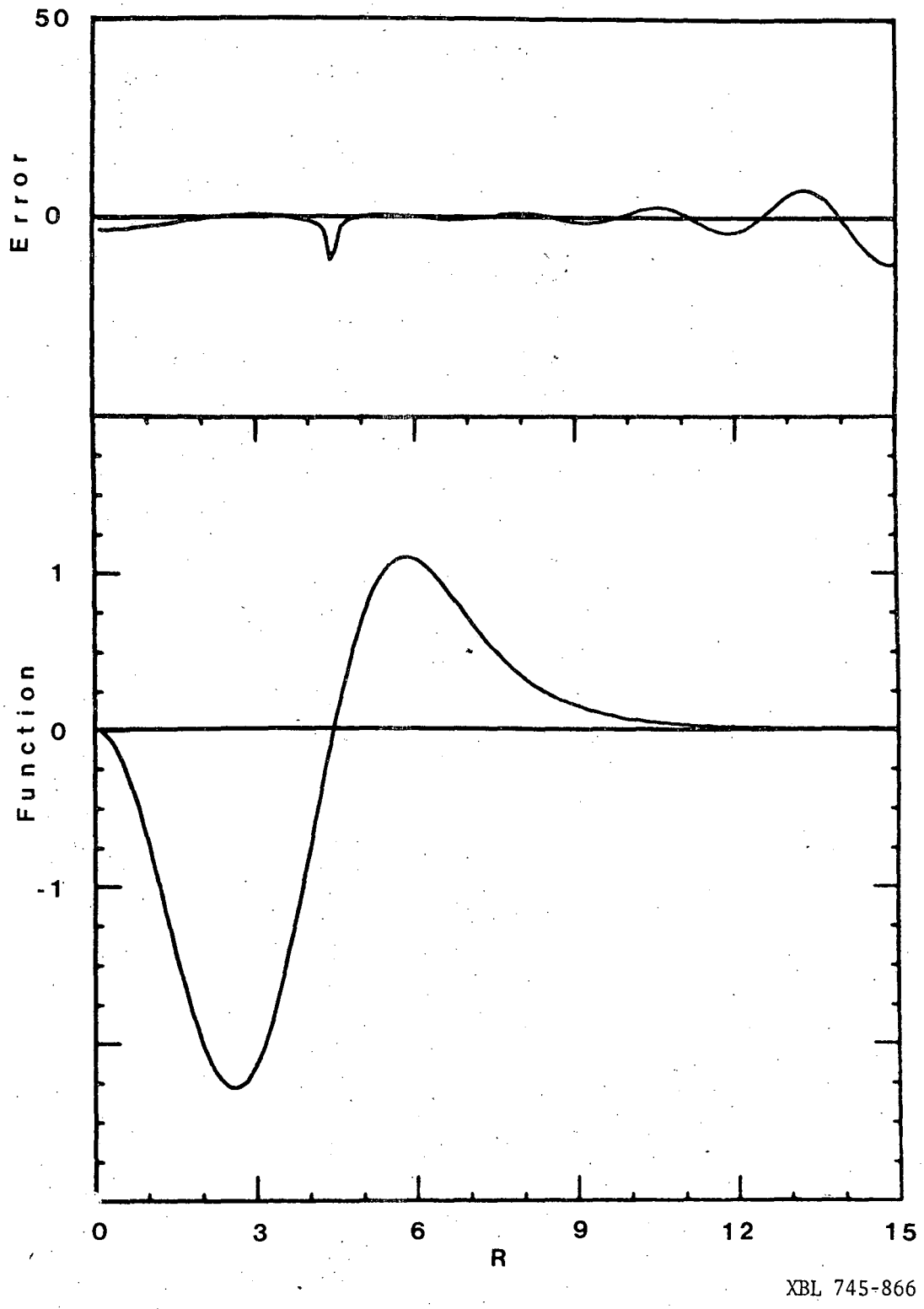

Fig. 4

A bound 2d state is shown together with the error with which it is represented by a 10 term Fourier-Bessel series.

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